CONTOUR METHODS AND PIROGOV SINAI THEORY FOR CONTINUOUS SPIN LATTICE MODELS

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Abstract. We developed an alternative version of the Pirogov–Sinai theory of low temperature continuous spin models, substantially simplified comparing to the original approach [DZ]. The new method adopted here is based on a transformation of a continuous spin model into a suitable discrete spin model (of a standard 'abstract Pirogov–Sinai' type). It uses a 'Gaussian transformation' (0) for a multiple well potential U and allows us to use methods of discrete spin theory [PS], [Z] and [ZR].

1. Introduction

This is an outline of the new approach to the problem originally solved in [DZ] (see also [ZG] for its summary). We reduce the generality of models treated in [DZ] but considerably simplify or modify the arguments of that paper. The new approach is still based on the same leading ideas and main ingredients, namely:

(0) elementary theory of Gaussian fields with positive mass Hamiltonians,
(1) perturbation theory around positive mass Gaussian fields,
(2) Pirogov–Sinai theory of models with discrete spin.

However, the exposition is now organised quite differently and is much simpler. Theorems 0, 1, 2 (and 3) related to the items above summarize our results. The main novel feature is a representation of a potential U of the following type

$$(0) \quad \exp(-U(x_i)) = \sum_{q\in Q} \exp(-U^q(x_i)) + w(x_i), \text{ where } |w(x_i)| := \exp(-U^*(x_i)).$$

Here, $i = (i_1, \dots, i_\nu) \in \mathbb{Z}^\nu$, $\nu \geq 2$ and $x_i \in \mathbb{R}$ is a spin configuration in the lattice point i. Furthermore $Q \subset \mathbb{R}$ is supposed to be a 'uniformly discrete' (possibly infinite) set; points $q \in Q$ label translation invariant 'locally ground' states $\{x_i = q\}$, $q \in Q$ of H. The terms $U^q(x_i)$ are suitable **quadratic**, positive definite, approximations of $U(x_i)$ around state $\{x_i = q,\, i \in A\}$, and $w(x_i)$ is a remainder,

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assumed to be **small**. See (17B) below. The last condition is, essentially, the only assumption on the potential U. It is valid for many **low temperature** situations.

We will use representation (0) for all $i \in \mathbb{Z}^\nu$. The perturbative term $w(x_i)$ will be treated below as an effect of a newly introduced artificial **energy barrier** $U^*(x_i) = -\log|w(x_i)|$. Decomposition (0) combines the ideas of high temperature expansion and of Fortuin–Kasteleyn type representation (used for the study of Potts models; see e.g. [KS]). One can visualise (0) as an attempt to approximate a given 'Gibbsian hilly landscape' $\{\exp(-U(x_i)), x_i \in \mathbb{R}\}$ for a given potential U by a **sum** of Gaussian (i.e. logarithmically quadratic) 'hills' $\exp(-U^q(x_i))$ where $U^q, q \in Q$ are suitably chosen quadratic approximations of U around the points $q \in Q$ of its **local minima**.

Note that if we wanted to have $w(x_i)$ positive, it would impose rather strong restrictions on U. In fact, as we will see below, nonpositivity of $w(x_i)$ causes no problems, and this general case can be handled practically without any change compared to the case of positive $w(x_i)$.

Notes. 1. I do not know whether decomposition (0) of factor $\exp(-U(x_i))$ was used before in rigorous statistical mechanics of continuous spin models. It greatly simplifies technicalities. One could call it a 'Gaussian transformation' of the Gibbsian factor $\exp(-U(x_i))$ around approximating family $\{\exp(-U^q(x_i))\}$.

One could use an analogous expression as (0) also for pair interactions, or for suitable 'blocks' of (both pair and single spin) interactions. Throughout this paper, we will not exploit such a more general approach. We leave it for a further study.

2. The special case of $w(x_i) \equiv 0$ offers substantial simplifications. Here, the reduction to a discrete spin model is particularly elegant. See the preprint [Ku].

3. The phase transitions in models with double well potentials were first studied in [GJS] (see also [GJ]) and then in [DS], by methods of reflection positivity. The paper [I] first used the idea of conversion to discrete spin models (in a context very different to that which is used here). The paper [DZ] (which arose independently of [I]) developed a general machinery of Pirogov–Sinai theory for these models.

Now we propose a method which is much simpler than these approaches (having in mind that a developed theory for discrete spin models exists). While the use of perturbative techniques (e.g. estimates of semiinvariants of Wick polynomials) around Gaussian fields is minimised in our new approach, we recommend the paper [M] and the book [MM] for a detailed information on these important techniques.

2. The Hamiltonian and its Quadratic Approximation on a Torus

The configuration space of the model under consideration is $X = \mathbb{R}^{\mathbb{Z}^\nu}$. [In principle, one could consider a configuration space $X = S^{\mathbb{Z}^\nu}$, where $S \subset \mathbb{R}$ is a suitable subset of $\mathbb{R}$, reasonably 'dense' in those parts of $\mathbb{R}$ where values of U are close to a local minimum.] Similarly, for any finite torus Λ (a quotient of $\mathbb{Z}^\nu$ with respect to a subgroup of a finite index) we consider the configuration space $\mathbb{R}^\Lambda$.

We prefer to work with these 'periodical boundary conditions' in contrast to the existing tradition in the Pirogov–Sinai theory. In fact, the corresponding formulas are simpler for the periodic than for the case of a constant boundary conditions $x_i \equiv q$ on Λ^c. The problem of a transition from finite tori to infinite volume limits deserves a further study, in the general context of abstract Pirogov–Sinai models.

We assume that the Hamiltonian of the model is given, for $x_\Lambda = \{x_i, i \in \Lambda\}$ as

$$(1) \qquad H(x_\Lambda) = H^{\rm pair}(x_\Lambda) + H^{\rm single}(x_\Lambda) \quad \text{with} \quad H^{\rm single}(x_\Lambda) = \sum_{i\in\Lambda} U(x_i).$$

Agreement. Writing the Gibbs factor as $\exp(-H(x))$ instead of its usual form $\exp(-h/T\ H(x))$ we make an agreement that the Boltzmann factor h/T is included into the Hamiltonian. To avoid confusion we should stress yet now that the Peierls condition proven below in Theorem 2 will be derived from the condition (17B) (and (17C)) which will be normally fulfilled for **low enough** temperatures only!

Below we always work on some finite torus Λ.

The potential field U is assumed to have a standard form of a **multiple–well**

$$U(x_i) = \tilde U(x_i) \prod_{q\in Q} (x_i - q)^2 + \hat U(x_i)\ ,\ \ x_i \in \mathbb{R}.$$

Here $\tilde U(x_i) > 0$ varies smoothly (and rather **slowly**) between the points of $q \in Q$ (which are **local minima** of function $x_i \in \mathbb{R} \mapsto \tilde U(x_i) \prod_{q\in Q}(x_i - q)^2$), and $\hat U(x_i)$ is small in some δ_q–neighborhood of q, where δ_q is **not** too small. Pictorially, this equation can be written, in the δ_q neighborhood of $q \in Q \subset \mathbb{R}$, as

$$(2) \qquad U(x_i) = U^q(x_i) + \text{small remainder}\ ,\quad \text{where} \quad U^q(x_i) = d_q(y-q)^2 + e_q$$

with some constants $d_q > 0$ and e_q. (See more precisely (17B).)

An example of a multiple well potential which is especially well suited to a transformation (0) around the temperature $T = 1$ is (see [Ku])

$$U(x_i) = -\log\Big(\sum_{q\in Q} \exp(-d_q(x_i-q)^2 - e_q)\Big).$$

The **pair interaction** part of the Hamiltonian is considered to be

$$(3) \qquad H^{\rm pair}(x_\Lambda) = (Bx_\Lambda, x_\Lambda) = \sum_{(i,j)\in\Lambda\times\Lambda} B_{i,j} x_i x_j, \quad x_\Lambda = (x_i, i\in\Lambda) \in \mathbb{R}^\Lambda.$$

Here $(\cdot\,,\cdot)$ denotes the scalar product in $\ell_2(\Lambda)$, and (Bx_Λ, x_Λ) is a quadratic form in $\{x_i,\ i\in\Lambda\}$ given by a symmetric matrix B. This quadratic form is moreover assumed to be invariant with respect to the space shifts of x in Λ and also with respect to all shifts $x_\Lambda \to x_\Lambda + \text{const}$. In other words, we require that (3) equals to

$$(3') \qquad H^{\rm pair}(x_\Lambda) = \sum_{\{i,j\}\subset\Lambda} b_{i-j}(x_i - x_j)^2$$

for suitable $b_i = b_{-i}$. Then $B_{i,j} = b_{i-j} = b_{j-i} = B_{j,i}$ for $j \neq i$ and $B_{i,i} = \sum_j b_j$ for all i. A finite-range condition $b_k = 0$ for $|k| > R$ will be also assumed. In principle, we could also incorporate (at the expense of simplicity) a remainder $\hat H^{\rm pair}(x)$ that is small in a neighborhood of all constant configurations $\{x_i = q\}$, $q \in Q$.

We add the diagonal part $D_q(x) = \sum_{i\in\Lambda} d_q(x_i - q)^2$ (from (2)) to (Bx_Λ, x_Λ). In sections II and III, we moreover put $e_q \equiv 0$ for the simplicity of notations. Write

$$(4) \qquad H^q(x_\Lambda) = (Bx_\Lambda, x_\Lambda) + D_q(x_\Lambda) = \sum_{\{i,j\}\subset\Lambda} b_{i-j}(x_i - x_j)^2 + \sum_{i\in\Lambda} d_q(x_i - q)^2.$$

Equivalently, $H^q(x_\Lambda)$ can be also defined as $(A\tilde{x}_\Lambda, \tilde{x}_\Lambda) = \sum_{i,j\in\Lambda} A_{i,j}\tilde{x}_i\tilde{x}_j$ where $\tilde{x}_i = x_i - q$ and the symmetric matrix A has entries $A_{i,j} = \alpha_{i-j}$, with

$$\alpha_k = -b_k, \ k \neq 0, \ \ \alpha_0 = \alpha_0^q = d_q + \sum_{\tilde{k}:\ \tilde{k}\neq 0} b_{\tilde{k}}.$$

We can interpret the operator $\{\tilde{x}_\Lambda \mapsto A\tilde{x}_\Lambda\}$ as a **convolution** $\{x_\Lambda \mapsto \alpha * x_\Lambda\}$, where $(\alpha * x_\Lambda)_i = \sum_{j\in\Lambda} \alpha_j x_{i-j}$ has a symmetric convolution kernel

$$\alpha = \alpha^q = \{\alpha_k^q = -b_k, \ \alpha_0^q = d_q + \sum_{j\neq 0} b_j\}.$$

Notice that with $d_q > 0$ the form $(A\tilde{x}_\Lambda, \tilde{x}_\Lambda)$ is **positive definite** which means that the Fourier transform $\hat{\alpha}^q(\lambda) = \sum_j \exp(2\pi i(\lambda, j))\alpha_j^q$ is positive for each $\lambda \in [0,1]^\nu$. When necessary, write A^q instead of A thus stressing its dependence on α_0^q.

