BORG-TYPE THEOREMS FOR MATRIX-VALUED SCHRÖDINGER OPERATORS

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Abstract. A Borg-type uniqueness theorem for matrix-valued Schrödinger operators is proved. More precisely, assuming a reflectionless potential matrix and spectrum a half-line $[0,\infty)$, we derive triviality of the potential matrix. Our approach is based on trace formulas and matrix-valued Herglotz representation theorems. As a by-product of our techniques, we obtain an extension of Borg's classical result from the class of periodic scalar potentials to the class of reflectionless matrix-valued potentials.

1. Introduction

The principal aim of this paper is to advocate a new proof of Borg's [7] uniqueness result for periodic one-dimensional Schrödinger operators in $L^2(\mathbb{R})$ and to extend it to general matrix-valued Schrödinger operators in $L^2(\mathbb{R})^m$. In order to describe Borg's result and our generalizations of it (see Section 4 for more details), we need a few preparations. Let $H^{p,q}(\mathbb{R})$, $p,q\in\mathbb{N}$ denote the standard Sobolev spaces and $\sigma(\,\cdot\,)$ abbreviate the spectrum. Assuming

$$q\in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R}) \text{ to be real-valued,} \tag{1.1}$$

$$\sigma(h)=[e_0,\infty) \text{ for some } e_0\in\mathbb{R}, \tag{1.2}$$

$$q \text{ periodic,} \tag{1.3}$$

with h on $H^{2,2}(\mathbb{R})$ denoting the usual self-adjoint realization of the differential expression $-\frac{d^2}{dx^2}+q(x)$ in $L^2(\mathbb{R})$, Borg [7] proved the uniqueness result

$$q(x)=e_0 \text{ for all } x\in\mathbb{R}. \tag{1.4}$$

(Actually, Borg only assumed $q\in L^2_{\mathrm{loc}}(\mathbb{R})$ and hence obtained $q=e_0$ a.e. in (1.4) but we will temporarily ignore this for simplicity of exposition.)

Next, consider matrix-valued Schrödinger operators H in $L^2(\mathbb{R})^m$, $m\in\mathbb{N}$ associated with differential expressions

$$-I_m\frac{d^2}{dx^2}+Q(x), \quad x\in\mathbb{R}, \tag{1.5}$$

where I_m is the identity matrix in $\mathbb{C}^m$, $Q\in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R})^{m\times m}$, and $Q(x)=Q(x)^*$ for all $x\in\mathbb{R}$.

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In Section 4 we will prove a uniqueness result of the type (1.4) for the matrix-valued operators H. Our main tool will be a trace formula of the type

$$Q(x) = E_0 I_m + \lim_{z\to i\infty} \int_{E_0}^{\infty} d\lambda\, z^2(\lambda - z)^{-2}(I_m - 2\Xi(\lambda, x)), \quad x \in \mathbb{R} \tag{1.6}$$

for Schrödinger operators. Here $\Xi(\lambda, x)$ is a self-adjoint $m \times m$ matrix satisfying

$$0 \le \Xi(\lambda, x) \le I_m \tag{1.7}$$

and E_0 denotes the infimum of the spectrum of H. Given the trace formula (1.6), our idea of extending Borg's uniqueness theorem to matrix-valued Schrödinger operators now becomes very simple. Suppose the analog of condition (1.2), that is,

$$\sigma(H) = [E_0, \infty) \text{ for some } E_0 \in \mathbb{R} \tag{1.8}$$

and instead of the periodicity condition (1.3) assume that for all $x \in \mathbb{R}$,

$$\Xi(\lambda, x) = \frac{1}{2} I_m \text{ for a.e. } \lambda \in [E_0, \infty). \tag{1.9}$$

Then the trace formula (1.6) immediately yields

$$Q(x) = E_0 I_m \tag{1.10}$$

and hence a desired generalization of Borg's result (1.4). We will show in Section 4 that periodicity of $Q(x)$, together with the assumption that H has uniform (maximal) spectral multiplicity $2m$, indeed implies condition (1.9). This recovers a generalization of Borg's theorem to periodic matrix-valued Schrödinger operators by Deprés [23], which partly motivated our present work. Consequently, assumption (1.9) is a proper extension of the periodicity requirement (1.3).

More generally, if for all $x \in \mathbb{R}$,

$$\Xi(\lambda, x) = \frac{1}{2} I_m \text{ for a.e. } \lambda \in \sigma_{\mathrm{ess}}(H) \tag{1.11}$$

($\sigma_{\mathrm{ess}}(\,\cdot\,)$ denoting the essential spectrum), we shall call $Q(x)$ a *reflectionless* potential following the traditional terminology in the scalar case $m = 1$ (cf. the discussion in Section 4). Thus reflectionless potentials $Q(x)$, in connection with the spectral assumption (1.8), are the prime candidates for Borg-type theorems.

Finally, we briefly sketch the content of each section. Section 2 provides the basic background results on matrix-valued Schrödinger operators. Following a series of papers by Hinton and Shaw [42]–[45], [47], and with a view toward a future treatment of Dirac-type operators, we treat Schrödinger operators as special Hamiltonian systems and briefly recall the corresponding Weyl-Titchmarsh and spectral theory. In Section 3 we derive new trace formulas for matrix-valued Schrödinger operators using appropriate Herglotz representation results for a diagonal Green's matrix as discussed in Section 2. In our principal Section 4 we finally derive the extension of Borg-type theorems to matrix-valued Schrödinger operators. We also provide a criterion for a potential to be reflectionless and close with an application to the case of periodic potentials.

Our results can be viewed as a first (and rather modest) step toward the construction of isospectral manifolds of certain classes of matrix-valued potentials for Schrödinger operators. Especially, one might think of the class of periodic (possibly reflectionless) potentials, see [11], [12]. Moreover, our results are relevant in the context of matrix-valued hierarchies of integrable evolution equations (i.e., soliton equations) and we refer the reader to [3], [13], [24], [25], [26], [27], [31], [70], [71],

[72], [84], and the vast literature therein. For related work on trace formulas, spectral properties of matrix-valued Schrödinger operators, and uniqueness theorems see, for instance, [1], [10], [11], [12], [14], [15], [16], [51], [52], [55], [66], [67], [68], [69], [74], [77], [78], [79], [80], [83], [85].

The present paper focuses on matrix-valued Schrödinger operators; correrspond-ing extensions to Dirac-type operators will appear elsewhere.

2. Matrix-Valued Schrödinger Operators

In this section we briefly recall the Weyl–Titchmarsh theory for matrix-valued Schrödinger operators. In view of a future treatment of Dirac operators we use a unifying approach representing Schrödinger operators as special cases of Hamiltonian systems and hence develop the theory from that point of view. Throughout this paper, all matrices will be considered over the field of complex numbers $\mathbb{C}$.

The basic assumption of this paper will be the following.

Hypothesis 2.1. *Fix $m \in \mathbb{N}$, $n = 2m$, and define the $n \times n$ matrix*

$$J = \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix}. \tag{2.1}$$

Suppose $Q = Q^* \in L^1_{\mathrm{loc}}(\mathbb{R})^{m\times m}$ and introduce the $n \times n$ matrices

$$A = \begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix}, \quad B(x) = \begin{pmatrix} -Q(x) & 0 \\ 0 & I_m \end{pmatrix}. \tag{2.2}$$

Given Hypothesis 2.1 we consider the Hamiltonian system

$$J\psi'(z,x) = (zA + B(x))\psi(z,x), \quad z \in \mathbb{C} \tag{2.3}$$

for a.e. $x \in \mathbb{R}$, where $z \in \mathbb{C}$ plays the role of a spectral parameter and $\psi(z,x)$ is assumed to satisfy

$$\psi(z,\,\cdot\,) \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R})^n. \tag{2.4}$$

Here I_p denotes the identity matrix in $\mathbb{C}^p$ for $p \in \mathbb{N}$, M^* the adjoint (i.e., complex conjugate transpose), M^t the transpose of the matrix M, and $\mathrm{AC}_{\mathrm{loc}}(\mathbb{R})$ denotes the set of locally absolutely continuous functions on $\mathbb{R}$. At times it will be convenient to consider an $n \times r$ solution matrix in (2.3), with $r = 1, \dots, n$, which will then be denoted by $\Psi(z,x)$ and assumed to satisfy $\Psi(z,\,\cdot\,) \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R})^{n\times r}$.

