SOLUTIONS, SPECTRUM, AND DYNAMICS FOR SCHRÖDINGER OPERATORS ON INFINITE DOMAINS

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ABSTRACT. Let $H_V^\Omega = -\Delta + V(x)$ be a Schrödinger operator defined on an unbounded domain $\Omega \subset \mathbb{R}^d$ with Dirichlet boundary conditions on $\partial\Omega$ (when $\Omega = \mathbb{R}^d$ there is no boundary condition, of course). Let $u(x, E)$ be a solution of the Schrödinger equation $(H_V^\Omega - E)u(x, E) = 0$, and let B_R denote a ball of radius R centered at zero. We show relations between the rate of growth of the L^2 norm $\|u(x, E)\|_{L^2(B_R \cap \Omega)}$ of such solutions, as $R \to \infty$, and continuity properties of spectral measures of the operator H_V^Ω. These results naturally lead to new criteria for the identification of various spectral properties. We also prove new fundamental relations between the rate of growth of L^2 norms of generalized eigenfunctions, dimensional properties of spectral measures, and dynamical properties of the corresponding quantum systems. We apply these results to study transport properties of some particular Schrödinger operators.

1. Introduction and main results

In this paper we investigate the relations between the rate of decay of solutions of Schrödinger equations, continuity properties of spectral measures of the corresponding operators, and dynamical properties of the corresponding quantum systems. The first main result of this paper shows that, in great generality, certain upper bounds on the rate of growth of L^2 norms of generalized eigenfunctions over expanding balls imply certain minimal singularity of the spectral measures. Consider an operator H_V^Ω defined by the differential expression

$$H_V^\Omega = -\Delta + V(x)$$

on some connected infinite domain Ω with a smooth boundary and with Dirichlet boundary conditions on $\partial\Omega$. The case of $\Omega = \mathbb{R}^d$ is not excluded; no boundary conditions are needed in this case. To every vector $\phi \in L^2(\Omega)$ we associate a spectral measure μ^ϕ in the usual way (namely, μ^ϕ is the unique Borel measure on $\mathbb{R}$ obeying $\int f(E)\, d\mu^\phi(E) = (f(H_V^\Omega)\phi, \phi)$ for any Borel function f). For any measure μ, we define the upper α-derivative $D^\alpha \mu(E)$ in the standard way:

$$D^\alpha \mu(E) = \limsup_{\delta \to 0} \frac{\mu(E - \delta, E + \delta)}{\delta^\alpha}.$$

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We denote by B_R the ball of radius R centered at the origin, and use the notation $\|f\|_{B_R}$ for the L^2 norm of the function f restricted to B_R. We denote by W^l_m the usual Sobolev spaces of functions f such that $D^l f$ exists in the distributional sense and $\int(|u|^m + |D^l u|^m)dx < \infty$. We say that $f(x) \in W^l_{m,\mathrm{loc}}(\Omega)$ if $f(x) \in W^l_m(\Omega \cap B_R)$ for every $R < \infty$. One of the main theorems that we prove here is the following:

Theorem 1.1. *Assume that the potential $V(x)$ belongs to L^∞_{loc} and is bounded from below, and Ω is a domain with piecewise smooth boundary. Suppose that there exists a distributional solution $u(x, E)$ of the generalized eigenfunction equation*

$$(H_V^\Omega - E)u(x, E) = 0 \tag{1}$$

satisfying the boundary conditions and such that for some α, $0 \le \alpha \le 1$, we have

$$\liminf_{R\to\infty} R^{-\alpha} \int\limits_{B_R\cap\Omega} |u(x, E)|^2\, dx < \infty. \tag{2}$$

Fix some compactly supported $\phi(x) \in L^2(\Omega)$ such that

$$\int\limits_\Omega \phi(x)u(x, E)\, dx \neq 0.$$

Then we have

$$D^\alpha \mu^\phi(E) > 0.$$

Remarks. 1. Notice that under our assumptions on the potential, we have $u \in W^2_{2,\mathrm{loc}}$ by standard results on Sobolev estimates for elliptic operators (see, e.g., [12]), and the boundary values for u are well-defined.

2. We chose not to formulate this Theorem for more general classes of potentials, domains, and boundary conditions in order to be able to give a transparent proof. Certainly, we can extend this theorem to wider classes of potentials and boundary conditions. The nature of the limitations will be clear from the proof and Stark operators example in Appendix 2. For instance, when $\Omega = \mathbb{R}^d$, we only ask that the negative part of the potential, V_-, belongs to the Kato class K^d (see, e.g., [3, 39] for the definition of Kato classes).

3. If we replace "$< \infty$" in (2) by "$= 0$", we obtain that $D^\alpha \mu^\phi(E) = \infty$.

Theorem 1.1 provides information on the pointwise behavior of spectral measures from rather simple and natural assumptions about the behavior of generalized eigenfunctions. From this theorem follow new criteria for the existence of absolutely continuous spectrum or singular continuous spectrum of given dimensional characteristics (see Section 2, and, in particular, Theorem 2.5 for more details). This contrasts the well-known result (see [5, 38, 39]) that existence of a polynomially bounded (but not

L^2) solution of (1) implies that the energy E belongs to the essential spectrum of H_V, but gives no further information on the structure of the essential spectrum. To the best of our knowledge, Theorem 1.1 is the first rigorous result providing a relation between the behavior of solutions and pointwise properties of the spectral measures for multidimensional Schrödinger operators.

A result analogous to Theorem 1.1 also holds for discrete Schrödinger operators defined on some $\Omega \subset \mathbb{Z}^d$ by

$$(h_v u)(n) = \sum_{|m-n|=1,\, m\in\Omega} u(m) + v(n)u(n).$$

We discuss this extension in Section 3. In Appendix 1, we also indicate that results similar to Theorem 1.1 hold for more general elliptic and higher order operators.

The motivation for seeking relations between the pointwise in energy behavior of solutions and properties of spectral measures comes from the fact that in many problems the solutions are among the objects we can hope to investigate. When we are interested in the fine structure of the spectrum of Schrödinger operators for which the methods of scattering theory are not applicable, there are very limited tools in higher dimensions which may be effectively used for spectral analysis. On the other hand, for one-dimensional Schrödinger operators the subordinacy theory created by Gilbert and Pearson [14, 15] and further extended by Jitomirskaya and Last [19, 20, 21] provides a powerful method for spectral analysis. The main results of the above mentioned papers give a necessary and sufficient link between the behavior of solutions and the singularity of the spectral measure. Subordinacy theory played an important role in many recent results in one-dimensional spectral theory [7, 9, 18, 19, 20, 21, 24, 26, 29, 35].

In this paper, we derive only a sufficient-type relation between the solutions and the spectrum, but in much greater generality. However, in contrast to subordinacy theory, which requires comparison of different solutions, we need information about only one solution—the one obeying the appropriate boundary conditions. We remark that for one-dimensional Schrödinger operators, the result of Theorem 1.1 can be derived from subordinacy theory [20, 21].

Our second major result in this paper establishes a fundamental relation between spectral properties, generalized eigenfunctions and quantum dynamics, and in particular, provides new bounds for the transport properties of quantum systems. We study the behavior of the time-averaged moments of the position operator X under the Schrödinger evolution. Pick some initial state ψ and consider

$$\langle\langle |X|^m \rangle\rangle_T = \frac{1}{T}\int_0^T |\langle |X|^m \exp(-iH_V^\Omega t)\psi, \exp(-iH_V^\Omega t)\psi\rangle|\, dt.$$

Recall that a measure μ is called α-continuous if it gives zero weight to any set of zero α-dimensional Hausdorff measure (we recall the definition of these measures in Section 2). Let us denote by $P_{\alpha c}$ the spectral projector on the $\alpha-$continuous spectral subspace, the set of all vectors ξ such that μ^ξ is α-continuous (see [30]). In particular, if μ^ψ has an α-continuous component (i.e., $P_{\alpha c}\psi \neq 0$), then the following lower bound holds [8, 16, 17, 30]:

$$\langle\langle |X|^m \rangle\rangle_T \geq C_m T^{\frac{m\alpha}{d}}$$

(here d is the space dimension and C_m is a constant depending on μ^ψ and m).

