On the magnetisation of a charged Bose gas in the canonical ensemble

H. D. Cornean

Institute of Mathematics of the Romanian Academy , P.O. Box 1-764, 70700 Bucharest, Romania;
e-mail cornean@barutu.fizica.unibuc.ro

June 24, 1999

Abstract

Consider a charged Bose gas without self-interactions, confined in a three dimensional cube of side length $L \geq 1$, submitted to a constant magnetic field $B \neq 0$ and in canonical conditions (the bulk density of particles ρ and the temperature T are fixed). If the canonical magnetisation is defined as the partial derivative after B of the reduced free energy, it is proven that it admits thermodynamical limit for all strictly positive ρ, T and B. It is also proven that the canonical magnetisation and the grand-canonical magnetisation (written in canonical conditions) are equal up to the surface order corrections.

1 Introduction

Much work has been done on the thermodynamical behaviour of large systems composed from independent quantum particles in the presence of external magnetic fields. As is well known, the fundamental problem consists in proving the existence of thermodynamical limit for the potentials and the state equations written at finite volume. In the particular case of the magnetisation (defined as the partial derivative after the magnetic field of the reduced free energy), one has to prove that the derivative commutes with the thermodynamical limit. Although the quantum canonical ensemble is (from the physical point of view) the most important one, most of the previous works were carried out either using the Maxwell-Boltzmann statistics or in the framework of quantum grand-canonical ensemble, because in those settings the properties of the relevant quantities can be obtained easier by studying the integral kernel of the Gibbs semigroup associated to the one particle problem. Moreover, one is able sometimes to go beyond the bulk terms and to investigate finite size effects.

Take for example the grand-canonical pressure of a quantum gas in a constant magnetic field. The rigorous proof of its thermodynamical limit goes back at least to Angelescu and Corciovei [A, A-C]; its surface correction (in the regime in which the fugacity is less than one) was obtained by Kunz [K]. As for the Maxwell-Boltzmann magnetisation, nice results were obtained by Macris, Martin and Pulé [M-M-P]; they wrote down even the corner corrections. Notice that in these papers the domain Λ was allowed to be more general, typically convex with piecewise smooth boundary.

Another result concerning the thermodynamical limit and the surface corrections for the magnetisation and the susceptibility of a Fermi gas at zero magnetic field was obtained by Angelescu, Bundaru and Nenciu [A-B-N 2]. Because this paper motivated our work, we are going to give some more details about it.

First of all, as in our setting, the domain was a rectangular parallelepiped and the magnetic field oriented after the third direction. They defined the grand-canonical magnetisation $m_\Lambda(\beta, z)$ (susceptibility $\chi_\Lambda(\beta, z)$) as the first (second) derivative after the magnetic field of the grand-canonical pressure at $B = 0$, for all $z \in \mathbf{C} \setminus (-\infty, -1]$. Their main result can be roughly stated as follows:

i. $m_\Lambda(\beta, z) = 0, \forall z \in \mathbf{C} \setminus (-\infty, -1]$;

ii. There exists $\chi_\infty(\beta, z)$ analytic in $\mathbf{C} \setminus (-\infty, -1]$ such that for any compact $K \subset \mathbf{C} \setminus (-\infty, -1]$ one has:

$$\lim_{\Lambda \to \infty} \sup_{z \in K} |\chi_\Lambda(\beta, z) - \chi_\infty(\beta, z)| = 0.$$

More than that, they gave even the surface correction for the susceptibility and proved that this expansion is uniform on compacts. Because the relation between the fugacity and the grand-canonical density of Fermi particles can be always inverted, they were able to express the grand-canonical susceptibility in terms of the canonical parameters ρ and β. Let us stress that $B = 0$ and Λ a rectangular parallelepiped were crucial ingredients in [A-B-N 2], the uniform convergence on compacts being obtained via a substantial use of the closed formula of the integral kernel of the Gibbs semigroup associated to the Dirichlet Laplacian.

In our paper, we are studying the "true" canonical problem for a Bose gas at nonzero magnetic field $B_0 > 0$ (in order to avoid the Bose condensation). Using a standard procedure (see [K-U-Z, H]) of expressing the canonical partition function in terms of the grand-canonical pressure (see (2.27)), we are able to transform the uniform convergence on compacts of the grand-canonical magnetisation (see Lemma 2.1) into a pointwise canonical convergence (β, ρ fixed and $L \to \infty$), given in Theorem 2.2. Moreover, we obtain that the canonical magnetisation m_L (see (2.29)) and the grand-canonical magnetisation written in canonical conditions (see (2.30)) are equal up to the surface order corrections.

2 Preliminaries and the results

Let $\Lambda = \left\{\mathbf{x} \in \mathbf{R}^3 | -\frac{L}{2} < x_j < \frac{L}{2},\, j \in \{1,2,3\}\right\}$, $L > 0$, be a cube with its side length equal to L. Then the "one particle" Hilbert space is $\mathcal{H}_{1,L} = \mathbf{L}^2(\Lambda)$; denote with $\mathcal{H}_{n,L}$ the proper subspace of $\otimes_{j=1}^{n}\mathcal{H}_{1,L} \cong \mathbf{L}^2(\Lambda^n)$ which contains all totally symmetric functions. Denote with $\mathcal{H}_{0,L} = \mathbf{C}$ the space with no particles; then the Fock space is defined as $\mathcal{F}_L = \bigoplus_{n\geq 0}\mathcal{H}_{n,L}$. In this space one can define the "number of particles" operator N_L as the unique self-adjoint extension of the multiplication with n on each $\mathcal{H}_{n,L}$.

Suppose now that the particles (each having an electric charge e) are submitted to a constant magnetic field $\mathbf{B} = B\mathbf{e_3}$, corresponding to a magnetic vector potential $B\mathbf{a} = \frac{B}{2}\mathbf{e_3} \wedge \mathbf{x}$. If c stands for the speed of light, define $\omega := (e/c)B$. Then the "one particle" Hamiltonian (denoted with $H_{1,L}(\omega)$) will be the Friederichs extension of the positive and symmetric operator $\frac{1}{2}\left(-\,\mathrm{i}\nabla - \omega\,\mathbf{a}\right)^2$ defined on $C_0^\infty(\Lambda)$.

Due to the regularity of Λ, $H_{1,L}(\omega)$ is essentially self-adjoint on

$$D = \left\{f \in C^2(\Lambda) \cap C^1(\overline{\Lambda}),\ f|_{\partial\Lambda} = 0,\ \Delta f \in \mathbf{L}^2(\Lambda)\right\}.$$

The Hamiltonian which corresponds to n particles reads as:

$$H_{n,L}(\omega) = \underbrace{H_{1,L}(\omega) \otimes \ldots \otimes I + \ldots + I \otimes \ldots \otimes H_{1,L}(\omega)}_{\text{"n" terms}}. \tag{2.1}$$

In the second quantisation, the Hamiltonian $H_L(\omega)$ which describes the system is defined as the unique self-adjoint operator in $\mathcal{F}_L$ whose restrictions to $\mathcal{H}_{n,L}$ coincide with $H_{n,L}(\omega)$.

If $T > 0$ stands for the temperature and $\mu \in \mathbf{R}$ for the chemical potential, then define $\beta = \frac{1}{k_B T} > 0$ and $z = \exp(\beta\,\mu)$ (the fugacity), where k_B is the Boltzmann constant. When working in the canonical ensemble, one considers that the bulk density of particles ρ is constant, therefore the number of particles is defined as $N(L) := \rho L^3$.

It is known (see [R-S 4]) that $H_{1,L}(\omega)$ is positive, has compact resolvent and therefore it has purely discrete spectrum with accumulation point at infinity. Moreover, from the min-max "principle" it follows:

$$\inf\sigma(H_{1,L}(\omega)) \geq \inf\sigma(H_{1,\infty}(\omega)) = \frac{\omega}{2}. \tag{2.2}$$

It is also known that the Gibbs semigroup $W_L(\beta,\omega) := \exp\left(-\beta H_{1,L}(\omega)\right)$ is trace class and admits an integral kernel $G_{\omega,L}(\mathbf{x},\mathbf{x}';\beta)$, which is continuous in both its "spatial" coordinates. The diamagnetic inequality at finite volume (see [B-H-L]) reads as:

$$|G_{\omega,L}(\mathbf{x},\mathbf{x}';\beta)| \leq G_{0,L}(\mathbf{x},\mathbf{x}';\beta) \leq \frac{1}{(2\pi\beta)^{3/2}}\exp\left(-\frac{|\mathbf{x}-\mathbf{x}'|^2}{2\beta}\right). \tag{2.3}$$

If $\mathcal{I}_1(\mathbf{L}^2(\Lambda))$ denotes the Banach space of the trace class operators, it follows that:

$$||W_L(\beta,\omega)||_{\mathcal{I}_1} = \mathrm{tr}\ W_L(\beta,\omega) \leq \frac{L^3}{(2\pi\beta)^{3/2}}. \tag{2.4}$$

Denote with $\{E_j(\omega)\}_{j\in\mathbf{N}}$ the set of the eigenvalues of $H_{1,L}(\omega)$. The canonical partition function for our system is:

$$Z_L(\beta,\rho,\omega) = \mathrm{tr}_{\mathcal{H}_{N(L),L}} \exp\left(-\beta H_{N(L),L}(\omega)\right). \tag{2.5}$$

If $\mu < 0$, the grand-canonical partition function reads as:

$$\Xi_L(\beta,z,\omega) = \mathrm{tr}_{\mathcal{F}_L} \exp\left[-\beta(H_L(\omega) - \mu N_L)\right] = \prod_{j=0}^{\infty}[1 - z\exp\left(-\beta E_j(\omega)\right)]^{-1}. \tag{2.6}$$

The link between them is given by:

$$\Xi_L(\beta,z,\omega) = \sum_{n=0}^{\infty} z^n \mathrm{tr}_{\mathcal{H}_{n,L}} \exp\left[-\beta H_{n,\Lambda}\right]. \tag{2.7}$$

Throughout the entire paper, by $\log z$ we will understand the logarithm function restricted to $\mathbf{C}\setminus(-\infty,0]$.

Let $\mathcal{C}$ be a contour which surrounds the origin, does not intersect the cut $[1,\infty)$ but contains the spectrum of the trace class operator zW_L, where $z \in \mathbf{C}\setminus[\exp(\beta\omega/2),\infty)$. Let $q(\xi) = \frac{1}{\xi}\log(1-\xi)$ be an analytic function in the interior of $\mathcal{C}$. Define the following bounded operator:

$$q(zW_L) = \frac{1}{2\pi\imath}\int_{\mathcal{C}} d\xi\, q(\xi)(\xi - zW_L)^{-1}. \tag{2.8}$$

It is easy to see that $\log(1 - zW_L) = zW_L \cdot q(zW_L)$ and using (2.6) one obtains:

$$\log \Xi_L(\beta,z,\omega) = -\mathrm{tr}\left(zW_L \cdot q(zW_L)\right). \tag{2.9}$$

Using the above expression, one can easily show that the grand-canonical potential (seen as a function of z) is analytic in $\mathbf{C}\setminus[\exp(\beta\omega/2),\infty)$. When $|z| < 1$, (2.9) becomes:

$$\log \Xi_L(\beta,z,\omega) = \sum_{n=1}^{\infty}\frac{z^n}{n}\mathrm{tr}\,(W_L^n) = \sum_{n=1}^{\infty}\frac{z^n}{n}\left(\int_{\Lambda} d\mathbf{x}\, G_{\omega,L}(\mathbf{x},\mathbf{x};n\beta)\right). \tag{2.10}$$

The grand-canonical pressure and density are defined as:

$$P_L(\beta,z,\omega) := \frac{1}{\beta L^3}\log \Xi_L(\beta,z,\omega) = -\frac{1}{\beta L^3}\sum_j \log\left(1 - ze^{-\beta E_j(\omega)}\right), \tag{2.11}$$

and

$$\rho_L(\beta,z,\omega) := \beta z\frac{\partial P_L}{\partial z}(\beta,z,\omega). \tag{2.12}$$

Let us remark that $\rho_L(\beta,x,\omega)$ is an increasing function if $0 < x < \exp(\beta\omega/2)$:

$$\frac{\partial \rho_L}{\partial x}(\beta,x,\omega) = 1/L^3 \mathrm{tr}[(1 - xW_L)^{-2}W_L] > 0. \tag{2.13}$$