Hypothesis 2.1 governs the case of matrix-valued Schrödinger operators. In fact, equation (2.3) becomes equivalent to

$$-\psi_1''(z,x) + Q(x)\psi_1(z,x) = z\psi_1(z,x), \tag{2.5}$$

$$\psi_2(z,x) = \psi_1'(z,x), \tag{2.6}$$

where

$$\psi(z,x) = (\psi_1(z,x), \psi_2(z,x))^t. \tag{2.7}$$

Here it is assumed that

$$\psi_j(z,\,\cdot\,) \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R})^m, \quad j = 1,2. \tag{2.8}$$

In order to recall the limit point and limit circle cases associated with the Hamiltonian system (2.3), we introduce the notation $(-\infty \le a < b \le \infty)$

$$L^2_A((a,b)) = \{\phi : (a,b) \to \mathbb{C}^n \mid \int_a^b dx\, (\phi(x), A\phi(x))_{\mathbb{C}^n} < \infty\}, \tag{2.9}$$

$$N(z,\infty) = \{\phi \in L^2_A((c,\infty)) \mid J\phi' = (zA+B)\phi \text{ a.e. on } (c,\infty)\}, \tag{2.10}$$

$$N(z,-\infty) = \{\phi \in L^2_A((-\infty,c)) \mid J\phi' = (zA+B)\phi \text{ a.e. on } (-\infty,c)\}, \tag{2.11}$$

for some $c \in \mathbb{R}$ and $z \in \mathbb{C}$. (Here $(\phi,\psi)_{\mathbb{C}^n} = \sum_{j=1}^n \bar{\phi}_j \psi_j$ denotes the standard scalar product in $\mathbb{C}^n$, writing $\chi \in \mathbb{C}^n$ as $\chi = (\chi_1,\dots,\chi_n)^t$, etc.) Both dimensions of the spaces in (2.10) and (2.11), $\dim_{\mathbb{C}}(N(z,\infty))$ and $\dim_{\mathbb{C}}(N(z,-\infty))$, are constant for $z \in \mathbb{C}_\pm = \{\zeta \in \mathbb{C} \mid \operatorname{Im}(\zeta) \gtrless 0\}$, see for instance [4], [56], who prove this fact for much more general Hamiltonian systems. Hence one defines the Hamiltonian system (2.3) to be in the limit point (l.p.) case at $\pm\infty$ if

$$\dim_{\mathbb{C}}(N(z,\pm\infty)) = n/2 \text{ for all } z \in \mathbb{C}\backslash\mathbb{R}, \tag{2.12}$$

and in the limit circle case (l.c.) case at $\pm\infty$ if

$$\dim_{\mathbb{C}}(N(z,\pm\infty)) = n \text{ for all } z \in \mathbb{C}. \tag{2.13}$$

Later on we will introduce Schrödinger operators H in $L^2(\mathbb{R})^m$ associated with (2.3) and Hypothesis 2.1 and see that the l.p. and l.c. notions for the Hamiltonian system (2.3) and the operators H coincide. From this point on we only consider the l.p. case at $\pm\infty$ and hence work with the following assumption.

Hypothesis 2.2. *In addition to Hypothesis* 2.1 *suppose the Hamiltonian system* (2.3) *to be in the l.p. case at* $\pm\infty$.

Next we briefly turn to Weyl–Titchmarsh theory associated with (2.3) and briefly recall some of the results developed by Hinton and Shaw in a series of papers devoted to spectral theory of (singular) Hamiltonian systems [42]–[45], [47] (see also [62], [63]). While they discuss (2.3) under much more general hypotheses on $A(x)$ and $B(x)$, we here confine ourselves to the special cases of matrix-valued Schrödinger systems governed by Hypothesis 2.2. Let $\Psi(z,x,x_0)$ be a normalized fundamental system of solutions of (2.3) at some $x_0 \in \mathbb{R}$, that is, $\Psi(z,x,x_0)$ satisfies

$$J\Psi'(z,x) = (zA + B(x))\Psi(z,x), \quad z \in \mathbb{C} \tag{2.14}$$

for a.e. $x \in \mathbb{R}$, and

$$\Psi(z,x_0,x_0) = I_n. \tag{2.15}$$

Moreover, we partition $\Psi(z,x,x_0)$ as

$$\Psi(z,x,x_0) = (\psi_{j,k}(z,x,x_0))_{j,k=1}^2 = \begin{pmatrix} \theta_1(z,x,x_0) & \phi_1(z,x,x_0) \\ \theta_2(z,x,x_0) & \phi_2(z,x,x_0) \end{pmatrix}, \tag{2.16}$$

where $\theta_j(z,x,x_0)$ and $\phi_j(z,x,x_0)$ for $j = 1,2$ are $m \times m$ matrices, entire with respect to $z \in \mathbb{C}$, and normalized according to (2.15), that is,

$$\theta_1(z,x_0,x_0) = \phi_2(z,x_0,x_0) = I_m, \quad \theta_2(z,x_0,x_0) = \phi_1(z,x_0,x_0) = 0. \tag{2.17}$$

(We recall that $\theta_2(z,x,x_0) = \theta_1'(z,x,x_0)$ and $\phi_2(z,x,x_0) = \phi_1'(z,x,x_0)$ in the present case of Schrödinger operators, cf. (2.6).) One can prove [42] that

$$\det(\phi_1(z,x,x_0)) \neq 0 \text{ for } x \in \mathbb{R}\backslash\{x_0\},\ z \in \mathbb{C}\backslash\mathbb{R} \tag{2.18}$$

so that

$$M_{\pm,R}(z,x_0) = -\phi_1(z,R,x_0)^{-1}\theta_1(z,R,x_0), \quad R \gtrless x_0,\ z \in \mathbb{C}\backslash\mathbb{R} \tag{2.19}$$

are well-defined. Due to the assumption of the l.p. case at $\pm\infty$, one obtains the existence of the following limits [42], [44], [45], [73], [81],

$$M_\pm(z,x_0) = \lim_{R\to\pm\infty} M_{\pm,R}(z,x_0), \quad z\in\mathbb{C}\backslash\mathbb{R}. \tag{2.20}$$

$M_+(z,x_0)$ (resp. $M_-(z,x_0)$) represent the half-line Weyl–Titchmarsh matrices associated with (2.3) and the interval (x_0,∞) (resp. $(-\infty,x_0)$). For later reference we summarize the principal results on $M_\pm(z,x_0)$ in the following theorem.

Theorem 2.3 ([2], [9], [37], [42], [43], [47], [61]). *Assume Hypothesis* 2.2 *and let* $z\in\mathbb{C}\backslash\mathbb{R}$, *and* $x_0\in\mathbb{R}$. *Then*
(i) $\pm M_\pm(z,x_0)$ *is a matrix-valued Herglotz function of rank* m. *In particular,*

$$\operatorname{Im}(\pm M_\pm(z,x_0)) > 0, \quad z\in\mathbb{C}_+, \tag{2.21}$$

$$M_\pm(\bar z,x_0) = M_\pm(z,x_0)^*, \tag{2.22}$$

$$\operatorname{rank}(M_\pm(z,x_0)) = m, \tag{2.23}$$

$$\lim_{\varepsilon\downarrow 0} M_\pm(\lambda+i\varepsilon,x_0) \text{ exists for a.e. } \lambda\in\mathbb{R}. \tag{2.24}$$

$\pm M_\pm(z,x_0)$ *and* $\mp M_\pm(z,x_0)^{-1}$ *have isolated poles of at most first order which are real and have a negative definite residue.*
(ii) $\pm M_\pm(z,x_0)$ *admit the representations*

$$\pm M_\pm(z,x_0) = F_\pm(x_0) + \int_{\mathbb{R}} d\Omega_\pm(\lambda,x_0)\,\Big(\frac{1}{\lambda-z} - \frac{\lambda}{1+\lambda^2}\Big) \tag{2.25}$$

$$= \exp\bigg(C_\pm(x_0) + \int_{\mathbb{R}} d\lambda\, \Xi_\pm(\lambda,x_0)\Big(\frac{1}{\lambda-z} - \frac{\lambda}{1+\lambda^2}\Big)\bigg), \tag{2.26}$$

where

$$F_\pm(x_0) = F_\pm(x_0)^*, \quad \int_{\mathbb{R}} \frac{\|d\Omega_\pm(\lambda,x_0)\|}{1+\lambda^2} < \infty, \tag{2.27}$$

$$C_\pm(x_0) = C_\pm(x_0)^*, \quad 0\le \Xi_\pm(\,\cdot\,,x_0)\le I_m \text{ a.e.} \tag{2.28}$$

Moreover,

$$\Omega_\pm((\lambda,\mu],x_0) = \lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\int_{\lambda+\delta}^{\mu+\delta} d\nu\, \operatorname{Im}(\pm M_\pm(\nu+i\varepsilon,x_0)), \tag{2.29}$$

$$\Xi_\pm(\lambda,x_0) = \lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\operatorname{Im}(\ln(\pm M_\pm(\lambda+i\varepsilon,x_0)) \text{ for a.e. } \lambda\in\mathbb{R}. \tag{2.30}$$

(iii) Define the $n\times m$ *matrices*

$$\begin{aligned}\Psi_\pm(z,x,x_0) &= \begin{pmatrix}\psi_{\pm,1}(z,x,x_0)\\ \psi_{\pm,2}(z,x,x_0)\end{pmatrix}\\ &= \begin{pmatrix}\theta_1(z,x,x_0) & \phi_1(z,x,x_0)\\ \theta_2(z,x,x_0) & \phi_2(z,x,x_0)\end{pmatrix}\begin{pmatrix} I_m\\ M_\pm(z,x_0)\end{pmatrix}.\end{aligned} \tag{2.31}$$

Then the m *columns of* $\Psi_\pm(z,x,x_0)$ *form a basis for* $N(z,\pm\infty)$ *and*

$$\operatorname{Im}(M_\pm(z,x_0)) = \operatorname{Im}(z)\int_{x_0}^{\pm\infty} dx\, \Psi_\pm(z,x,x_0)^* A \Psi_\pm(z,x,x_0). \tag{2.32}$$