Recall that for a wide class of Schrödinger operators, one has a generalized eigenfunction expansion theorem (see, e.g., [5, 29, 39]). In particular, for every ψ there is a unique unitary map U_ψ from the cyclic subspace $\mathcal{H}_\psi$, generated by the vector ψ and the operator H_V, to $L^2(\mathbb{R}, d\mu^\psi(E))$. This map sends ψ to a function equal to 1 everywhere and realizes a unitary equivalence $U_\psi H_V|_{\mathcal{H}_\psi} U_\psi^{-1} = E$, where E stands for the operator of multiplication by E. The operator U_ψ is an integral operator with kernel $u(x, E)$, where the $u(x, E)$'s, for each fixed E, solve (1) and are called generalized eigenfunctions. We will say that the $u(x, E)$'s correspond to ψ if they constitute the kernel of the unitary map U_ψ described above. Note that they are only defined a.e. w.r.t. μ^ψ. We prove the following theorem, which holds in both the discrete and continuous settings:

Theorem 1.2. *Let ψ be a vector for which there exists a Borel set $S \subset \mathbb{R}$ of positive μ^ψ measure, such that the restriction of μ^ψ to S is α-continuous and, in addition, the generalized eigenfunctions $u(x, E)$ for all $E \in S$ satisfy*

$$\limsup_{R\to\infty} R^{-\gamma} \|u(x, E)\|^2_{B_R} < \infty \tag{3}$$

for some γ such that $0 < \gamma < d$. Then, for any $m > 0$, there exists a constant C_m such that

$$\langle\langle |X|^m \rangle\rangle_T \geq C_m T^{\frac{m\alpha}{\gamma}} \tag{4}$$

for all $T > 0$.

Remarks. 1. Theorem 1.2 is somewhat related to (although it does not coincide with) some recent heuristic results by Ketzmerick et. al. [23].

2. It may be seen from Theorem 1.1 that we cannot have $\gamma < \alpha$, since it would follow that the upper γ-derivative of the spectral measure is positive on too large a set (see Corollary 2.6). The physical reason is that when V is bounded from below, the velocity is bounded, and the propagation rate is at most ballistic. However, the range of applicability of Theorem 1.2 is wider than that of Theorem 1.1. In particular, it is applicable to operators with strongly negative potentials, such as Stark operators, which exhibit faster-than-ballistic transport. See Appendix 2.

The somewhat striking aspect of Theorem 1.2 is that for a fixed non-zero spectral dimension, faster decay of $u(x, E)$ leads to faster transport. Theorem 1.2 shows that the behavior of the generalized eigenfunctions plays an important role in determining dynamical properties of quantum systems. We apply Theorem 1.2 to investigate the dynamics in the random decaying potentials model studied in [26]. When weakly coupled, these systems have (almost surely) some singular continuous spectrum with local dimensions that depend on the energy, but we show that the dynamical spreading of wavepackets, for any energy region where the spectrum is continuous, is almost ballistic with probability one. More precisely, we show that for almost every realization, we have for every $\epsilon > 0$ a bound of the form

$$\langle\langle |X|^m \rangle\rangle_T \geq C_{m,\epsilon} T^{m(1-\epsilon)}.$$

The paper is organized as follows. In Section 2 we prove Theorem 1.1 and its corollaries, rendering new spectral criteria. In Section 3 we sketch the argument for similar results in the discrete setting. In Section 4 we consider some simple examples, in particular, showing that the result of Theorem 1.1 provides only a sufficient but not necessary criterion for positivity of the derivative of the spectral measure. It is, however, an optimal result in the sense that one cannot in general say more by looking only at the rate of growth of the L^2 norm (Section 5). It remains an interesting open question to find additional properties of solutions that determine the spectrum (or other important characteristics of the operator, such as transport properties) completely. In Section 5 we study the relationship between solutions, spectral dimension, and quantum dynamics, in particular proving Theorem 1.2. In the appendices, we indicate further possible generalizations for elliptic and higher order operators and consider dynamics for strongly perturbed one-dimensional Stark operators. The example of Stark operators provides another illustration of the relationship between the behavior of solutions and transport properties.

2. Solutions and spectrum: Continuous case

We begin the proof of Theorem 1.1 with the following simple observation:

Lemma 2.1 *Let A be a self-adjoint operator acting on a Hilbert space $\mathcal{H}$ and fix a vector $\phi \in \mathcal{H}$. Let $z \in \mathbb{C} \setminus \mathbb{R}$. Then*

$$\operatorname{Im} z \|(A-z)^{-1}\phi\|_{\mathcal{H}}^2 = \operatorname{Im}\left((A-z)^{-1}\phi, \phi\right).$$

Proof. Consider the spectral representation associated with a vector ϕ and perform a straightforward computation:

$$\operatorname{Im}\left(\int\limits_R \frac{d\mu^\phi}{t-z}\right) = \operatorname{Im} z \int\limits_R \frac{d\mu^\phi}{|t-z|^2}.$$

$\square$

The first idea in the proof of Theorem 1.1 is to estimate from below $\operatorname{Im}((H_V^\Omega - E - i\epsilon)^{-1}\phi, \phi)$ as $\epsilon \to 0$. Such an estimate is equivalent to an estimate on the upper α-derivative of the spectral measure by the following lemma:

Lemma 2.2. *Let $Q_\mu^\beta(E)$ denote*

$$Q_\mu^\beta(E) = \limsup_{\epsilon\to 0} \epsilon^\beta \operatorname{Im}\left(\int \frac{d\mu(t)}{t-E-i\epsilon}\right).$$

Then

$$D^\alpha\mu(E) \le C_1 Q_\mu^{1-\alpha}(E) \le C_2 D^\alpha \mu(E),$$

where C_1, C_2 are positive constants depending only on α.

Proof. The proof is a direct computation. For details, we refer to [11], Lemmas 3.2 and 3.3. $\square$

To derive an estimate on the imaginary part of the Borel transform, we will use Lemma 2.1, namely estimates from below on the norm of the function

$$\theta(x, E+i\epsilon) = (H_V^\Omega - E - i\epsilon)^{-1}\phi(x)$$

over balls of radius of order $\frac{1}{\epsilon}$ as ϵ goes to zero over some properly chosen sequence.

The last technical lemmas that we need for the proof concern estimation of the W_2^1 norms of $u(x,E)$ and $\theta(x,z)$ in terms of their L^2 norms.

Lemma 2.3. *Let $\Omega \subset \mathbb{R}^d$ be a domain with piecewise smooth boundary. Suppose that the potential V belongs to L^∞_{loc} and is bounded from below, and let H_V^Ω denote an operator with Dirichlet boundary conditions on $\partial\Omega$. Suppose that the function $g(x,z)$ satisfies Dirichlet boundary conditions and*

$$(H_V^\Omega - z) g(x,z) = \phi(x),$$

where $\phi \in L^2(\Omega)$ is compactly supported and real-valued, and z is in general complex. Then

$$\|g\|_{W_2^1(B_R\cap\Omega)} \le C(z, V_-)\left(\|g\|_{L^2(B_{R+1}\cap\Omega)} + \|\phi\|_{L^2(\Omega)}\right). \tag{5}$$

The constant in (5) depends only on the lower bound on V and on z, and may be chosen uniformly for z in any compact set.

Proof. The proof is standard and we provide it for the sake of completeness. See, for example, [3, 39] for detailed exposition of similar results and further references. Throughout the proof, we assume that the function g is sufficiently smooth to justify integration by parts (local W_2^2 is sufficient). Clearly this is the case under our assumptions on V (see, e.g., [12]). To prove the bound (5) with the constant independent of R, let

$$g(x,z) = g_1(x,z) + ig_2(x,z),$$

where g_1, g_2 are real-valued. For any $\psi \in C^\infty(\Omega)$ such that $1 \geq \psi(x) \geq 0$, $\psi(x) = 1$ when $x \in B_R \cap \Omega$, $\psi(x) = 0$ when $x \notin B_{R+1} \cap \Omega$, we have

$$\begin{aligned} \int\limits_{B_R\cap\Omega} (\nabla g_1)^2\,dx \;&\leq\; \int\limits_{B_{R+1}\cap\Omega} \psi(\nabla g_1)^2\,dx \;=\; \int\limits_{\partial(\Omega\cap B_{R+1})} \psi\frac{\partial g_1}{\partial n} g_1\,d\sigma - \\ &- \int\limits_{B_{R+1}\cap\Omega} (\nabla\psi)(\nabla g_1)g_1\,dx - \int\limits_{B_{R+1}\cap\Omega} \psi g_1\Delta g_1\,dx, \end{aligned} \tag{6}$$

where $d\sigma$ is the surface measure on $\partial(\Omega \cap B_{R+1})$ induced from $\mathbb{R}^d$. The first term vanishes because g_1 vanishes on $\partial\Omega$ and ψ vanishes on $(\partial B_R) \cap \Omega$. Furthermore, by Green's formula

$$2\int\limits_{B_{R+1}\cap\Omega} (\nabla\psi)(\nabla g_1)g_1\,dx = \int\limits_{\partial(B_{R+1}\cap\Omega)} \frac{\partial\psi}{\partial n}(g_1)^2\,d\sigma - \int\limits_{B_{R+1}\cap\Omega} \Delta\psi(g_1)^2\,dx. \tag{7}$$