The proof of the thermodynamical limit for these two quantities goes back at least to Angelescu and Corciovei [A, A-C]. Because this result plays a very important role in our work, we'll reproduce it here. In order to do that, let's define (we are treating only the case when $\omega > 0$):

$$P_\infty(\beta,z,\omega) := \omega\frac{1}{(2\pi\beta)^{3/2}}\sum_{k=0}^{\infty} g_{3/2}\left(ze^{-(k+1/2)\omega\beta}\right), \tag{2.14}$$

and

$$\rho_\infty(\beta, z, \omega) := \beta z \frac{\partial P_\infty}{\partial z}(\beta, z, \omega) = \beta\omega \frac{1}{(2\pi\beta)^{3/2}} \sum_{k=0}^{\infty} g_{1/2}\left(z e^{-(k+1/2)\omega\beta}\right), \tag{2.15}$$

where $g_\sigma(\zeta)$ are the usual Bose functions:

$$g_\sigma(\zeta) = \frac{\zeta}{\Gamma(\sigma)} \int_0^\infty dt\, \frac{t^{\sigma-1}e^{-t}}{1-\zeta e^{-t}}, \tag{2.16}$$

analytic in $\mathbf{C} \setminus [1, \infty)$ and if $|\zeta| < 1$, they are given by the following expansion:

$$g_\sigma(\zeta) = \sum_{n=1}^{\infty} \frac{\zeta^n}{n^\sigma}.$$

Then the following result is true (see [A, A-C]):

Theorem 2.1 *Let $K \subset \mathbf{C} \setminus [\exp(\beta\omega/2), \infty)$ be a compact set. Then the grand-canonical pressure and density admit thermodynamical limit i.e.:*

$$\lim_{L\to\infty} \sup_{z\in K} |P_L(\beta, z, \omega) - P_\infty(\beta, z, \omega)| = 0, \tag{2.17}$$

and

$$\lim_{L\to\infty} \sup_{z\in K} |\rho_L(\beta, z, \omega) - \rho_\infty(\beta, z, \omega)| = 0. \tag{2.18}$$

First of all, because P_L and ρ_L are analytic functions, then via Cauchy integral formula it follows that all their complex derivatives admit a limit which is uniform on compacts. In particular:

$$\lim_{L\to\infty} \sup_{z\in K} \left| \frac{\partial \rho_L}{\partial z}(\beta, z, \omega) - \frac{\partial \rho_\infty}{\partial z}(\beta, z, \omega) \right| = 0. \tag{2.19}$$

It can be seen from (2.15) that $\lim_{x\nearrow e^{\beta\omega/2}} \rho_\infty(\beta, x, \omega) = \infty$, which means that the Bose condensation is absent when a nonzero magnetic field is present. A very important consequence of the theorem is that the relation between the fugacity and density can be inverted for all temperatures; namely, if $0 < x_\infty(\beta, \rho, \omega) < e^{\beta\omega/2}$ is the unique real and positive solution of the equation $\rho_\infty(\beta, x, \omega) = \rho$ and if $x_L(\beta, \rho, \omega)$ is the unique real and positive solution which solves $\rho_L(\beta, x, \omega) = \rho$, then $\lim_{L\to\infty} x_L = x_\infty$.

Let us perform the Legendre transform at finite volume:

$$\tilde{f}_L(\beta, \rho, \omega) := -P_L(\beta, x_L(\beta, \rho, \omega), \omega) + \frac{\rho}{\beta} \log x_L(\beta, \rho, \omega). \tag{2.20}$$

A straightforward result is that $\tilde{f}_L$ has the following limit:

$$f_\infty(\beta, \rho, \omega) := -P_\infty(\beta, x_\infty(\beta, \rho, \omega), \omega) + \frac{\rho}{\beta} \log x_\infty(\beta, \rho, \omega). \tag{2.21}$$

Let us denote with $\frac{\partial W_L}{\partial \omega}(\beta, \omega_0)$ the following integral which converges in $B(\mathbf{L}^2(\Lambda))$ (see [A-B-N 1]):

$$-\int_0^\beta d\tau\, W_L(\beta - \tau, \omega_0)\, [\mathbf{a} \cdot (\mathbf{p} - \omega_0 \mathbf{a})] W_L(\tau, \omega_0). \tag{2.22}$$

A particular case of the problem treated in [A-B-N 1] is that $\frac{\partial W_L}{\partial \omega}(\beta, \omega_0)$ is even trace class and moreover, for $\delta\omega$ sufficiently small one has:

$$||W_L(\beta,\omega_0+\delta\omega)-W_L(\beta,\omega_0)-\delta\omega\frac{\partial W_L}{\partial\omega}||_{\mathcal{I}_1}=\mathcal{O}(\delta\omega^2). \tag{2.23}$$

Using for $P_L(\beta,z,\omega_0)$ the following expression:

$$-\frac{1}{\beta L^3}\,\mathrm{tr}[\log(1-zW_L(\beta,\omega_0))],$$

then the estimate (2.23) justifies the definition of the grand-canonical magnetisation:

$$\Gamma_L(\beta,z,\omega_0):=-\frac{e}{c}\,\frac{\partial P_L}{\partial\omega}(\beta,z,\omega_0)=-\frac{e\;z}{c\beta L^3}\;\mathrm{tr}\left[(1-zW_L(\beta,\omega_0))^{-1}\frac{\partial W_L}{\partial\omega}\right]. \tag{2.24}$$

From its definition, one can easily see that Γ_L has the same domain of analyticity in z. Let us define now the natural candidate for its thermodynamical limit:

$$\Gamma_\infty(\beta,z,\omega_0):=-\frac{e}{c}\;\frac{\partial P_\infty}{\partial\omega}(\beta,z,\omega_0). \tag{2.25}$$

Our main technical result is contained in the following lemma:

Lemma 2.1 *Let $K\subset\mathbf{C}\setminus[\exp(\beta\omega/2),\infty)$ be a compact set. Then the grand-canonical magnetisation admits thermodynamical limit i.e.:*

$$\lim_{L\to\infty}\sup_{z\in K}|\Gamma_L(\beta,z,\omega)-\Gamma_\infty(\beta,z,\omega)|=0. \tag{2.26}$$

Let us go back to the canonical ensemble. From (2.7), (2.11) and (2.5), one can write down an useful representation of the canonical partition function:

$$Z_L(\beta,\rho,\omega)=\frac{1}{2\pi\imath}\int_{C_1}d\xi\;\frac{1}{\xi}\left[\frac{\exp\left(\frac{\beta}{\rho}P_L(\beta,\xi,\omega)\right)}{\xi}\right]^{N(L)}, \tag{2.27}$$

where C_1 is a contour which surrounds the origin and avoids the cut. The reduced free energy reads as:

$$f_L(\beta,\rho,\omega):=-\frac{1}{\beta L^3}\log Z_L(\beta,\rho,\omega). \tag{2.28}$$

The canonical magnetisation is defined as follows:

$$m_L(\beta,\rho,\omega):=\frac{e}{c}\frac{\partial f_L}{\partial\omega}(\beta,\rho,\omega). \tag{2.29}$$

We expect that m_L should be close to the following quantity:

$$\tilde m_L(\beta,\rho,\omega):=\frac{e}{c}\frac{\partial\tilde f_L}{\partial\omega}(\beta,\rho,\omega)=\Gamma_L(\beta,x_L(\beta,\rho,\omega),\omega), \tag{2.30}$$

which converges to $\Gamma_\infty(\beta,x_\infty(\beta,\rho,\omega),\omega)$.

We are able now to give our main result:

Theorem 2.2 *Fix $0<\delta<1/2$. For all strictly positive temperatures, bulk densities and magnetic fields, the canonical magnetisation $m_L(\beta,\rho,\omega)$ admits thermodynamical limit. Moreover, there exist two positive constants $C_\delta(\beta,\rho,\omega)$ and $L_\delta(\beta,\rho,\omega)$ such that for all $L\geq L_\delta$ one has:*

$$|m_L-\tilde m_L|\leq C_\delta\;L^{-3/2+\delta}. \tag{2.31}$$

Remark. It is clear that $\tilde m_L$ is a much more convenient quantity. If (at least for dilute gases) one would be able to write down an expansion for $\tilde m_L$ which takes into account the surface corrections:

$$\tilde m_L=m_\infty+\frac{1}{L}m_S+o(1/L), \tag{2.32}$$

then the estimate (2.31) would imply that the same expansion is also true for m_L.

3 The proof of Theorem 2.2

At this point, we'll consider that Lemma 2.1 is true and we'll give its proof in the next section. In order to simplify the notations, we'll drop the dependence on β, ρ and ω but we'll reintroduce it when needed.

The main idea of the proof consists in isolating the principal part of the integral from (2.27). Although this procedure is far from being new (in the physical literature it is known as the Darwin-Fowler method; see [K-U-Z, H] and references therein), we decided to give a rather detailed proof in order to have a clearer image of the remainder from (2.31). First of all, let's choose the contour C_1 as follows:

$$C_1 := \{x_L e^{\imath\phi},\ \phi \in [-\pi, \pi]\}. \tag{3.1}$$

Using (2.20), the formula (2.27) can be rewritten as:

$$\begin{aligned} Z_L &= \exp(-\beta \tilde{f}_L L^3) \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi\ \exp\left[\frac{N(L)\beta(P_L(x_L e^{\imath\phi}) - P_L(x_L))}{\rho}\right] e^{-\imath N(L)\phi} = \\ &= \exp(-\beta \tilde{f}_L L^3) N(L)^{-1/2} \mathcal{A}_L, \end{aligned} \tag{3.2}$$

where $\mathcal{A}_L(\beta, \rho, \omega)$ is given by:

$$\mathcal{A}_L = \frac{\sqrt{N(L)}}{2\pi} \int_{-\pi}^{\pi} d\phi\ e^{\left[N(L)\beta/\rho(\Re P_L(x_L e^{\imath\phi}) - P_L(x_L))\right]} e^{\imath N(L)\left[\beta \Im P_L(x_L e^{\imath\phi})/\rho - \phi\right]}. \tag{3.3}$$

Denote with $\tilde{p}_\infty(\phi)$ the function given by $\Re P_\infty(x_\infty e^{\imath\phi})$, and with $\tilde{p_L}(\phi)$ the function given by $\Re P_L(x_L e^{\imath\phi})$. We'll show that $\frac{d^2\tilde{p}_\infty}{d\phi^2}(0) < 0$ and moreover, the following limit is true:

$$\lim_{L\to\infty} \mathcal{A}_L := \mathcal{A}_\infty = \sqrt{-\frac{\rho}{2\pi\beta \frac{d^2\tilde{p}_\infty}{d\phi^2}(0)}}. \tag{3.4}$$

First of all, let's remark that $\overline{P_L(\xi)} = P_L(\overline{\xi})$, where the over line means complex conjugation. Let $0 < \epsilon < \frac{e^{\beta\omega/2} - x_\infty}{2}$ be a fixed positive number, and let C_ϵ be a circle centred in x_∞ with radius ϵ. If ξ belongs to the interior of C_ϵ then the Cauchy integral formula gives ($L \leq \infty$):

$$P_L(\xi) = \frac{1}{2\pi\imath} \int_{C_\epsilon} d\zeta\ \frac{P_L(\zeta)}{\zeta - \xi}. \tag{3.5}$$

Take $0 < \delta < 1/2$ and define $\phi_L := N(L)^{-1/2+\delta/3}$. Then for L big enough and $|\phi| \leq \phi_L$, one has that $x_L e^{\imath\phi}$ belongs to the interior of $C_{\epsilon/2}$ and (3.5) implies:

$$\tilde{p}_L(\phi) = \frac{1}{4\pi\imath} \int_{C_\epsilon} d\zeta\ P_L(\zeta) \left(\frac{1}{\zeta - x_L e^{\imath\phi}} + \frac{1}{\zeta - x_L e^{-\imath\phi}}\right). \tag{3.6}$$

Theorem 2.1 implies now that:

$$\lim_{L\to\infty} \frac{d^2\tilde{p_L}}{d\phi^2}(0) = \frac{d^2\tilde{p_\infty}}{d\phi^2}(0) \quad \text{and} \quad \forall|\phi| \leq \phi_L, \quad \left|\frac{d^3\tilde{p_L}}{d\phi^3}(\phi)\right| \leq \text{const}. \tag{3.7}$$

Denote with $s_L(\phi)$ the function $\Im P_L(x_L e^{\imath\phi})$. In a similar way, one can obtain that for all $|\phi| \leq \phi_L$ one has:

$$\left|\frac{d^3 s_L}{d\phi^3}(\phi)\right| \leq \text{const}. \tag{3.8}$$

Finally, let us prove (3.4). This will be done in two steps:

1. The first one consists in showing that the contribution to $\mathcal{A}_L$ coming from the region $|\phi| \geq \phi_L$ is exponentially small in L; this will come from the fact that $\tilde{p}_L(\phi)$ is an even function, is decreasing on the interval $[0, \pi]$ and has a nondegenerate maximum in 0;
2. The second one consists in a more careful study of the case in which $|\phi| \leq \phi_L$; the most important thing here is showing that the oscillations of the imaginary part are small.