Assumption. In the following, we will assume that

$$(5) \qquad \sum_{j\neq 0} |\alpha_j^q| \leq \kappa^q \, \alpha_0^q \qquad \text{where} \ \ \kappa^q \ll 1.$$

Proposition. *Any positive definite, finite range convolution kernel α defined above satisfies the property (5) when expressed in suitable new coordinates ξ_Λ. The required change of coordinates is given by a convolution with a symmetric kernel*

$$(6) \qquad \xi_\Lambda = K * \tilde{x}_\Lambda \ \ ; \ \ K = \{K_i; \ |i| \leq R'\}.$$

Proof. It suffices to take K suitably close to the (infinite range) kernel $K^\infty = \sqrt{\alpha}$ diagonalizing the quadratic form $(A\tilde{x}_\Lambda, \tilde{x}_\Lambda) = (\alpha * \tilde{x}_\Lambda, \tilde{x}_\Lambda) = (K^\infty * K^\infty * \tilde{x}_\Lambda, \tilde{x}_\Lambda) = (K^\infty * \tilde{x}_\Lambda, K^\infty * \tilde{x}_\Lambda)$. The terms of the kernels K^∞ and $(K^\infty)^{-1}$ decay more quickly than the inverse of any polynomial of i. Really, the Fourier transform of K^∞ is an infinitely differentiable positive function. (It is the square root of the above trigonometric polynomial $\hat{\alpha}^q(\lambda)$.) We now cut suitably K^∞ by taking $K_i = K_i^\infty$ for $|i| \leq R'$ and $K_i = 0$ otherwise. We put $\hat{K} = K^\infty - K$. We have

$$(A\tilde{x}_\Lambda, \tilde{x}_\Lambda) = (\alpha * \tilde{x}_\Lambda, \tilde{x}_\Lambda) = (K^{-1} * \alpha * K^{-1}\xi_\Lambda, \xi_\Lambda) = (\xi_\Lambda, \xi_\Lambda) - (\Delta * \xi_\Lambda, \xi_\Lambda)$$

where $\Delta = K^{-1} * \hat{K} + \hat{K} * K^{-1} - K^{-1} * \hat{K} * \hat{K} * K^{-1}$. Then we can write K^{-1} as $K^{-1} = (K^\infty - \hat{K})^{-1} = (K^\infty)^{-1} * (J + \sum_{n=1}^{\infty} L^n)$ where $L = \hat{K} * (K^\infty)^{-1}$. Notice that $\hat{K}_i = 0$ for $|i| > R'$ and so L decays like $|L_i| \leq \varepsilon_{L,d} \, |i|^{-d}$ with a small constant $\varepsilon_{L,d}$ and $d > \nu$. It is then easy to show also that $|L_i^2| \leq \varepsilon_{L^2,d} \, |i|^{-d}$ with $\varepsilon_{L^2,d} \ll \varepsilon_{L,d}$

etc. and analogously that $|(\sum_n L^n)_i| \le \varepsilon\ |i|^{-d}$ with a small $\varepsilon = \varepsilon_{d,R'}$. Thus one establishes (5) for $J - \Delta$ (J identity operator), for enough large R'.

The fact that K in (6) has a **finite** range is important. Namely, the mapping $\tilde{x}_\Lambda \mapsto \xi_\Lambda = K * \tilde{x}_\Lambda$ increases the range of interactions of the model, from the value R to the value $R + R'$. In fact, the value $R + R'$ must then be used – instead of R – in the definition of precontours! For reasons of simplicity, we study below **only** the case when (5) is directly valid; like in our canonical **example**

$$H^q(x_\Lambda) = (A\tilde{x}_\Lambda, \tilde{x}_\Lambda) = \sum_{\{i,j\}\subset\Lambda : |i-j|\le 1} b_{j-i}(\tilde{x}_i - \tilde{x}_j)^2 + d_q \sum_{i\in\Lambda} (\tilde{x}_i)^2\ ,\ \tilde{x}_i = x_i - q$$

giving the value $\kappa^q = \sum_j |b_j| / (\sum_j b_j + d_q) \ll 1$ for a suitably big d_q.

3. Auxiliary results for Positive Definite Quadratic Hamiltonians

Because of the special form of our quadratic Hamiltonians (defined as Toeplitz quadratic forms (Ax, x) with a finite range convolution kernel; we put $q = 0$ $e_q = 0$ everywhere in this chapter and we do not indicate the dependence of A on q here)

$$H(x_\Lambda) = (Ax_\Lambda, x_\Lambda) \quad \text{where} \quad Ax_\Lambda = x_\Lambda * \alpha\ ;\quad \alpha = \{\alpha_t,\ t \in \Lambda\}\ ,\ \alpha_t = \alpha_{-t}$$

the results stated below will be simpler than that of [DZ]. The Hamiltonian $H(x_\Lambda)$ will denote a purely quadratic function everywhere in this chapter.

Expression of the partition function on torus.

Denote by μ_Λ the Gaussian measure on R^Λ (Λ is a torus) having the density

$$\frac{d\mu_\Lambda(x_\Lambda)}{dx_\Lambda} = \frac{\exp(-H(x_\Lambda))}{Z(\Lambda)}\quad ;\quad Z(\Lambda) = \int_{\mathbb{R}^\Lambda} \exp(-H(x_\Lambda))\ dx_\Lambda \tag{7}$$

where the Gaussian partition function is expressed as $Z(\Lambda) = \pi^{\frac{|\Lambda|}{2}} (\det A)^{-1/2}$ i.e.

$$2 \log Z(\Lambda) - \log\pi\ |\Lambda| = -\log\det A = -\ \mathrm{Tr}(\log A) = -\ \mathrm{Tr}\log(I - W) - \log\alpha_0 |\Lambda| \tag{8}$$

where I is the identity matrix and $W = I - A/\alpha_0$ has zero diagonal. We are using here the formula $\det\exp\tilde{A} = \exp\mathrm{Tr}\,\tilde{A}$, for $A = \exp\tilde{A}$. This can be written, with the help of the transition weights $w_{ij} = w_{i-j}$ given, for any $i \in \mathbb{Z}^\nu, i \ne 0$, as

$$w_i = \alpha_i/\alpha_0 \qquad (\text{where} \quad \sum_{i\ne 0} |w_i| < \kappa < 1\ )$$

in the following way, just by taking the Taylor expansion of the logarithm of $I - W$

$$-\ \mathrm{Tr}\ \log(I - W) = \sum_{n\ge 2} 1/n\ \mathrm{Tr}\, W^n = \sum_P w_P \quad \text{where} \quad w_P = \prod_{\{i,j\}\in P} w_{j-i} \tag{9}$$

and where the last sum is over all **closed** oriented paths[1] in Λ with 'steps' (i,j) such that $i \ne j$. Define then the **free energy** of this Gaussian model (the sum being taken over all closed paths P containing a selected point, say 0)

$$-h = 1/2\ \big(\log\pi - \log\alpha_0 + \sum_{P\ni 0} \frac{w_P}{|P|}\ \big). \tag{10}$$

[1] Notice that the term $1/n$ disappeared in $\sum_P w_P$ (because each P is taken $|P|$ times there).

Computation of the conditioned minima for quadratic Hamiltonians.

In this section, we develop some further preparatory material, to be summarised later in Theorem 0. Let $M \subset \Lambda$. The forthcoming formulas are valid even for an infinite Λ, however $M^c \equiv \Lambda \setminus M$ has to be finite. Denote by $\bar{x}_M$ the configuration **minimising** $H(x_\Lambda)$ under a fixed $x_{M^c} = \bar{x}_{M^c}$. An elementary computation (consider the derivative of the expression $f(t) = H(t\bar{x}_i \cup \bar{x}_{M\setminus i} \cup \bar{x}_{M^c})$ and put $f'(1) = 0$!) shows that the relation $\bar{x}_i = (W\bar{x})_i$ holds for each $i \in M$. Then, by **iterating** this expression (until reaching values $\bar{x}_i$ on the set M^c), we can express the values $\bar{x}_i, i \in M$ and also the value of the quadratic form $(A\bar{x}_\Lambda, \bar{x}_\Lambda) = \alpha_0((I-W)\bar{x}_\Lambda, \bar{x}_\Lambda)$ by the following formulas (the weights w_P are given by (9))

$$(11) \qquad \bar{x}_i = \sum_P w_P\ \bar{x}_{e(P)} \ \text{ and } \ (A\bar{x}_\Lambda, \bar{x}_\Lambda) = \alpha_0\ \big(\sum_{i\in M^c} x_i^2 - \sum_P w_P\ \bar{x}_{s(P)}\bar{x}_{e(P)}\big).$$

Here, the first sum is over all (nonclosed) oriented paths P starting in $i \in M$ and reaching the set M^c only at its **end point** $e(P)$. The second sum in the expression of $(A\bar{x}, \bar{x})$ is over all oriented paths P intersecting M^c **exactly twice**: at the starting point $s(P)$ and at the end point $e(P)$. The rest of the path P (possibly empty) is spent in M. To derive the **second formula** in (11) from the first one, substitute the above expression (11) of $\bar{x}_i$ into $(A\bar{x}_\Lambda, \bar{x}_\Lambda) = \alpha_0((I-W)\bar{x}, \bar{x})$:

1) Consider first the $-(W\bar{x}, \bar{x})$ term i.e. all the possible decompositions of the non oriented path P (from M^c back to M^c) into a triple $P = P_1 \cup b \cup P_2$, $b = \{s(P_1), s(P_2)\}$ with oriented paths P_1, P_2 going from $s(P_i) \in M$ to M^c or being empty. Each such triple contributes twice the minus sign.

2) One then has to add, to $-(W\bar{x}, \bar{x})$, also the $(\bar{x}, \bar{x})$ term which can be expressed as the contribution, with the coefficient 2 (1 in the case when $P_1 = P_2$), of all possible pairs $P = P_1 \cup P_2$ of oriented paths P_1, P_2 going from a point $s(P_1) = s(P_2) \in M$ to M^c. Taking in account the signs of all these terms, the net contribution of a path $P, |P| = n$, is equal to $(-2n + 2(n-1))w_P = -2w_P$ (and similarly for the special case when $P_1 = P_2$ for some $P_1 = P_2$). So it is equal to $-w_P$ if we consider the **two** possible orientations of such a path P.

Conditioned Gaussian fields, the correlation decay. Summary.

The forthcoming Theorem 0 summaries everything what we will need to know on the partition functions of Gaussian Gibbsian fields defined by above quadratic forms (Ax_Λ, x_Λ) where $A = \alpha_0(I-W))$, and on the correlation decay of the corresponding conditioned Gaussian fields. We remark in advance that the formulas (13), (13') and (14) below are valid universally (with weights w_P given by (9)) but they will be really useful later only under the condition (5).

Given a subset M of a torus Λ consider the 'conditioned' Hamiltonian

$$(12) \qquad H(x_M|x_{M^c}) = H(x_\Lambda) - H(x_{M^c} \cup 0_M) = \sum_{i,j\in\Lambda:\{i,j\}\cap M\neq\emptyset} A_{i,j}\, x_i x_j$$

defined for any $x_\Lambda = x_M \cup x_{M^c} \in \mathbb{R}^\Lambda$. For $q \neq 0$, one has to write q_M instead of 0_M and $x_i - q$ instead of x_i in the definition of $H^q(x_M|x_{M^c})$. Introduce also the

conditional probabilities $\mu_{\bar{x}_{M^c}}(x_M)$, by (7). We note that an alternative to (12)

$$\tilde{H}^q(x_M|x_{M^c}) = \tilde{H}^q(x_\Lambda) - \tilde{H}^q(x_{M^c}) = \sum_{\{i,j\}\not\subseteq M^c} b_{i-j}(\tilde{x}_i - \tilde{x}_j)^2 + \sum_{i\in M} d_q(\tilde{x}_i)^2$$

where $\tilde{x}_i = x_i - q$ and $\tilde{H}^q(x_M) = \sum_{\{i,j\}\subset M} b_{i-j}(\tilde{x}_i - \tilde{x}_j)^2 + \sum_{i\in M} d_q(\tilde{x}_i)^2$ will be more natural in section 8 but not here where we are working with the quadratic form (Ax_Λ, x_Λ).[2] However, we have $\tilde{H}^q(x_\Lambda) \equiv H^q(x_\Lambda)$. We have also the relation

$$H^q(x_M \cup q_{M^c}) = H^q(x_M|\, q_{M^c}) = \tilde{H}^q(x_M) + \tilde{H}^q(q_{M^c}|\, x_M) = (A\tilde{x}_M, \tilde{x}_M).$$