The boundary term in this equality is also equal to zero. Substituting (7) into (6), we find

$$\begin{aligned} \int\limits_{B_R\cap\Omega} (\nabla g_1)^2\,dx \;&\leq\; \frac{1}{2}\int\limits_{B_{R+1}\cap\Omega} \Delta\psi(g_1)^2\,dx \\ &+ \int\limits_{B_{R+1}\cap\Omega} \psi g_1\left((\operatorname{Re} z - V)g_1 + \phi - (\operatorname{Im} z)g_2\right)\,dx. \end{aligned}$$

Therefore,

$$\|g_1\|^2_{W_2^1(B_R)} \leq C_\psi \left(\|\phi\|^2_{L^2} + (2(1+|z|) + \|V_-\|_{L^\infty})\|g_1\|^2_{L^2(B_{R+1})} + \operatorname{Im} z\|g_2\|^2_{L^2(B_{R+1})}\right).$$

A similar estimate holds for g_2. Combining these two estimates, we obtain the result of the lemma. □

Remarks. 1. We have not tried to determine the most general classes of potentials and boundary conditions for which the lemma holds. With slightly more technical effort, we can treat some other boundary conditions, such as Neumann, for instance.

2. For the case of the whole space, the lemma is true under the assumption that $V_- \in K^d$, the Kato class, which allows singularities in the negative part of the

potential (see [39] for the definition and properties of potentials from these classes). This result follows from the technique developed in [3, 39], which uses Brownian motion to derive subsolution estimates implying bounds like in Lemma 2.3. Although [3, 39] consider only real z (and homogeneous equation), it is not hard to see that their arguments extend to give results like (5).

We now introduce an important object in our consideration. Suppose S is a domain with piecewise smooth boundary and f, g belong to $W^2_{2,\mathrm{loc}}(S)$. We denote by $W_{\partial S}[f,g]$ the following expression

$$W_{\partial S}[f,g] = \int\limits_{\partial S} (f(t)\frac{\partial g}{\partial n}(t) - \frac{\partial f}{\partial n}(t)g(t))\,d\sigma(t), \tag{8}$$

where σ is the surface measure induced from $\mathbb{R}^d$ and $\frac{\partial}{\partial n}$ is the derivative in the outer normal direction. The definition makes sense for $W^2_{2,\mathrm{loc}}$ functions by Sobolev trace theorems (see, e.g., [13]). The notation W stresses the fact that in one dimension, the corresponding expression is related to the Wronskian of two functions (precisely, it is the difference of the Wronskians taken at the endpoints of the interval S). We will abuse verbal notation and call the expression (8) the Wronskian of f and g over ∂S for the rest of this paper. The final lemma we need is

Lemma 2.4. *Suppose that two functions f, g are locally W^2_2 and satisfy Dirichlet boundary condition on $\partial\Omega$. Then for every R*

$$\int\limits_0^R |W_{\partial(B_r\cap\Omega)}[f,g]|\,dr \le \|f\|_{W^1_2(B_R\cap\Omega)}\|g\|_{W^1_2(B_R\cap\Omega)}.$$

Proof. We have $W_{\partial\Omega\cap B_R}[f,g] = 0$ since f and g satisfy the boundary conditions. Next note that

$$\int\limits_0^R |W_{\partial B_r\cap\Omega)}[f,g]|\,dr \le \int\limits_{B_R\cap\Omega} (|f||\nabla g| + |\nabla f||g|)\,dx \le \|f\|_{W^1_2(B_R\cap\Omega)}\|g\|_{W^1_2(B_R\cap\Omega)}.$$

We used the Cauchy-Schwartz inequality in the last step. □

Now we are ready to prove Theorem 1.1.

Proof. An interplay of the scales in space and in the spectral parameter plays an important role in the analysis. Let us assume that

$$\int\limits_\Omega \phi(x)u(x,E)\,dx = c \ne 0.$$

Take sufficiently large R_0, such that $\mathrm{supp}\phi \subset B_{R_0}$. By Green's formula we have

$$E \int\limits_{B_{R_0}\cap\Omega} \theta(x, E+i\epsilon)u(x,E)\,dx = W_{\partial(B_{R_0}\cap\Omega)}[\theta, u] + \int\limits_{B_{R_0}\cap\Omega} H_V^\Omega \theta(x, E+i\epsilon)u(x,E)\,dx =$$

$$= W_{\partial(B_{R_0}\cap\Omega)}[\theta, u] + (E+i\epsilon) \int\limits_{B_{R_0}\cap\Omega} \theta(x, E+i\epsilon)u(x,E)\,dx + \int\limits_{B_{R_0}\cap\Omega} \phi(x)u(x,E)\,dx.$$

In the above computation we used the definition of $\theta(x,z)$ and the fact that the function u satisfies $(H_V^\Omega - E)u = 0$. Hence, we obtain

$$W_{\partial(B_{R_0}\cap\Omega)}[\theta, u] = -c - i\epsilon \int\limits_{B_{R_0}\cap\Omega} \theta(x, E+i\epsilon)u(x,E)\,dx. \tag{9}$$

Let us integrate equation (9) from R_0 to some larger value of R :

$$\int\limits_{R_0}^{R} |W_{\partial(B_r)\cap\Omega}[\theta, u]|\,dr \geq |c|(R-R_0) - \epsilon \int\limits_{R_0}^{R} dr \left| \int\limits_{B_r\cap\Omega} \theta(x, E+i\epsilon)u(x,E)\,dx \right|.$$

Using Lemmas 2.3, 2.4, we see that

$$C^2(\|\theta(x, E+i\epsilon)\|_{L_2(B_{R+1}\cap\Omega)} + \|\phi\|_{L^2})\|u\|_{L_2(B_{R+1}\cap\Omega)} \geq$$

$$|c|(R-R_0) - \epsilon \int\limits_0^R dr \|\theta(x, E+i\epsilon)\|_{L_2(B_r\cap\Omega)}\|u\|_{L_2(B_r\cap\Omega)}. \tag{10}$$

According to the assumption (2) of the theorem, there exists a sequence $R_n \to \infty$, such that

$$\|u\|_{L_2(B_{R_n}\cap\Omega)} \leq C_1 R_n^{\frac{\alpha}{2}}. \tag{11}$$

Let us set $\epsilon_n = \frac{C_2}{R_n}$, and pick $R+1 = R_n$ and $\epsilon = \epsilon_n$ in formula (10). We obtain

$$(C^2 + C_2)(\|\theta(x, E+i\epsilon_n)\|_{L_2(B_{R_n}\cap\Omega)} + \|\phi\|_{L^2})\|u\|_{L_2(B_{R_n}\cap\Omega)} \geq |c|(R_n - R_0 - 1).$$

Substituting (11) into the last inequality, we find that there exists some constant C_3 such that for n large enough, we have

$$\|\theta(x, E+i\epsilon_n)\|_{L_2(B_{R_n}\cap\Omega)} \geq C_3 R_n^{1-\frac{\alpha}{2}} - \|\phi\|_{L^2}. \tag{12}$$

Now it remains to invoke Lemma 2.1 and note that

$$\mathrm{Im}\,((H_V - E - i\epsilon_n)^{-1}\phi, \phi) \geq \epsilon_n \|\theta(x, E+i\epsilon_n)\|^2_{L_2(B_{R_n})}$$

for every n. Using the estimate (12) and the relation between R_n and ϵ_n, we find

$$\mathrm{Im}\,((H_V - E - i\epsilon_n)^{-1}\phi, \phi) \geq C_4 \epsilon_n^{\alpha-1}$$

for sufficiently small ϵ_n The application of Lemma 2.2 now completes the proof. □

Remarks. 1. Theorem 1.1 also holds for wider classes of potentials and boundary conditions. The restrictions of the classes come from Lemma 2.3, the necessary estimate on the energy norms. With the help of smooth mollifiers to justify integration by parts, Theorem 1.1 can be extended to the classes to which one can extend Lemma 2.3.

2. We also note that the same argument as in the proof implies that $D^\alpha \mu^\phi(E) = \infty$ if instead of (2) in the assumption of Theorem 1.2 we suppose that

$$\liminf_{R\to\infty} R^{-\alpha}\|u(x,E)\|^2_{B_R} = 0.$$

We will use this fact in the proof of Corollary 2.6 below.