In order to describe the Green's matrix associated with (2.3) on $\mathbb{R}$, we assume the hypotheses of Theorem 2.3 and introduce

$$K(z,x,x') = \Psi_\mp(z,x,x_0)(M_-(z,x_0)-M_+(z,x_0))^{-1}\Psi_\pm(\bar z,x',x_0)^*,$$

$$x \lessgtr x', \; z \in \mathbb{C}\backslash\mathbb{R} \tag{2.33}$$

and

$$M(z,x_0) = \frac{1}{2}(K(z,x_0,x_0+0) + K(z,x_0,x_0-0)) \tag{2.34}$$

$$= \begin{pmatrix} N_-(z,x_0)^{-1} & \frac{1}{2}N_-(z,x_0)^{-1}N_+(z,x_0) \\ \frac{1}{2}N_+(z,x_0)N_-(z,x_0)^{-1} & M_\pm(z,x_0)N_-(z,x_0)^{-1}M_\mp(z,x_0) \end{pmatrix}, \quad z \in \mathbb{C}\backslash\mathbb{R}, \tag{2.35}$$

with

$$N_\pm(z,x_0) = M_-(z,x_0) \pm M_+(z,x_0). \tag{2.36}$$

Next let $\phi \in L^2_A(\mathbb{R})$ and consider

$$J\psi'(z,x) = (zA + B(x))\psi(z,x) + A\phi(x), \quad z \in \mathbb{C}\backslash\mathbb{R} \tag{2.37}$$

for a.e. $x \in \mathbb{R}$. Then (2.37) has a unique solution $\psi(z,\cdot\,) \in L^2_A(\mathbb{R}) \cap \mathrm{AC}_{\mathrm{loc}}(\mathbb{R})^n$ given by [42], [44]

$$\psi(z,x) = \int_{\mathbb{R}} dx'\, K(z,x,x')A\phi(x'). \tag{2.38}$$

Let H be the matrix-valued Schrödinger operator in $L^2(\mathbb{R})^m$,

$$H = -I_m \frac{d^2}{dx^2} + Q, \tag{2.39}$$

$$\mathrm{dom}(H) = \{g \in L^2(\mathbb{R})^m \mid g, g' \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R})^m, \, (-I_m g'' + Qg) \in L^2(\mathbb{R})^m\}.$$

In the following we associate the operator H in $L^2(\mathbb{R})^m$ with the Hamiltonian system (2.3). Hypothesis 2.2 then renders H to be self-adjoint in $L^2(\mathbb{R})^m$. (Equivalently, the differential expressions $-I_m d^2/dx^2 + Q(x)$ is in the l.p. case at $\pm\infty$.)

Denoting by $\rho(H)$, $\sigma(H)$, $\sigma_p(H)$, $\sigma_{\mathrm{ess}}(H)$, $\sigma_{\mathrm{ac}}(H)$, and $\sigma_{\mathrm{sc}}(H)$ the resolvent set, spectrum, point spectrum (i.e., the set of eigenvalues), essential spectrum, absolutely and singularly continuous spectrum of H, respectively, one can summarize the connections between $M(z,x_0)$ and the various spectra of H as follows.

Theorem 2.4 ([2], [9], [37], [42], [43], [47]). *Assume Hypothesis* 2.2, $z \in \mathbb{C}\backslash\mathbb{R}$, *and* $x_0 \in \mathbb{R}$. *Then*
(i) $M(z,x_0)$ *is a matrix-valued Herglotz function of rank* n *with representations*

$$M(z,x_0) = F(x_0) + \int_{\mathbb{R}} d\Omega(\lambda,x_0)\left(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2}\right) \tag{2.40}$$

$$= \exp\left(C(x_0) + \int_{\mathbb{R}} d\lambda\, \Upsilon(\lambda,x_0)\left(\frac{1}{\lambda-z} - \frac{\lambda}{1+\lambda^2}\right)\right), \tag{2.41}$$

where

$$F(x_0) = F(x_0)^*, \quad \int_{\mathbb{R}} \frac{\|d\Omega(\lambda,x_0)\|}{1+\lambda^2} < \infty, \tag{2.42}$$

$$C(x_0) = C(x_0)^*, \quad 0 \le \Upsilon(\,\cdot\,,x_0) \le I_n \;\; a.e. \tag{2.43}$$

Moreover,

$$\Omega((\lambda,\mu],x_0) = \lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0} \frac{1}{\pi}\int_{\lambda+\delta}^{\mu+\delta} d\nu\, \mathrm{Im}(M(\nu + i\varepsilon, x_0)), \tag{2.44}$$

$$\Upsilon(\lambda,x_0) = \lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\mathrm{Im}(\ln(M(\lambda+i\varepsilon,x_0))) \;\; \textit{for a.e. } \lambda \in \mathbb{R}. \tag{2.45}$$

(ii) $z \in \rho(H)$ *if and only if* $M(z, x_0)$ *is holomorphic near* z. *In this case,*

$$((H-z)^{-1}f)(x) = \int_{\mathbb{R}} dx'\, K_{1,1}(z,x,x')f(x'), \quad z \in \mathbb{C}\backslash\sigma(H),\ f \in L^2(\mathbb{R})^m. \tag{2.46}$$

(Here $K_{1,1}(z,x,x')$ *denotes the left upper* $m\times m$ *submatrix of* $K(z,x,x')$*, i.e., we write* $K(z,x,x') = (K_{j,k}(z,x,x'))_{j,k=1}^2$*.) Moreoever,* $\lambda_0 \in \rho(H)\cap\mathbb{R}$ *if and only if there is an* $\varepsilon > 0$ *such that*

$$\Omega(\lambda_0+\varepsilon, x_0) - \Omega(\lambda_0-\varepsilon, x_0) = 0. \tag{2.47}$$

(iii) For all $\lambda \in \mathbb{R}$*,*

$$\lim_{\varepsilon\downarrow 0} \varepsilon\, \mathrm{Im}(M(\lambda+i\varepsilon, x_0)) = \Omega(\lambda+0,x_0) - \Omega(\lambda-0,x_0) \geq 0. \tag{2.48}$$

(iv) $\lambda_0 \in \sigma_p(H)$ *if and only if*

$$\lim_{\varepsilon\downarrow 0} \varepsilon\, \mathrm{Im}(M(\lambda_0+i\varepsilon, x_0)) = \Omega(\lambda_0+0,x_0) - \Omega(\lambda_0-0,x_0) \neq 0. \tag{2.49}$$

(v)

$$\sigma(H) = \mathrm{supp}(d\Omega(\,\cdot\,,x_0)), \tag{2.50}$$

$$\sigma_{\mathrm{ac}}(H) = \mathrm{supp}(d\Omega_{\mathrm{ac}}(\,\cdot\,,x_0)), \tag{2.51}$$

$$\sigma_{\mathrm{sc}}(H) = \mathrm{supp}(d\Omega_{\mathrm{sc}}(\,\cdot\,,x_0)), \tag{2.52}$$

$$\overline{\sigma_p(H)} = \mathrm{supp}(d\Omega_{\mathrm{pp}}(\,\cdot\,,x_0)). \tag{2.53}$$

Here $\mathrm{supp}(\,\cdot\,)$ denotes the topological (i.e., smallest closed) support and $d\Omega = d\Omega_{\mathrm{pp}} + d\Omega_{\mathrm{sc}} + d\Omega_{\mathrm{ac}}$ represents the Lebesgue decomposition of $d\Omega$ into its pure point (pp), singularly continuous (sc) and absolutely continuous (ac) parts.

3. Trace Formulas

In this section we derive trace formulas for the matrix-valued Schrödinger systems studied in Section 2. Throughout this section we will assume the limit point case at $\pm\infty$ and hence adopt Hypothesis 2.2.

In the Schrödinger case at hand, the inhomogeneous term in (2.37) is of the type $A\phi = (\phi_1, 0)^t$ and $\psi = (\psi_1, \psi_2)^t = (\psi_1, \psi_1')^t$. Hence $B(x) = \left(\begin{smallmatrix} -Q(x) & 0 \\ 0 & I_m \end{smallmatrix}\right)$ is of a very special nature and only the $m\times m$ submatrix $B_{1,1} = -Q(x)$ contains information on $Q(x)$. Thus, we will focus on the $m\times m$ submatrix $M_{1,1}(z,x_0)$ of $M(z,x_0)$ in (2.35) and (2.40)–(2.43). By (2.33) and (2.46) one infers that the Green's matrix $G(z,x,x')$ of H is given by

$$\begin{aligned} G(z,x,x') &= K_{1,1}(z,x,x') \\ &= \psi_{\mp,1}(z,x,x_0)(M_-(z,x_0) - M_+(z,x_0))^{-1}\psi_{\pm,1}(\bar{z},x',x_0)^*, \\ &\qquad\qquad x \lesseqqgtr x',\ z \in \mathbb{C}\backslash\mathbb{R}, \end{aligned} \tag{3.1}$$

where the $m\times m$ matrices $\psi_{\pm,j}(z,x,x_0)$ are defined in (2.31), that is,

$$\psi_{\pm,j}(z,x,x_0) = \theta_j(z,x,x_0) + \phi_j(z,x,x_0)M_\pm(z,x_0), \quad j = 1,2. \tag{3.2}$$

Moreover, since (block) diagonal elements of matrix-valued Herglotz functions are (lower-dimensional) matrix-valued Herglotz functions (see, e.g., [37]), $M_{1,1}(z,x_0) = G(z,x_0,x_0)$ is an $m\times m$ matrix-valued Herglotz function satisfying

$$G(z,x_0,x_0) = M_{1,1}(z,x_0) = (M_-(z,x_0) - M_+(z,x_0))^{-1} \tag{3.3}$$

$$= F_{1,1}(x_0) + \int_{\mathbb{R}} d\Omega_{1,1}(\lambda, x_0)\,\big(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2}\big) \tag{3.4}$$

$$= \exp\bigg(E(x_0) + \int_{\mathbb{R}} d\lambda\, \Xi(\lambda, x_0)\,\big(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2}\big)\bigg), \tag{3.5}$$

where

$$F_{1,1}(x_0) = F_{1,1}(x_0)^*, \quad \int_{\mathbb{R}} \frac{\|d\Omega_{1,1}(\lambda, x_0)\|}{1+\lambda^2} < \infty, \tag{3.6}$$

$$E(x_0) = E(x_0)^*, \quad 0 \leq \Xi(\lambda, x_0) \leq I_m \text{ a.e.}, \tag{3.7}$$

and

$$\Omega_{1,1}((\lambda, \mu], x_0) = \lim_{\delta \downarrow 0} \lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \int_{\lambda+\delta}^{\mu+\delta} d\nu\, \mathrm{Im}(G(\nu + i\varepsilon, x_0)), \tag{3.8}$$

$$\Xi(\lambda, x_0) = \lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \mathrm{Im}(\ln(G(\lambda + i\varepsilon, x_0))) \text{ for a.e. } \lambda \in \mathbb{R}. \tag{3.9}$$

Next we discuss the asymptotic expansion of $G(z, x, x)$ as $|z| \to \infty$. It will be convenient to start with $M_\pm(z, x)$. In the following we denote $M(z) = O(|z|^\alpha)$ or $o(|z|^\alpha)$ as $|z| \to \infty$, whenever $\|M(z)\| = O(|z|^\alpha)$ or $o(|z|^\alpha)$ as $|z| \to \infty$ for an appropriate matrix norm.