Theorem 0. *Assume (5). Write again $H^q \equiv H$ and put $q = 0, e_q = 0$. Denote by $\bar{x}_M$ the configuration minimising $H(x_\Lambda)$ under $x_{M^c} = \bar{x}_{M^c}$. Then $H(x_M \cup \bar{x}_{M^c}) - H(\bar{x}_\Lambda) = H(x_M|\bar{x}_{M^c}) - H(\bar{x}_M|\bar{x}_{M^c}) = H((x-\bar{x})_M \cup 0_{M^c})$ and*

$$(13)\quad Z(M|\bar{x}_{M^c}) = \int_{\mathbb{R}^M} \exp(-H(x_M|\bar{x}_{M^c}))\, dx_M = \exp(-H(\bar{x}_M|\bar{x}_{M^c}))Z(M|0_{M^c}).$$

Moreover we have the following expression of the right hand side of (13) in terms of random walks, with the same notations as in (9) and (12), see also (11)

$$(13')\qquad H(\bar{x}_M|\bar{x}_{M^c}) = -\alpha_0\ \big(\sum_{P:P\cap M\neq\emptyset} w_P\ \bar{x}_{s(P)}\bar{x}_{e(P)}\big)$$

where the sum is over all oriented paths P from M^c to M^c crossing (at least in one point) the set M. The partition function $Z(M|\,0_{M^c})$ is expressed as the sum over all oriented paths P **not** *intersecting M^c, resp. intersecting* **both** *M and M^c*

$$(14)\qquad \log Z(M|\,0_{M^c}\,) = 1/2\ \big((\log\pi - \alpha_0)\ |M| + \sum_P w_P\big) = -h|M| - \sum_{\tilde{P}} \tilde{w}_{\tilde{P}}$$

where the free energy h is from (10) and $\tilde{w}_P$ are related to w_P (see (9)) as

$$\tilde{w}_P = \frac{|P\cap M|}{2|P|} w_P\ ;$$

this notation will be used also below. The symbol $|P\cap M|$ denotes the number of visits of P in M. The first sum in (14) is over all closed oriented paths P, below called **loops** *(in comparison to the* **semiloops** *P of (13')) in M. The second sum is over loops $\tilde{P}$ intersecting both M and M^c. The values $\bar{x}_i$, $i\in M$ are expressed by (11) and estimated, for the case of interaction range*[3] *R*

$$(15)\qquad |\bar{x}_i| \le (1-\kappa)^{-1}\kappa^{\frac{\mathrm{dist}(i,M^c)}{R}}\ \|\,\tilde{x}_{M^c}\,\|_\infty\ .$$

[2]Notice that $H^q(x_M|x_{M^c})$ is nonzero, in contrary to $\tilde{H}^q(x_M|x_{M^c})$, for a constant $x_\Lambda \neq q_\Lambda$. A confusion could arise here as the concepts of 'pair' and 'single spin' interactions change its meaning when expanding $\sum b_{i-j}(x_i - x_j)^2 = (\sum_j b_j)\sum(\tilde{x}_i)^2 - 2\sum b_{i-j}\tilde{x}_i\tilde{x}_j$.

[3]We will not optimise this bound. After all, (11) is more detailed.

Proof. This is just a recapitulation of the arguments used when deriving (8)–(11). The relation $H(x_M \cup \bar{x}_{M^c}) = H(\bar{x}_M \cup \bar{x}_{M^c}) + H(x_M - \bar{x}_M \,|\, 0_{M^c})$ follows easily if we notice that $H(t(x_M - \bar{x}_M) + \bar{x}_\Lambda)$ attains its minimum (as a function of $t \in \mathbb{R}$) at $t = 0$. The sum over paths in the formula (14) emerges like in (9):

$$2 \log Z(M|\ 0\ ) - (\log \pi - \log \alpha_0)|\Lambda| = - \log \det(I - W_M) = - \operatorname{Tr} \log(I - W_M),$$

where $W_M = I - A_M/\alpha_0$. This is a notation analogous to that of (9) and the symbol A_M denotes the quadratic form (Ax, x) restricted to the configuration space R^M. Finally we have the expression, with the same weights w_P as in (9)

$$- \operatorname{Tr} \log(I - W_M) = \sum_n 1/n \ \operatorname{Tr} W_M^n = \sum_P w_P \tag{16}$$

where the sum is over all closed oriented paths **in the volume** M. The expression of $H(\bar{x}_M | \bar{x}_{M^c})$ used in (13') then follows from (11) and (12).

Namely, by (12) we have the relation $H(\bar{x}_M|\bar{x}_{M^c}) = (A\bar{x}_\Lambda, \bar{x}_\Lambda) - (A\bar{x}_{M^c}, \bar{x}_{M^c})$. Then we can express $(A\bar{x}_\Lambda, \bar{x}_\Lambda)$ by (11) and subtract the term $(A\bar{x}_{M^c}, \bar{x}_{M^c})$ from it. Only contributions of nontrivial paths 'crossing' M will remain. Finally, to get (15) one notices that the summation over all paths P from i to M^c gives

$$\sum_P |w_P| \ \leq \sum_{k:\ Rk > \operatorname{dist}(i, M^c)} (\sum_{j: |j| \leq R} |w_j|)^k < \ (1 - \kappa)^{-1} \kappa^{\frac{\operatorname{dist}(i, M^c)}{R}} .$$

4. Conditions on the Potential U.

Now we are again studying a general nonquadratic Hamiltonian (1). We will discuss in more detail the decomposition (0). Recall that we assume (for simplicity) that the **pair** interactions in (1) are purely **quadratic** ones.[4] Recall that we take the decomposition (0) with suitable $U^q(x_i) = e_q + d_q(x_i - q)^2$ (see (2)). The approximations U^q are now with $e_q \neq 0$ in general. In contrast to the 'temperature independent' considerations of the preceding sections the fact that we have a **sufficiently low** temperature will be crucial now, and this will be expressed in terms of the 'potential' $U^*(x_i) = - \log |w(x_i)|$ assumed to be enough large:

Assumptions on $U^*(x)$ and the approximating Hamiltonians H^q.

Recall that around each $q \in Q$ we have the approximate relation $U(x_i) \doteq U^q(x_i)$. More precisely see (17B) below (we will in fact formulate such a requirement in conjunction with the condition on the 'sufficient height of the hills' of U, between all the pairs of nearest neighbors $q < q'$ in $Q \cup \{\infty\} \cup \{-\infty\}$). Our formulation will be given directly in terms of U^*. (After all, if we are outside of the 'wells' of U then either both U and U^* are 'safely big' or we have $U^* \doteq U$. Then we may write U in (17B).) Denote by $\delta_q \equiv 1/2 \operatorname{dist}(q, Q \setminus \{q\})$ and assume that the quantity

$$\delta = \min_{q \in Q} \delta_q = \frac{1}{2} \min_{q, q' \in Q:\ q \neq q'} |q' - q| \tag{17A}$$

[4] For nonquadratic pair interactions, a similar expansion to (0) can be applied in principle. Usually, one would have only one quadratic term there but the method applies also to models with more complicated interactions $\Phi_{\{i,j\}}(x_i, x_j) = \tilde{U}(x_i - x_j)$ where $\tilde{U}$ has **several** local minima.

is not too small (see (17C) below). We need a precise condition on 'sufficient energy barriers': Introduce an auxiliary function $\rho(x_i) = \min\{\rho_q(x_i), \rho_{q'}(x_i)\}$ in any interval (q, q') where $q < q'$ are nearest neighbors from $Q \cup \{\infty\} \cup \{-\infty\}$ and

$$\rho_q(x_i) = (\log_2(1 + |x_i - q|/\delta_q) + 1)^\nu \qquad (\rho_{\pm\infty} \equiv \infty)$$

Assume that for each $q < r < r' < q'$ such that $\rho(r) = \rho(r') \geq d$ (we take $r', r = \pm\infty$ if $q', q = \pm\infty$) we have a bound, with a **small** ε

$$\text{(17B)} \qquad \int_r^{r'} \exp(-U^*(x_i))\, dx_i \;<\; \varepsilon^d \,.$$

The conditions (17A) and (17B) summarise our requirements on U.

Note. The condition (17B) gives also a precise meaning to the statement $U(x_i) \doteq U^q(x_i)$ used in the vicinity of q. All the 'important' (taking in account also the widths of the corresponding 'wells') local minima of U should be represented by some $q \in Q$. Usual examples of U with 'well separated' local minima $q \in Q$ (even e.g. $U(x_i) = \sin \pi x_i$, $Q = \mathbb{Z}$) satisfy (17B) for enough low temperatures, and it suffices to check (17B) for $r = q$ and $r' = q'$. In fact, (17B) is an energy/entropy contest condition and to have an idea how it could be violated imagine that the 'hill' of $U^*(\doteq U)$ between q and q' looks like a 'low and extremely long plateau'.

Conditions (17 A,B) will enable to prove the validity of the **Peierls condition**, if complemented by another assumption of positive definiteness, ' on $(1, 1, \ldots, 1)^\perp$', of the pair Hamiltonian $H^{\text{pair}}(x_\Lambda) \equiv \sum_{\{i,j\}\in\Lambda} b_{i-j}(x_i - x_j)^2$: Assume that

$$\text{(17C)} \qquad H^{\text{pair}}(x_\Lambda) > \alpha \sum_{i\in\Lambda} \sum_{j:|j-i|=1} (x_i - x_j)^2$$

for some α such that $\alpha\delta^2$ (see (17A)) is enough **large**. (This is an assumption on low temperature, too. Namely α is proportional to the inverse temperature β.)

Let us go back to the Hamiltonian H^q. From (5) we get the following bound for $H^q(x_\Lambda)$. Below we assume that also all α_0^q are enough large:[5]

$$\text{(17D)} \quad \alpha_0^q(1 + \kappa^q) \sum_{i\in\Lambda} |x_i - q|^2 \;\geq\; H^q(x_\Lambda) - e_q|\Lambda| \;\geq\; \alpha_0^q(1 - \kappa^q) \sum_{i\in\Lambda} |x_i - q|^2.$$

For any $\Lambda \supset \tilde{M} \supset M$ and any configuration x_M consider the quantity

$$H_e^q(\tilde{M}|x_M) = \min_{x_\Lambda}(H^q(\tilde{x}_\Lambda) - e_q|\Lambda|)$$

where the minimum is over all extensions x_Λ of x_M such that $x_i = q$ for each $i \in (\tilde{M})^c$. Then we have a bound[6], more useful below than (17D)

$$\text{(17E)} \qquad C^q\alpha_0^q \sum_{i\in M} |x_i - q|^2 \geq\; H_e^q(\tilde{M}|x_M) \;\geq \tilde{C}^q\alpha_0^q \sum_{i\in M} |x_i - q|^2$$

with suitable constants $C^q = C^q(\kappa^q) \doteq 1$ and $\tilde{C}^q = \tilde{C}^q(\kappa^q) \doteq 1$.

[5]Notice that $\alpha_0^q - d_q \asymp \alpha$ with $d_q > 0$ (see (2)); δ is temperature independent.

[6]See (13') for more detailed information on $H_e^q(\tilde{M}|x_M)$. Hence estimates of C^q and $\tilde{C}^q$.

5. Expression of $Z(\Lambda)$ using (thick) precontours

Write the partition function of the model with Hamiltonian (1)

$$Z(\Lambda) = \int_{\mathbb{R}^\Lambda} \exp(-H(x_\Lambda))\, dx_\Lambda \tag{18}$$

by inserting the expression (0) into each Gibbs factor

$$\exp(-H(x_\Lambda)) = \prod_{\{i,j\}} \exp(-b_{i-j}(x_i - x_j)^2) \prod_{i\in\Lambda} \exp(-U(x_i)).$$

Denote $Q^* = Q \cup \{*\}$, i.e. enrich the reference set Q by another value $*$ in the expression (0). In other words, write $\exp(-U(x)) = \sum_{q\in Q^*} \exp(-U^q(x))$. Then expand all the parentheses. We get the following formula.