The next question that we would like to discuss is a sufficient condition for the existence of the various components of the spectrum. Let us recall the definition of Hausdorff measures and dimension. For $\alpha \in [0,1]$ and any $S \subset \mathbb{R}$, the α-dimensional Hausdorff measure of S is defined by

$$h^\alpha(S) = \lim_{\delta\to 0} \inf_{\delta-\text{covers}} \sum_{\gamma=1}^{\infty} |I_\gamma|^\alpha,$$

where I_γ are the intervals constituting the cover. The Hausdorff dimension of a set S is the infimum of all values of α such that $h^\alpha(S) = 0$. First, we are going to prove

Theorem 2.5. *Let H_V^Ω be a Schrödinger operator, with V and Ω satisfying the same conditions as in Theorem 1.1. Suppose that for a measurable set S of positive h^α measure, for each $E \in S$, there exists a non-trivial solution $u(x,E)$ of the generalized eigenfunction equation (1) satisfying the boundary conditions such that*

$$\liminf_{R\to\infty} R^{-\alpha}\|u(x,E)\|^2_{B_R} < \infty.$$

Then there exists a vector $\varphi \in L^2(R^n)$ such that $\mu^\varphi(S_1) > 0$ for any $S_1 \subset S$ of positive h^α measure. In particular, if $\alpha = 1$, we have absolutely continuous spectrum filling the set S.

Remark. In many applications, particularly in one dimension, one applies a reasoning different from the one suggested by Theorem 2.5 to derive existence of various dimensional spectral components from results like Theorem 1.1. One proves the existence of solutions as in (2) for a.e. E, and then uses rank-one perturbation arguments (see, e.g., [20, 26]).

Proof. Recall that for every self-adjoint operator there is an associated spectral measure of maximal type, μ, such that for every ψ and any measurable set S, $\mu^\psi(S) > 0$ implies $\mu(S) > 0$. A vector χ is of the maximal type if for any measurable set S,

$\mu^\chi(S) > 0$ given that $\mu(S) > 0$. We will show that for any $S_1 \subset S$ of positive α-dimensional Hausdorff measure, there exists a vector ψ with $\mu^\psi(S_1) > 0$. By the standard argument for the existence of vectors of maximal type (see e.g. [6]), this would imply existence of the vector φ as in the theorem. Pick some ball B_{R_0} such that $\|u(x,E)\|_{L^2(B_{R_0}\cap\Omega)} \neq 0$ for energies E in a subset S_2 of S_1 of positive h^α measure (it is easy to see that such a ball exists, because of the σ-additivity of h^α). We remark that for a wide class of operators H_V^Ω, an arbitrary ball will do because of the unique continuation (solutions $u(x,E)$ cannot vanish identically on any ball), but there is no need to invoke these results. Pick a basis $\{\psi_n(x)\}_{n=1}^\infty$ in the Hilbert space $L^2(B_{R_0}\cap\Omega)$. Since $\{\psi_n\}$ forms a basis, for every $E \in S_2$ there exists an n such that

$$\int\limits_{B_{R_0}\cap\Omega} \psi_n(x) u(x,E)\, dx \neq 0.$$

Consider the functions $D^\alpha\mu^{\psi_n}$ on the set S_2. By Theorem 1.1, for every $E \in S_2$ there exists an n such that $D^\alpha\mu^{\psi_n}(E) > 0$. In particular, by σ-additivity of h^α, there exists an n_0 such that $D^\alpha\mu^{\psi_{n_0}}(E) > 0$ for every E in a set $S_{n_0} \subset S_2$ of positive h^α measure. By the results of Rogers-Taylor theory (see [36], Theorem 63), it follows that the measure $\mu^{\psi_{n_0}}$ gives positive weight to the set S_{n_0}, and hence to the set S_1. The case of the absolutely continuous spectrum corresponds to $\alpha = 1$; in this case the application of Rogers-Taylor theory may be replaced by a well-known fact that a measure gives positive weight to a set of positive Lebesgue measure when its derivative is positive a.e. in this set. □

From Theorem 2.5 (or, essentially, from its proof and the remark after the proof of Theorem 1.1) immediately follows:

Corollary 2.6. *For any α, the set S of energies E for which there exists a solution $u(x,E)$ satisfying*

$$\liminf_{R\to\infty} R^{-\alpha}\|u(x,E)\|_{B_R}^2 = 0 \tag{13}$$

has zero h^α measure.

Remark. The fact that there may be only countably many values of E (counting multiplicities) for which equation (1) has L^2 solutions satisfying the boundary conditions, is an obvious consequence of the separability of the Hilbert space $L^2(\Omega)$. This corollary may be viewed as a less trivial generalization for slower rates of decay.

Proof. Suppose that S has positive h^α measure. By the remark after the proof of Theorem 1.1, (13) implies that $D^\alpha\mu^\phi(E) = \infty$ for every $E \in S$ and finitely supported ϕ such that $\int u(x,E)\phi(x) \neq 0$. Proceeding as in the proof of Theorem 2.5, we can find a vector φ such that $D^\alpha\mu^\varphi(E) = \infty$ for any E in some set of positive h_α measure.

This is not possible by Rogers-Taylor (see [36], Theorem 67) and therefore gives a contradiction. We remark that for $\alpha = 1$, this argument reduces to the well-known statement that a finite Borel measure μ^ψ cannot have an infinite derivative on a set of positive Lebesgue measure. $\square$

We would like to end this section by drawing a link with the well-known results of Rellich [34] and Kato [22] who showed, respectively, that for the free Laplacian and the Laplacian with a short-range perturbation (i.e., a potential which satisfies $|V(x)| \le C(1+|x|)^{-1-\epsilon}$), there are no solutions satisfying (13) with $\alpha = 1$ for *any energy.* Corollary 2.6 shows that for a much larger class of potentials, such solutions are still in some sense "exceptional" and can only occur on a set of energies of zero Lebesgue measure.

3. Solutions and spectrum: Discrete case

In this section, we consider discrete Schrödinger operators. All the results of the previous section extend to the discrete setting. In fact, the proofs are easier due to the absence of the Sobolev estimates issue, and there are no restrictions on potential.

Let Ω be some connected infinite domain in $\mathbb{Z}^d$. We define the Schrödinger operator h_v^Ω on $L^2(\Omega)$ with Dirichlet boundary conditions by

$$h_v^\Omega f(n) = \sum_{|n-m|=1,\, m\in\Omega} f(m) + v(n)f(n).$$

It is easy to check that the operator defined in this way is self-adjoint.

We need an analog of the Green's formula in the discrete setting. For any domain $S \subset Z^d$ let us denote by ∂S the set of points outside S which have a point of S within a unit distance. We have for any two functions f, g

$$\sum_{n\in S}(h_v^\Omega f(n)g(n) - f(n)h_v^\Omega g(n)) = \sum_{m\in\partial S}\left(f(m)\sum_{l\in N_S(m)} g(l) - g(m)\sum_{l\in N_S(m)} f(l)\right),$$

where $N_S(m)$ denotes the set of neighbors of the point $m \in \partial S$ lying in S (so that $|m-n| = 1$ for any $n \in N_S(m)$). We will say therefore that the analog of the Wronskian over ∂S of two functions is, in the discrete setting,

$$w_{\partial S}[f,g] = \sum_{m\in\partial S}\left(f(m)\sum_{l\in N_S(m)} g(l) - g(m)\sum_{l\in N_S(m)} f(l)\right).$$

For convenience, in all considerations for the discrete case, we replace the balls B_R with cubes C_R. The point $n = (n_1, \dots n_d)$ of the lattice belongs to C_R if and only if $|n_i| \le R$ for all $i = 1, \dots, d$.

We now formulate and prove an analog of Theorem 1.1 in the discrete case.

Theorem 3.1. *Suppose that there exists a solution $u(n, E)$ of the generalized eigenfunction equation*

$$(h_v^\Omega - E)u(n, E) = 0 \tag{14}$$

satisfying the Dirichlet boundary conditions on $\partial\Omega$. Suppose that for some α, $0 \le \alpha \le 1$, we have

$$\liminf_{R\to\infty} R^{-\alpha} \sum_{n \in C_R \cap \Omega} |u(n, E)|^2 \, dx < \infty. \tag{15}$$

Fix some vector ϕ of compact support such that

$$\sum_n u(n, E)\phi(n) \neq 0.$$

Then we have

$$D^\alpha \mu^\phi(E) > 0.$$

In particular, if $u(n_0, E) \neq 0$, then

$$D^\alpha \mu^{\delta_{n_0}}(E) > 0.$$

(here δ_{n_0} is a function equal to 1 at n_0 and 0 otherwise).