Theorem 3.1. [18, Theorem 4.7] *Suppose Hypothesis* 2.2. *In addition, assume $Q^{(N)} \in L^1_{\mathrm{loc}}(\mathbb{R})^{m\times m}$ for some $N \in \mathbb{N}_0$ and let $C_\varepsilon \subset \mathbb{C}_+$ be the sector along the positive imaginary axis with vertex at zero and opening angle ε with $0 < \varepsilon < \pi/2$. Then $M_\pm(z, x)$ has the asymptotic expansion as $|z| \to \infty$ in C_ε of the form* $(\mathrm{Im}(z^{1/2}) \geq 0,\ z \in \mathbb{C})$

$$M_\pm(z, x) \underset{\substack{|z|\to\infty \\ z \in C_\varepsilon}}{=} \begin{cases} \pm i I_m z^{1/2} + o(|z|^{1/2}) & \text{for } N = 0, \\ \pm i I_m z^{1/2} + \sum_{k=1}^N m_{\pm,k}(x) z^{-k/2} + o(|z|^{-N/2}) & \text{for } N \in \mathbb{N}. \end{cases} \tag{3.10}$$

The expansion (3.10) *is uniform with respect to* $\arg(z)$ *for* $|z| \to \infty$ *in* C_ε *and uniform in* $x \in \mathbb{R}$ *as long as* x *varies in compact intervals. The expansion coefficients* $m_{\pm,k}(x)$ *can be recursively computed from*

$$m_{\pm,1}(x) = \pm\frac{1}{2i} Q(x), \quad m_{\pm,2}(x) = \frac{1}{4} Q'(x),$$

$$m_{\pm,k+1}(x) = \pm\frac{i}{2}\bigg(m'_{\pm,k}(x) + \sum_{\ell=1}^{k-1} m_{\pm,\ell}(x) m_{\pm,k-\ell}(x)\bigg), \quad k \geq 2. \tag{3.11}$$

We briefly sketch a derivation of the recursion (3.11). Let

$$\hat\Psi(z, x, x_0) = \begin{pmatrix} \psi_{-,1}(z, x, x_0) & \psi_{+,1}(z, x, x_0) \\ \psi_{-,2}(z, x, x_0) & \psi_{+,2}(z, x, x_0) \end{pmatrix}, \quad z \in \mathbb{C}\backslash\mathbb{R} \tag{3.12}$$

be the fundamental system of solutions of (2.14) as defined in (2.31) and observe that

$$\psi_{+,2}(z, x, x_0) = \psi'_{+,1}(z, x, x_0) \tag{3.13}$$

in the Schrödinger operator case. Hence any nonnormalized solutions $\tilde\psi_{\pm,1}(z, \cdot\,) \in L^2((\pm\infty, c))^{m\times m}$, $c \in \mathbb{R}$ of $-\psi_1'' + Q\psi_1 = z\psi_1$ for $z \in \mathbb{C}\backslash\mathbb{R}$ are of the type

$$\tilde\psi_{\pm,1}(z, x) = \psi_{\pm,1}(z, x, x_0) C_\pm \tag{3.14}$$

for some nonsingular $m \times m$ matrices $C_\pm$. Thus,

$$\tilde\psi_{\pm,1}(z,x_0) = C_\pm, \quad \tilde\psi'_{\pm,1}(z,x_0) = M_\pm(z,x_0)C_\pm \tag{3.15}$$

by (2.17) and (2.31). In particular,

$$M_\pm(z,x_0) = \tilde\psi'_{\pm,1}(z,x_0)\tilde\psi_{\pm,1}(z,x_0)^{-1}, \quad z \in \mathbb{C}\backslash\mathbb{R} \tag{3.16}$$

is independent of the normalization chosen for $\tilde\psi_{\pm,1}(z,x_0)$. Varying the reference point $x_0 \in \mathbb{R}$ then yields the standard Riccati-type equation,

$$M'_\pm(z,x) + M_\pm(z,x)^2 = Q(x) - zI_m. \tag{3.17}$$

Existence of the asymptotic expansion (3.10) under the conditions imposed on $Q(x)$ is a highly nontrivial matter and proved separately in [18]. The recursion relation for the coefficients $m_{\pm,k}(x)$ in (3.11) then follows by inserting (3.10) into (3.17).

Since $G(z,x,x) = M_{1,1}(z,x)$, Theorem 3.1 and (3.3) then yield an analogous asymptotic expansion for the diagonal Green's matrix $G(z,x,x)$ of H. In fact, one obtains the following result.

Theorem 3.2 ([18]). *Assume the hypotheses in Theorem* 3.1. *Then $G(z,x,x)$ has an asymptotic expansion in C_ε of the form* ($\mathrm{Im}(z^{1/2}) \geq 0$ *for* $z \in \mathbb{C}$)

$$G(z,x,x) \underset{\substack{|z|\to\infty\\ z\in C_\varepsilon}}{=} \frac{i}{2}\sum_{k=0}^{N} G_k(x)z^{-k-1/2} + o(|z|^{-N-1/2}), \tag{3.18}$$

where

$$G_0(x) = I_m, \quad G_1(x) = \frac{1}{2}Q(x), \ \ etc. \tag{3.19}$$

The expansion (3.18) *is uniform with respect to* $\arg(z)$ *for* $|z| \to \infty$ *in* C_ε *and uniform in* $x \in \mathbb{R}$ *as long as x varies in compact intervals.*

Proof. The existence of the asymptotic expansion (3.18) is clear from (3.3) and (3.10). The actual expansion coefficients then can be determined from (3.3) and (3.11). □

The trace formula for $Q(x)$ is then derived as follows.

Theorem 3.3. *In addition to Hypothesis* 2.2 *suppose that $Q \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R})^{m\times m}$ and $E_0 = \inf(\sigma(H)) > -\infty$. Then*

$$Q(x) = E_0 I_m + \lim_{z\to i\infty}\int_{E_0}^{\infty} d\lambda\, z^2(\lambda - z)^{-2}(I_m - 2\Xi(\lambda,x)), \quad x \in \mathbb{R}. \tag{3.20}$$

Proof. By (3.5) and (3.9) one infers

$$\frac{d}{dz}\ln(G(z,x,x)) = \int_{E_0}^{\infty} d\lambda\,(\lambda - z)^{-2}\Xi(\lambda,x). \tag{3.21}$$

By (3.18), (3.19), and the uniformity of the asymptotic expansion (3.18) with respect to $\arg(z)$ as $|z| \to \infty$ in C_ε, which permits its differentiation in z, one derives

$$-\frac{d}{dz}\ln(G(z,x,x)) \underset{z\to i\infty}{=} \frac{1}{2}I_m z^{-1} + \frac{1}{2}Q(x)z^{-2} + o(|z|^{-2}). \tag{3.22}$$

Thus,

$$-\frac{d}{dz}\ln(G(z,x,x)) = \frac{1}{2}I_m(z - E_0)^{-1} + \frac{1}{2}\int_{E_0}^{\infty} d\lambda\,(\lambda - z)^{-2}(I_m - 2\Xi(\lambda,x))$$

$$\underset{z\to i\infty}{=} \frac{1}{2}I_m z^{-1} + \frac{1}{2}Q(x)z^{-2} + o(|z|^{-2}) \tag{3.23}$$

proves (3.20). □

In the scalar case $m = 1$, Theorem 3.3 was first derived in [36]. Subsequent extensions in the case $m = 1$ and their applications to KdV invariants appeared in [33], [34], [39], [78], [79]. For an abstract approach to trace formulas based on perturbation theory and the theory of self-adjoint extensions of symmetric operators we refer to [35]. The case of matrix-valued Schrödinger operators was briefly sketched in [34]. A different kind of trace formula, based on scattering-theoretic concepts for short-range matrix-valued potentials, appeared in [71]. This reference also contains a variety of applications to matrix-valued completely integrable evolution equations. Moreover, a trace formula for matrix-valued Schrödinger operators H on a finite interval with Dirichlet boundary conditions at the endpoints was briefly discussed in [74].