Let us prove 1. First of all, because $\tilde{p}_L(\phi) = \tilde{p}_L(-\phi)$, it is sufficient to look only at $\phi \in [\phi_L, \pi]$. From (2.11) one has:

$$\tilde{p}_L(\phi) = -\frac{1}{2\beta L^3} \sum_j \log \left[\left(1 - x_L \cos\phi \, e^{-\beta E_j}\right)^2 + x_L^2 \sin^2\phi \, e^{-2\beta E_j} \right], \tag{3.9}$$

and:

$$\frac{d\tilde{p}_L}{d\phi}(\phi) = -\frac{x_L \sin\phi}{\beta L^3} \sum_j \frac{e^{-\beta E_j}}{|1 - x_L e^{\imath\phi} \, e^{-\beta E_j}|^2} \, . \tag{3.10}$$

This means that $\tilde{p}_L$ is decreasing on $[0, \pi]$. Its second derivative in 0 reads as:

$$\frac{d^2\tilde{p}_L}{d\phi^2}(0) = -\frac{x_L}{\beta L^3} \sum_j \frac{e^{-\beta E_j}}{|1 - x_L \; e^{-\beta E_j}|^2} \, . \tag{3.11}$$

For L big enough, one has that $x_L \geq x_\infty/2$. The above equality implies:

$$\frac{d^2\tilde{p}_L}{d\phi^2}(0) \leq -\frac{x_\infty}{2\beta} \frac{1}{L^3} \mathrm{trW_L}. \tag{3.12}$$

To obtain an uniform estimate in L, we employ (see [K]):

$$\frac{1}{L^3} \mathrm{trW_L} = \frac{1}{(2\pi\beta)^{3/2}} \frac{\omega\beta/2}{\sinh(\omega\beta/2)} \, (1 + \mathcal{O}(1/L)). \tag{3.13}$$

Then (3.7), (3.12) and (3.13) imply that for L sufficiently large:

$$\frac{d^2\tilde{p}_L}{d\phi^2}(0) \leq -\frac{x_\infty}{4\beta} \frac{1}{(2\pi\beta)^{3/2}} \frac{\omega\beta/2}{\sinh(\omega\beta/2)} \quad \text{and} \quad \frac{d^2\tilde{p}_\infty}{d\phi^2}(0) < 0. \tag{3.14}$$

From (3.14) and (3.7) it follows that for all $\phi \in [\phi_L, \pi]$ one has:

$$\tilde{p}_L(\phi) - P_L(x_L) \leq \tilde{p}_L(\phi_L) - \tilde{p}_L(0) \leq \frac{1}{4} \frac{d^2\tilde{p}_\infty}{d\phi^2}(0)\phi_L^2, \tag{3.15}$$

or in other words,

$$\exp\left\{ \frac{N(L)\beta}{\rho} [\tilde{p}_L(\phi) - P_L(x_L)] \right\} \leq \exp\left[\frac{\beta}{4\rho} \frac{d^2\tilde{p}_\infty}{d\phi^2}(0) N(L)^{2\delta/3} \right], \tag{3.16}$$

therefore the contribution to $\mathcal{A}_L$ coming from $|\phi| \geq \phi_L$ is exponentially small.

Let us study now the region $|\phi| \leq \phi_L$. We are mainly interested in the behaviour of the imaginary part of the pressure (see (3.8)). Similarly as in (3.9), one has:

$$s_L(\phi) = -\frac{1}{\beta L^3} \sum_j \arg\left(1 - x_L e^{\imath\phi} e^{-\beta E_j}\right) = \frac{1}{\beta L^3} \sum_j \arctan \frac{x_L \sin\phi \; e^{-\beta E_j}}{1 - x_L \cos\phi \; e^{-\beta E_j}}. \tag{3.17}$$

By direct computation, one obtains:

$$s_L(0) = 0, \quad \frac{ds_L}{d\phi}(0) = \frac{\rho_L(x_L)}{\beta} = \frac{\rho}{\beta} \quad \text{and} \quad \frac{d^2 s_L}{d\phi^2}(0) = 0. \tag{3.18}$$

Together with (3.8), one has:

$$\frac{\beta}{\rho}s_L(\phi)-\phi=\mathcal{O}(\phi_L^3). \tag{3.19}$$

Let's take a look at the following integral:

$$\frac{\sqrt{N(L)}}{2\pi}\int_{-\phi_L}^{\phi_L}d\phi\; e^{\left[N(L)\beta/\rho(\Re P_L(x_Le^{\imath\phi})-P_L(x_L))\right]}e^{\imath N(L)\left[\beta\Im P_L(x_Le^{\imath\phi})/\rho-\phi\right]}. \tag{3.20}$$

Changing the variable in $t=\sqrt{N(L)}\phi$ one obtains:

$$\frac{1}{2\pi}\int_{-N(L)^{\delta/3}}^{N(L)^{\delta/3}}dt\; e^{\frac{\beta}{2\rho}\frac{d^2\tilde{p}_L}{d\phi^2}(0)\;t^2}e^{\mathcal{O}[N(L)^{-1/2+\delta}]}. \tag{3.21}$$

Using (3.14) and (3.7) it is easy to show that the above integral converges to (3.4).

From (2.28) and (3.2), the reduced free energy reads as:

$$f_L=\tilde{f}_L+\frac{\log N(L)}{2\beta L^3}-\frac{\log\mathcal{A}_L}{\beta L^3}\;. \tag{3.22}$$

From (2.29) and (2.30) it follows:

$$m_L=\tilde{m}_L-\frac{e}{\beta cL^3}\,\frac{1}{\mathcal{A}_L}\,\frac{\partial\mathcal{A}_L}{\partial\omega}, \tag{3.23}$$

or:

$$m_L-\tilde{m}_L=\frac{N(L)^{1/2}}{2\pi\,\mathcal{A}_L}\int_{-\pi}^{\pi}d\phi\; e^{\left[\frac{N(L)\beta}{\rho}\left(P_L(x_Le^{\imath\phi})-P_L(x_L)\right)\right]}\left[-\frac{e}{c}\frac{d}{d\omega}\left(P_L(x_Le^{\imath\phi})-P_L(x_L)\right)\right], \tag{3.24}$$

where:

$$-\frac{e}{c}\frac{d}{d\omega}\left(P_L(x_Le^{\imath\phi})-P_L(x_L)\right)=\Gamma_L(x_Le^{\imath\phi})-\Gamma_L(x_L)-\frac{e}{c}\left[\frac{\partial P_L}{\partial z}(x_Le^{\imath\phi})\frac{\partial x_L}{\partial\omega}e^{\imath\phi}-\frac{\partial P_L}{\partial z}(x_L)\frac{\partial x_L}{\partial\omega}\right]. \tag{3.25}$$

From (2.12) it follows that $-\frac{e}{c}\frac{\partial\rho_L}{\partial\omega}(\beta,z,\omega)=\beta z\frac{\partial\Gamma_L}{\partial z}(\beta,z,\omega)$. This implies that:

$$-\frac{e}{c}\,\frac{\partial x_L}{\partial\omega}=-\beta x_L\frac{\partial\Gamma_L}{\partial z}(x_L)\left[\frac{\partial\rho_L}{\partial z}(x_L)\right]^{-1}.$$

Lemma 2.1 and Theorem 2.1 imply (via Cauchy integral formula) that the above quantity remains bounded when L goes to infinity. Performing a similar analysis of the integral from (3.24) as that one made for $\mathcal{A}_L$, it follows that the contribution coming from the region $|\phi|\geq\phi_L$ is exponentially small in L. When we're looking at the region $|\phi|\leq\phi_L$, the factor $N(L)^{1/2}$ disappears when we're changing the variable, therefore $m_L-\tilde{m}_L$ will behave as the quantity from (3.25), and this one behaves like $\phi_L\sim L^{-3/2+\delta}$. The proof of (2.31) is now completed.

4 The proof of Lemma 2.1

The idea of the proof is borrowed from [A]. Namely, the uniform convergence on compacts included in $\mathcal{D}:=\mathbf{C}\setminus[e^{\beta\omega_0/2},\infty)$ will be obtained via Vitali Theorem applied to the sequence of analytical functions $\Gamma_L(\beta,z,\omega_0)$ when L is going to infinity. Therefore, we have to make the following three steps:

I. To prove the uniform boundness in L of the functions $\Gamma_L(\beta,z,\omega_0)$ on any compact $K\subset\mathcal{D}$;

II. To identify the limit ($\Gamma_\infty(\beta,z,\omega_0)$ in our case) and to prove that it has the same domain of analyticity $\mathcal{D}$;

III. To prove the existence of a set $\mathcal{D}_0\subset\mathcal{D}$ having at least one point of accumulation, such that $\Gamma_L(\beta,z,\omega_0)$ has pointwise convergence to $\Gamma_\infty(\beta,z,\omega_0)$ on $\mathcal{D}_0$.

4.1 The proof of I.

Here is the "nontrivial part" of our paper. In order to see where the main difficulty is, let's take a look to (2.24). One could try a bound on $\Gamma_L(\beta, z, \omega)$ using the following inequality (see also (2.22)):

$$\sup_{z\in K}\left|\mathrm{tr}\left[(1-zW_L)^{-1}\frac{\partial W_L}{\partial\omega}\right]\right| \leq \left\|\frac{\partial W_L}{\partial\omega}\right\|_{\mathcal{I}_1}\sup_{z\in K}\left\|(1-zW_L)^{-1}\right\|_{B(\mathbf{L}^2)} \leq C(\beta,K,\omega)\left\|\frac{\partial W_L}{\partial\omega}\right\|_{\mathcal{I}_1}.$$

Unfortunately, the linear growth of the vector potential $\mathbf{a}$ leads to a bad trace norm estimate for $\frac{\partial W_L}{\partial\omega}$, which behaves like L^4 and not like L^3 as needed. In the case of the pressure [A-C], this kind of problem do not appear (see (2.9), (2.8) and (2.4)), because for the Gibbs semigroup the trace and the trace norm are both growing like L^3.

Because the proof is rather lengthy, we will try to outline in a few words our strategy. First of all, having (2.8) in mind, let's define $(\omega_0, \beta, \tau > 0)$:

$$g_{\omega_0}(\xi, z; \beta, \tau) := [\xi - zW_L(\beta,\omega_0)]^{-1} zW_L(\tau,\omega_0), \tag{4.1}$$

where ξ belongs to the contour $\mathcal{C}$. Then the pressure will admit the following representation (see (2.9)):

$$P_L(\beta, z, \omega_0) = -\frac{1}{2\pi i}\int_{\mathcal{C}} d\xi\; q(\xi)\frac{1}{L^3}\,\mathrm{tr}[g_{\omega_0}(\xi, z; \beta, \beta)], \tag{4.2}$$

where the norm convergent integral from (2.8) commutes with the trace from (2.9). Let us argue now that we can choose the same contour of integration if ω varies in a small interval $\Omega := [\omega_0, \omega_1]$, $0 < \omega_0 < \omega_1$. In other words, we search for a $\mathcal{C}$ included in $\mathbf{C}\setminus[1,\infty)$ such that for all $L > 1$, $z \in K$, $\omega \in \Omega$ and $\xi \in \mathcal{C}$ one has the following bound:

$$\|[\xi - zW_L(\beta,\omega)]^{-1}\| \leq M < \infty. \tag{4.3}$$

Suppose now that every $z \in K$ verifies the following condition:

$$\mathrm{dist}\{z, [\,e^{\frac{\beta\omega_0}{2}}, \infty\,)\} = \delta > 0, \quad |z| \leq d < \infty. \tag{4.4}$$

We know that the spectrum of $W_L(\beta,\omega)$ is included in $[0, e^{-\frac{\beta\omega_0}{2}}]$ for all $L > 1$ and $\omega \in \Omega$. We claim that $\mathcal{C}$ can be chosen as the reunion $\mathcal{C}_1 \cup \mathcal{C}_2$, where $\mathcal{C}_2$ is given by $(\eta > 0)$:

$$\{(1+t, \pm\eta)|\; -\eta \leq t \leq 2d\} \cup \{(1-\eta, t)| -\eta \leq t \leq \eta\},$$

and $\mathcal{C}_1$ is chosen such that if $\xi \in \mathcal{C}_1$, then $|\xi| \geq 2d+1$. It is not difficult to prove that by choosing η sufficiently small, one has the inequality:

$$\sup_{z\in K}\sup_{\xi\in\mathcal{C}}\sup_{0\leq r\leq e^{-\frac{\beta\omega_0}{2}}} |(\xi - z\,r)^{-1}| \leq M < \infty, \tag{4.5}$$

and via the Spectral Theorem, (4.3) takes place.