Proposition 1. *The partition function $Z(\Lambda)$ can be expressed as*

$$Z(\Lambda) = \sum_{y=y_\Lambda \in (Q^*)^\Lambda} \int_{\mathbb{R}^\Lambda} \exp(-H_y(x_\Lambda)) \prod_{i:y_i=*} w(x_i)\; dx_\Lambda, \tag{19}$$

where the (translation noninvariant!) Hamiltonian $H_y(x_\Lambda)$ is given by the formula

$$H_y(x_\Lambda) = \sum_{\{i,j\}\subset\Lambda} b_{i-j}(x_i - x_j)^2 + \sum_{i\in\Lambda: y_i \neq *} U^{y_i}(x_i)$$

and where the summation and integration is over all pairs (x_Λ, y_Λ) composed of a **configuration** $x_\Lambda \in \mathbb{R}^\Lambda$ *and a* **'coloring'** *(Q^*–valued configuration) $y_\Lambda \in (Q^*)^\Lambda$.*

Proof. This is just the result of expanding the parentheses that appear when (0) is inserted into (18). Of course, the leading idea here is that the most important contribution to $Z(\Lambda)$ comes only from those pairs (x_Λ, y_Λ) where the coloring y_Λ 'mimics, as best as possible' the configuration x_Λ – but we emphasise that the integral and the sum here is really over **all** the pairs $(x_\Lambda, y_\Lambda) \in \mathbb{R}^\Lambda \times (Q^*)^\Lambda$.

In the following, we impose another simplifying **assumption** namely that $R = 1$. It is straightforward to generalise the considerations below to the general case of interactions having a range R (resp. $R + R'$ if a transformation (6) is used).[7]

Precontours.

Say that $i \in \Lambda$ is a q–correct point of y_Λ, $q \in Q$, if $y_j = q$ holds for all j such that $|j - i| \leq 1$. Say that $i \in \Lambda$ is **incorrect** if it is q–correct for no $q \in Q$. Thus, incorrectness of i means **either** that y_i has the value $*$ **or** that the value $x_i = q$ is not shared by all nearest neighbours of i. It is a property of the 'coloring' y_Λ.

[7] One has to start with writing $|j - i| \leq R$ in the definition below. However, the requirements on temperature are much stronger for large R. The case of fast decaying infinite range interactions could be also dealt with, like in [YP]. The proper choice of a value R (interactions of a range bigger than R are then treated as small perturbations) deserves a detailed discussion.

A restriction of x_Λ to a connected component G of the collection of all incorrect points of $y_\Lambda \in (Q^*)^\Lambda$ will be called a **precontour** of the pair (x_Λ, y_Λ), denoted by a symbol γ. The set G will be called the **support** of γ, denoted by $G = \operatorname{supp}\gamma$. The configuration y_G, denoted more precisely also as y_G^γ is the **colour** of γ (it can be extended uniquely to Λ such that all points of G^c are correct) while $x_i, i \in G$ are the **values** of γ. Define the **Gibbs weight** of a precontour $\gamma = (x_G, y_G)$

$$(20)\quad w_\gamma = \prod_{i\in \operatorname{supp}\gamma : y_i \neq *} \exp(-U^{y_i}(x_i)) \prod_{i \in \operatorname{supp}\gamma : y_i = *} w(x_i) \quad \exp(-H^{\text{pair}}(x_G \cup y_{G^c}))$$

where $H^{\text{pair}}(x_M \cup x_{M^c}) = \sum_{\{i,j\}\subset\Lambda} b_{i-j}(x_i - x_j)^2$ is from (3').

Thus, w_γ depends both on x_G and $y_{G\cup\partial G^c}$. (While the support of a precontour is wholly determined by the colour y, the value w_γ, possibly negative, depends also on the values $x_{\operatorname{supp}\gamma} \in \mathbb{R}^{\operatorname{supp}\gamma}$ of γ.) Notice that the colour of $\partial \operatorname{supp}\gamma = \{i \in \operatorname{supp}\gamma : \operatorname{dist}(i, \operatorname{supp}\gamma^c) = 1\}$ is locally constant, taking some value $q \in Q$.

Say that $\{\gamma_n\}$ is an **allowed collection** of precontours if it exhausts all precontours of some pair (x_Λ, y_Λ). Clearly, the property of being allowed is determined only by the properties of the coloring y_Λ on the boundary of the union $\cup_n \operatorname{supp}\gamma_n$. Outside that set, the corresponding 'correct' values of $q \in Q$ are determined uniquely. For any $q \in Q$ we denote by $\Lambda^q = \Lambda^q(\{\gamma_n\}) = \{i \in (\cup_n \operatorname{supp}\gamma_n)^c : y_i = q\}$ the collection of **all** q**–correct points** of the allowed collection $\{\gamma_n\}$ ($\equiv$ the collection of all q–correct points of the colour $y_\Lambda^{\{\gamma_n\}}$ induced by $\{\gamma_n\}$ on Λ).

Denote the values of γ_n in $i \in \partial(\Lambda^q)^c$ below as x_i. We then have the following

Proposition 2. *The partition function Z_Λ is expressed, using (20), as*

$$(21)\qquad Z(\Lambda) = \sum_{G, y_\Lambda} \int_{\mathbb{R}^G} \prod_n w(\gamma_n) \prod_{q\in Q} Z^q(\Lambda^q | x_{\partial(\Lambda^q)^c})\ dx_G \qquad \textit{where}$$

$$Z^q(M|x_{\partial(M^c)}) = \int_{\mathbb{R}^M} \exp(-H^q(x_M|x_{\partial M^c})\,dx_M. \ \ (\textit{We have } \gamma_n = (x_{\operatorname{supp}\gamma_n}, y_{\operatorname{supp}\gamma_n}.)$$

The sum is over all possible $G = \cup_n \operatorname{supp}\gamma_n$ and $y_\Lambda = y_\Lambda^{\{\gamma_n\}}$. The partition function $Z^q(\Lambda^q|x_{\partial(\Lambda^q)^c})$ is a Gaussian one. The integration dx_G is over all $x_i \in \mathbb{R},\ i \in G$.

Proof. See the relation (19) and the notation (20). We will denote the relative Hamiltonian $H^q(x_{\Lambda^q}|x_{(\Lambda^q)^c})$ below also as $H^q(x_{\Lambda^q}|x_{\partial(\Lambda^q)^c})$. Notice that we have a disjoint decomposition $\Lambda = G \cup \cup_q \Lambda^q$. Using the correctness of y outside of G we get the following relation[8] which obviously proves (21) from (19)

$$\tilde{H}_y^Q(x_\Lambda) = H^{\text{pair}}(x_G \cup y_{G^c}) + \sum_{i\in G: y_i \neq *} U^{y_i}(x_i) + \sum_q H^q(x_{\Lambda^q}|x_G).$$

Here, $H^q(x_{\Lambda^q}|x_G) \equiv H^q(x_{\Lambda^q}|x_{\partial(\Lambda^q)^c})$ and $H_y^Q(x_\Lambda)$ denotes an 'inhomogeneous' Hamiltonian (1) with U replaced by U^q or 0 according to whether $y_i = q$ or $y_i = *$.

[8]While using (12), do not attemt to introduce any $H^{\text{pair}}(x_M|x_{M^c})$. This would cause confusion.

Now, we will **insert the expressions** (13), (14) for Gaussian partition functions $Z^q(\Lambda^q|x_{(\Lambda^q)^c})$ into[9] the formula (21). Say that a path P crosses the set Λ^q if it intersects both Λ^q as well as its complement. Denote such a path by a symbol P^q, and associate to such a path the value w_{P^q} given by the formula (9), for the quadratic Hamiltonian H^q.[10] The forthcoming formulas will be used for $M \subset \Lambda^q$.

Proposition 3. *Denote by* $H^q_e(\bar{x}_M|\bar{x}_{M^c}) = H^q(\bar{x}_M|\bar{x}_{M^c}) - e_q|M|$. *Then*

$$(22)\qquad Z^q(M|\bar{x}_{M^c}) = \exp\big(-h^q|M| - H^q_e(\bar{x}_M|\bar{x}_{M^c}) + \sum_{P^q : P^q \cup M^c \neq \emptyset} \tilde{w}^q_{P^q}\big)$$

where $h_q = e_q + \log\pi - \log\alpha_q - \sum_{P^q \ni 0} \frac{w^q_P}{|P^q|}$ *is the free energy (10) and the correction terms* $\tilde{w}_{\tilde{P}} = -\frac{|P^q \cap M|}{2|P^q|}\ w^q_{P^q}$ *are taken from (14), for the Hamiltonian* H^q.[11]

Proof. It follows immediately from (13) and (14).

Notation. Below, it will be convenient to write x_{M^c} instead of $\bar{x}_{M^c}$. The quantities $H^q(\bar{x}_M|x_{M^c})$ and $H^q_e(\bar{x}_M|x_{M^c})$ will then have the same meaning as with $\bar{x}_{M^c}$.

We are now quite close to our goal – to realise that the partition functions (21) are (after the substitution of the expressions (22)) just partition functions of a suitable 'abstract Pirogov–Sinai model'. However, the quantities $H^q_e(\bar{x}_{\Lambda^q})|\bar{x}_{(\Lambda^q)^c})$ and the perturbing terms $\tilde{w}_P$ obscure the picture. They have to be added to the 'precontour energies' $E(\gamma) = \log|w_\gamma|$ given by (20) but we have to control the fact that $H^q_e(\bar{x}_{\Lambda^q}|x_{(\Lambda^q)^c})$ is **not** additive as the function $\{\gamma_n\}$.

Small corrections to the additivity could be handled in the same way as the terms $\tilde{w}_P$ but we may have to deal also with a possible 'strong nonadditiveness' emerging if the values of spins x_i on the boundary $\partial(\Lambda^q)^c$ differ very much from their 'bottom' value q. The remedy is to glue together the precontours which are 'too close' to each other, the meaning of 'closeness' depending also on how much the spins x_i on the boundary $\{i \in \partial \cup_i \operatorname{supp}\gamma_n, \operatorname{dist}(i, \Lambda^q) = 1\}$ differ from q:

Protection zones and thick precontours.

We take the correlation length of our quadratic Hamiltonian H^q equal to 1 (because of the assumed smallness of κ^q). Define the **protection zone** $\tilde{\Lambda}^q$ of the boundary $\partial(\Lambda^q)^c$ as the union of balls, with centers in $\partial(\Lambda^q)^c \subset \cup_n \operatorname{supp}\gamma_n$[12]

$$(23)\qquad \tilde{\Lambda}^q = \cup_{i\in(\partial\Lambda^q)^c}\ B^q_i \ \text{ where }\ B^q_i = \{j \in \Lambda^q : |j - i| < 5\ (\log(|x_j - q|/\delta_q) + 1)\}.$$

Denote by $\Lambda^q_{\text{inn}} = \Lambda^q \setminus \tilde{\Lambda}^q$ the complement of the protection zone $\tilde{\Lambda}^q$ in Λ^q. Let us call Λ^q_{inn} as the set of all **inner** correct (or simply inner) points of Λ^q.

Define now thick precontours of x as follows. **Supports** of these thick precontours are defined as connected components of the set $\Lambda \setminus \cup_q \Lambda^q_{\text{inn}}$. Denote them by

[9] We do not employ here the expression (13') of $H(\bar{x}_M|x_{M^c})$ (for the minimising configurations $\bar{x}$ in Λ^q, under a given $\{\gamma_n\}$). This will be exploited later (27) in a context of thick precontours.

[10] Two identical paths P^q, $P^{q'}$ will be treated as different ones if $q \neq q'$ in formulas (22).

[11] $H^q(q_M|q_{M^c}) = e_q\ |M|$ is the minimum, over $x_M \in \mathbb{R}^M$, of $H^q(x_M|q_{M^c})$.