Remark 3.4. Assuming the hypotheses of Theorem 3.1, on infers

$$-\frac{d}{dz}\ln(G(z,x,x)) \underset{z\to i\infty}{=} \sum_{k=0}^{N} R_k(x)z^{-k-1} + o(|z|^{-N-1}),$$
$$R_0(x) = \frac{1}{2}I_m, \quad R_1(x) = \frac{1}{2}Q(x), \text{ etc.,} \tag{3.24}$$

and derives in a similar fashion the higher-order trace formulas (see [39] for the special scalar case $m = 1$)

$$R_k(x) = \frac{1}{2}E_0^k + k \lim_{z\to i\infty} \int_{E_0}^{\infty} d\lambda\, z^{k+1}(\lambda - z)^{-k-1}(-\lambda)^{k-1}(\frac{1}{2}I_m - \Xi(\lambda,x)),$$
$$k = 1,\dots,N,\ x \in \mathbb{R}. \tag{3.25}$$

4. Borg-Type Uniqueness Theorems

In 1946 Borg [7] proved, among a variety of other inverse spectral theorems, the following result.

Theorem 4.1 ([7]). *Let $q \in L^2_{\text{loc}}(\mathbb{R})$ be real-valued and periodic. Let $h = -\frac{d^2}{dx^2} + q$ be the associated self-adjoint Schrödinger operator in $L^2(\mathbb{R})$ (cf. (2.39) for $m = 1$) and suppose that $\sigma(h) = [e_0, \infty)$ for some $e_0 \in \mathbb{R}$. Then*

$$q(x) = e_0 \text{ for a.e. } x \in \mathbb{R}. \tag{4.1}$$

Traditionally, uniqueness results such as Theorem 4.1 are called Borg-type theorems. (However, this terminology is not uniquely adopted and hence a bit unfortunate. Indeed, inverse spectral results on finite intervals recovering the potential coefficient(s) from several spectra, were also pioneered by Borg in his celebrated paper [7], and hence are also coined Borg-type theorems in the literature, see, e.g., [68, Sect. 6].) The purpose of this section is to develop a new strategy of proof for such theorems based on trace formulas and prove extensions to matrix-valued situations. In order to explain our strategy, we provide a quick proof of Theorem 4.1 (assuming $q \in \text{AC}_{\text{loc}}(\mathbb{R})$ instead of $q \in L^2_{\text{loc}}(\mathbb{R})$).

Proof of Theorem 4.1 Suppose $q \in \text{AC}_{\text{loc}}(\mathbb{R})$ is real-valued and periodic. Then by standard Floquet theory, $g(z,x,x)$, the diagonal Green's function of h, is well-known to be purely imaginary on $\sigma(h)^o$ (A^o the open interior of a set $A \subseteq \mathbb{R}$).

Hence, introducing $\xi(\lambda,x) = \pi^{-1}\lim_{\varepsilon\downarrow 0}\operatorname{Im}(\ln(g(\lambda+i\varepsilon,x,x))$ for a.e. $\lambda\in\mathbb{R}$, one infers for all $x\in\mathbb{R}$,

$$\xi(\lambda,x) = \frac{1}{2}, \quad \lambda\in\sigma(h)^o. \tag{4.2}$$

Since $\sigma(h) = [e_0,\infty)$ by hypothesis, the trace formula (3.20) for $q(x)$ and (4.2) yield

$$q(x) = e_0 + \lim_{z\to i\infty}\int_{e_0}^{\infty} d\lambda\, z^2(\lambda-z)^{-2}(1-2\xi(\lambda,x)) = e_0, \quad x\in\mathbb{R}. \tag{4.3}$$

□

A closer examination of the proof shows that periodicity of $q(x)$ is not the point for the uniqueness result (4.1). The key ingredient (besides $\sigma(h) = [e_0,\infty)$ and q real-valued) is clearly the fact (4.2), that is, for all $x\in\mathbb{R}$,

$$\xi(\lambda,x) = 1/2 \text{ for a.e. } \lambda\in\sigma_{\text{ess}}(h) \tag{4.4}$$

($\sigma_{\text{ess}}(\,\cdot\,)$ the essential spectrum).

Real-valued periodic potentials are known to satisfy (4.4) but so are certain classes of real-valued quasi-periodic and almost-periodic potentials $q(x)$ (see, e.g., [8], [20], [21], [57], [58], [59], [60], [61], [82]). In particular, the class of real-valued algebro-geometric finite-gap potentials $q(x)$ (a subclass of the set of real-valued quasi-periodic potentials) is a prime example satisfying (4.4) without necessarily being periodic. Traditionally, potentials $q(x)$ satisfying (4.4) are called *reflectionless* (see [8], [20], [21], [60]).

Remark 4.2. We note that real-valuedness of q is an essential assumption in Theorem 4.1. Indeed, $q(x) = \exp(ix)$, $x\in\mathbb{R}$, is well-known to lead to the half-line spectrum $\sigma(h) = [0,\infty)$, with $h = -\frac{d^2}{dx^2}+q$ in $L^2(\mathbb{R})$ defined on the standard Sobolov space $H^{2,2}(\mathbb{R})$. A detailed treatment of a class of examples of this type can be found in [29], [30], [40], [75], [76]. Moreover, the example of complete exponential localization of the spectrum of a discrete Schrödinger operator with a quasi-periodic real-valued potential having two basic fequencies and no gaps in its spectrum illustrates the importance of the reflectionless property of q in Theorem 4.1.

Taking the quick proof of Theorem 4.1 as the point of departure for our extension of Borg-type results to matrix-valued Schrödinger operators, we now use the reflectionless situation described in (4.4) as the model for the subsequent definition.

Definition 4.3. Assume Hypothesis 2.2 and $Q\in\mathrm{AC}_{\mathrm{loc}}(\mathbb{R})^{m\times m}$.
Then the matrix-valued potential $Q(x)$ is called *reflectionless* if for all $x\in\mathbb{R}$,

$$\Xi(\lambda,x) = \frac{1}{2}I_m \text{ for a.e. } \lambda\in\sigma_{\text{ess}}(H). \tag{4.5}$$

Since hardly any confusion can arise, we will also call H reflectionless if (4.5) is satisfied.

Given Definition 4.3, we turn to a Borg-type uniqueness theorem and formulate the analog of Theorem 4.1 for (reflectionless) matrix-valued Schrödinger operators.

Theorem 4.4. *Assume Hypothesis* 2.2 *and* $Q\in\mathrm{AC}_{\mathrm{loc}}(\mathbb{R})^{m\times m}$. *Suppose that* $Q(x)$ *is reflectionless and* $\sigma(H) = [E_0,\infty)$. *Then*

$$Q(x) = E_0 I_m \textit{ for all } x\in\mathbb{R}. \tag{4.6}$$

Proof. By hypothesis, $\Xi(\lambda,x) = (1/2)I_m$ for a.e. $\lambda\in[E_0,\infty)$ and all $x\in\mathbb{R}$. Thus the trace formula (3.20) yields (4.6). □

In the remainder of the section we will show that the case of periodic $Q(x)$ is covered by Theorem 4.4 under appropriate uniform multiplicity assumptions on $\sigma(H)$. Among other results this then recovers a recent theorem by Dépres [23] for matrix-valued periodic Schrödinger operators.

In order to discuss Floquet theory for H we adopt the following assumptions for the remainder of this section.

Hypothesis 4.5. *In addition to Hypothesis* 2.1 *suppose that* $Q \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R})^{m\times m}$ *is periodic, that is, there is an* $\omega > 0$ *such that* $Q(x+\omega) = Q(x)$ *for all* $x \in \mathbb{R}$.

Since by Hypothesis 4.5, $Q \in L^\infty(\mathbb{R})^{m\times m}$, the corresponding periodic Hamiltonian system (2.3) is in the l.p. case at $\pm\infty$.

We briefly review a few basic facts from Floquet theory for Hamiltonian systems of the type $J\psi'(z,x) = (zA + B(x))\psi(z,x)$ with $B(x)$ periodic of period $\omega > 0$. For a detailed treatment of Floquet theory, relevant in our context, we refer to [22], [28, pp. 1486–1498], [32], [41], [55], [65], [64], [66], [67], [77], [83], [85], [86], [87], [88], and the literature therein. Recalling the notation introduced in (2.16) and (2.17) one considers the monodromy matrix

$$\Phi(z,x_0) = \begin{pmatrix} \theta_1(z,x_0+\omega,x_0) & \phi_1(z,x_0+\omega,x_0) \\ \theta_2(z,x_0+\omega,x_0) & \phi_2(z,x_0+\omega,x_0) \end{pmatrix}, \quad z \in \mathbb{C}. \tag{4.7}$$

Denoting its eigenvalues by $\rho_j(z)$, that is,

$$\sigma(\Phi(z,x_0)) = \{\rho_j(z)\}_{j=1,\dots,n}, \tag{4.8}$$

it is a well-known fact that $\sigma(\Phi(z,x_0))$, unlike $\Phi(z,x_0)$, is independent of the chosen reference point $x_0 \in \mathbb{R}$. Moreover,

$$\det(\Phi(z,x_0)) = 1, \quad z \in \mathbb{C}. \tag{4.9}$$

One then obtains the following characterization of the spectrum of H,

$$\sigma(H) = \{\lambda \in \mathbb{R} \mid |\rho_j(\lambda)| = 1 \text{ for some } j \in \{1,2,\dots,n\}\}. \tag{4.10}$$

In particular,

$$|\rho_j(z)| \neq 1 \text{ for all } z \in \mathbb{C}\backslash\sigma(H). \tag{4.11}$$