If we manage to prove the existence of a numerical constant $c(\beta, K, \omega_0)$ such that for all $\delta\omega \in (0, \omega_1 - \omega_0)$ and $z \in K$ to have:

$$\sup_z\sup_{\delta\omega}\frac{1}{\delta\omega}|P_L(\beta, z, \omega_0 + \delta\omega) - P_L(\beta, z, \omega_0)| \leq c(\beta, K, \omega_0), \tag{4.6}$$

then the sequence of the magnetisations will be bounded by the same constant. This estimate is straightforward if a stronger one takes place:

$$\sup_z\sup_\xi\sup_{\delta\omega}\frac{1}{L^3\delta\omega}|\mathrm{tr}[g_{\omega_0+\delta\omega}(\xi, z; \beta, \beta)] - \mathrm{tr}[g_{\omega_0}(\xi, z; \beta, \beta)]| \leq C(\beta, K, \omega_0). \tag{4.7}$$

Our main task will consist in constructing a trace class operator $A_{\omega_0+\delta\omega}(\xi, z; \beta)$ having the following two properties: $\mathrm{tr}A_{\omega_0+\delta\omega}(\xi, z; \beta) = \mathrm{tr}g_{\omega_0}(\xi, z; \beta, \beta)$ (i.e. its trace is *not* depending on $\delta\omega$) and moreover,

$$\sup_z\sup_\xi\sup_{\delta\omega}\frac{1}{L^3\delta\omega}\|g_{\omega_0+\delta\omega}(\xi, z; \beta, \beta) - A_{\omega_0+\delta\omega}(\xi, z; \beta)\|_{\mathcal{I}_1} \leq C'(\beta, K, \omega_0), \tag{4.8}$$

which would clearly end the problem.

We will see that $A_{\omega_0+\delta\omega}(\xi,z;\beta)$ can be chosen as the product $\tilde{g}_{\omega_0+\delta\omega}(\xi,z;\beta,\beta/2)S_{\omega_0+\delta\omega}(\beta/2)$ where the first term is bounded, the second one is trace class and moreover:

$$\sup_z \sup_\xi \|g_{\omega_0+\delta\omega}(\xi,z;\beta,\beta/2) - \tilde{g}_{\omega_0+\delta\omega}(\xi,z;\beta,\beta/2)\|_{B(\mathbf{L}^2)} \leq C_1(\beta,K,\omega_0)\delta\omega, \tag{4.9}$$

and

$$\|W_L(\beta/2,\omega_0+\delta\omega) - S_{\omega_0+\delta\omega}(\beta/2)\|_{\mathcal{I}_1} \leq C_2(\beta,\omega_0)L^3\delta\omega. \tag{4.10}$$

In particular, (4.9) and (4.10) imply that:

$$\|S_{\omega_0+\delta\omega}(\beta/2)\|_{\mathcal{I}_1} \leq C(\beta,\omega_0)L^3 \quad \text{and} \quad \sup_z \sup_\xi \|\tilde{g}_{\omega_0+\delta\omega}(\xi,z;\beta,\beta/2)\|_{B(\mathbf{L}^2)} \leq C_3(\beta,K,\omega_0).$$

Using these estimates together with the following identity (see (4.1)):

$$g_{\omega_0+\delta\omega}(\xi,z;\beta,\beta) = g_{\omega_0+\delta\omega}(\xi,z;\beta,\beta/2)W_L(\beta/2,\omega_0+\delta\omega),$$

the proof of (4.8) follows easily.

The rest of this subsection is dedicated to the rigorous proofs of these estimates and will be structured in a sequence of technical propositions. We start with a well known result, given without any other comments:

Proposition 4.1 *The Dirichlet Laplacian defined in Λ admits a trace class semigroup which has an integral kernel given by the following formula:*

$$G_{0,L}(\mathbf{x},\mathbf{x}';\beta) = \prod_{j=1}^{3} g_{0,L}(x_j,x_j';\beta), \tag{4.11}$$

where the "one dimensional" kernels read as:

$$\begin{aligned} g_{0,L}(x,x';\beta) &= \frac{1}{(2\pi\beta)^{1/2}}\sum_{m\in\mathbf{Z}}\left\{\exp\left[-\frac{(x-x'+2mL)^2}{2\beta}\right] - \exp\left[-\frac{(x+x'-2mL-L)^2}{2\beta}\right]\right\} := \\ &:= g_{0,\infty}(x,x';\beta) + \zeta_{0,L}(x,x';\beta). \end{aligned} \tag{4.12}$$

Using the previous proposition, one can write:

$$G_{0,L}(\mathbf{x},\mathbf{x}';\beta) = G_{0,\infty}(\mathbf{x},\mathbf{x}';\beta) + Z_{0,L}(\mathbf{x},\mathbf{x}';\beta), \tag{4.13}$$

where:

$$G_{0,\infty}(\mathbf{x},\mathbf{x}';\beta) = \frac{1}{(2\pi\beta)^{3/2}}\exp\left(-\frac{|\mathbf{x}-\mathbf{x}'|^2}{2\beta}\right). \tag{4.14}$$

The purpose of the next proposition is to give a few properties of smoothness and localisation of the reminder $Z_{0,L}(\mathbf{x},\mathbf{x}';\beta)$:

Proposition 4.2 *For all $\beta > 0$ and $L > 1$, there exist two positive numerical constants c_1 and c_2 such that:*

i. $\displaystyle \left|\frac{\partial Z_{0,L}}{\partial x_j}\right|(\mathbf{x},\mathbf{x}';\beta) \leq c_1\,\frac{1+\beta}{\beta^{1/2}}\,G_{0,\infty}(\mathbf{x},\mathbf{x}';c_2\beta);$

ii. $\displaystyle \max\left\{\left|\frac{\partial Z_{0,L}}{\partial \beta}\right|(\mathbf{x},\mathbf{x}';\beta),\ \left|\frac{\partial^2 Z_{0,L}}{\partial x_j\partial x_k'}\right|(\mathbf{x},\mathbf{x}';\beta)\right\} \leq c_1\frac{1+\beta}{\beta}G_{0,\infty}(\mathbf{x},\mathbf{x}';c_2\beta);$

iii. $|Z_{0,L}|(\mathbf{x},\mathbf{x}';\beta) \leq c_1\,(1+\beta)\,G_{0,\infty}(\mathbf{x},\mathbf{x}';c_2\beta).$

Proof. For $x, x' \in (-L/2, L/2)$ define:

$$\begin{aligned} \zeta_1(x,x';\beta) &= \frac{1}{\sqrt{2\pi\beta}} \sum_{m\in\mathbf{Z}\setminus\{0\}} \exp\left(-\frac{(x-x'+2mL)^2}{2\beta}\right), \\ \zeta_2(x,x';\beta) &= \frac{1}{\sqrt{2\pi\beta}} \sum_{m\in\mathbf{Z}} \exp\left(-\frac{[x-x'-(2m+1)L]^2}{2\beta}\right). \end{aligned} \tag{4.15}$$

It is clear that if one would obtain uniform estimates in L and β for these two quantities, the same would be true for $Z_{0,L}$. A very useful estimate is the following:

$$\forall t \geq 0, \quad te^{-t} = 2(t/2)e^{-t/2}e^{-t/2} \leq 2e^{-t/2}. \tag{4.16}$$

Then we have the inequalities:

$$\begin{aligned} \left|\frac{\partial \zeta_1}{\partial x}\right| (x,x';\beta) &\leq \frac{\text{const.}}{\beta} \sum_{m\neq 0} \exp\left(-\frac{(x-x'+2mL)^2}{4\beta}\right), \\ \left|\frac{\partial^2 \zeta_1}{\partial x \partial x'}\right| (x,x';\beta) &\leq \frac{\text{const.}}{\beta^{3/2}} \sum_{m\neq 0} \exp\left(-\frac{(x-x'+2mL)^2}{4\beta}\right), \end{aligned} \tag{4.17}$$

and similarly for ζ_2. At this point we have to control the summation over m. Let's prove the following inequality:

$$\sum_{m\neq 0} \exp\left(-\frac{(x-x'+2mL)^2}{4\beta}\right) \leq c_1(1+\beta)\exp\left(-\frac{(x-x')^2}{c_2\beta}\right). \tag{4.18}$$

Because $|x-x'| < L$ one has ($|m|, L \geq 1$):

$$(x-x'+2mL)^2 = (x-x')^2 + 4mL(x-x') + 4m^2L^2 \geq (x-x')^2 + 4(|m|-1). \tag{4.19}$$

Therefore:

$$\begin{aligned} \sum_{m\neq 0} \exp\left(-\frac{(x-x'+2mL)^2}{4\beta}\right) &\leq \exp\left(-\frac{(x-x')^2}{4\beta}\right) 2\left[1 + \sum_{m\geq 1}\exp\left(-\frac{1}{\beta}m\right)\right] = \\ &= 2\exp\left(-\frac{(x-x')^2}{4\beta}\right)\left[1 + \frac{1}{2}\frac{\frac{1}{2\beta}}{\sinh\left(\frac{1}{2\beta}\right)} 2\beta \exp\left(-\frac{1}{2\beta}\right)\right] \leq 2(1+\beta)\exp\left(-\frac{(x-x')^2}{4\beta}\right). \end{aligned} \tag{4.20}$$

In order to control ζ_2, one has to study the following quantity:

$$A := \sum_{m\in\mathbf{Z}} \exp\left(-\frac{[x+x'-(2m+1)L]^2}{4\beta}\right). \tag{4.21}$$

Denote with $\xi = x + L/2$ and $\xi' = x' + L/2$; then $0 < \xi, \xi' < L$, $x - x' = \xi - \xi'$ and $(\xi+\xi')^2 \geq (\xi-\xi')^2$. It follows:

$$[x+x'-(2m+1)L]^2 = [\xi+\xi'-2(m+1)L]^2 = (\xi+\xi')^2 - 4(m+1)L(\xi+\xi') + 4(m+1)^2L^2. \tag{4.22}$$

If $m \leq -1$, then:

$$[x+x'-(2m+1)L]^2 \geq (x-x')^2 + 4(|m|-1).$$

If $m = 0$, then:

$$[x+x'-(2m+1)L]^2 = [(L/2-x)+(L/2-x')]^2 \geq (x-x')^2.$$

If $m \geq 1$, then:

$$[x + x' - (2m+1)L]^2 \geq (\xi + \xi')^2 + 4L^2(m+1)(m-1) \geq (x - x')^2 + 4(m-1),$$

and we can repeat the summation procedure used in (4.18). Putting all these together, the proof is completed.

The next proposition is a variant of the perturbation theory for self-adjoint Gibbs semigroups (see [H-P]). Instead of starting with a perturbation of its generator, we start with an approximation of the semigroup. Although simple, this proposition contains the main technical core of our paper.