[12] The constant 5 is enough 'generous'. This will help us later, when discussing the Peierls condition. The exponential decay (15) of $\bar{x}$ will guarantee ε in (23') of the order $\approx e^{-5}$.

symbols G. For any such G denote by G^* its **core** which is defined as the union $\cup_n \operatorname{supp} \gamma_n$ of supports of all precontours γ_n of x contained in the 'thick' set G.

Thick precontour Γ is now defined as a pair (y_G, x_{G^*}) where $x_{G^*} = \cup_n \gamma_n$ denotes the restriction of x to G^*. Denote them by symbols[13] Γ. We will also use the notations $G = \operatorname{supp} \Gamma$ and $G^* = \operatorname{supp}^* \Gamma$ (the 'core' of $\operatorname{supp} \Gamma$).

Later, we will rewrite (21) in the following, formally similar to (21) but more convenient form (25). Notice first that for any thick precontour Γ we have a bound $|\bar{x}_i - q| \le \tilde{\varepsilon}$ for any $i \in \partial(\Lambda^q)^c$, if we take the configuration $\bar{x}_{\Lambda^q}$ minimising the quadratic Hamiltonian $H^q(x_{\Lambda^q}|x_{G^*})$ under a fixed x_{G^*}. In other words,

$$(23') \qquad |\bar{x}_i - y_i| \le \varepsilon \quad \text{for any} \quad i \in \partial \operatorname{supp} \Gamma$$

where $\tilde{\varepsilon}$ $(< \delta)$ depends on the choice of a constant 5 in the definition of Γ.

Having any thick precontour $\Gamma = (G, x_{G^*})$, where $G^* = \cup_n \operatorname{supp} \gamma_n$ (the union is over all precontours γ_n forming Γ) and $G = G^* \cup \cup_q \tilde{\Lambda}^q$ (the last union is over all protection zones $\tilde{\Lambda}^q$) we define, using (20), the **Gibbs factor** of Γ

$$(24) \qquad w_\Gamma = \prod_n w_{\gamma_n} \prod_q \exp(-H^q(\bar{x}_{\tilde{\Lambda}^q} | \cup_n \gamma_n \cup q_{\Lambda^q_{\text{inn}}})).$$

where $\tilde{\Lambda}^q$ is the q–th protection zone and $\bar{x}_{\Lambda^q}$ is the minimising configuration.[14]

Here and below we use notations like $H^q(x_{\Lambda^q}|\gamma)$ instead of $H^q(x_{\Lambda^q}|(x_{\operatorname{supp}\gamma}))$.

Notation: Restricted partition functions $Z_{\mathcal{M}}(\Lambda)$. In the following, we will introduce a general symbol $Z_{\mathcal{M}}(\Lambda)$ for partition functions taken over all configurations satisfying some limitation $\mathcal{M}$. Such a limitation will be always expressed in terms of values of x_i in the incorrect points of a pair (x_Λ, y_Λ). It can be therefore equivalently formulated in the language of (thick) precontours of a pair (x_Λ, y_Λ). A basic example of such a limitation $\mathcal{M}$ will be a condition on the maximal possible **diameter** of the thick precontours of (x_Λ, y_Λ).[15] Often we will have $\mathcal{M} = \emptyset$.

Proposition 4. *Then we have, by (21),(13) and (24), the expression*

$$(25) \qquad Z_{\mathcal{M}}(\Lambda) = \int \prod_n w_{\Gamma_n} \prod_q Z^q(\Lambda^q | q_{(\Lambda^q)^c}) \prod_q \exp(-H^q_e(\bar{x}_{\Lambda^q_{\text{inn}}} | \bar{x}_{\tilde{\Lambda}^q})) \prod_n \, d\Gamma^*_n$$

where $G = \cup_n \operatorname{supp} \Gamma^*_n$. See Proposition 3 for the definition of $H^q_e(\bar{x}_{\Lambda^q_{\text{inn}}}|\bar{x}_{\tilde{\Lambda}^q})$; these terms are small and so we have an approximate relation

$$(25') \qquad Z_{\mathcal{M}}(\Lambda) \doteq \int_{\mathbb{R}^G} \prod_n w_{\Gamma_n} \prod_q Z^q(\Lambda^q | q_{(\Lambda^q)^c}) \prod_n \, d\Gamma^*_n.$$

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[13]**Contours** will be later constructed from thick precontours and denoted by symbols $\mathbf{\Gamma}$.

[14]Using (23'), we will see that the value $H^q(\bar{x}_{\tilde{\Lambda}^q}|x_{G^*} \cup q_{\Lambda^q_{\text{inn}}}) + e_q\,|\Lambda^q_{\text{inn}}|$ in (24), with the boundary condition q on $\partial\Lambda^q_{\text{inn}} = (\partial\tilde{\Lambda}^q)^c \setminus G^*$, is a good approximation of of $H^q(\bar{x}_{\Lambda^q}|x_{G^*})$.

[15]Such a limitation will depend also on the 'external colour' of Γ and on $\{x_i, i \in \partial G^*\}$.

The symbol $\prod_n d\Gamma_n^*$ means summation and integration over all possible $y_\Lambda = y_\Lambda^{\{\Gamma_n\}}$ and all possible values $x_i \in \mathbb{R}, i \in \operatorname{supp}^* \Gamma_n$ where $\{\Gamma_n\}$ is an allowed collection of thick precontours (each Γ_n being written as $(y_{\operatorname{supp}\Gamma_n}, x_{\Gamma_n^*})$) satisfying $\mathcal{M}$.

Proof. The first relation is just a rearranged expression (21), taking in account the notation (24) and the relation, for any decomposition $M \cup M' \cup M'' = \Lambda$, $H^q(x_{M''}|x_M \cup x_{M'}) + H^q(x_{M'}|x_M \cup q_{M''}) = H^q(x_{M^c}|x_M)$. (Take $M'' = \Lambda^q_{\text{inn}}$, $M' = \tilde\Lambda^q$ and $M = G \cup \cup_{q' \neq q} \Lambda^{q'}$.) The second relation follows from (25) and (23'), see more precisely the definition of a **contour** given below and the relation (27).

6. Loops and semiloops. Contours

Applying Theorem 0: Short and long semiloops anchored in $(\Lambda^q)^c$.

Now we will give another, more detailed expression (27) of $Z_{\mathcal{M}}(\Lambda)$ which will be obtained by inserting the expressions (13') and (14) of Theorem 0 into (25).

Before doing so, let us introduce some further terminology. Let us call the closed oriented paths P^q as **loops**, and oriented paths 'from M^c to M^c through M' as **semiloops** 'anchored' in M^c. While the weights $\tilde w_{P^q}$ of the loops P^q in (14) are always small, so that their contributions can be studied by the 'high temperature expansion' $\exp(\tilde w_{P^q}) = 1 + w^*_{P^q}$, the weights $w_{P^q}\ (x-q)_{s(P^q)}(x-q)_{e(P^q)}$ corresponding to short semiloops P^q are **not** always small. They will be treated differently:

Let $s(P^q)$ resp. $e(P^q)$ be the starting resp. the ending point of the semiloop P^q in Λ^q (i.e. of an oriented path P^q going through Λ^q from $s(P^q) \in \partial(\Lambda^q)^c$ to $e(P^q) \in \partial(\Lambda^q)^c$). Say that P^q is **short** resp. **long** if any point of P^q has a distance at most $5\ \log(|x_i - q|/\delta_q + 1)$ from a suitable point i of the set $\partial(\Lambda^q)^c$ resp. if there is a point of P^q having a bigger distance than $5(\log|x_i - q|/\delta_q + 1)$ from any point i of that set. In other words, short semiloops are precisely those ones living wholly in the 'short' protection zones $\tilde\Lambda^q$ constructed in the definition of a 'thick' Γ.

For long semiloops P^q, the term $(x-q)_{s(P^q)}(x-q)_{e(P^q)}$ is 'sufficiently damped' by the weight w_{P^q}, so that the whole term $w_{P^q}\ (x-q)_{s(P^q)}(x-q)_{e(P^q)}$ is small. On the other hand, the sum over all short semiloops S^q 'going from $(\Lambda^q)^c$ back'

$$(26)\qquad \sum_{S^q} w_{S^q}\ (x-q)_{s(S^q)}(x-q)_{e(S^q)} = -H^q_e(\bar x_{\tilde\Lambda^q}|x_{\partial(\Lambda^q)^c} \cup q_{\Lambda^q\setminus\tilde\Lambda^q})$$

gives another expression for the minimum of the quadratic, relative energy term $H^q_e(x_{\tilde\Lambda^q}|x_{\partial(\Lambda^q)^c} \cup q_{\Lambda^q_{\text{inn}}})$ defined in (22). Here, $\tilde\Lambda^q = \Lambda^q \setminus \Lambda^q_{\text{inn}}$ is the protection zone i.e. the collection of all points of Λ^q whose distance to $\partial(\Lambda^q)^c$ is 'short' i.e. smaller than $5(\log(|x_i - q|/\delta_q + 1)$ for some i. Recall that the boundary condition $x_{\partial(\Lambda^q)^c}$ is determined by the collection of the values of precontours $\gamma_{n,j}$ of the system.

Using (22) and separating long and short semiloops we rewrite (25) as follows:

Proposition 5. *Denote by $\bar x_{\Lambda^q}$ the configuration minimising $H^q(\cdot\,|x_{\partial(\Lambda^q)^c})$. Then*

$$(27)\quad Z_{\mathcal{M}}(\Lambda) = \int \prod_n w_{\Gamma_n} \prod_q \exp(-h_q|\Lambda^q|) \prod_q \exp(\Delta^q(\{\Gamma_n\})) \prod_n d\Gamma_n^* \qquad \textit{where}$$

$$\Delta(\{\Gamma_n\}) = \sum_{L^q \in \mathcal{L}^q} w_{L^q}(x-q)_{s(L)}(x-q)_{e(L)} + \sum_{P^q \in \mathcal{P}^q} \tilde w_{P^q}\ ;\ \tilde w_{P^q} = \frac{|P^q \cap (\Lambda^q)^c|}{|P^q|} w_{P^q}.$$

We write $\int g(\{\Gamma_n\})\prod_n d\Gamma_n^*$ instead of a more precise $\sum_{y_\Lambda}\int g(\{\Gamma_n\})\prod_{n,j} d\,x_{G_{n,j}}$ here. The sum is over all possible colorings $y_\Lambda = y_\Lambda^{\{\Gamma_n\}}$ (for some allowed collection $\{\Gamma_n\}=\{\gamma_{n,j}\}$) and the integral is over all possible values $\{x_{G_{n,j}}, G_{n,j}=\operatorname{supp}\gamma_{n,j}\}$ of the 'core' $\{\Gamma_n^*\}$, satisfying $\mathcal{M}$. The symbol $\mathcal{L}^q$ denotes the collection of all **long semiloops** *L^q going from $\partial(\Lambda^q)^c$ back (through Λ^q) and $\mathcal{P}^q$ denotes the collection of all* **loops** *P^q intersecting both Λ^q and $(\Lambda^q)^c$.*

Proof. See (25) and (22). The term $H_e^q(\bar{x}_{\Lambda^q}|x_{\partial(\Lambda^q)^c})$, extracted from the partition functions Z^q in (22) is now decomposed into the 'short' sum (26) (added to the energy of Γ_n, because its terms may be dangerously large) and the remainder

$$(26') \qquad H_e^q(\bar{x}_{\Lambda^q_{\rm inn}}|\bar{x}_{\tilde{\Lambda}^q}) = -\sum_{L^q\in\mathcal{L}^q} w_{L^q}(x-q)_{s(L^q)},(x-q)_{e(L^q)}$$

which is a sum of small terms indexed by 'long' semiloops L^q. (We were using the conditioned Hamiltonian of the type (12) everywhere; notice that then the condition $x_i=q,\ i\in\Lambda^q_{\rm inn}$ has the same effect as the empty boundary condition.) Thus, (27) appears when we substitute (26') and (14) into (25).