Proof. The argument repeats the proof of Theorem 1.1, except that we do not need Lemma 2.3. The analog of Lemma 2.4 is proven directly by the observation that

$$w_{\partial(\Omega \cap C_r)}[f, g] = w_{\partial C_r \setminus \partial\Omega}[f, g],$$

and

$$\sum_{r=1}^{R} |w_{\partial C_r \setminus \partial\Omega}[f, g]| \le d\|f\|_{L^2(C_{R+1}\cap\Omega)}\|g\|_{L^2(C_{R+1}\cap\Omega)}.$$

□

Remark. As in the continuous case, the result is also true for more general boundary conditions.

We also have an analog of Theorem 2.5:

Theorem 3.2. *Suppose that for each energy E in some measurable set S of positive h^α measure, there exists a non-trivial solution $u(n, E)$ of equation (14) satisfying Dirichlet boundary conditions and having the property (15). Then there exists a vector $\phi \in L^2(\mathbb{Z}^d)$, such that $\mu^\phi(S_1) > 0$ for any set $S_1 \subset S$ of positive h^α measure. In particular, if $\alpha = 1$, the set S is an essential support of the absolutely continuous* *part of the measure μ^ϕ restricted to S.*

The proof of this theorem is the same as the proof of Theorem 2.5.

4. Examples and discussion

The purpose of this section is purely illustrative — to show where the solutions we are studying are known to occur. However, these observations will also partly lead us to the issue which is the topic of the next section: the relationships between generalized eigenfunctions, spectrum, and dynamics. In addition, we show that the criteria given by Theorems 1.1 and 3.1 are sufficient but not, in general, necessary for the positivity of the derivatives of spectral measures. We give an explicit example to confirm this statement.

Our first remark is that solutions $u(x,E)$ satisfying

$$\liminf_{R\to\infty} R^{-1}\|u(x,E)\|^2_{B_R} < \infty$$

exist for every energy $E \neq 0$ in the spectrum in the case of the free Laplacian operator in $\mathbb{R}^d$ or in the cylinder with Dirichlet boundary conditions. In the cylinder case, we may take

$$u(x,E) = \exp(i\sqrt{E-E_l}x_1)Z_{\bar{l}}(x_2,\dots x_d),$$

where x_1 is the coordinate along the rotation axis, E_l is any eigenvalue (less then E) of the Laplace operator with Dirichlet boundary conditions on the $d-1$-dimensional ball, and $Z_{\bar{l}}(x_2,\dots,x_d)$ is any eigenfunction corresponding to this eigenvalue. In the free case, we can take any function

$$u(x,E) = r^{-\frac{d}{2}+1}J_\nu(\sqrt{E}r)Y_{\bar{l}}(\bar{\theta}),$$

where $Y_{\bar{l}}$ is any of the spherical harmonics corresponding to the eigenvalue $E_l = l(l+d-2)$ of the Laplace-Beltrami operator on the d-dimensional sphere, and J_ν is a Bessel function (without singularity at the origin) with ν defined by $\nu^2 = l(l+d-2)+(\frac{d}{2}-1)^2$. Note that for large r,

$$J_\nu(\sqrt{E}r) \sim Cr^{-\frac{1}{2}}\cos\left(\sqrt{E}r - \frac{2\pi\nu-\pi}{4}\right)(1+o(1)).$$

See, for example, [4, 44] for more information on spherical harmonics and Bessel functions.

Using the results of Agmon theory and related estimates on the Fourier transform (see [1] or [33], and [2]), it is straightforward to show that the existence for every $E \in (0,\infty)$ of solutions with the rate of growth of the L^2 norm as in (2) with $\alpha = 1$ extends to perturbations of the free Laplacian by short range potentials, $|V(x)| \leq C(1+|x|)^{-1-\epsilon}$, if C is sufficiently small. In one dimension, it was recently shown [7, 35] that such solutions exist for a.e. $E \in (0, \infty)$ for any potential V satisfying $|V(x)| \leq C(1+|x|)^{-\frac{1}{2}-\epsilon}$. This implies that the absolutely continuous spectrum of the free operator in one dimension is stable under all perturbations decaying at this rate. This result is optimal: there are potentials which satisfy $|V(x)| \leq C(1+|x|)^{-\frac{1}{2}}$ and lead to purely singular spectrum in $(0, \infty)$. The corresponding question about the borderline decay for the stability of the absolutely continuous spectrum is open in higher dimensions, with any power in $[1, \frac{1}{2}]$ a possible candidate, in principle. We conjecture

Conjecture I. *Suppose that H_V is a Schrödinger operator in $\mathbb{R}^d$ for which $|V(x)| \leq C(1+|x|)^{-\frac{1}{2}-\epsilon}$, $\epsilon > 0$. Then the absolutely continuous spectrum of the operator H_V fills the whole positive semi-axis.*

This conjecture would in particular follow from

Conjecture II. *Under the conditions of the previous conjecture, for a.e. $E \in (0, \infty)$ there exists a solution $u(x, E)$ of the generalized eigenfunction equation satisfying* (2) *with $\alpha = 1$.*

Our next example concerns Schrödinger operators with periodic potentials. Let $V(x)$ be a smooth periodic potential of period one in all variables $x_1, \dots, x_d$. Given E in the spectrum of H_V, consider the boundary value problem

$$\begin{aligned} &(H_V - E)b(x, E) = 0, \\ &\frac{\partial^j b}{\partial x_l^j}\Big|_{x_l=1} = \exp(i\theta_l)\frac{\partial^j b}{\partial x_l^j}\Big|_{x_l=0}, \quad l = 1, \dots d, \;\; j = 0, 1. \end{aligned} \tag{16}$$

The set of all values of $\theta \in [0, 2\pi)^d$ for which there exist solutions of the boundary value problem (16) is called the real (physical) Fermi surface F_E. From well-known results on spectral properties of periodic differential operators (see [28]), it follows that for all but a countable set of energies in the spectrum (exceptional points corresponding to band edges), we can find solutions $u(x, E)$ of the generalized eigenfunction equation (1) of the following type:

$$u(x, E) = \int_S b(x, \theta, E)\gamma(\theta)d\sigma, \tag{17}$$

where $S \subset F_E$ is a piece of an analytic $(d-1)$-dimensional surface, $\gamma(\theta)$ is a $C_0^\infty(S)$-function and $b(x, \theta, E)$ are Bloch functions satisfying (16)

$$b(x, \theta, E) = \exp(i\theta x)f(x, \theta, E),$$

where $f(x,\theta,E)$ is periodic with period one in all directions in x, continuous in x, and analytic (as an $L^2([0,1)^d)$ vector) in $\theta \in S$. We claim that $u(x,E)$ satisfies

$$\liminf_{R\to\infty} R^{-1}\|u(x,E)\|^2_{B_R} \leq \infty.$$

This can be shown in a way similar to the proof of this property in the case of Fourier transforms of measures supported on $(d-1)$-dimensional smooth surfaces (see [2]). Represent the equation of the surface S as $\theta_d = s(\theta_1,\dots,\theta_{d-1})$ (we can assume that S is small enough and θ_d is chosen so that this is possible). Then we can rewrite (17) as

$$u(x,E) = \int\limits_{S'} \exp(i\theta' x' + is(\theta')x_d) f(x,\theta',E)\gamma'(\theta')d\theta',$$

where the integration is now over the projection S' of S on the hyperplane $\theta_d = 0$, θ' denotes first $d-1$ coordinates, and γ' includes the Jacobian from the change of variables. Fix the value of x_d and integrate over the cube C'_R in the other coordinates $x' = x_1,\dots,x_{d-1}$:

$$\begin{aligned}\int\limits_{C'_R} |u(x,E)|^2 dx' &= \int\limits_{S'}\int\limits_{S'} \gamma'(\theta')\gamma'(\tilde\theta')\exp(i(s(\theta')-s(\tilde\theta'))x_d) \\ &\quad \int\limits_{C'_R} \exp(i(\theta'-\tilde\theta')x') f(x',\theta',E) f(x',\tilde\theta',E)\, dx' d\theta' d\tilde\theta'.\end{aligned}$$

Without loss of generality, take R to be an integer. Then we obtain

$$\int\limits_{C'_R} |u(x,E)|^2 dx' = \int\limits_{S'}\int\limits_{S'} d\theta' d\tilde\theta' \left(\prod_{j=1}^{d-1} \frac{\sin(R+\frac12)(\theta_j - \tilde\theta'_j)}{\sin\frac12(\theta_j-\tilde\theta'_j)}\right)\psi(\theta',\tilde\theta'), \tag{18}$$

where

$$\begin{aligned}\psi(\theta',\tilde\theta') &= \gamma'(\theta')\gamma'(\tilde\theta')\exp(i(s(\theta')-s(\tilde\theta'))x_d) \\ &\quad \int\limits_{C'_1} f(x',\theta',E) f(x',\tilde\theta',E)\exp(i(\theta'-\tilde\theta')x')\, dx'.\end{aligned}$$

Due to the properties of f and γ, the function ψ is smooth and hence the right-hand side in (18) converges as $R\to\infty$ to the constant

$$C = \int\limits_{S'} d\theta' \psi(\theta',\theta') = \int\limits_{S'} d\theta' |\gamma'(\theta')|^2 \left(\int\limits_{C'_1} |f(x',\theta'),E|^2\, dx'\right).$$

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FIGURE 1. The spiral domain

Therefore integrating in x_d from $-R$ to R, we obtain

$$\int_{B_R} |u(x,E)|^2 dx \leq CR,$$

as claimed.