Proposition 4.3 *Let $\mathcal{H} := \mathbf{L}^2(\Lambda)$ and let H be a self-adjoint and positive operator having the domain D. Fix $\beta_0 > 0$. Suppose that there exists an application $0 < \beta \leq \beta_0 \to S(\beta) \in B(\mathcal{H})$ with the following properties:*

A. $\sup_{0<\beta\leq\beta_0} ||S(\beta)|| \leq c_1 < \infty$;

B. *It is strongly differentiable,* $\mathrm{Ran}S(\beta) \subset D$ *and* $s - \lim_{\beta\searrow 0} S(\beta) = 1$;

C. *There exists a normly continuous application $0 < \beta \leq \beta_0 \to R(\beta) \in B(\mathcal{H})$ such that $||R(\beta)|| \leq c_2/\beta^\alpha$ where $0 \leq \alpha < 1$ and:*

$$\frac{\partial S}{\partial \beta} f + HS(\beta)f = R(\beta)f. \tag{4.23}$$

Then the following two affirmations are true:

i. *The sequence of operators $T_n(\beta) := \int_{1/n}^{\beta-1/n} d\tau\ \exp[-(\beta-\tau)H]R(\tau)$, $n > [1/\beta]$ converges in norm to $T(\beta)$;*

ii. *The following equality takes place in $B(\mathcal{H})$:*

$$\exp(-\beta H) = S(\beta) - T(\beta). \tag{4.24}$$

Proof. i. The norm convergence is assured by the integrability condition imposed on the norm of $R(\beta)$. Moreover, when β is near zero:

$$\sup_n ||T_n(\beta)|| \leq c(\alpha)\beta^{1-\alpha}, \tag{4.25}$$

therefore the same thing is true for $T(\beta)$.

ii. Let $0 < \beta_1 < \beta < \beta_0$. If $n > [1/\beta_1]$ and $\phi \in \mathcal{H}$, define the vector:

$$\psi_n(\beta) := \exp(-\beta H)\phi - S(\beta)\phi + T_n(\beta)\phi. \tag{4.26}$$

From (4.25) and condition A it follows:

$$\lim \psi_n(\beta) := \psi(\beta) = \exp(-\beta H)\phi - S(\beta)\phi + T(\beta)\phi \quad \text{and} \quad \sup_{n,\beta} ||\psi_n(\beta)|| \leq \text{const}. \tag{4.27}$$

Define $f_n(\beta) = ||\psi_n(\beta)||^2$ and $f(\beta) = ||\psi(\beta)||^2$. From the strong convergence to one of $S(\beta)$ when β goes to zero and from the norm convergence to zero of $T(\beta)$, it follows that $\lim_{\beta\searrow 0} f(\beta) = 0$. If we manage to prove that $f(\beta)$ is decreasing, then it would be identically zero and this would end the proof.

Notice that $T_n(\beta)$ is normly differentiable and:

$$\frac{\partial T_n}{\partial \beta}\phi + HT_n(\beta)\phi = \exp\left(-\frac{1}{n}H\right)R(\beta - 1/n)\phi. \tag{4.28}$$

The positivity of H implies:

$$\frac{\partial f_n}{\partial \beta} \leq 2\, \Re[\langle \psi_n(\beta), \exp\left(-\frac{1}{n}H\right)R(\beta - 1/n)\phi - R(\beta)\phi\rangle]. \tag{4.29}$$

Using $f_n(\beta) = f_n(\beta_1) + \int_{\beta_1}^{\beta} d\tau\ \frac{\partial f_n}{\partial \tau}$, one has:

$$0 \leq f_n(\beta) \leq f_n(\beta_1) + \text{const} \int_{\beta_1}^{\beta} d\tau\ ||\exp\left(-\frac{1}{n}H\right)R(\tau - 1/n)\phi - R(\tau)\phi||. \tag{4.30}$$

Because the above integrand is bounded on the domain of integration and has pointwise convergence to zero, from the dominated convergence theorem we conclude that the whole integral converges to zero. Taking the limit, it follows that $f(\beta)$ is decreasing and therefore is identically zero.

Define now the "magnetic phase":

$$\varphi(\mathbf{x},\mathbf{x}') = \mathbf{x}\cdot\mathbf{a}(\mathbf{x}') = \frac{1}{2}\mathbf{e_3}\cdot(\mathbf{x}'\wedge\mathbf{x}). \tag{4.31}$$

We will use this quantity as a local gauge transformation (see [C-N] for another use of this idea). Its main property is the diagonalisation of the magnetic vector potential: $\mathbf{a}(\mathbf{x}) - \mathbf{a}(\mathbf{x}-\mathbf{x}') = \nabla_{\mathbf{x}}\varphi(\mathbf{x},\mathbf{x}')$. To see how this transformation acts (at a formal level) on $H_{1,L}(\omega)$, let's notice the following equation:

$$[-\imath\nabla_{\mathbf{x}} - \omega\mathbf{a}(\mathbf{x})]\, e^{\imath\omega\varphi(\mathbf{x},\mathbf{x}')} = e^{\imath\omega\varphi(\mathbf{x},\mathbf{x}')}\,[-\imath\nabla_{\mathbf{x}} - \omega\mathbf{a}(\mathbf{x}-\mathbf{x}')]\,. \tag{4.32}$$

Proposition 4.4 *The bounded operator $S(\beta)\in\mathbf{L}^2(\Lambda)$ given by the integral kernel $e^{\imath\omega\varphi(\mathbf{x},\mathbf{x}')}G_{0,L}(\mathbf{x},\mathbf{x}';\beta)$, verifies the conditions from Proposition 4.3, having the following properties:*

i. $s - \lim_{\beta\searrow 0} S(\beta) = \mathbf{1}$;

ii. *The application* $(0,\infty)\ni\beta\longmapsto S(\beta)$ *is strongly differentiable and unniformly bounded*;

iii. $\mathrm{Ran}S(\beta)\in\mathrm{Dom}(H_{1,L}(\omega))$ *and* $\dfrac{\partial S}{\partial\beta}(\beta)f + H_{1,L}(\omega)S(\beta)f = R(\beta)f,$

where $R(\beta)$ is given by the following integral kernel:

$$R(\mathbf{x},\mathbf{x}';\beta) = e^{\imath\omega\varphi(\mathbf{x},\mathbf{x}')}[\omega^2\mathbf{a}^2(\mathbf{x}-\mathbf{x}')G_{0,L}(\mathbf{x},\mathbf{x}';\beta) + 2\imath\omega\mathbf{a}(\mathbf{x}-\mathbf{x}')\cdot\nabla_{\mathbf{x}}Z_{0,L}(\mathbf{x},\mathbf{x}';\beta)].$$

Proof. i. We will use throughout the whole section a few well known results, which are given without proof. Let us begin with an useful criterion for an integral operator to be bounded (see [S]):

Let A be an integral operator, given by a continuous integral kernel $A(\mathbf{x},\mathbf{x}')\in C(\overline{\Lambda}\times\overline{\Lambda})$. If the following inequality holds:

$$\max\left[\sup_{\mathbf{x}\in\Lambda}\int_\Lambda d\mathbf{x}'\,|A(\mathbf{x},\mathbf{x}')|,\ \sup_{\mathbf{x}'\in\Lambda}\int_\Lambda d\mathbf{x}\,|A(\mathbf{x},\mathbf{x}')|\right]\le \mathrm{C}<\infty, \tag{4.33}$$

then the operator norm of A in $B(\mathbf{L}^2(\Lambda))$ is bounded by C.

If c, β and $\tau<\beta$ are three strictly positive numbers, then we have the following three identities:

$$\int_{\mathbf{R}^3} d\mathbf{y}\,\exp\left(-\frac{|\mathbf{x}-\mathbf{y}|^2}{c(\beta-\tau)}\right)\exp\left(-\frac{|\mathbf{y}-\mathbf{x}'|^2}{c\tau}\right) = (\pi c)^{\frac{3}{2}}\left[\frac{(\beta-\tau)\tau}{\beta}\right]^{\frac{3}{2}}\exp\left(-\frac{|\mathbf{x}-\mathbf{x}'|^2}{c\beta}\right), \tag{4.34}$$

$$\int_{\mathbf{R}^3} d\mathbf{y}\,\frac{1}{(\pi c\beta)^{\frac{3}{2}}}\exp\left(-\frac{|\mathbf{x}-\mathbf{y}|^2}{c\beta}\right) = 1, \tag{4.35}$$

$$\left(\int_{\mathbf{R}^3} d\mathbf{y}\,|G_{0,\infty}(\mathbf{x},\mathbf{y};\beta)|^2\right)^{\frac{1}{2}} = (2\pi\beta)^{-\frac{3}{4}}. \tag{4.36}$$

A very useful inequality will be the next one ($t>0,\ n\ge 1$):

$$t^n\exp\left(-\frac{t^2}{\beta}\right) = (2\beta)^{n/2}\exp\left(-\frac{t^2}{2\beta}\right)\left(\frac{t^2}{2\beta}\right)^{n/2}\exp\left(-\frac{t^2}{2\beta}\right)\le \mathrm{const}(n)\beta^{\frac{n}{2}}\exp\left(-\frac{t^2}{2\beta}\right). \tag{4.37}$$

Let us get back to the proof of i. Firstly, because $|e^{\imath\omega\varphi(\mathbf{x},\mathbf{x}')}G_{0,L}(\mathbf{x},\mathbf{x}';\beta)|\le G_{0,\infty}(\mathbf{x},\mathbf{x}';\beta)$, it follows that $S(\beta)$ obeys the condition (4.33) with $\mathrm{C}\le 1$, which means that is uniformly bounded in $\beta>0$.

Let's show now that the operator given by the integral kernel $(e^{\imath\omega\varphi(\mathbf{x},\mathbf{x}')}-1)G_{0,L}(\mathbf{x},\mathbf{x}';\beta)$ converges in norm to zero. Let's remark first that $|\varphi(\mathbf{x},\mathbf{x}')|\leq L\;|\mathbf{x}-\mathbf{x}'|$ and moreover:

$$|(e^{\imath\omega\varphi(\mathbf{x},\mathbf{x}')}-1)|\leq\omega|\varphi(\mathbf{x},\mathbf{x}')|\leq\omega L\;|\mathbf{x}-\mathbf{x}'|. \tag{4.38}$$

Then (using (4.37) with $n=1$):

$$|(e^{\imath\omega\varphi(\mathbf{x},\mathbf{x}')}-1)G_{0,L}(\mathbf{x},\mathbf{x}';\beta)|\leq \text{const}\;L\;\beta^{1/2}G_{0,\infty}(\mathbf{x},\mathbf{x}';2\beta), \tag{4.39}$$

therefore its operator norm behaves in zero like $\beta^{1/2}$ at least. The proof of i is now straightforward.

ii. We will prove that the application is in fact normly differentiable. For $\beta>0$ and $\delta\beta$ sufficiently small, one has:

$$S(\beta+\delta\beta)f-S(\beta)f=\delta\beta\int_\Lambda d\mathbf{x}'\;e^{\imath\omega\varphi(\cdot,\mathbf{x}')}\frac{\partial G_{0,L}}{\partial\beta}(\cdot,\mathbf{x}';\beta)f(\mathbf{x}')+\frac{(\delta\beta)^2}{2}\int_\Lambda d\mathbf{x}'\;e^{\imath\omega\varphi(\cdot,\mathbf{x}')}\frac{\partial^2 G_{0,L}}{\partial\beta^2}(\cdot,\mathbf{x}';\tilde{\beta})f(\mathbf{x}'), \tag{4.40}$$

where $\tilde{\beta}$ is situated between β and $\beta+\delta\beta$. It is not difficult now to see that the "operator derivative" is an integral operator whose integral kernel is the derivative after β of the initial one.

iii. Denote with D_0 the common domain of essentially self-adjoint-ness of the operators $H_{1,L}(\omega)$, $\omega\geq 0$:

$$D_0=\{\psi\in C^2(\Lambda)\cap C^1(\overline{\Lambda})|\;\psi|_{\partial\Lambda}=0,\;\Delta\psi\in\mathbf{L}^2(\Lambda)\}. \tag{4.41}$$

The action of $H_{1,L}(\omega)$ on a function from D_0 is as follows:

$$[H_{1,L}(\omega)\psi](\mathbf{x})=-(\Delta\psi)(\mathbf{x})+2\imath\omega\mathbf{a}(\mathbf{x})\cdot(\nabla\psi)(\mathbf{x})+\omega^2\mathbf{a}^2(\mathbf{x})\psi(\mathbf{x}). \tag{4.42}$$

Let $f\in C_0^\infty(\Lambda)$. After an integration by parts, using (4.32) and the fact that $G_{0,L}(\mathbf{x},\mathbf{x}';\beta)$ solves the heat equation in the interior of Λ, one obtains:

$$\begin{aligned}\langle H_{1,L}(\omega)\psi,S(\beta)f\rangle &= \int_{\Lambda^2}d\mathbf{x}'\;d\mathbf{x}\;\overline{\psi(\mathbf{x})}f(\mathbf{x}')e^{\imath\omega\varphi(\mathbf{x},\mathbf{x}')}[-\Delta_{\mathbf{x}}+2\imath\omega\mathbf{a}(\mathbf{x}-\mathbf{x}')\cdot\nabla_{\mathbf{x}}+\omega^2\mathbf{a}^2(\mathbf{x}-\mathbf{x}')]G_{0,L}(\mathbf{x},\mathbf{x}';\beta)\\ &= -\langle\psi,S'(\beta)f\rangle+\langle\psi,R(\beta)f\rangle.\end{aligned} \tag{4.43}$$

The result is easily obtained after a density argument and with the remark that:

$$\mathbf{a}(\mathbf{x}-\mathbf{x}')\cdot\nabla_{\mathbf{x}}G_{0,\infty}(\mathbf{x},\mathbf{x}';\beta)=0. \tag{4.44}$$

Finally, let us notice that the norm of $R(\beta)$ is independent of L and is integrable in zero. To do that, one has to use the estimates from Proposition 4.2, (4.37) and the criterion from (4.33).