For any long semiloop L^q going 'from $(\Lambda^q)^c$ back' define now the quantity

$$(28)\ \ w^*_{L^q}\equiv w^*_{L^q}((x-q)_{s(L^q)},(x-q)_{e(L^q)}) = \exp(w_{L^q}\ (x-q)_{s(L^q)}(x-q)_{e(L^q)})-1.$$

Let us use the notations P^q resp. L^q for loops resp. long semiloops also in the following definition, which summarises all the previous constructions of the paper.

Note. The terms $w_{S^q}\ (x-q)_{s(S^q)}(x-q)_{e(S^q)}$ for short semiloops S^q were already summed in (26). They are added to the energy of the precontours.

Contours.

They will be defined as suitable triples $\mathbf{\Gamma}=(\mathcal{A},\mathcal{C},\mathcal{L})$ where $\mathcal{A}$ is an allowed collection $\{\Gamma_n\}$ of **thick precontours**, $\mathcal{C}=\cup_q\mathcal{C}^q$ is some collection of **loops** (oriented paths, 'cycles') in Λ such that each P^q from the system $\mathcal{C}^q$ intersects both the sets Λ^q and $(\Lambda^q)^c$, and $\mathcal{L}=\cup_q\mathcal{L}^q$ where $\mathcal{L}^q$ is a collection of **long semiloops** L^q in Λ^q, which start and end in $(\Lambda^q)^c$. The condition on the whole system $\mathbf{\Gamma}=(\mathcal{A},\mathcal{C},\mathcal{L})$ is that we require the **connectedness** of the support

$$\operatorname{supp}\mathbf{\Gamma} := (\cup_{\Gamma\in\mathcal{A}}\operatorname{supp}\Gamma)\cup(\cup_{P\in\mathcal{C}}\operatorname{supp}P)\cup(\cup_{L\in\mathcal{L}}\operatorname{supp}L).$$

Contours will be denoted by symbols $\mathbf{\Gamma}$. We will use also another notation $\operatorname{supp}^*\mathbf{\Gamma}=\cup_{\Gamma_n}\operatorname{supp}^*\Gamma_n$ for the support of the **core** $\mathbf{\Gamma}^*$ of $\mathbf{\Gamma}$. We associate to each contour $\mathbf{\Gamma}$ its **Gibbs weight** $w_{\mathbf{\Gamma}}$ determined, with the help of w_{Γ_n}, as follows. Put

$$(29) \qquad w_{\mathbf{\Gamma}} = \prod_{\Gamma_n\in\mathcal{A}} w_{\Gamma_n}\prod_{q\in Q}\prod_{P^q\in\mathcal{C}^q} w^*_{P^q}\prod_{L^q\in\mathcal{L}^q} w^*_{L^q}$$

where the weights w_Γ are given by (24) (see also (20), (26) for more details). The weights $w^*_{L^q}$ are given by (28) and the weights $w^*_{P^q}$ by (compare also (14))

$$(30) \qquad w^*_{P^q} = \exp(\tilde{w}_{P^q})-1 = \exp(\frac{|P\cap\Lambda^q|}{2|P|}w_{P^q})-1.$$

In later investigations, we will write $w_{\mathbf{\Gamma}}$ in the exponential form

$$(29') \qquad \exp(-E(\mathbf{\Gamma})) := |w_{\mathbf{\Gamma}}| = \prod_{\Gamma_n\in\mathcal{A}}|w_{\Gamma_n}|\prod_{P^q\in\mathcal{C}^q}|w^*_{P^q}|\prod_{L^q\in\mathcal{L}^q}|w^*_{L^q}|.$$

7. Frames. The Abstract Pirogov Sinai Model

The following modification of the notion of a contour – which will be used systematically below – is now only a 'cosmetic' one. Namely, we will see in (33) that the values x_i, $i \in \operatorname{supp} \mathbf{\Gamma}^*$ can be integrated out in the expressions (27) of partition functions $Z_{\mathcal{M}}(\Lambda)$.[16] This suggests to apply the following factorisation of contours $\mathbf{\Gamma}$, by 'forgetting' all the values of $\mathbf{\Gamma}^*$ (but not the colours of $\mathbf{\Gamma}^*$):

By a **frame** of a contour $\mathbf{\Gamma}$ we will mean a pair $\mathcal{F} = (\operatorname{supp} \mathcal{F}, y_{\mathcal{F}})$ where $\operatorname{supp} \mathcal{F} = \operatorname{supp} \mathbf{\Gamma}$ and $y_{\mathcal{F}}$ is the colour of $\mathcal{F}$: $y_{\mathcal{F}} = \{y_i, i \in \operatorname{supp} \mathbf{\Gamma}\} \in (Q \cup *)^{\operatorname{supp} \mathbf{\Gamma}}$. Put

$$w_{\mathcal{F}} = w_{(\operatorname{supp} \mathcal{F}, y_{\mathcal{F}})} = \int_{\mathbf{\Gamma} \in \mathcal{F}} w_{\mathbf{\Gamma}} \; d\mathbf{\Gamma}^* \tag{31}$$

where the integral is over all the values of the cores $\mathbf{\Gamma}^*$ of contours $\mathbf{\Gamma}$ having the same frame $\mathcal{F} = (\operatorname{supp} \mathcal{F}, y_{\mathcal{F}})$. We introduce also the notation $\operatorname{supp}^* \mathcal{F} = \operatorname{supp}^* \mathbf{\Gamma}$.

Note. The paths P^q, L^q live in the Gaussian model given by H^q; they have a constant colour q. On the other hand, the 'inner' colour of a (thick) precontour can be different from the outer one. Frames still carry the information about a possible change of a colour inside $\mathbf{\Gamma}$. The values $w_{\mathcal{F}}$ depend also on these colours.

Signed frames.

Below. we will work only with the frames. Namely, these objects will be the 'contours' of our new, 'abstract Pirogov–Sinai model'. Notice that neither $w_{\mathbf{\Gamma}}$ nor $w_{\mathcal{F}}$ have to be positive. Introduce a concept of a **sign** of a frame $\mathcal{F}$

$$w_{\mathcal{F}} = \operatorname{sign} \mathcal{F} \; \exp(-E(\mathcal{F})) \tag{32}$$

where $\operatorname{sign}(\mathcal{F}) = \pm$. The accompanying quantity $E(\mathcal{F})$ is the **energy** of $\mathcal{F}$.

See the discussion below (33). We mention in advance that the value of $\operatorname{sign} \mathcal{F}$ has to be kept also for the **contour functional** $F(\mathcal{F})$ of $\mathcal{F}$, an important concept of the discrete spin Pirogov–Sinai theory (see [S], [PS], [ZR]) used to study (33).

Note. Most papers from Pirogov–Sinai theory do not consider signed contours. However, it is not difficult to adapt e.g. [ZR] for such a case.

One could introduce (and this is a much more serious problem) even a more general notion of a **complex** modulus $\operatorname{sign} \mathcal{F} \in \{z : |z| = 1\}$ of a contour (frame) as an attribute of $\mathcal{F}$ appearing always together with the (real) energy $E(\mathcal{F})$. (Thus introducing a general concept of a **complex Hamiltonian** for abstract Pirogov–Sinai models.) This problem deserves a separate paper. The general complex modulus $\operatorname{sign}(\mathcal{F})$ will change its value appropriately when going from $E(\mathcal{F})$ to the contour functional $F(\mathcal{F})$. What is now lost is the **probabilistic** interpretation of some' results; however the definition of quantities like h_q can be carried also to this complex case. Such an approach would yield appropriate piecewise analyticity results for the phase diagram. We plan to write a new variant of the paper [ZA] where these questions would be handled in the new language of [ZR].

Let us finish the promised transcription, of our continuous spin model, to an 'abstract Pirogov–Sinai model'. ¿From (27), (29) we have the following conclusion:

[16] In (21) and (25), the values of $\mathbf{\Gamma}^*$ were needed also as an information on **boundary conditions** for Gaussian partition functions $Z^q(\Lambda^q | x_{(\Lambda^q)^c})$. In (27), the only objects depending on the **values** of $\gamma_{n,j}$ are $\operatorname{supp} \Gamma_n$ and also the terms $w^*_{L^q}((x-q)_{s(L)}, (x-q)_{e(L)})$.

Theorem 1. *Any partition function $Z_{\mathcal{M}}(\Lambda)$ can be expressed as*

$$Z_{\mathcal{M}}(\Lambda) = \int \prod_n w_{\mathbf{\Gamma}_n} \prod_{q\in Q} \exp(-h^q|\Lambda^q|))\ d\{\mathbf{\Gamma}_n^*\} \ = \sum_{\{\mathcal{F}_n\}} \prod_n w_{\mathcal{F}_n} \prod_{q\in Q} \exp(-h^q|\Lambda^q|) =$$

$$= \sum_{\{\mathcal{F}_n\}} \prod_n \operatorname{sign} \mathcal{F}_n \exp(-E(\mathcal{F}_n))) \prod_q \exp(-h^q\,|\Lambda^q|) \tag{33}$$

where $w_{\mathbf{\Gamma}}$ resp. $w_{\bar{\mathcal{F}}}$ are given as above in (29) resp. (31). The integral resp. sum in (33) is over all allowed collections of contours $\mathbf{\Gamma}_n$ resp. frames $\mathcal{F}_n$ obeying $\mathcal{M}$.

Proof. This follows directly from Proposition 5 and from the definition of contours (first relation) and frames (second relation). Then we use the notations (32).

Notes. 1. We note that the true **energy barrier** of a frame $\mathcal{F}$ (with respect to the constants q on Λ^q) is not $E(\mathcal{F})$ but $E(\mathcal{F}) + h|\operatorname{supp}^* \mathcal{F}|$ where h is something like $h = \min h_q$. Namely, take in mind the 'holes' $\operatorname{supp}^* \mathcal{F}_n$ in (33) i.e. the relation

$$\Lambda = \cup_q \Lambda^q \cup_n \operatorname{supp} \mathcal{F}_n^*.$$

In usual examples, the differences between various h_q are small and one can assume that $h_q \equiv 0$ when discussing the validity of the Peierls condition.

2. Having established the Peierls condition, the technology of the discrete spin Pirogov–Sinai theory ([ZR]) can be used. The limitation $\mathcal{M}$ can have a meaning that the 'external' thick precontours of an allowed configuration do not have an 'astronomical' size; being 'considerably smaller' than the size of Λ. Such an assumption is also useful if we want to have an intuitively obvious definition of 'external contours' on a torus. However, the choice of a selected point '∞' in Λ, see Theorem 3, is formally simpler.

Then we get an analogy of usual q–diluted partition functions which are now more precisely taken over configurations with 'nonastronomical' contours only (resp. taken over configurations having the prescribed value $y_\infty = q$). The limitation $\mathcal{M}$ can have also some more subtle meaning namely the 'metastability' of any considered configuration. To express such a property ([Z], [ZR]) of the 'abstract Pirogov–Sinai model' in the language of original configurations $x_\Lambda \in \mathbb{R}^\Lambda$ (and 'colorings' y) is slightly cumbersome. Fortunately, the problems of an interpretation of the partition functions (33)) appear for 'nonstable' values of q only (and for very large volumes); otherwise we usually have **no** limitation $\mathcal{M}$ in the expressions (33).

8. The Peierls Condition

It remains to check the Peierls condition for the energies $E(\mathcal{F})$ in (33). Then we will really have an 'abstract Pirogov–Sinai model' and the analysis of the events which can be formulated in terms of the partition functions (33) will be possible. See the forthcoming section for some interpretation. With regard to (31), (29') we establish below in (38) suitable upper bounds for the activities $|w_{\mathbf{\Gamma}}|$ and $|w_{\bar{\mathcal{F}}}|$:

Investigation of the structure of the contour and frame energy.