Let $\Psi(z,x,x_0)$ denote the normalized fundamental system (2.16) of the Hamiltonian system (2.14) with $\Psi(z,x_0,x_0) = I_n$, then

$$\Psi(z,x+\omega,x_0) = \Psi(z,x,x_0)\Phi(z,x_0), \quad z \in \mathbb{C} \tag{4.12}$$

by periodicity of Q. Since by hypothesis H is self-adjoint, there exists a fundamental system $\hat\Psi(z,x,x_0)$ of (2.14) of the following type (cf. (2.31)),

$$\hat\Psi(z,x,x_0) = \begin{pmatrix} \psi_{-,1}(z,x,x_0) & \psi_{+,1}(z,x,x_0) \\ \psi_{-,2}(z,x,x_0) & \psi_{+,2}(z,x,x_0) \end{pmatrix}, \quad z \in \mathbb{C}\backslash\sigma(H), \tag{4.13}$$

with $\psi_{\pm,1}(z,x_0,x_0) = I_m$, where

$$\psi_{\pm,1}(z,\cdot\,,x_0) \in L^2((R,\pm\infty))^{m\times m} \text{ for all } R \in \mathbb{R},\ z \in \mathbb{C}\backslash\sigma(H). \tag{4.14}$$

Thus,

$$\hat\Psi(z,x+\omega,x_0) = \hat\Psi(z,x,x_0)\hat\Phi(z,x_0), \quad z \in \mathbb{C}\backslash\sigma(H) \tag{4.15}$$

and hence, as in the proof of Theorem 3.1 (cf. (3.15)), one infers that $\hat{\Phi}(z,x_0)$ must be of the form

$$\hat{\Phi}(z,x_0)=\begin{pmatrix}\rho_-(z,x_0) & 0\\ 0 & \rho_+(z,x_0)\end{pmatrix} \tag{4.16}$$

for nonsingular $m\times m$ matrices $\rho_\pm(z,x_0)$. Thus,

$$\psi_{\pm,j}(z,x+\omega,x_0)=\psi_{\pm,j}(z,x,x_0)\rho_\pm(z,x_0),\quad j=1,2. \tag{4.17}$$

Next, noticing

$$\hat{\Psi}(z,x,x_0)=\Psi(z,x,x_0)C(z,x_0),\quad z\in\mathbb{C}\backslash\sigma(H) \tag{4.18}$$

for some nonsingular $n\times n$ matrix $C(z,x_0)$ one infers

$$\hat{\Phi}(z,x_0)=C(z,x_0)^{-1}\Phi(z,x_0)C(z,x_0),\quad z\in\mathbb{C}\backslash\sigma(H) \tag{4.19}$$

and hence

$$\sigma(\hat{\Phi}(z,x_0))=\sigma(\Phi(z,x_0)),\quad z\in\mathbb{C}\backslash\sigma(H). \tag{4.20}$$

We observe from (4.14), (4.16), (4.17), and (4.20) that $\sigma(\Phi(z,x_0))$ can be partioned as

$$\sigma(\Phi(z,x_0))=\{\rho_j(z)\}_{j=1,\dots,n}=\sigma(\rho_-(z,x_0))\cup\sigma(\rho_+(z,x_0)), \tag{4.21}$$

$$\sigma(\rho_\pm(z,x_0))=\{\rho_{\pm,j}(z)\}_{j=1,\dots,m}, \tag{4.22}$$

where (cf. (4.9) and (4.11))

$$0\neq|\rho_{\pm,j}(z)|\lessgtr 1 \text{ for } z\in\mathbb{C}\backslash\sigma(H),\ j=1,\dots,m. \tag{4.23}$$

Hence

$$\psi_{\pm,1}(z,x_0+\omega,x_0)=\rho_\pm(z,x_0)=\theta_1(z,x_0+\omega,x_0)+\phi_1(z,x_0+\omega,x_0)M_\pm(z,x_0) \tag{4.24}$$

yields

$$\begin{aligned}M_\pm(z,x_0)&=\psi_{\pm,2}(z,x_0,x_0)\\&=\phi_1(z,x_0+\omega,x_0)^{-1}\big(\rho_\pm(z,x_0)-\theta_1(z,x_0+\omega,x_0)\big),\\&\qquad\qquad z\in\mathbb{C}\backslash(\sigma(H)\cup\sigma(H^D_{x_0}))\end{aligned} \tag{4.25}$$

and

$$\begin{aligned}\psi_{\pm,j}(z,x,x_0)=&\theta_j(z,x,x_0)\\&+\phi_j(z,x,x_0)\phi_1(z,x_0+\omega,x_0)^{-1}(\rho_\pm(z,x_0)-\theta_1(z,x_0+\omega,x_0)),\\&\qquad z\in\mathbb{C}\backslash(\sigma(H)\cup\sigma(H^D_{x_0})),\ j=1,2,\end{aligned} \tag{4.26}$$

where

$$\sigma(H^D_{x_0})=\{z\in\mathbb{C}\mid\det(\phi_1(z,x_0+\omega,x_0))=0\}. \tag{4.27}$$

One can show that $\sigma(H^D_{x_0})$ is the spectrum of a self-adjoint operator $H^D_{x_0}$ (associated with a Dirichlet-type boundary condition $\psi_1(z,x_0+\omega)=\psi_1(z,x_0)=0$, $\psi_1(z,\cdot\,)\in\mathrm{AC}([x_0,x_0+\omega])^m$) and hence $\sigma(H^D_{x_0})\subset\mathbb{R}$. Combining (2.16), (4.7), (4.12), (4.17), and (4.26) yields

$$\hat{\Phi}(z,x_0)^2-\begin{pmatrix}\theta_1+\phi_1\phi_2\phi_1^{-1} & 0\\ 0 & \theta_1+\phi_1\phi_2\phi_1^{-1}\end{pmatrix}\hat{\Phi}(z,x_0)$$

$$+\begin{pmatrix} \phi_1\phi_2\phi_1^{-1}\theta_1 - \phi_1\theta_2 & 0 \\ 0 & \phi_1\phi_2\phi_1^{-1}\theta_1 - \phi_1\theta_2 \end{pmatrix} = 0, \quad (4.28)$$

where θ_j, ϕ_j are evaluated at the point $(z, x_0+\omega, x_0)$, that is,

$$\phi_j = \phi_j(z, x_0+\omega, x_0), \quad \theta_j = \theta_j(z, x_0+\omega, x_0), \quad j=1,2. \quad (4.29)$$

Equation (4.28) is equivalent to

$$\begin{aligned} &\rho_\pm(z,x_0)^2 - \big(\theta_1(z,x_0+\omega,x_0) + \phi_1(z,x_0+\omega,x_0)\phi_2(z,x_0+\omega,x_0) \times \\ &\qquad \times \phi_1(z,x_0+\omega,x_0)^{-1}\big)\rho_\pm(z,x_0) \\ &+ \phi_1(z,x_0+\omega,x_0)\phi_2(z,x_0+\omega,x_0)\phi_1(z,x_0+\omega,x_0)^{-1}\theta_1(z,x_0+\omega,x_0) \\ &- \phi_1(z,x_0+\omega,x_0)\theta_2(z,x_0+\omega,x_0) = 0, \quad z \in \mathbb{C}\backslash(\sigma(H)\cup\sigma(H^D_{x_0})). \end{aligned} \quad (4.30)$$

In anticipation of (4.33), which will be proven next, $\rho_\pm(z,x_0)$ can be extended to $\rho_\pm(\lambda+i0,x_0) = \lim_{\varepsilon\downarrow 0}\rho_\pm(\lambda+i\varepsilon,x_0)$ by continuity for all $\lambda\in\sigma(H)^o$. Hence we will in the following extend the domain of validity of (4.16) and (4.28)–(4.30) to all $z\in\sigma(H)^o$ (agreeing to take normal limits to the real line in $\mathbb{C}_+$).

Theorem 4.6. *Suppose Hypothesis* 4.5. *If H has uniform spectral multiplicity $2m$, then for all $x\in\mathbb{R}$ and all $\lambda\in\sigma(H)^o$,*

$$M_+(\lambda+i0,x) = M_-(\lambda+i0,x)^* = M_-(\lambda-i0,x). \quad (4.31)$$

In particular, $M_-(z,x)$ is the analytic continuation of $M_+(z,x)$ (and vice versa) through $\sigma(H)^o$.

Proof. By general Floquet theory, $\sigma(H)$ consists of a countable union of closed intervals on $\mathbb{R}$, possibly separated by gaps in between. Moreover, H is bounded from below and it has no eigenvalues,

$$\sigma_p(H) = \emptyset, \quad \sigma(H) = \sigma_{\mathrm{ess}}(H) = \sigma_c(H). \quad (4.32)$$

In fact, one can show that $\sigma(H)$ is purely absolutely continuous, $\sigma(H)=\sigma_{\mathrm{ac}}(H)$ (as is also clear from the analytic continuation of $M_\pm(\lambda+i0,x)$ through $\sigma(H)^o$ implied by (4.31)), but we omit the details. The assumptions of uniform (maximal) spectral multiplicity $n=2m$ of $\sigma(H)$ guarantees the existence of n eigenvalues $\rho_j(\lambda)$ of the monodromy matrix $\Phi(\lambda,x_0)$ with $|\rho_j(\lambda)|=1$ for $j=1,\dots,n$ for $\lambda\in\sigma(H)^o$, in particular, $\Phi(\lambda,x_0)$ is unitary and hence diagonalizable for $\lambda\in\sigma(H)^o$.