Our last example in this section shows that the criteria for the positivity of derivatives of spectral measures, given by Theorems 1.1 and 3.1, provide a sufficient, but, in general, not necessary condition. The example is especially simple and transparent in the discrete setting. Let us consider the discrete plane $\mathbb{Z}^2$ and let Ω be an infinite "spiral" in this plane (see Figure 1; we marked by $\times$ the points which do not belong to the domain). Consider h_0^Ω defined on the spiral with Dirichlet boundary conditions. By inspection, we see that h_0^Ω acts on $l^2(\Omega)$ as a free one-dimensional Jacobi matrix. Hence the spectrum is absolutely continuous in $[-2,2]$, and for every E in this interval, there exists an explicitly computable unique solution $u(n,E)$ of the generalized eigenfunction equation satisfying the boundary conditions:

$$u(n,E) = \sin\left(\cos^{-1}\left(\frac{E}{2}\right)n\right).$$

This is a standard discrete plane wave. If we measure the linear distance N along the spiral, the square of the l^2 norm of this solution grows as N. However in $\mathbb{Z}^2$, we have

$$\|u(x,E)\|^2_{B_R\cap\Omega} \sim R^2.$$

Hence in this case, we cannot find solutions as in Theorem 1.1.

We remark that [40] contains an example of a *bounded* spiral "jelly roll" domain on which the Laplace operator with Neumann boundary conditions has absolutely continuous spectrum. In this case, for a.e. E in the spectrum, the norm of solutions becomes infinite for finite R.

5. Solutions and dynamics

In this section, we prove Theorem 1.2 and apply it to study quantum dynamics in the random decaying potentials model studied in [10] and more recently in [26, 27].

The previous section provided us with several examples of operators with absolutely continuous spectrum and solutions satisfying the condition (2) in Theorem 1.1 for $\alpha = 1$, and one example of an operator with absolutely continuous spectrum, but without such solutions. For the former three, the transport is ballistic for every vector (i.e., $\langle\langle |X|^m\rangle\rangle_T \sim T^m$); for the latter, it is easy to see that the transport is not ballistic (it is diffusive in $\mathbb{Z}^2$). Theorem 1.2 indicates that this is not a coincidence.

The proof of Theorem 1.2 is an extension of the proof of Theorem 6.1 of [30], and it is essentially the same in both the discrete and continuous settings. We will use a discrete notation which formally only covers the discrete case, but the continuous case follows from it in a totally straightforward manner (which essentially amounts to replacing n by x and some summations by integrals). We note that in [30] the continuous case (Theorem 6.2 of [30]) is getting an independent treatment, based on semi-group kernel inequalities. This is not needed here, since we assume the existence of eigenfunction expansions with suitable properties. This allows our Theorem 1.2 to cover some cases, such as Stark operators, that are excluded from Theorem 6.2 of [30].

Recall that a measure μ is called uniformly α-Hölder continuous (denoted UαH) if there exists a constant C such that for every interval I with $|I| \le 1$ we have

$$\mu(I) \le C|I|^\alpha. \tag{19}$$

α-continuous measures (recall that this means measures giving zero weight to all sets of zero h^α measure) can be approximated by UαH measures in the following sense:

Theorem (Rogers-Taylor [37]**).** *A finite Borel measure μ on $\mathbb{R}$ is α-continuous if and only if for every $\epsilon > 0$, there exist two mutually singular Borel measures μ_1^ϵ and μ_2^ϵ, such that $\mu = \mu_1^\epsilon + \mu_2^\epsilon$, where μ_1^ϵ is UαH and $\mu_2^\epsilon(\mathbb{R}) < \epsilon$.*

For UαH measures, we can study dynamics with the aid of the following Strichartz estimate:

Theorem (Strichartz [43]**).** *Let μ be a finite UαH measure, and for each $f \in L^2(\mathbb{R}, d\mu)$ denote*

$$\widehat{f\mu}(t) = \int \exp(-ixt) f(x)\, d\mu(x).$$

Then there exists a constant C_1, depending only on μ (more precisely, only on C in (19)), such that for any $f \in L^2(\mathbb{R}, d\mu)$ and $T > 0$

$$\langle |\widehat{f\mu}|^2 \rangle_T < C_1 \|f\|^2 T^{-\alpha},$$

where $\|f\|$ is the L^2 norm of f.

We now prove Theorem 1.2.

Proof. Without loss of generality, assume $\|\psi\| = 1$. We first establish the existence of a Borel set $\tilde{S} \subset S$, for which the following three properties are true:
(i) $\|P_{\tilde{S}}\psi\| > 0$.
(ii) The restriction of the spectral measure μ^ψ to $\tilde{S}$ is UαH.
(iii) There exists a constant C_2, such that for each $E \in \tilde{S}$ and $R > 0$, the corresponding generalized eigenfunction $u(n, E)$ satisfies

$$\sum_{|n|<R} |u(n,E)|^2 < C_2 R^\gamma.$$

We shall establish (i)–(iii) in two stages. First, note that the function

$$f(E) \equiv \sup_{R>0} R^{-\gamma} \sum_{|n|<R} |u(n,E)|^2$$

is a measurable function of E, which, by (3), is finite everywhere on S. Thus, since $S = \bigcup_{k=1}^\infty \{E \in S \mid f(E) < k\}$, there is clearly a Borel subset $S_1 \subset S$ of positive μ^ψ measure and a constant C_2, such that $f(E) < C_2$ for any $E \in S_1$. That is, property (iii) holds for S_1. Next, since the restriction of μ^ψ to S_1 is α-continuous, the above mentioned Rogers-Taylor theorem implies that there is a Borel subset $\tilde{S} \subset S_1$ of positive μ^ψ measure (so property (i) holds) such that the restriction of μ^ψ to $\tilde{S}$ is UαH (so property (ii) holds).

Let us now denote $\psi_1 = P_{\tilde{S}}\psi$, $\psi_2 = P_{\mathbb{R}\setminus\tilde{S}}\psi$, where $P_\cdot$ denotes the spectral projection over the corresponding set. Then $\psi = \psi_1 + \psi_2$, and ψ_1, ψ_2 are mutually orthogonal so $1 = \|\psi\|^2 = \|\psi_1\|^2 + \|\psi_2\|^2$. Let P_{R_T} be the projector on the set of sites n with $|n| \leq R_T$. R_T is a function of the time parameter T to be chosen later. Given any vector φ, we will routinely use the notation $\varphi(t) = \exp(-ih_v^\Omega t)\varphi$.