We investigate below the energy $E(\mathcal{F})$ of a frame $\mathcal{F}$ of a contour $\mathbf{\Gamma} = (\mathcal{A}, \mathcal{C}, \mathcal{L})$ where $\mathcal{A} = \{\mathit{\Gamma}_i\}$ is a collection of thick precontours and $\mathcal{C} = \{\mathcal{C}^q\}$ and $\mathcal{L} = \{\mathcal{L}^q\}$ are systems of loops and long semiloops. Write the Gibbs factor (29) $w_{\mathbf{\Gamma}}$ as

$$w_{\mathbf{\Gamma}} = \prod_i w_{\mathit{\Gamma}_i} \prod_{q\in Q} \exp(\prod_{P^q\in\mathcal{C}^q} \prod_{L^q\in\mathcal{L}^q} w^*_{P^q}\ w^*_{L^q} = \tag{34}$$

$$= \prod_{i,j} w_{\gamma_{i,j}} \prod_q \exp(-H^q(\bar{x}_{\tilde{\Lambda}^q} | \cup_{i,j}\gamma_{i,j} \cup q_{\Lambda^q_{\mathrm{inn}}}) \prod_{P^q\in\mathcal{C}^q} \prod_{L^q\in\mathcal{L}^q} w^*_{P^q}\ w^*_{L^q}$$

where $\{\gamma_{i,j}\}$ is the collection of precontours contained in the thick precontour $\mathit{\Gamma}_i$ and $\tilde{\Lambda}^q$ is the 'q–short neighborhood' (protection zone) of $\partial(\Lambda^q)^c$ introduced in the definition (23) of thick precontours. The configuration $\bar{x}_{\tilde{\Lambda}}$ optimizes the value of $H^q(x_{\Lambda^q} | \cup_{i,j}\gamma_{i,j})$. The quantities w^*_L resp. w^*_P were defined in (28) resp. (29').

The formula (34) follows from (29) and (24), and it can be also written, using auxiliary notations $E(\mathbf{\Gamma}) = -\log|w_{\mathbf{\Gamma}}|$, $E(P) = -\log|w^*_P|$, $E(L) = -\log|w^*_L|$, in the following, more compact form (y denotes the colour induced by $\mathbf{\Gamma}$)

$$\exp(-E(\mathbf{\Gamma})) = \exp(-H_y(\bar{x}_G|y_{G^c}) \prod_{P^q\in\mathcal{C}^q} \exp(-E(P^q)) \prod_{L^q\in\mathcal{L}^q} \exp(-E(L^q)). \tag{35}$$

Here, G is defined as $G = G^* \cup \tilde{\Lambda}$ with the core $G^* := \cup_{i,j} \operatorname{supp}\gamma_{i,j}$ (union over all precontours of $\mathbf{\Gamma}$) and with $\tilde{\Lambda}$ being given as follows: $\tilde{\Lambda} = \cup_q \tilde{\Lambda}^q$ where $\tilde{\Lambda}^q$ denotes the q–th protection zone of G^*. Alternatively, $\tilde{\Lambda}^q$ is equal to the union of 'truncated' (excluding start- and endpoints) supports of all short semiloops in Λ^q going from ∂G^* back to ∂G^* through Λ^q.

The configuration $\bar{x}$ and the Hamiltonian $H_y(\bar{x}_G|y_{G^c})$ are defined as follows:

(1) The configuration $\bar{x}_G$ is chosen to be equal to the values of $\cup_{i,j}\gamma_{i,j}$ on G^*. It is extended to the set $G = G^* \cup \cup_q \tilde{\Lambda}^q$ in such a way that it minimises, in each $\tilde{\Lambda}^q$, the quadratic Hamiltonian $H^q(x_{\tilde{\Lambda}^q}|x_{G^*} \cup q_{\Lambda^q_{\mathrm{inn}}})$.[17]

(2) The pair Hamiltonian $H^{\mathrm{pair}}(\bar{x}_G|y_{G^c})$ is used in (35). For any $k \in G$ we add thepotential $U^{y_k}(\bar{x}_k)$ where $y_\Lambda = \{y_k,\ k \in \Lambda\}$ denotes the colour induced by $\{\gamma_{i,j}\}$. In particular for any $k \in G \setminus G^*$ which is a q–correct point of y_Λ we add the potential $U^q(\bar{x}_i)$ when defining $H_y(\bar{x}_G|y_{G^c})$.

Thus, to check the validity of the Peierls condition we have to analyze the behaviour of the three types of 'energies': the 'physical' quantity $H_y(\bar{x}_G|y_{G^c})$ and also the 'artificial' quantities $E(P), P \in \mathcal{C}$ and $E(L), L \in \mathcal{L}$. While it is rather straightforward to check the Peierls condition for the latter two terms (for small constants κ_q, and with slight modifications needed for the long semiloops L^q, compared to the case of loops P^q, see (45)) the investigation of $H(\bar{x}_G|y_{G^c})$ requires more care.

We prove the required bounds for $H_y(\bar{x}_G|y_{G^c})$ only in the special case $R = 1$. The notation $\tilde{H}_y(x_G)$ used below for $H_y(x_G|y_{G^c})$ suggests that now it is more convenient to think in terms of interactions $\sum_{i,j} b_{i-j}(x_i - x_j)^2$ and relative Hamiltonians $\tilde{H}^q(x_M|x_{M^c})$ instead of quadratic forms $\sum_{i,j} A^q_{i,j}(x_i - q)(x_j - q)$ and (12).

[17] This new $\bar{x}$ is 'slightly improved' (w.r. to the boundary condition $q_{\Lambda^q_{\mathrm{inn}}}$) compared to (34).

Notations. Say that $i \in \Lambda$ is a $q,*$–**correct point** of a coloring y_Λ if the values $y_j,\ |j-i| \le 1$ are equal either to q, at least for one such j, or to $*$. Say that i is a **wild** point of y_Λ if it is $q,*$–correct for **no** $q \in Q \cup \{*\}$. Decompose further the collection $\Lambda^{q,*}$ of all $q,*$–correct points of y_Λ into $\Lambda^{q,*}_q \cup \Lambda^{q,*}_*$, according to whether $y_i = q$ or $y_i = *$. Finally, say that x_i where $i \in \Lambda^{q,*}_*$ is q–**like** if $x_i \in (q', q'')$ (where $q' < q < q''$ are nearest neighbors from $Q \cup \{\infty\} \cup \{-\infty\}$) otherwise '$q$–redundant'.

Theorem 2. *Consider the model (1), (2), (3) with $e_q \equiv 0$. Let a connected set G be given, $G = \cup_{q\in Q} G^q \cup G^*$ such that G^* is the set of all incorrect points of some $y = y_\Lambda \in (Q^*)^\Lambda$ and all points of G^q are q–correct. Let x_{G^*} be a configuration on G^*. Decompose G^* into 'wild' resp. '$*,*$–correct' resp. '$q,*$–correct' parts*

$$G^* = G^w \cup G^{*,*} \cup \cup_{q\in Q} G^{q,*}\ , \quad G^{q,*} = G^{q,*}_q \cup G^{q,*}_* . \tag{36}$$

Assume that the union of 'protecting balls' B^q_j with centers in $G^{q,*}_*$ and with radiuses $5(\log(|x_j - q|/\delta_q) + 1)$ covers the set G^q, for each $q \in Q$. Extend x_{G^*} in an optimal way to G so that the minimal value of the expression

$$\tilde H_y(x_G) = \sum_{\{i,j\}\subset G\cup\partial(G^c)} b_{i-j}(x_i - x_j)^2 + \sum_{q\in Q}\sum_{j\in G^q} U^q(x_j) + \sum_{j\in G^*} U^{y_i}(x_j) \tag{37}$$

is attained, under the boundary condition $x_{\partial G^c} = y_{\partial G^c}$ (compare (35).[18] *Fix the value y_i in a selected point ('∞') of Λ and take the sum and integral over all possible (y_Λ, x_{G^*}) giving the same 'incorrect' set G^* and 'protection zones' G^q. Then the assumptions (17 B), (17C), (17E) imply a bound (with $\tilde\varepsilon \asymp \varepsilon_{(17B)} + \exp(-\alpha)$)*

$$\sum_{y_\Lambda} \int \exp(-\tilde H_y(x_G))\ dx_{G^*} \le \tilde\varepsilon^{\ |G|}. \tag{38}$$

Proof. One has to understand the contributions of G^w, $G^{q,*}$ and $G^{*,*}$ to the energy (37). The idea is that for $G^* = G^{q,*} \cup G^{*,*}$, the sum of $U^*(x_i)$ over $i \in G^{q,*}_*$ gives sufficient lower bound for the energy (37) if all x_i are q–like. If $G^w \ne \emptyset$, another big energy term comes from the 'wild' quadratic terms in (37), and we start with the estimate of the quadratic part of (37): For $x_{G^c} = y_{G^c}$ write (37) as

$$\tilde H_y(x_G) = \tilde H^Q_y(x_G) + U^*_*(x_{G^*}) + U^*_w(x_{G^*}) \text{ where } U^*_*(G^*) = \sum_{i\in G^*\setminus G^w}^{y_i=*} U^*(x_i), \tag{39}$$

$$U^*_w(x_{G^*}) = \sum_{i\in G^w}^{y_i=*} U^*(x_i) \text{ and } \tilde H^Q_y(x_G) = \sum_{i\in G: y_i\ne *} U^{y_i}(x_i) + \sum_{\{i,j\}\not\subseteq G^c} b_{i-j}(x_i - x_j)^2 .$$

Assume for clarity of the argument given below that $y_i \ne *$ for each $i \in G^w$. (Then $U^*_w \equiv 0$.) The generalization to a general y_{G^w} is not difficult. If $y_k = *$ for some $k \in G^w$ (e.g. if there is a jump from $i \in G^{q,*}_q$ to $j \in G^{q',*}_{q'}$ with $q' \ne q$ and with $|k-i| =$

[18] Namely, for $\partial G^q \setminus \partial(G^*)^c = \cup_q \partial\Lambda^q_{\text{inn}}$ we have assumed that $x_{\partial G^c} = y_{\partial G^c}$.

$|k-j|=1$) then the term $U^*_w(x_{G^*})$ must be also employed in these estimates. (It is helpful –and possible– to imagine that $H^{\rm pair}(x_\Lambda) = \sum_{|i-j|\le 2}(x_i-x_j)^2$ i.e. to employ 'artificial next nearest neighbour interactions' in such a case.)

For any $q, q' \in Q$, $q \neq q'$ we have a bound, with some $0<\gamma<1$ and $C \asymp 1/2\nu$

$$(40)\quad \alpha(x_i-x_j)^2 + 1/2\nu\ (U^q(x_i)+U^{q'}(x_j)) > \gamma(q-q')^2 + C(x_i-q-x_j+q')^2.$$

Put $G' = G\setminus G^w$. Using the notation (37) also for G^w decompose $\tilde H^Q_y(x_G)$ as

$$\tilde H^Q_y(x_G) = \tilde H^Q_y(x_{G'}|x_{(G')^c}) \ + \tilde H^Q_y(x_{G^w})\ ;\ \tilde H^Q_y(x_{G^w}) = \gamma \tilde H^Q_y(y_{G^w}) + \phi_y(x_{G^w})$$

(take $(q-*)^2 \equiv 0$) where $\tilde H^Q(x_M|x_{M^c}) = \sum_{\{i,j\}\not\subseteq M^c} b_{i-j}(x_i-x_j)^2 + \sum_{i\in M} U^{y_i}(x_i)$. The remainder $\phi_y(x_{G^w}) \equiv \tilde H^Q_y(x_{G^w}) - \gamma\tilde H^Q_y(y_{G^w})$ is quickly growing for $\gamma<1$. Notice that $\tilde H^Q_y(y_{G^w}) \equiv \tilde H^Q_y(y_{G^w}\cup y_{(G^w)^c})$ and $\tilde H^Q_y(y_\Lambda) = \tilde H^{\rm pair}_y(y_\Lambda)$ because $e_q\equiv 0$. By (40), $\phi_y(x_{G^w}) > C|x_{G^w} - \bar y_{G^w}|^2$ for a suitable $\gamma<1$ and $C>0$. Thus, write

$$(41)\qquad \int \exp(-\tilde H^Q_y(x_{G^w}))dx_{G^w} = \exp(-\gamma \tilde H^Q_y(y_G)) \int \exp(-\phi_y(x_{G^w}))\ dx_{G^w}.$$

$\exp(C'|G^w|)$. Summing over all possible 'wild' y_{G^w} yields a term of the order $\exp(-\gamma\alpha\delta^2|G^w|)$. Namely, $\exp(-\gamma\tilde H^Q_y(y_G)) \le \exp(-\gamma\alpha|\delta|^2|G^w|)$ by (17C). Therefore, the sum over y_{G^w} of the integrals (41) is bounded as $\varepsilon^{|G^w|}$ with $\lim_{\alpha\to\infty}\varepsilon = 0$.