Next, suppose that $\psi_1(\lambda,\cdot,x_0),\dots,\psi_n(\lambda,\cdot,x_0)\in L^\infty(\mathbb{R})^n$ for $\lambda\in\sigma(H)^o$, with $\psi_j(\lambda,x_0,x_0)=(\delta_{j,1},\dots,\delta_{j,n})^t$ for $j=1,\dots,n$ are n linearly independent normalized solutions of (2.3). We claim that

$$\sigma_p(H^D_{x_0})\cap\sigma(H)^o = \emptyset \quad (4.33)$$

since eigenfunctions of $H^D_{x_0}$ for $\lambda\in\sigma(H)^o$ would necessarily be linear combinations of $\psi_1(\lambda,x,x_0),\dots,\psi_n(\lambda,x,x_0)$. However, none of them can lie in $L^2(\mathbb{R})^n$ since the fundamental matrix $\Psi(\lambda,x,x_0)=(\psi_1(\lambda,x,x_0),\dots,\psi_n(\lambda,x,x_0))$, $\lambda\in\sigma(H)^o$ satisfies

$$\Psi(\lambda,x+\omega,x_0) = \Psi(\lambda,x,x_0)\Phi(\lambda,x_0), \quad \lambda\in\sigma(H)^o \quad (4.34)$$

(cf. (4.12)) with $\Phi(\lambda,x_0)$ unitary. Since $\sigma(H^D_{x_0})\subset\mathbb{R}$, (4.33) implies the existence of $\phi_1(z,x_0+\omega,x_0)^{-1}$ for $z\in\mathbb{C}\backslash\sigma(H^D_{x_0})$.

In the following we denote by $\mathcal{D}\subset\mathbb{C}$ the discrete set of points (i.e., countable without finite limit points) where $\hat\Phi(z,x_0)$ is not diagonalizable.

By our hypothesis of uniform spectral multiplicity n of H, $\hat{\Phi}(z,x_0)$ is diagonalizable for all $z \in \mathbb{C}\backslash\partial\sigma(H)$ (∂A denoting the boundary of a subset $A \subseteq \mathbb{R}$) and we conclude that $\mathcal{D} \subseteq \partial\sigma(H)$. By a well-known argument, see, for instance [28, Lemma XIII.7.63], one infers that all points in $\partial\sigma(H)$ are branch points for the eigenvalues $\rho_j(z)$ of $\Phi(z,x_0)$ and hence $\mathcal{D} = \partial\sigma(H)$. Thus we obtain upon diagonalizing $\hat{\Phi}(z,x_0)$ in (4.28) that

$$\tilde{\Phi}(z,x_0)^2 - E(z,x_0)\tilde{\Phi}(z,x_0) + F(z,x_0) = 0, \quad z \in \mathbb{C}\backslash(\partial\sigma(H) \cup \sigma(H^D_{x_0})), \tag{4.35}$$

where

$$\tilde{\Phi}(z,x_0) = \begin{pmatrix} \rho_1(z) & 0 & \dots & 0 \\ 0 & \rho_2(z) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \rho_n(z) \end{pmatrix} \tag{4.36}$$

denotes the diagonalization of $\hat{\Phi}(z,x_0)$, and $E(z,x_0)$, $F(z,x_0)$ are analytic in $z \in \mathbb{C}\backslash\sigma(H)$ with a continuous extension to $\sigma(H)^o$. Thus each $\rho_j(z)$ satisfies a quadratic equation and one introduces a canonical set of cuts $\{\mathcal{C}_k\}_{k\in I}$ along $\sigma(H)$ (I a finite or countably infinite index set), joining the branch points, that is, all points in $\partial\sigma(H)$ (as well as $+\infty$ and possibly $-\infty$ in case I is finite)

$$\sigma(H) = \bigcup_{k\in I} \mathcal{C}_k. \tag{4.37}$$

In this manner, each $\rho_j(z)$ becomes an analytic function on a (fixed) two-sheeted Riemann surface (glued together crosswise along these cuts in a standard manner). In particular, $\{\rho_1(z),\dots,\rho_n(z)\}$ can now be split into pairs $\{\rho_{1,+}(z),\rho_{1,-}(z),\dots,$ $\rho_{m,+}(z),\rho_{m,-}(z)\}$ such that $\rho_{k,-}(z)$ represents the analytic continuation of $\rho_{k,+}(z)$ (and vice versa), whenever z crosses tranversally through one of the cuts.

Next, pick a $\lambda_0 \in \sigma(H)^o$, that is, $\lambda_0 \in \mathcal{C}^o_{k_0}$ for some $k_0 \in I$ and pick a $\rho_{j_0,+}(z)$ for z in a sufficiently small neighborhood $U_+(\lambda_0) \cap \mathbb{C}_+$ (or $U_-(\lambda_0) \cap \mathbb{C}_-$) of λ_0. Suppose $|\rho_{j_0,+}(z)| > 1$ for z along a path in $U_+(\lambda_0)$ transversally approaching $\mathcal{C}^0_{k_0}$ and intersecting $\mathcal{C}^o_{k_0}$ at λ_0. By analyticity of $\rho_{j_0,+}(z)$, the analytic continuation $\tilde{\rho}_{j_0,+}(z) = \rho_{j_0,-}(z)$ of $\rho_{j_0,+}(z)$ will satisfy $|\tilde{\rho}_{j_0,+}(z)| < 1$ in an appropriate neighborhood $V_-(\lambda_0) \cap \mathbb{C}_-$ (or $V_+(\lambda_0) \cap \mathbb{C}_+$) of λ_0. Hence we may identify $\{\rho_{j,\pm}(z)\}_{j=1,\dots,m}$ with $\{\rho_{\pm,j}(z)\}_{j=1,\dots,m}$ in (4.21)–(4.23). Thus upon possibly reordering the eigenvalues along the diagonal in (4.36) we may write

$$\tilde{\Phi}(z,x_0) = \begin{pmatrix} \tilde{\rho}_-(z) & 0 \\ 0 & \tilde{\rho}_+(z) \end{pmatrix}, \tag{4.38}$$

where

$$\tilde{\rho}_\pm(z) = \begin{pmatrix} \rho_{\pm,1}(z) & 0 & \dots & 0 \\ 0 & \rho_{\pm,2}(z) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \rho_{\pm,m}(z) \end{pmatrix} \tag{4.39}$$

such that $\tilde{\rho}_-(z)$ is the analytic continuation of $\tilde{\rho}_+(z)$ through the interior of the cuts $\cup_{k\in I}\mathcal{C}^o_k$ and hence through $\sigma(H)^o$. Since the similarity transformations connecting $\tilde{\rho}_\pm(z)$ and $\rho_\pm(z,x_0)$ can be chosen as matrices whose column vectors are the eigenvectors $e_{\pm,j}(z)$ of $\rho_\pm(z,x_0)$, and $e_{\pm,j}(z)$ have the same branching behavior as the associated eigenvalues $\rho_{\pm,j}(z)$ (see, e.g., [5, Sect. 6.1]), one infers that $\rho_-(z,x_0)$ is the analytic continuation of $\rho_+(z,x_0)$ through $\sigma(H)^o$. By (4.25), $M_-(z,x_0)$ is the analytic continuations of $M_+(z,x_0)$ through $\sigma(H)^o$. Consequently,

$$M_-(\lambda \mp i0,x_0) = M_-(\lambda \pm i0,x_0)^* = M_+(\lambda \pm i0,x_0), \quad \lambda \in \sigma(H)^o. \tag{4.40}$$

Since $x_0 \in \mathbb{R}$ was arbitrary we obtain (4.31). □

The next result proves necessary and sufficient conditions for Q to be reflectionless. It is modeled after Lemma 3.3 in [38] in the context of Jacobi operators and we provide a formulation that anticipates extensions to the non-periodic case following [82], to be discussed elsewhere.

Theorem 4.7. *Assume Hypothesis* 4.5 *and that H has uniform spectral multiplicity* $2m$. *Let $\Sigma \subseteq \sigma(H)^o$. Then the following conditions are equivalent and each of them holds.*
(i) For all $x \in \mathbb{R}$ and all $\lambda \in \Sigma$,

$$\Xi(\lambda,x) = \frac{1}{2} I_m. \tag{4.41}$$

(ii) For some $x_0 \in \mathbb{R}$ and all $\lambda \in \Sigma$,

$$G(\lambda+i0,x_0,x_0) = -G(\lambda+i0,x_0,x_0)^*, \tag{4.42}$$

$$G'(\lambda+i0,x_0,x_0) = -G'(\lambda+i0,x_0,x_0)^*. \tag{4.43}$$

(Here $G'(\lambda+i0,x_0,x_0) = \frac{d}{dx}G(\lambda+i0,x,x)|_{x=x_0}$.)
(iii) For some $x_0 \in \mathbb{R}$ and all $\lambda \in \Sigma$,

$$M_+(\lambda+i0,x_0) = M_-(\lambda+i0,x_0)^*. \tag{4.44}$$

Proof. In the following, let $x, x_0 \in \mathbb{R}$ and $\lambda \in \Sigma$. By Theorem 4.6, the normal limits $M_\pm(\lambda+i0,x_0)$ and $(M_-(\lambda+i0,x_0) - M_+(\lambda+i0,x_0))^{-1}$ exist for all $\lambda \in \Sigma$ and condition (iii) holds. By (3.9), (4.41) is equivalent to

$$G(\lambda+i0,x,x) = -G(\lambda+i0,x,x)^* \tag{4.45}$$

and hence taking $x = x_0$ implies (4.42). By (2.16), (3.1), and (3.2), one infers

$$\begin{aligned} G(z,x,x) &= \psi_{\mp,1}(z,x,x_0)G(z,x_0,x_0)\psi_{\pm,1}(\bar z,x,x_0)^* \\ &= (\theta_1(z,x,x_0) + \phi_1(z,x,x_0)M_\mp(z,x_0))G(z,x_0,x_0) \times \\ &\quad \times (\theta_1(z,x,x_0) + M_\pm(z,x_0)\phi_1(z,x,x_0)), \quad z \in \mathbb{C}_+, \end{aligned} \tag{4.46}$$

and

$$\begin{aligned} \frac{d}{dx}G(z,x,x) &= (\theta_1'(z,x,x_0) + \phi_1'(z,x,x_0)M_\mp(z,x_0))G(z,x_0,x_0) \times \\ &\quad \times (\theta_1(z,x,x_0) + M_\pm(z,x_0)\phi_1(z,x,x_0)) \\ &\quad + (\theta_1(z,x,x_0) + \phi_1(z,x,x_0)M_\mp(z,x_0))G(z,x_0,x_0) \times \\ &\quad \times (\theta_1'(z,x,x_0) + M_\pm(z,x_0)\phi_1'(z,x,x_0)), \quad z \in \mathbb{C}_+. \end{aligned} \tag{4.47}$$

Hence, $G(\lambda+i0,x,x)$ and $G'(\lambda+i0,x,x)$ exist for all $\lambda \in \Sigma$, and we may differentiate (4.45) with respect to $x \in \mathbb{R}$ to obtain

$$\frac{d}{dx}G(\lambda+i0,x,x) = -\frac{d}{dx}G(\lambda+i0,x,x)^*. \tag{4.48}$$

Taking $x = x_0$ in (4.48) then implies (4.43) and hence we have shown that (i) implies (ii).