We have

$$\begin{aligned}
\langle \|P_{R_T}\psi_1(t)\|^2\rangle_T &= \sum_{|n|\le R_T} \frac{1}{T}\int\limits_0^T \left|\int \exp(-iEt)\overline{u(n,E)}\,d\mu^{\psi_1}(E)\right|^2 dt \\
&\le C_1 T^{-\alpha} \sum_{|n|<R_T} \int |u(n,E)|^2 d\mu^{\psi_1}(E) \\
&\le C_1\|\psi_1\|^2 \left(\sup_{E\in\tilde S} \sum_{|n|<R_T} |u(n,E)|^2\right) T^{-\alpha}
\end{aligned}$$

by Strichartz theorem, and so

$$\langle \|P_{R_T}\psi_1(t)\|^2\rangle_T \le C_2C_1\|\psi_1\|^2 R_T^\gamma T^{-\alpha}. \tag{20}$$

For each $T>0$, we now define

$$R_T = \left(\frac{\|\psi_1\|^2\,T^\alpha}{64\,C_2C_1}\right)^{1/\gamma},$$

such that we have

$$\langle \|P_{R_T}\psi_1(t)\|^2\rangle_T < \frac{\|\psi_1\|^4}{64},$$

and thus

$$\begin{aligned}
\langle \|P_{R_T}\psi(t)\|^2\rangle_T &\le \left\langle (\|P_{R_T}\psi_1(t)\| + \|P_{R_T}\psi_2(t)\|)^2\right\rangle_T \\
&\le \left\langle (\|P_{R_T}\psi_1(t)\| + \|\psi_2\|)^2\right\rangle_T \\
&\le \left(\sqrt{\langle \|P_{R_T}\psi_1(t)\|^2\rangle_T} + \|\psi_2\|\right)^2 \\
&< \left(\frac{\|\psi_1\|^2}{8} + \|\psi_2\|\right)^2 \\
&= \frac{\|\psi_1\|^4}{64} + \|\psi_2\|^2 + \frac{1}{4}\|\psi_2\|\|\psi_1\|^2 \\
&< \|\psi_2\|^2 + \frac{1}{2}\|\psi_1\|^2 \\
&= 1 - \frac{1}{2}\|\psi_1\|^2.
\end{aligned}$$

Since

$$\langle \|P_{R_T}\psi(t)\|^2\rangle_T + \langle \|(1-P_{R_T})\psi(t)\|^2\rangle_T = 1,$$

we obtain

$$\langle \|(1-P_{R_T})\psi(t)\|^2\rangle_T > \frac{1}{2}\|\psi_1\|^2,$$

which implies

$$\langle\langle |X|^m \rangle\rangle_T > \frac{1}{2}\|\psi_1\|^2 R_T^m = \frac{\|\psi_1\|^2}{2}\left(\frac{\|\psi_1\|^2}{64\, C_2 C_1}\right)^{m/\gamma} T^{\alpha m/\gamma},$$

proving (4). □

Note that the above proof does not attempt to provide optimal estimates. We could (by allowing various constants to grow) choose ψ_1 to have a norm that is arbitrarily close to that of $P_S\psi$, and $R_T \sim T^{\alpha/\gamma}$ so that $\langle \|(1 - P_{R_T})\psi(t)\|^2\rangle_T$ is larger than something arbitrarily close to $\|P_S\psi\|^2$. This means that there is a component of the wave packet of size corresponding to $\|P_S\psi\|$ that is spreading on average at a rate of at least $T^{\alpha/\gamma}$.

We now apply Theorem 1.2 to investigate dynamics for the following model. Let $v_\omega(n)$ be independent random variables such that

$$E(v_\omega(n)) = 0, \quad E(v_\omega(n)^2)^{\frac{1}{2}} = \lambda n^{-\frac{1}{2}}, \text{ and } \sup_\omega |v_\omega(n)| \leq C n^{-\frac{1}{3}-\delta}, \ \delta > 0. \tag{21}$$

For example, if we take i.i.d. random variables $a_\omega(n)$ with uniform distribution in $[-\sqrt{3}, \sqrt{3}]$, then $v_\omega(n) = \lambda n^{-\frac{1}{2}} a_\omega(n)$ satisfy all the conditions. The half-line random Schrödinger operators h_ω with, say, Dirichlet boundary conditions at zero and potential v_ω exhibit very rich spectral structure. Such operators where studied by Delyon, Simon, and Souillard [10], and more recently by Kotani and Ushiroya [27], and by Kiselev, Last, and Simon [26]. Our study here is based mainly on the results of the last paper. In particular, the following has been proven in [26]:

Theorem (KLS [26]). *For all ω, the essential spectrum of h_ω is $[-2,2]$. If $|\lambda| < 2$, then for a.e. ω, h_ω has purely singular continuous spectrum in $\{E \mid |E| < (4-\lambda^2)^{\frac{1}{2}}\}$ and only dense pure point spectrum in $\{E \mid (4-\lambda^2)^{\frac{1}{2}} < |E| < 2\}$.*

For a.e. ω and $E \in (-2,2)$

$$\lim_{n\to\infty} \frac{\log \|T_E(n,0)\|}{\log n} = \frac{\lambda^2}{8 - 2E^2}, \tag{22}$$

and there exists an initial condition $\theta(\omega)$ at zero such that

$$\lim_{n\to\infty} \frac{\log \|T_E(n,0)u_{\theta(\omega)}\|}{\log n} = -\frac{\lambda^2}{8 - 2E^2}, \tag{23}$$

where $u_{\theta(\omega)}$ is the 2-vector corresponding to the boundary condition $\theta(\omega)$ at 0, and $T_E(n,0)$ is the transfer matrix from 0 to n at energy E.

This theorem implies that for a.e. ω and $E \in (-\sqrt{4-\lambda^2}, \sqrt{4-\lambda^2})$, the spectral measure μ (corresponding to the vector δ_1) has local Hausdorff dimension

$$\alpha(E,\lambda) = \frac{4-E^2-\lambda^2}{4-E^2} \tag{24}$$

at energy E, in the sense that for any $\epsilon > 0$, there is a δ so that $\mu(A) = 0$ if A is a subset of $(E-\delta, E+\delta)$ of Hausdorff dimension less than $\alpha(E,\lambda)-\epsilon$, and there is a subset B of Hausdorff dimension less than $\alpha(E,\lambda)+\epsilon$ such that $\mu((E-\delta,E+\delta)\setminus B) = 0$. These properties of the spectral measure follow from (22), (23) by subordinacy theory [19, 20]. See [26] for details.

Remark. The KLS theorem also provides an example indicating that the criterion of Theorem 1.1 is optimal in the sense that one cannot, in general, say more by looking at the rate of growth of the L^2 norm. Indeed, by (22), for a.e. ω, all solutions $\tilde{u}(n,E)$ for every energy E in the continuous spectrum satisfy

$$R^{-\rho}\|\tilde{u}(n,E)\|^2_{B_R} \leq C,$$

for any

$$\rho > 1 + \frac{\lambda^2}{4-E^2}$$

and all R. In particular, for every $\rho > 1$ we can take λ sufficiently small to ensure the existence of an interval I_ρ around $E = 0$ such that for a.e. ω all solutions (and in particular the one obeying the boundary condition) satisfy

$$R^{-\rho}\|\tilde{u}(n,E)\|^2_{B_R} < C$$

for $E \in I_\rho$. Yet for a.e. $E \in I_\rho$, we have $D\mu(E) = 0$ since the measure is purely singular. This shows that no condition of type (2) with $\alpha > 1$ leads in general to pointwise estimates on the derivatives of spectral measures.

This remark sounds trivial in one dimension, but it is straightforward — using the analysis of [26] for the continuous analog $V_\omega(x)$ of the family of random potentials we study — to give a similar example which works in any dimension (in the continuous case). Set $H_{V_\omega} = -\Delta + \lambda V_\omega(r)$ with spherically symmetric potential. Using spherical symmetry, one shows that the spectrum of H_{V_ω} is purely singular with probability one. However, for every $\rho > 1$, there are solutions for a.e. ω and all energies E sufficiently large such that (2) holds with $\alpha = \rho$.

The following theorem shows that as long as our operators h_ω have some continuous spectrum (which may be of arbitrarily small dimension), their transport properties are arbitrarily close to ballistic.

Theorem 5.1. *Consider the family h_ω of random Schrödinger operators defined on $\mathbb{Z}^+$ with potential $\lambda v_\omega(n)$, where $\lambda < 2$ and the potential satisfies* (21)*. Then for a.e. ω, for every ψ such that $P_c(\omega)\psi \neq 0$ (where $P_c(\omega)$ is the projector on the continuous* *spectrum of the operator h_ω) we have that for every $\epsilon > 0$ and $m > 0$ there is a positive constant $C_{\epsilon,m,\omega}$ such that for any $T > 0$*

$$\langle\langle |X|^m\psi(t), \psi(t)\rangle\rangle_T \geq C_{\epsilon,m,\omega}T^{m(1-\epsilon)}. \tag{25}$$

Proof. By the results of the Gilbert-Pearson theory, the spectral measure μ is supported on the set of the energies E for which the decaying solution (23) satisfies the boundary condition (namely, $\theta(\omega)$ coincides with the Dirichlet boundary condition). Moreover, these decaying solutions, which we will denote by $u(n,E)$, are exactly the generalized eigenfunctions in the sense of Theorem 1.2, if we normalize them by setting $u(1,E) = 1$.