In order to perform a similar perturbative treatment of the semigroup near a nonzero magnetic field, we need the estimate from the next proposition:

Proposition 4.5 *Let $\mathbf{n}$ be a unit vector in $\mathbf{R}^3$. Then there exist three positive numerical constants s, c_4 and c_5 such that for all $\omega\in\Omega$, $\mathbf{x},\mathbf{x}'\in\Lambda$ and $\beta>0$, one has the following uniform estimate in L:*

$$|\mathbf{n}\cdot(-\imath\nabla-\omega\mathbf{a}(\mathbf{x}))G_{\omega,L}(\mathbf{x},\mathbf{x}';\beta)|\leq c_4\;\frac{(1+\beta)^s}{\beta^2}\;\exp\left[-\frac{|\mathbf{x}-\mathbf{x}'|^2}{c_5\beta}\right]. \tag{4.45}$$

Proof. Proposition 4.3 allows us to write down the following integral equation:

$$W_L(\beta,\omega)=S(\beta)-\int_0^\beta d\tau\;W_L(\beta-\tau,\omega)\;R(\tau). \tag{4.46}$$

Because $S(\beta)$ and W_L are self-adjoint, one can rewrite (4.46) as:

$$W_L(\beta,\omega)=S(\beta)-\int_0^\beta d\tau\;R^*(\tau)W_L(\beta-\tau,\omega). \tag{4.47}$$

In terms of integral kernels, (4.47) reads as:

$$G_{\omega,L}(\mathbf{x},\mathbf{x}';\beta) = S(\mathbf{x},\mathbf{x}';\beta) - \int_0^\beta d\tau \int_\Lambda d\mathbf{y}\; R^*(\mathbf{x},\mathbf{y},\tau) G_{\omega,L}(\mathbf{y},\mathbf{x}',;\beta-\tau), \tag{4.48}$$

where the equality is between continuous functions in $C(\Lambda\times\Lambda)$ and the integral in τ has to be understood as "$\int_\epsilon^{\beta-\epsilon}$" in the limit $\epsilon\searrow 0$.

Because the kernel of R^* reads as $R^*(\mathbf{x},\mathbf{y};\tau) = \overline{R(\mathbf{y},\mathbf{x};\tau)}$, by direct computation one can obtain the estimate (see Proposition 4.2):

$$|\mathbf{n}\cdot(-\imath\nabla - \omega\mathbf{a}(\mathbf{x}))R^*(\mathbf{x},\mathbf{x}';\tau)| \leq c_4' \,\frac{(1+\tau)^{s'}}{\tau^2}\exp\left[-\frac{|\mathbf{x}-\mathbf{x}'|^2}{c_5'\tau}\right], \tag{4.49}$$

where one has to apply (4.32) then to use the estimates from Proposition 4.2; finally, introducing (4.49), (2.3), (4.34) and (4.37) in (4.48) and because the singularity in τ is integrable, the result for $G_{\omega,L}$ is straightforward.

Take $\omega = \omega_0 + \delta\omega \in \Omega$. The analogous of Proposition 4.4 at nonzero magnetic field is the following:

Proposition 4.6 *The bounded operator denoted with $S_\omega(\beta)$ and given by the kernel $e^{\imath\delta\omega\varphi(\mathbf{x},\mathbf{x}')}G_{\omega_0,L}(\mathbf{x},\mathbf{x}';\beta)$ has the following properties:*

i. $(0,\infty)\ni\beta\longmapsto S_\omega(\beta)$ *is strongly differentiable and* $s-\lim_{\beta\searrow 0} S_\omega(\beta) = \mathbf{1}$*;*

ii. $\mathrm{Ran}S_\omega(\beta)\in Dom(H_{1,L}(\omega))$ *and* $\frac{\partial}{\partial\beta}S_\omega(\beta)f + H_{1,L}(\omega)S_\omega(\beta)f = R_\omega(\beta)f$*, where $R_\omega(\beta)$ is given by the following integral kernel:*

$$R_\omega(\mathbf{x},\mathbf{x}';\beta) = e^{\imath(\delta\omega)\varphi(\mathbf{x},\mathbf{x}')}\left[(\delta\omega)^2\mathbf{a}^2(\mathbf{x}-\mathbf{x}')G_{\omega_0,L}(\mathbf{x},\mathbf{x}';\beta) + 2(\delta\omega)\mathbf{a}(\mathbf{x}-\mathbf{x}')\cdot(\imath\nabla_{\mathbf{x}} + \omega_0\mathbf{a}(\mathbf{x}))G_{\omega_0,L}(\mathbf{x},\mathbf{x}';\beta)\right]. \tag{4.50}$$

Proof. i. Rewriting the integral kernel of $S_\omega(\beta)$ as:

$$S_\omega(\mathbf{x},\mathbf{x}';\beta) = G_{\omega,L}(\mathbf{x},\mathbf{x}';\beta) + \left[e^{\imath(\delta\omega)\varphi(\mathbf{x},\mathbf{x}')} - 1\right]G_{\omega,L}(\mathbf{x},\mathbf{x}';\beta), \tag{4.51}$$

using the diamagnetic inequality one can reproduce the argument done at (4.39) in order to prove that the second term converges in norm to zero. Clearly, the first one converges strongly to one.

If $\{\psi_j\}$ and $\{E_j\}$ denote the sets of eigenvectors and eigenvalues of $H_{1,L}(\omega)$, then:

$$G_{\omega,L}(\mathbf{x},\mathbf{x}';\beta) = \sum_j e^{-\beta E_j}\psi_j(\mathbf{x})\overline{\psi_j}(\mathbf{x}'), \tag{4.52}$$

where the series is absolutely and uniformly convergent on $\overline{\Lambda}\times\overline{\Lambda}$. This can be seen from the fact that the semigroup is trace class and that the eigenfunctions belong to D_0 and admit the estimate:

$$|\psi_j|(\mathbf{x}) \leq \mathrm{const}(L)\;(E_j+1), \tag{4.53}$$

obtained from the fact that the resolvent $[H_{1,L}(\omega)+1]^{-1}$ is bounded between $\mathbf{L^2}(\Lambda)$ and $\mathbf{L}^\infty(\Lambda)$. It follows that uniformly in Λ:

$$|G_{\omega,L}(\cdot,\cdot;\beta+\delta\beta) - G_{\omega,L}(\cdot,\cdot;\beta) - \frac{\partial G_{\omega,L}}{\partial\beta}(\cdot,\cdot;\beta)| \leq \mathrm{const}(\delta\beta)^2, \tag{4.54}$$

which is sufficient for the strong differentiability (see also (4.40)).

ii. One has to make the same steps as in the proof of the third point of Proposition 4.4. As for the norm of $R(\beta)$, let's see that is independent of L and is integrable in zero: from (4.50), (4.45), (4.37) and (2.3), one can obtain an estimate on the kernel of $R_\omega(\beta)$ of the following form:

$$|R_\omega(\mathbf{x},\mathbf{x}';\beta)| \leq c_9\delta\omega(1+\beta)^s G_{0,\infty}(\mathbf{x},\mathbf{x}';c_{10}\beta), \tag{4.55}$$

which implies that its $B(\mathbf{L}^2(\Lambda))$ norm is bounded by a constant multiplied with $\delta\omega$ (see (4.33)).

We will give now without proof a result which gives sufficient conditions for an operator defined in $B(\mathbf{L^2}(\Lambda))$ to be trace class:

Proposition 4.7 *Let $\{T_n\}$ a sequence of trace class operators, converging to T in $B(\mathbf{L}^2(\Lambda))$. If $\sup_n ||T_n||_{\mathcal{I}_1} \leq c < \infty$, then $T \in \mathcal{I}_1$ and $||T||_{\mathcal{I}_1} \leq c$.*

Remark. A consequence of this proposition is the following: suppose that an operator T is defined by a $B(\mathbf{L}^2(\Lambda))$-norm Riemann integral on the interval $[a, b]$, with a trace class integrand $S(t)$. If $\int_a^b dt\ ||S(t)||_{\mathcal{I}_1} \leq c < \infty$, then T is trace class and:

$$||T||_{\mathcal{I}_1} \leq c, \quad \mathrm{tr}\, T = \int_a^b dt\ \mathrm{tr}\, S(t).$$

Denote now with $\tilde{R}(\omega, \beta)$ the bounded operator with the kernel:

$$\tilde{R}(\mathbf{x}, \mathbf{x}'; \beta) = 2\, \mathbf{a}(\mathbf{x} - \mathbf{x}') \cdot (-\imath\nabla - \omega_0 \mathbf{a}(\mathbf{x})) G_{\omega_0, L}(\mathbf{x}, \mathbf{x}'; \beta). \tag{4.56}$$

Amongst other things, the following proposition proves (4.10):

Proposition 4.8 *Take $\beta > 0$ and $\omega = \omega_0 + \delta\omega \in \Omega$.*
i. *The operator $S_\omega(\beta)$ is trace class and moreover, there exists a positive numerical constant c such that:*

$$||W_L(\beta, \omega) - S_\omega(\beta)||_{B(\mathbf{L}^2(\Lambda))} \leq c\ |\delta\omega| \quad \text{and} \tag{4.57}$$

$$||W_L(\beta, \omega) - S_\omega(\beta)||_{\mathcal{I}_1} \leq c\ \delta\omega\ L^3. \tag{4.58}$$

ii. For all $\mathbf{x}, \mathbf{x}' \in \Lambda$ and uniformly in L, one has:

$$G_{\omega, L}(\mathbf{x}, \mathbf{x}'; \beta) = \tag{4.59}$$

$$= \ e^{\imath\delta\omega\varphi(\mathbf{x}, \mathbf{x}')} G_{\omega_0, L}(\mathbf{x}, \mathbf{x}'; \beta) + \delta\omega \int_0^\beta d\tau \int_\Lambda d\mathbf{y}\ e^{\imath\delta\omega\varphi(\mathbf{x}, \mathbf{y})} G_{\omega_0, L}(\mathbf{x}, \mathbf{y}; \beta - \tau) e^{\imath\delta\omega\varphi(\mathbf{y}, \mathbf{x}')} \tilde{R}(\mathbf{y}, \mathbf{x}'; \tau) + \mathcal{O}((\delta\omega)^2).$$

Proof. i. We know that as bounded operators we have:

$$W_L(\beta, \omega) = S_\omega(\beta) - \int_0^\beta d\tau\ W_L(\beta - \tau, \omega) R_\omega(\tau). \tag{4.60}$$

As we have seen, the $B(\mathbf{L}^2(\Lambda))$ norm of $R_\omega(\tau)$ is bounded by a constant multiplied with $\delta\omega$ (see (4.55)), and its Hilbert-Schmidt norm is bounded by (see (4.36):

$$||R_\omega(\tau)||_{\mathcal{I}_2} \leq \mathrm{const}\ \delta\omega\ \frac{(1+\tau)^s}{\tau^{3/4}} L^{3/2}. \tag{4.61}$$

For the semigroup, we know that $||W_L||_{B(\mathbf{L}^2(\Lambda))} \leq 1$ and from (2.3) and (4.36) it follows that:

$$||W_L(\beta - \tau, \omega)||_{\mathcal{I}_2} \leq \mathrm{const} \frac{1}{(\beta - \tau)^{3/4}} L^{3/2}. \tag{4.62}$$

With the help of the well known inequality $||A\ B||_{\mathcal{I}_1} \leq ||A||_{\mathcal{I}_2} ||B||_{\mathcal{I}_2}$, it follows that in both situations ($B(\mathbf{L}^2)$ and $\mathcal{I}_1(\mathbf{L}^2)$) the singularities in τ are integrable (see the previous Remark), and the needed bounds follow easily.
ii. The formula (4.59) is obtained from (4.60) by isolating the term which contains $\delta\omega$.