Next we prove that (ii) implies (iii). We could immediately invoke (4.31), but prefer to show a simple argument that permits extensions to non-periodic cases to be discussed elsewhere. By (3.3), (4.45) is equivalent to

$$M_-(\lambda+i0,x)-M_+(\lambda+i0,x)=-(M_-(\lambda+i0,x)-M_+(\lambda+i0,x))^* \tag{4.49}$$

and hence to

$$\mathrm{Re}(M_+(\lambda+i0,x))=\mathrm{Re}(M_-(\lambda+i0,x)). \tag{4.50}$$

Thus, (4.42) is equivalent to

$$\mathrm{Re}(M_+(\lambda+i0,x_0))=\mathrm{Re}(M_-(\lambda+i0,x_0)) \text{ for all } \lambda\in\Sigma. \tag{4.51}$$

In order to exploit (4.43) one computes from (3.3) and (4.47),

$$\begin{aligned} G'(z,x_0,x_0)&=M_\mp(z,x_0)(M_-(z,x_0)-M_+(z,x_0))^{-1}\\ &\quad+(M_-(z,x_0)-M_+(z,x_0))^{-1}M_\pm(z,x_0). \end{aligned} \tag{4.52}$$

Consequently, (4.43) is equivalent to

$$\begin{aligned} &(M_\mp(\lambda+i0,x_0)-M_\pm(\lambda+i0,x_0)^*)(M_-(\lambda+i0,x_0)-M_+(\lambda+i0,x_0))^{-1}\\ &\quad=(M_-(\lambda+i0,x_0)-M_+(\lambda+i0,x_0))^{-1}(M_\mp(\lambda+i0,x_0)^*-M_\pm(\lambda+i0,x_0)). \end{aligned} \tag{4.53}$$

A simple manipulation in (4.53) (adding and subtracting $M_\pm(\lambda+i0,x_0)$) then yields

$$\begin{aligned} &(M_+(\lambda+i0,x_0)-M_+(\lambda+i0,x_0)^*)(M_-(\lambda+i0,x_0)-M_+(\lambda+i0,x_0))^{-1}\\ &\quad=(M_-(\lambda+i0,x_0)-M_+(\lambda+i0,x_0))^{-1}(M_-(\lambda+i0,x_0)^*-M_-(\lambda+i0,x_0)) \end{aligned} \tag{4.54}$$

and thus

$$\begin{aligned} &(M_-(\lambda+i0,x_0)-M_+(\lambda+i0,x_0))\mathrm{Im}(M_+(\lambda+i0,x_0))\\ &\quad=-\mathrm{Im}(M_-(\lambda+i0,x_0))(M_-(\lambda+i0,x_0)-M_+(\lambda+i0,x_0)) \text{ for all } \lambda\in\Sigma. \end{aligned} \tag{4.55}$$

Taking into account (4.51) in (4.55) results in

$$(\mathrm{Im}(M_+(\lambda+i0,x_0)))^2=(\mathrm{Im}(M_-(\lambda+i0,x_0)))^2 \tag{4.56}$$

and hence in

$$\mathrm{Im}(M_+(\lambda+i0,x_0))=-\mathrm{Im}(M_-(\lambda+i0,x_0)) \text{ for all } \lambda\in\Sigma, \tag{4.57}$$

since $\pm M_\pm(\lambda+i0,x_0)$ are Herglotz matrices, implying $\pm\mathrm{Im}(M_\pm(\lambda+i0,x_0))>0$. Combining (4.51) and (4.57) yields (4.44) and hence (iii).

Given (iii) and $M_\pm(z)^*=M_\pm(\bar z)$ one computes from (4.46)

$$G(\lambda+i0,x,x)^*=-G(\lambda+i0,x,x), \tag{4.58}$$

and hence

$$\Xi(\lambda,x)=\frac{1}{2}I_m \text{ for all } x\in\mathbb{R} \text{ and all } \lambda\in\Sigma, \tag{4.59}$$

since $\theta_1(\lambda,x,x_0)$ and $\phi_1(\lambda,x,x_0)$ are self-adjoint for all $(\lambda,x,x_0)\in\mathbb{R}^3$. Thus (iii) implies (i). ☐

Theorem 4.7 extends to more general situations (not necessarily periodic ones) as is clear from the corresponding results in [38], [57], [58], [59], [60], [82] in the scalar case $m = 1$ (replacing the phrase "for all $\lambda \in \Sigma$" by "for a.e. $\lambda \in \Sigma$", etc.). For the corresponding matrix-valued case we refer to [61].

Combining Theorems 4.6 and 4.7, one finally obtains the following result.

Theorem 4.8. *Suppose Hypothesis* 4.5*. If H has uniform spectral multiplicity $2m$, then H is reflectionless and for all $x \in \mathbb{R}$ and all $\lambda \in \sigma(H)^o$,*

$$\Xi(\lambda, x) = \frac{1}{2} I_m. \tag{4.60}$$

Corollary 4.9. *Assume Hypothesis* 4.5*. If H has uniform spectral multiplicity $2m$ and $\sigma(H) = [E_0, \infty)$ for some $E_0 \in \mathbb{R}$, then*

$$Q(x) = E_0 I_m \text{ for all } x \in \mathbb{R}. \tag{4.61}$$

Remark 4.10. The assumption of uniform (maximal) spectral multiplicity $n = 2m$ in Corollary 4.9 is an essential one. Otherwise, one can easily construct nonconstant potentials $Q(x)$ such that the associated operator H has overlapping band spectra and hence spectrum a half-line. For such a construction it suffices to consider the case where $Q(x)$ is a diagonal matrix.

Remark 4.11. Corollary 4.9 for matrix-valued Schrödinger operators (assuming $Q \in L^\infty(\mathbb{R})^{m\times m}$ to be periodic) has been proved by Dépres [23] using an entirely different approach based on a detailed Floquet analysis. Dépres' result partly motivated our work once it became clear that trace formulas would be a most natural tool for Borg-type uniqueness results and such theorems are naturally considered in the context of reflectionless rather than periodic potentials. Different proofs of Borg's Theorem 4.1 (i.e., the scalar case $m = 1$ in Corollary 4.9) have also been obtained in [53], [54], [57]. Moreover, it should be stressed that Theorem 4.6 is well-known in the special case $m = 1$. (For $m = 1$ the hypothesis of uniform spectral multiplicity 2 is automatically fulfilled as a simple consequence of $\det(\Phi(z, x_0)) = 1$.) In fact, (4.31) for $m = 1$ is proved, for instance, in [46], [49], [53], [54], [57], [58], [59], [82]. The reflectionless property (4.51) in the general matrix case where $m \in \mathbb{N}$ has also been isolated in [61] in connection with a study of stochastic Schrödinger and Jacobi operators on strips in terms of Lyapunov exponents.

Remark 4.12. In the scalar case $m = 1$, the trace formula (3.20) (more precisely, its heat kernel variant using a heat kernel regularization as opposed to our resolvent regularization) was proved in [39] under the general condition $V_+ \in L^1_{\text{loc}}(\mathbb{R})$, $\sup_{n\in\mathbb{N}}(\int_{-n}^{n} dx\, V_-(x)) < \infty$ (where $V_\pm = (|V| \pm V)/2$) for all Lebesgue points x for V (cf. also [79] in this context). Together with the reflectionless property (4.2) for periodic potentials this slightly extends Borg's original Theorem 4.1 to the case where $V \in L^1_{\text{loc}}(\mathbb{R})$ is real-valued and periodic. We expect a similar extension of Theorems 3.3 to hold for x in the intersection of the Lebesgue points for $Q_{j,k}$ but did not work out the details.

At the end we emphasize that all results presented in this paper also apply to matrix-valued finite-difference Hamiltonian systems. We refer the reader to [19] in this direction.

Finally, Borg-type uniqueness theorems for Hamiltonian systems are just a beginning. There is a natural extension of Borg's Theorem 4.1 to self-adjoint periodic

Schrödinger operators with one gap in its spectrum, that is, $\sigma(H) = [E_0, E_1] \cup [E_2, \infty)$, with $E_1 < E_2$. This extension is due to Hochstadt [50] and the resulting potential $q(x)$ becomes twice the elliptic Weierstrass function. Details on a matrix-valued extension of Hochstadt's uniqueness theorem will appear elsewhere [6].

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