Fix ω such that the results of the KLS theorem hold. (23) implies that the generalized eigenfunctions $u(n,E)$ of the operator h_ω satisfy

$$\limsup_{R\to\infty} R^{-\gamma}\|u(n,E)\|^2_{B_R} \leq \infty \tag{26}$$

for every $\gamma > \alpha(E,\lambda)$ given by (22). Pick an open energy interval $I = (E_1, E_2) \subset (-\sqrt{4-\lambda^2}, \sqrt{4-\lambda^2})$, such that $0 \notin I$, and $\mu^\psi(I) > 0$. Let $\alpha_1 = \alpha(E_1,\lambda)$, $\alpha_2 = \alpha(E_2,\lambda)$. $\alpha(E,\lambda)$ is monotone on I. Assume, without loss, that $\alpha_1 < \alpha_2$. The restriction of μ^ψ to I is α_1-continuous, and by (26), $\limsup_{R\to\infty} R^{-\alpha_2}\|u(n,E)\|^2_{B_R} < \infty$ for any generalized eigenfunction $u(n,E)$ with $E \in I$. Thus, by Theorem 1.2, for each $m > 0$ there is a constant $C_{m,I,\omega}$ such that for all $T > 0$

$$\langle\langle |X|^m\psi(t), \psi(t)\rangle\rangle_T \geq C_{m,I,\omega}T^{\frac{m\alpha_1}{\alpha_2}}.$$

Since $P_c(\omega)\psi \neq 0$, we can clearly choose such an interval I with $\frac{\alpha_1}{\alpha_2} > 1-\epsilon$ and $\mu^\psi(I) > 0$. Thus, Theorem 5.1 follows. □

Remark. By using an extension of the proof of Theorem 1.2, one can show that there is actually a component of the wave packet of size corresponding to $\|P_c(\omega)\psi\|$ that is spreading on average at a rate which is arbitrarily close to ballistic. More explicitly, one can show that for a.e. ω, for every $\epsilon > 0$ and $\rho > 0$ there exists a constant $C_{\omega,\rho,\epsilon}$ such that if $R_T = C_{\omega,\rho,\epsilon}T^{1-\epsilon}$, then

$$\langle\|P_{R_T}\psi(t)\|^2\rangle_T \leq \|\psi - P_c(\omega)\psi\|^2 + \rho. \tag{27}$$

This easily yields Theorem 5.1 and is thus a stronger statement.

Appendix 1. Generalizations

The whole proof of Theorem 1.1 readily extends to more general settings. Namely, we can replace the operator H_V^Ω with general uniformly elliptic self-adjoint operator Γ such that

$$\Gamma = (\partial_l - iA_l(x))a_{lk}(x)(\partial_k - iA_k(x)) + V(x)$$

provided that a_{lk}, A_l and V are "nice enough" (for example, bounded and sufficiently smooth). The proof for this case is very similar. The Green's formula leads us to consider the following modified Wronskian:

$$W_{\partial S}[f,g] = \int_{\partial S} (\cos(\overline{n}, x_l) a_{lk}((\partial_k - iA_k)u)\overline{v} - u\cos(\overline{n}, x_k)\overline{(\partial_l - iA_l)a_{lk}v}) d\sigma$$

It is clear that under our assumptions, the analog of Lemma 2.4 holds. The estimate of Lemma 2.3 also holds with the constant independent of R by the standard Sobolev estimates for bounded sufficiently smooth coefficients (see, e.g., [13, 31]). The rest of the proof does not change.

A similar remark applies to some higher order operators and systems. In particular, in one dimension, a self-adjoint half-line differential operator of order $2n$ is given by the expression

$$(Lf)(x) = (-1)^n (p_0 f^{(n)})^{(n)} + (-1)^{n-1}(p_1 f^{(n-1)})^{(n-1)} + \cdots + p_n f$$

and a set of self-adjoint boundary conditions at zero. The analog of the Wronskian in this case is determined by integration by parts:

$$W_{L,x}[f,g] = \sum_{j=1}^{n} \sum_{m=1}^{j} (-1)^m \left((p_j f^{(j)})^{(m-1)} g^{(j-m)} - f^{(j-m)} (p_j g^{(j)})^{(m-1)} \right), \tag{28}$$

where all values are taken at the point x. The analog of the Sobolev estimates of Lemma 2.3 is now the claim that for a solution of $(L-E)u = \phi$,

$$\|u\|_{W_2^m(B_R)} \le C\|u\|_{L_2(B_{R+1})}$$

holds for $m \le 2n-1$. Such estimates (in fact for $m \le 2n$) are well-known to hold for operators with bounded sufficiently smooth coefficients (see, e.g., [31]). The analog of Lemma 2.4 follows directly from (28); the rest of the proof of Theorem 1.1 does not change.

In particular, we have

Theorem A.1. *Let L denote the self-adjoint differential operator of order $2n$ with bounded sufficiently smooth (say, infinitely differentiable) coefficients. Suppose that for every E in a set S of positive Lebesgue measure, there exists a bounded solution $u(x,E)$ of the generalized eigenfunction equation*

$$(L-E)u = 0$$

satisfying the boundary conditions. Suppose that for a compactly supported function $\phi \in L^2$, we have

$$\int u(x,E)\phi(x)dx \neq 0$$

for a.e $E \in S$. Then the absolutely continuous part of the spectral measure μ^ϕ fills S (so that $\mu^\phi(S_1) > 0$ for any $S_1 \subset S$ of positive Lebesgue measure).

Remark. Of course, we can also allow for ϕ which are not L^2, but from the Sobolev space $H_{-2}(H_V)$, such as the δ function and its derivatives up to $2n-1$, which are often used in the setting of one-dimensional differential operators. The spectral measure is not finite in this case, but nothing else changes.

Theorem A.1 follows from the above discussion and proof of Theorem 2.5. This result may be viewed as a sort of an analog of [41, 42] for the higher order case. It is typical, though, that our condition involves only one solution ([41, 42] requires all solutions to be bounded) because the possible multiplicity of the spectrum makes it unreasonable to demand all solutions to be bounded (in higher order cases) to get absolutely continuous spectrum. On the other hand, our result does not guarantee pure absolute continuity.

Appendix 2. One-dimensional perturbed Stark operators

In this appendix, we make a remark concerning dynamical properties of a certain class of perturbed Stark operators. We denote by $H_{V,S}$ the operator defined on the whole axis by the differential expression

$$-\frac{d^2}{dx^2} - x + V(x).$$

Our results are based on the theorem proved in [25]:

Theorem. *Suppose that $|V(x)| \leq C(1+|x|)^{-\frac{1}{3}-\epsilon}$, or V is bounded and has a derivative V' which is bounded and Hölder continuous. Then the whole axis $(-\infty, \infty)$ is an essential support of the absolutely continuous part of the spectral measure μ. Moreover, for a.e. $E \in R$, there exist two linearly independent solutions $u_\pm(x,E)$, such that*

$$u_\pm(x,E) = x^{-\frac{1}{4}} \exp(\pm i(\frac{2}{3}x^{\frac{3}{2}} + f_\pm(x,E)))(1+o(1))$$

as $x \to +\infty$, where $|f'_\pm(x,E)| \leq C(1+x)^{-\frac{1}{2}}$.

Stark operators do not fit into the framework provided by Theorem 1.1 because of the strong negative part of the potential (and resulting failure of Lemma 2.3). Indeed, for a.e. energy E here, we have a solution $u(x,E)$ which satisfies $R^{-\frac{1}{2}}\|u(x,E)\|_{B_R} \leq C(E)$, which, if Theorem 1.1 were true, would imply $D^{\frac{1}{2}}(E) > 0$ a.e. E. It should be possible to prove an analog of Theorem 1.1 for some perturbed Stark operators taking into account that instead of the Sobolev estimates of Lemma 2.3, one rather has

$$\|\nabla u\|_{B_R}^2 \leq CR\|u\|_{B_R}^2.$$

However, the criterion of Theorem 1.2 applies, giving immediately

Theorem A.2. *Under the conditions of the previous theorem, for every vector ψ with non-zero projection on the absolutely continuous subspace, we have*

$$\langle\langle |X|^m \psi(t), \psi(t) \rangle\rangle_T \geq CT^{2m}.$$

We note that there are examples [32] of potentials V satisfying

$$|V(x)| \leq C(x)(1+|x|)^{-\frac{1}{3}},$$

where $C(x)$ tends to infinity as $x \to \infty$, but arbitrarily slowly, such that for a corresponding Stark operator, there is a dense set of eigenvalues embedded in the absolutely continuous spectrum. Theorem A.2 shows that such potentials, nevertheless, do not slow down dynamics corresponding to the absolutely continuous component.

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