The following proposition gives sufficient conditions for a trace class integral operator such that its trace to be equal to the integral of the kernel's diagonal (see [R-S 1]):

Proposition 4.9 *Let $T \in \mathcal{I}_1(\mathbf{L}^2(\Lambda))$, given by the integral kernel $T(\mathbf{x}, \mathbf{x}') \in C(\overline{\Lambda}, \overline{\Lambda})$. Then:*

$$\mathrm{tr}\, T = \int_\Lambda d\mathbf{x}\ T(\mathbf{x}, \mathbf{x}). \tag{4.63}$$

In what follows, our goal is to prove (4.9). Fix $\beta, \tau > 0$. Let $0 < \omega_0 < \omega_1$ and let $\Omega = [\omega_0, \omega_1]$. Clearly, $g_\omega(\xi, z; \beta, \tau)$ is trace class and admits a continuous integral kernel given by the following series, which is absolutely and uniformly convergent on $\Lambda \times \Lambda$ (see also (4.52)):

$$T_\omega(\mathbf{x}, \mathbf{x}') = \sum_j [\xi - z\exp(-\beta E_j)]^{-1} z \exp(-\tau E_j)\psi_j(\mathbf{x})\overline{\psi_j}(\mathbf{x}'), \tag{4.64}$$

where the dependence on ξ, z, β and τ was droped in order to simplify the notations.

Let's start with the equation satisfied by $T_\omega(\mathbf{x}, \mathbf{x}')$:

Proposition 4.10 *As continuous functions, one has the following equality:*

$$\xi T_\omega(\mathbf{x}, \mathbf{x}') - z \int_\Lambda d\mathbf{y}\; G_{\omega,L}(\mathbf{x}, \mathbf{y}; \beta) T_\omega(\mathbf{y}, \mathbf{x}') = z G_{\omega,L}(\mathbf{x}, \mathbf{x}'; \tau). \tag{4.65}$$

Proof. This identity is nothing but the rewriting in terms of integral kernels of the following operator identity:

$$[\xi - z W_L(\beta, \omega)] g_\omega(\xi, z; \beta, \tau) = z W_L(\tau, \omega). \tag{4.66}$$

We will prove now that uniformly in L, one has that $|T_\omega(\mathbf{x}, \mathbf{x}')| \sim e^{-\alpha|\mathbf{x}-\mathbf{x}'|}$ for some positive α. To do that, we need first a few definitions: let $\rho(\mathbf{x}) := (1 + \mathbf{x}^2)^{1/2}$ and $\alpha \geq 0$. It is known that the partial derivatives up to order n of ρ are bounded by a numerical constant and moreover:

$$\forall \mathbf{x} \in \mathbf{R}^3,\; e^{\pm\alpha\rho(\mathbf{x})} e^{\mp\alpha|\mathbf{x}|} \leq \mathrm{const}(\alpha). \tag{4.67}$$

Fix $\mathbf{x}_0 \in \Lambda$. Denote with $A(\alpha)$ the multiplication operator with $e^{-\alpha\rho(\cdot - \mathbf{x}_0)}$. Then $A(\alpha)$ and $A^{-1}(\alpha) = A(-\alpha)$ are bounded operators and invariate D_0 (see 4.41)). Then takes place the following proposition:

Proposition 4.11 i. *The operator $A(\alpha) W_L(\tau, \omega) A(-\alpha)$ belongs to $B(\mathbf{L}^2)$, with a norm which is uniformly bounded in L, $\mathbf{x}_0 \in \Lambda$, $\omega \in \Omega$ and $0 < \tau < \beta$;*

ii. *The operator $A(\alpha) W_L(\beta, \omega) A(-\alpha)$ belongs to $B(\mathbf{L}^2, \mathbf{L}^\infty)$, with a norm which is uniformly bounded in L, ω and $\mathbf{x}_0$.*

Proof. i. The inequality (4.67) allows us to replace A with the multiplication operator given by $e^{\alpha|\mathbf{x}-\mathbf{x}_0|}$. Let $\psi \in \mathbf{L}^2(\Lambda)$ and define:

$$\phi := A(\alpha) W_L(\tau, \omega) A(-\alpha)\psi, \quad \phi(\mathbf{x}) = \int_\Lambda d\mathbf{y}\; e^{-\alpha\rho(\mathbf{x}-\mathbf{x}_0)} G_{L,\omega}(\mathbf{x}, \mathbf{y}; \tau) e^{\alpha\rho(\mathbf{y}-\mathbf{x}_0)} \psi(\mathbf{y}). \tag{4.68}$$

Let's remark the following elementary estimate: for $0 < \tau < \beta$ one has:

$$e^{\alpha|\mathbf{x}-\mathbf{x}'|} \exp\left[-\frac{|\mathbf{x}-\mathbf{x}'|^2}{4\tau}\right] \leq \mathrm{const}(\alpha, \beta). \tag{4.69}$$

Applying (4.67), the triangle inequality, (2.3) and (4.69) one obtains:

$$|\phi|(\mathbf{x}) \leq \mathrm{const}(\alpha) \int_\Lambda d\mathbf{x}'\; e^{\alpha|\mathbf{x}-\mathbf{x}'|} |G_{\omega,L}(\mathbf{x}, \mathbf{x}'; \tau)|\; |\psi|(\mathbf{x}') \leq \mathrm{const}(\alpha, \beta) \int_\Lambda d\mathbf{x}'\; G_{0,\infty}(\mathbf{x}, \mathbf{x}'; 2\tau) |\psi|(\mathbf{x}'). \tag{4.70}$$

The result follows from (4.35) and (4.33).

ii. We apply the Schwartz inequality in (4.70) with $\tau = \beta$ and then use (4.36).

Proposition 4.12 *Under the same conditions as above, there exists a sufficiently small $0 < \alpha < 1$ such that the following inequalities are true in $B(\mathbf{L}^2(\Lambda))$, uniformly in L, ω and $\mathbf{x}_0$:*

i. $||W_L(\beta, \omega) - A(\alpha) W_L(\beta, \omega) A(-\alpha)|| \leq \alpha\, \mathrm{const}(\beta)$*;*

ii. *Uniformly in $\xi \in \mathcal{C}$ and $z \in K$ one has:* $||A(\alpha)[\xi - z W_L(\beta, \omega)]^{-1} A(-\alpha)||_{B(\mathbf{L}^2(\Lambda))} \leq \mathrm{const}(\beta)$.

Proof. i. Let $S(\beta) = A(\alpha)W_L(\beta,\omega)A(-\alpha)$. We will see that $S(\beta)$ obeys the conditions of Proposition 4.3. From Proposition 4.11 follows condition A. Then clearly, $S(\beta)$ is strongly differentiable, has its range included in the domain of $H_{1,L}(\omega)$ and converges strongly to one. Denote with $B := H_{1,L}(\omega) - A(\alpha)H_{1,L}(\omega)A(-\alpha)$, or in other form:

$$B = 2\imath\alpha(\mathbf{p} - \omega\mathbf{a}) \cdot \nabla\rho(\cdot - \mathbf{x}_0) + \alpha(\Delta\rho)(\cdot - \mathbf{x}_0) - \alpha^2|\nabla\rho(\cdot - \mathbf{x}_0)|^2. \tag{4.71}$$

By direct computation, one obtains:

$$R(\beta) = B\ A(\alpha)W_L(\beta,\omega)A(-\alpha), \quad \left\| B[H_{1,L}(\omega) + 1]^{-1/2} \right\| \leq \text{const}, \tag{4.72}$$

where $||R(\beta)|| \leq \text{const}(L)/\sqrt{\beta}$ and even if the constant behaves badly with L, the norm is integrable in zero.
In conclusion:

$$W_L(\beta,\omega) - S(\beta) = \int_0^\beta d\tau\ W_L(\beta - \tau,\omega)R(\tau) = \int_0^\beta d\tau\ W_L(\beta - \tau,\omega)BS(\tau), \tag{4.73}$$

where the integral converges in norm. But uniformly in L and $\mathbf{x}_0$ there exists a numerical constant such that:

$$||W_L(\beta - \tau,\omega)B|| \leq \alpha \frac{\text{const}}{\sqrt{\beta - \tau}}. \tag{4.74}$$

From (4.74), (4.73) and point i of Proposition 4.11, the needed estimate follows.
ii. Using point i., the estimate (4.3) and the identity (α small enough):

$$\begin{aligned} & A(\alpha)[\xi - zW_L(\beta,\omega)]^{-1}A(-\alpha) = [\xi - zA(\alpha)W_L(\beta,\omega)A(-\alpha)]^{-1} = \\ = & \sum_{j\geq 0}[\xi - zW_L(\beta,\omega)]^{-1}z^j \left\{[A(\alpha)W_L(\beta,\omega)A(-\alpha) - W_L(\beta,\omega)][\xi - zW_L(\beta,\omega)]^{-1}\right\}^j, \end{aligned}$$

the result follows.

Corollary 4.1 *The operator $A(\alpha)g_\omega(\xi,z;\beta,\tau)A(-\alpha)$ belongs to $B(\mathbf{L}^2,\mathbf{L}^\infty)$ if α is small enough, and uniformly in ξ, z, $\mathbf{x}_0$, ω and L one has:*
i. $||A(\alpha)g_\omega(\xi,z;\beta,\tau)A(-\alpha)|| \leq \text{const}(\beta,\tau)$;
ii. $\int_\Lambda d\mathbf{y}\ e^{2\alpha|\mathbf{y}-\mathbf{x}_0|}|T_\omega(\mathbf{x}_0,\mathbf{y})|^2 \leq \text{const}(\beta,\tau)$;
iii. $e^{\alpha|\mathbf{x}-\mathbf{y}|}|T_\omega(\mathbf{x},\mathbf{y})| \leq \text{const}(\beta,\tau)$.

Proof. i. It is an immediate consequence of Propositions 4.11 and 4.12.
ii. Let $\phi = A(\alpha)g_\omega(\xi,z;\beta,\tau)A(-\alpha)\psi$, where ϕ is bounded and continuous. From i. it follows:

$$|\phi(\mathbf{x}_0)| = |\int_\Lambda d\mathbf{y}\ e^{\alpha\rho(\mathbf{y}-\mathbf{x}_0)}T_\omega(\mathbf{x}_0,\mathbf{y})\psi(\mathbf{y})| \leq \text{const}(\beta,\tau)||\psi||_{\mathbf{L}^2}, \tag{4.75}$$

and the result follows from the representation theorem of linear and continuous functionals on $\mathbf{L}^2$.
iii. Rewrite the identity $g_\omega(\xi,z;\beta,\tau) = g_\omega(\xi,z;\beta,\tau/2)W_L(\tau/2,\omega)$ in terms of integral kernels, use (4.67), (2.3), ii., the triangle and Schwartz inequalities, and the proof is completed.