So we may conclude: Having suitable Peierls bounds for the case $G'=G$ we can get them from (17C) also for a general $G^w \neq \emptyset$, if α is enough large.

Peierls bounds for the case $G^w=\emptyset$ and $G^{*,*}=\emptyset$.

We are essentially left in the case when $G^* = G^{q,*}$ for some q. Estimate, by (39)

$$(42)\qquad \exp(-\tilde H^Q_y(x_G) - U^*(x_{G^*})) \text{ where } \ \tilde H^Q_y(x_G) = \tilde H^{\rm pair}(x_G) + \sum_{y_i=q} U^q(x_i).$$

The smallness of (38) will now be obtained with the help of (17B) and the fact that the points with $y_i = *$ are 'dense' in G^*. The difficulty is that $|G^q|$ may be much bigger than $|G^*|$. Denote by N a minimal possible collection of points $i\in G^*$ 'responsible for G^q' such that $y_i=*$ for each $i\in N$ and the union of all

$$B^q_i = \{j : |j-i| \le 5(\log(|x_i-q|/\delta_q)+1)\}$$

covers $G : G\subset \cup_{i\in N}\ B^q_i$. Say that N is a 'net' of G^*. Fix a coloring y, allowing given sets G^q and G^*. Consider the integral $\int_{(N)} \exp(-\tilde H_y(x_G))\ dx_{G^*}$ over all x_{G^*} allowing the choice N of a net of G^q. Assume that $G^{*,*}=\emptyset$ for a simplicity. Then

$$(43)\qquad \int_{(N)} \exp(-\tilde H_y(x_G))\ dx_{G^*} = \int_{(N)} \exp(-U^*(x_{G^*}) - H^Q_y(x_G))\ dx_{G^*} \le$$

$$\le C^{|G^*|}\int_{(N)} \exp(-U^*(x_{G^*}))\ dx_N \le \prod_{i\in N}(\sum_{B^q_i}(C\varepsilon)^{({\rm diam}\, B^q_i)^\nu}) \ \le\ \prod_{i\in N}(\sum_{B^q_i}(C'\varepsilon)^{|B^q_i|})$$

which is $\leq (C''\varepsilon)^{|G|}$ if we sum over all possible B_i^q. Really, take a function, e.g.

$$\tilde{U}^*(x_i) = U^*(x_i) + \frac{1}{2\nu}((x_i - x_j)^2 + U^q(x_j)) \quad ; \quad i \in G_*^{q,*} \tag{44}$$

with $|j-i| = 1$, $j = q$, to control also the 'redundant' case $x_i \in (q', q'')^c$. Then $\tilde{U}^*(x_i)$ grows quickly in $(q', q'')^c$ and so we get, by (17B), the second bound in (43). The factor $C^{|G^*|}$ bounds the integral $\int \exp(-\tilde{H}_y^Q(x_G))dx_G^*$. Summing over all possible N adds another factor $2^{|G^*|}$ not changing the nature of this estimate.

Finally, the region $G^{*,*} \neq \emptyset$ is controlled by using the fact that U^* is 'uniformly large' by (17B). Here, again, the quadratic term $\tilde{H}_y^Q(x_{G^{*,*}}|x_{(G^{*,*})^c})$ helps to control (using (17C)) the integral over $x_{G^{*,*}}$. This concludes the proof of Theorem 2.

We need finally also some **estimates of the terms w_P^* and w_L^*.** For long semiloops L^q, the term $(x-q)_{s(L^q)}(x-q)_{e(L^q)}$ has a much smaller logarithm than $(w_L^q)^{-1}$ and so the estimates of the terms $w_{L^q}\ (x-q)_{s(L^q)}(x-q)_{e(L^q)})$ and $w_{L^q}^*$ can be done in a similar way as for the (configuration independent) terms w_{P^q} and $w_{P^q}^*$. We need crucially the assumption that all κ^q from (5) are small. Then, writing $\kappa^q = \exp(-K^q)$ estimate the sum over all paths $P^q \ni i$ of a length n by

$$\sum_{P^q \ni i : |P^q| = n} |w_{P^q}| \leq \exp((C - K^q)n) \tag{45}$$

which gives the necessary complement to (38) for κ^q small. We conclude:

Corollary. *For any collection $\{\mathcal{F}\}$ of frames with a fixed colour at a given point of Λ and with a fixed support* $\operatorname{supp}\mathcal{F}$*,*

$$\sum_{\mathcal{F}} \exp(-E(\mathcal{F})) \leq \exp(-\tau|\operatorname{supp}\mathcal{F}|) \quad \textit{where } \tau = C\min\{-\log\tilde{\varepsilon}, -\log\kappa\}. \tag{46}$$

9. Conclusion. Example.

Thus we have an 'abstract Pirogov–Sinai model' whose 'contours' (frames) satisfy (46). Such models are studied systematically e.g. in [ZR] (for the case $\operatorname{sign}\mathbf{\Gamma} \equiv 1$ only) and so we can use the available technology here, to expand the partition function (33). What is needed is also a suitable **interpretation** of these expansions, in terms of the original continuous spin model.

This is rather straightforward for **stable** values $q \in Q$. Stability means the existence of the 'q–like' Gibbs state of the discrete spin ensemble on $\mathbb{Z}^\nu$ (constructed for the model of Theorem 1). For a more detailed information on this crucial notion see [Z] and [ZR]. It is essential that this is equivalent also to the 'stability' of q in the **continuum spin** model (1) i.e. to the existence of a Gibbs state on $\mathbb{R}^{\mathbb{Z}^\nu}$ whose almost all configurations have values 'mostly from' the q–th well of U.

More precisely we have the following result, formulated for any torus and for the following special ensemble, corresponding to the choice of a 'q–like boundary condition in an infinity': Select a point in Λ, denote it by ∞ and consider the modified Hamiltonian $H_\infty^q(x_\Lambda)$ defined by (1) but with the term $U(x_\infty)$ replaced by the **quadratic term** $U^q(x_\infty)$; formally $H_\infty^q(x) = H(x) + U^q(x_\infty) - U(x_\infty)$.

Such a choice of a **fuzzy boundary condition** q defined in terms of a modified Hamiltonian H^q_∞ will be convenient here, and one has to supplement it by a convenient expression of the idea that 'x_i resides in the q– th well' (to characterize the behaviour of a typical configuration in the q–th Gibbs state). Consider the following 'smooth indicator of being outside of the q–th well' namely

$$(47)\quad f(x_i) = \exp(U(x_i))\big(\sum_{q'\in Q\setminus\{q\}} \exp(-U^{q'}(x_i)) + w(x_i)\big) = 1 - \frac{\exp(-U^q(x_i))}{\exp(-U(x))}.$$

Theorem 3. *Let q be stable. Then the expectation of the indicator (47) is expressed by the formula, where μ^q is the Gibbs measure on $\mathbb{R}^\Lambda$ corresponding to H^q_∞*

$$(48)\qquad \int f(x_i)d\mu^q(x_i) = \frac{Z^q_\infty(\Lambda|*)}{Z^q_\infty(\Lambda)} = 1 - \frac{Z^q_\infty(\Lambda|\dagger)}{Z^q_\infty(\Lambda)}.$$

The partition functions $Z^q_\infty(\Lambda)$ and $Z^q_\infty(\Lambda|*)$ resp. $Z^q_\infty(\Lambda|\dagger)$ are of the type (18), given as integrals/sums over all (x_Λ, y_Λ) satisfying the additional condition $y_\infty = q$ and also, for the numerators, $y_i \neq q$ (case $*$) resp. $y_i = q$ (case $\dagger$). They are expressed as partition functions (33), $\mathcal{M} = \emptyset$ and can be expanded like

$$(49)\qquad \frac{Z^q_\infty(\Lambda|\dagger)}{Z^q_\infty(\Lambda)} = \exp(\sum_{\mathcal{C}}^{\dagger} n_{\mathcal{C}} w_{\mathcal{C}}) \approx 1 - \omega$$

with $\omega = \exp(-C\tau)$. The sum $\sum^\dagger_{\mathcal{C}} n_{\mathcal{C}} w_{\mathcal{C}}$ is taken over all clusters $\mathcal{C}$ of frames $\mathcal{F}$ **not touching** *the point ∞ but* **touching** *the point i (in such a way that $y_i(\mathcal{F}) \neq q$ for some $\mathcal{F} \in \mathcal{C}$). The weights $w_{\mathcal{C}}$ are given as products $\prod_{\mathcal{F}\in\mathcal{C}} w_{\mathcal{F}}$ (see (32)) over frames $\mathcal{F} \in \mathcal{C}$. The combinatorial coefficients $n_{\mathcal{C}}$ grow at most as $C^{\sum_{\mathcal{F}} |\operatorname{supp}\mathcal{F}|}$.*

The proof of (48) is straightforward. The partition functions (49) are then analyzed by cluster expansion method. See [ZR] for the discussion of the Pirogov–Sinai technology. See also [NOZ] showing a simple approach to one important method (Cammarota, Dobrushin, Kotecky, Preiss, Seiler, ...) of cluster expansions.

Thus, a typical configuration x_Λ in the ensemble given by $Z^q_\infty(\Lambda)$ has a small value of $f(x_i)$ ('x_i lives in the q–th well of U') for most $i \in \Lambda$. One could estimate also the effect of the boundary condition q. Here, it is given in a single point ∞ only – but yet it determines the prevailing colour q of Λ if q is stable ! However the influence, at i, of the **position** of ∞ decays exponentially with $\operatorname{dist}(i, \infty)$.

Our method allows also an infinite set Q of reference spins q. However, then the 'stable' values of q may not exist at all. More precisely, 'finite volume stability' (observed in a finite torus Λ for at least one $q \in Q$) may disappear in the thermodynamical limit, for **any** $q \in Q$. An example of such a situation with an interesting, 'instable' behaviour in large enough volumes (under any constant boundary condition $x_i = q$) is the following model, with $U_1 \equiv 1$ and $U_2(x_i) = ax_i, a \in \mathbb{R}$:

Example. Consider the model[19]

$$(50)\quad H(x_\Lambda) = \sum_{|i-j|=1} b(x_i - x_j)^2 \;\; ; \;\; U(x_i) = d\, \cos(x_i U_1(x_i)) + U_2(x_i) \;,\; b > 0$$

[19]For $U_1 \equiv 1$ and $U_2 \equiv 0$ this is the sine-Gordon representation of a two dimensional Coulomb gas. See [BF], [FS] and the references given there for more information.

with slowly changing U_j, at low enough temperature. To guarantee the existence of a stable q, one needs special conditions on U_j. Take $U_1 = 1$, $U_2 = 0$. Using our method, one can show a coexistence of infinitely many extremal Gibbs states, located around the wells of U, and an exponential decay of correlations in these Gibbs states. I thank to D. Brydges who explained me the relevance of this example.

The aim of this paper was to reformulate the model (1) into the language of discrete spin models and to check the Peierls condition. The further development and interpretation of the results thus obtained surely deserves an additional study.

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