Let $\delta\omega > 0$ be such that $\omega = \omega_0 + \delta\omega \in \Omega$. Define the bounded operator $\tilde{g}_\omega(\xi,z;\beta,\tau)$ given by the following integral kernel:

$$\tilde{T}_\omega(\mathbf{x},\mathbf{x}') := e^{\imath\delta\omega\varphi(\mathbf{x},\mathbf{x}')}T_{\omega_0}(\mathbf{x},\mathbf{x}'). \tag{4.76}$$

From (4.3) and (4.57) it follows that if $\delta\omega$ is sufficiently small then there exists a numerical constant such that uniformly in L:

$$\sup_\xi \sup_z ||[\xi - zS_\omega(\beta)]^{-1}|| \leq \text{const} \tag{4.77}$$

and:

$$\sup_\xi \sup_z ||[\xi - zS_\omega(\beta)]^{-1} - [\xi - zW_L(\beta,\omega)]^{-1}|| \leq \text{const}\ \delta\omega. \tag{4.78}$$

We give now an important property of $\tilde{g}_\omega(\xi,z;\beta,\tau)$:

Proposition 4.13 *Under the above conditions, there exists a numerical constant such that if $\delta\omega$ is small enough, then uniformly in ξ, z and L, the following $B(\mathbf{L}^2)$ estimate takes place:*

$$||[\xi - zS_\omega(\beta)]^{-1} zS_\omega(\tau) - \tilde{g}_\omega(\xi, z; \beta, \tau)|| \leq \text{const } \delta\omega. \tag{4.79}$$

Proof. The integral kernel of the operator $[\xi - zS_\omega(\beta)]\tilde{g}_\omega(\xi, z; \beta, \tau)$ is given by:

$$\xi \tilde{T}_\omega(\mathbf{x}, \mathbf{x}') - z \int_\Lambda d\mathbf{y} \; S_\omega(\mathbf{x}, \mathbf{y}; \beta) \tilde{T}_\omega(\mathbf{y}, \mathbf{x}'). \tag{4.80}$$

Let's notice the following property of the magnetic phase:

$$\varphi(\mathbf{x}, \mathbf{y}) + \varphi(\mathbf{y}, \mathbf{x}') = \varphi(\mathbf{x}, \mathbf{x}') + \text{fl}(\mathbf{x}, \mathbf{y}, \mathbf{x}'), \tag{4.81}$$

where $\text{fl}(\mathbf{x}, \mathbf{y}, \mathbf{x}') = 1/2 \, \mathbf{B} \cdot [(\mathbf{y} - \mathbf{x}') \wedge (\mathbf{x} - \mathbf{y})]$. Using (4.81) and (4.65) in (4.80) we obtain:

$$[\xi - zS_\omega(\beta)]\tilde{g}_\omega(\xi, z; \beta, \tau) = zS_\omega(\tau) + R, \tag{4.82}$$

where R is an integral operator given by:

$$-z \; e^{i\delta\omega\varphi(\mathbf{x}, \mathbf{x}')} \int_\Lambda d\mathbf{y} \; (e^{i\delta\omega \; \text{fl}(\mathbf{x}, \mathbf{y}, \mathbf{x}')} - 1) \; G_{\omega_0, L}(\mathbf{x}, \mathbf{y}; \beta) T_{\omega_0}(\mathbf{y}, \mathbf{x}'). \tag{4.83}$$

Because $|e^{i\delta\omega \; \text{fl}(\mathbf{x}, \mathbf{y}, \mathbf{x}')} - 1| \leq \delta\omega |\mathbf{x} - \mathbf{y}| \, |\mathbf{y} - \mathbf{x}'|$, denoting with P the operator given by $|\mathbf{x} - \mathbf{y}| \, |zG_{\omega_0, L}(\mathbf{x}, \mathbf{y}; \beta)|$ and with Q the operator corresponding to $|\mathbf{y} - \mathbf{x}'| \, |T_{\omega_0}(\mathbf{y}, \mathbf{x}')|$, it follows that $||R|| \leq \delta\omega ||P|| \, ||Q||$. Using (2.3), the point iii. of Corollary 4.1, (4.33) and (4.77), the proof follows.

The proof of (4.9) is easy now. From the following chain of equalities:

$$\begin{aligned} & g_\omega(\xi, z; \beta, \beta/2) - \tilde{g}_\omega(\xi, z; \beta, \beta/2) = [(\xi - zW_L(\beta, \omega))^{-1} - (\xi - zS_\omega(\beta))^{-1}] zW_L(\beta/2, \omega) + \\ + \quad & (\xi - zS_\omega(\beta))^{-1} z[W_L(\beta/2, \omega) - S_\omega(\beta/2)] + (\xi - zS_\omega(\beta))^{-1} zS_\omega(\beta/2) - \tilde{g}_\omega(\xi, z; \beta, \beta/2), \end{aligned} \tag{4.84}$$

using (4.57), (4.79), (4.78) and (4.77) the result follows.

The last thing remained here is proving that the trace class operator $A_\omega(\xi, z; \beta) = \tilde{g}_\omega(\xi, z; \beta, \beta/2) S_\omega(\beta/2)$ has its trace equal to tr $g_{\omega_0}(\xi, z; \beta, \beta)$. Indeed, because A_ω fulfils the conditions of Proposition 4.9 and noticing that $\varphi(\mathbf{x}, \mathbf{x}') = -\varphi(\mathbf{x}', \mathbf{x})$, one can write:

$$\text{tr} A_\omega(\xi, z; \beta) = \int_{\Lambda^2} d\mathbf{x} \, d\mathbf{x}' \; T_{\omega_0}(\xi, z; \beta, \beta/2; \mathbf{x}, \mathbf{x}') G_{\omega_0, L}(\mathbf{x}', \mathbf{x}; \beta/2) = \int_\Lambda d\mathbf{x} \; T_{\omega_0}(\xi, z; \beta, \beta; \mathbf{x}, \mathbf{x}) = \text{tr} g_{\omega_0}(\xi, z; \beta, \beta). \tag{4.85}$$

4.2 The proof of II. and III.

The analyticity of $\Gamma_\infty(\beta, z, \omega_0)$ in $\mathcal{D}$ follows from the bound (see (2.16)): $|g_\sigma(\zeta)| \leq \text{const}(\sigma, K)|\zeta|$ where K is some compact in $\mathbf{C} \setminus [1, \infty)$.

In what follows, we will prove that if $z \in \mathcal{D}_0 := \{|z| < 1\}$, then:

$$\lim_{L \to \infty} \Gamma_L(\beta, z, \omega_0) = \Gamma_\infty(\beta, z, \omega_0). \tag{4.86}$$

Because $|z| < 1$, the grand-canonical pressure will be (see (2.10)):

$$P_L(\beta, z, \omega) = \sum_{n=1}^\infty \frac{z^n}{n} \frac{1}{\beta L^3} \text{tr} W_L(n\beta, \omega) = \sum_{n=1}^\infty \frac{z^n}{n} \left(\frac{1}{\beta L^3} \int_\Lambda d\mathbf{x} \; G_{\omega, L}(\mathbf{x}, \mathbf{x}; n\beta) \right). \tag{4.87}$$

Under the same conditions, the magnetisation reads as:

$$\Gamma_L(\beta, z, \omega_0) = -\frac{e}{c}\sum_{n=1}^{\infty}\frac{z^n}{n}\frac{1}{\beta L^3}\,\mathrm{tr}\left[\frac{\partial W_L}{\partial \omega}\right](n\beta, \omega_0). \tag{4.88}$$

An important quantity is the integral kernel of the "whole space" semigroup:

$$G_{\omega,\infty}(\mathbf{x},\mathbf{x}';\beta) = \frac{e^{\imath\omega\varphi(\mathbf{x},\mathbf{x}')}}{(2\pi\beta)^{3/2}}\frac{\omega\beta/2}{\sinh(\omega\beta/2)}\exp\left\{-\frac{1}{2\beta}\left[\frac{\omega\beta/2}{\tanh(\omega\beta/2)}(\mathbf{e_3}\wedge(\mathbf{x}-\mathbf{x}'))^2 + (\mathbf{e_3}\cdot(\mathbf{x}-\mathbf{x}'))^2\right]\right\}. \tag{4.89}$$

By direct computation, one can obtain a similar estimate for $G_{\omega,\infty}(\mathbf{x},\mathbf{x}';\beta)$ as that one from (4.45), for all $\mathbf{x},\mathbf{x}' \in \mathbf{R}^3$. Denote with:

$$g(\beta,\omega) = G_{\omega,\infty}(\mathbf{x},\mathbf{x};\beta) = \frac{1}{(2\pi\beta)^{3/2}}\frac{\omega\beta/2}{\sinh(\omega\beta/2)}.$$

Then it is easy to see that if $|z| < 1$:

$$P_\infty(\beta,z,\omega_0) = \sum_{n=1}^{\infty}\frac{z^n}{n}\frac{g(n\beta,\omega_0)}{\beta}, \quad \Gamma_\infty(\beta,z,\omega_0) = -\frac{e}{\beta c}\sum_{n=1}^{\infty}\frac{z^n}{n}\frac{\partial g}{\partial\omega}(n\beta,\omega_0).$$

A particular case of the result from [M-M-P] is the following:

$$\lim_{L\to\infty}\frac{1}{L^3}\,\mathrm{tr}\left[\frac{\partial W_L}{\partial\omega}\right](\beta,\omega_0) = \frac{\partial g}{\partial\omega}(\beta,\omega_0). \tag{4.90}$$

It follows that if we manage to prove the existence of a positive function f with at most polynomial growth such that:

$$\left|\frac{1}{L^3}\,\mathrm{tr}\left[\frac{\partial W_L}{\partial\omega}\right]\right|(n\beta,\omega_0) \leq f(n\beta), \tag{4.91}$$

then (4.86) would be true.

The next corollary is a direct consequence of Proposition 4.8 ii.:

Corollary 4.2 *Under the conditions of Proposition 4.8 one has:*

$$\lim_{\delta\omega\searrow 0}\frac{1}{\delta\omega}\int_\Lambda d\mathbf{x}\,[G_{\omega_0+\delta\omega,L}(\mathbf{x},\mathbf{x};\beta) - G_{\omega_0,L}(\mathbf{x},\mathbf{x};\beta)] = \mathrm{tr}\left[\frac{\partial W_L}{\partial\omega}\right](\beta,\omega_0) = \tag{4.92}$$

$$= 2\int_0^\beta d\tau\int_{\Lambda^2} d\mathbf{x}\,d\mathbf{y}G_{\omega_0,L}(\mathbf{x},\mathbf{y};\beta-\tau)\mathbf{a}(\mathbf{y}-\mathbf{x})\cdot(-\imath\nabla_\mathbf{y} - \omega_0\mathbf{a}(\mathbf{y}))G_{\omega_0,L}(\mathbf{y},\mathbf{x};\tau).$$

Use (2.3), (4.45), (4.37) and (4.34) in (4.92) and (4.91) follows. The proof of (4.86) is now completed.

Acknowledgements:

The author thanks Professors Angelescu, Bundaru and Nenciu for their encouragements and fruitful discussions.

References

[A] Angelescu, N.: Ph.D thesis, I.F.A., Bucharest, 1976

[A-B-N 1] Angelescu, N., Bundaru, M., Nenciu, G.: On the perturbation of Gibbs semigroups. Commun. Math. Phys. **42**, 29-30 (1975)

[A-B-N 2] Angelescu, N., Bundaru, M., Nenciu, G.: On the Landau diamagnetism. Commun. Math. Phys. **42**, 9-28 (1975)

[A-C] Angelescu, N., Corciovei, A.: On free quantum gases in a homogeneous magnetic field. Rev. Roum. Phys. **20**, 661-671 (1975)

[B-H-L] Broderix, K., Hundertmark, D., Leschke, H.: Continuity properties of Schrodinger semigroups with magnetic fields. Mathematical-Physics preprint archive of University of Texas at Austin.

[C-N] Cornean, H.D., Nenciu, G.: On eigenfunction decay of two dimensional magnetic Schrödinger operators. Commun. Math. Phys. **192**, 671-685 (1998)

[H] Huang, K.: Statistical mechanics. New York-London: John Wiley & Sons, Inc. 1963

[H-P] Hille, E., Phillips, R.S.: Functional integral and semigroups. Amer. Math. Soc., Providence, Rhode Island, 1957.

[K-U-Z] Kac, M., Uhlenbeck, G.E., Ziff, R.M.: The ideal Bose-Einstein gas, revisited. Phys. Rep. **32C** n^o 4, 169-248 (1977)

[K] Kunz, H.: Surface orbital magnetism. J. Stat. Phys. **76**, 183-207 (1994)

[M-M-P] Macris, N., Martin, Ph.A., Pulé, J.V.: Large volume asymptotics of Brownian integrals and orbital magnetism. Ann. I.H.P. Phys. Theor. **66**, 147-183 (1997)

[R-S 1,4] Reed, M., Simon, B.: Methods of modern mathematical physics **I, IV**. New York: Academic Press 1975

[S] Simon, B.: Schrödinger semigroups. Bull. Amer. Math. Soc. (N. S.) **7**, 447-510 (1982)