Comparison of finite volume canonical and grand canonical Gibbs measures under a mixing condition

N.Cancrini [1] and F. Martinelli [2]

[1] Dipartimento di Energetica, Università dell'Aquila, Italy and INFM Unità di Roma "La Sapienza"
e-mail: nicoletta.cancrini@roma1.infn.it

[2] Dipartimento di Matematica, Università di Roma Tre, Italy
e-mail: martin@mat.uniroma3.it

Abstract. We envisage a simple approach to prove sharp bounds between the finite volume (multi) canonical and (multi) grand canonical expectation of local observables of a lattice discrete spin model by assuming a mixing property for the Gibbsian specifications. We then apply our results to various situations including mean field perturbations of the Ising model. Our method does not rely upon a local limit theorem and applies also to the dilute Ising model in the Griffiths phase for which the mixing condition fails.

Key Words: Gibbs measures, equivalence of ensembles, Gibbs conditioning principle.

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1. Introduction

Throughout this paper μ_Λ^τ will denote the grand canonical Gibbs measure of a finite range, discrete lattice spin model in a given fixed finite volume $\Lambda \subset \mathbb{Z}^d$ with some boundary condition τ (see e.g. [Geo]). Let $N_\Lambda = \sum_{x\in\Lambda} h_x$, where $h_x = h(\eta(x))$ is some real function of the single spin $\eta(x)$ with a lattice distribution. The typical example one should keep in mind is a lattice gas with $N_\Lambda = \#$ of particles in Λ. Let N be a possible value of N_Λ and assume that $N = \mu_\Lambda^\tau(N_\Lambda)$. For a lattice gas model that means that the chemical potential was conveniently chosen from start as a function of N, Λ and τ (*warning*: the fact that the chemical potential depends on Λ and τ has no consequences on e.g. the DLR property of the measure since Λ and τ are kept fixed once and for all). Then, under a suitable mixing condition on the Gibbsian specification of the model, we compare the marginals of μ_Λ^τ and ν_Λ^τ, the latter being μ_Λ^τ conditioned on the event $N_\Lambda = N$, on subsets of Λ whose size may increase with $|\Lambda|$. One of the main results of the present paper, see theorem 4.1, is that, uniformly in N, the variational distance between the two marginals on $\Delta \subset \Lambda$ is bounded from above by $C\frac{|\Delta|}{|\Lambda|}$, provided that $|\Delta| \le |\Lambda|^{1-\epsilon}$, $\epsilon \ll 1$. For certain simple geometrical shapes, e.g. cubes, and "normal" *i.e.* of $O(1)$ values of the density $\rho := \frac{N}{|\Lambda|}$, the bound can be pushed up to $|\Delta| = o(|\Lambda|)$ (see remark at the end of theorem 4.1).

The above problem belongs to a more general and widely studied class of problems related to the so–called "Gibbs conditioning principle" (see [DZ] and references therein) or "equivalence of ensembles" (see e.g. [DT], [L] and [SZ]). In our case, classical results on the equivalence of ensembles state that the above variational distance should vanishes as $|\Lambda| \to \infty$ and Δ is kept *fixed*. In the last few years and because of very different motivation, people have tried to improve on the classical results by comparing the marginal of canonical and grand canonical Gibbs measures on increasingly (with Λ) large "sub–systems" (see e.g. [LY], [DZ] and [CZ]).

Our own motivation for studying such a problem comes actually from a quite different subject, namely the theory of stochastic spin exchange dynamics reversible w.r.t. the canonical measure ν_Λ^τ. In this context it is rather essential, for example in the study of the relaxational properties of the dynamics, to have a very detailed control of canonical expectations and canonical covariances uniformly in the conditioning value N. We refer the reader to [LY], [Y], [BZ] and in particular to [CM1] and [CM2]. Another quite recent but very interesting area of research where such a question comes into play are the theory of renormalization group pathologies, particularily the so–called block spin transformation [BCO], and the problem of mean field perturbations of finite range lattice models discussed in [CZ].

Most of the usual approaches to prove the equivalence of ensembles are either based on the proof of a local central limit theorem (see e,g [K] and [DT], [LY], [Y]) or on the theory of large deviations (see e.g. [L], [SZ] and [DZ]). Recently other methods, like entropy bounds using Pinsker's inequality (see [CZ] and [LPS]) or simple estimates on the logarithmic generating function using the complete analiticity of the interaction [BCO], have been introduced.

The method based on CLT is quite involved from a technical point of view, particularily if one wants to keep track of the dependence on Δ and N of the various estimates (see [Y] where however some of the arguments do not seem to be complete). The method of entropy bounds is well suited for quite general type of questions but it does not seem to give the best bounds, at least in our specific context, because Pinsker inequality gives a bound of the *square* of the variational distance of two measures in terms of their relative entropy. Finally the methods based on the complete analiticity of the model look rather simple and natural but they pose quite severe restriction on how large the size of the subset Δ can be.

Here we decided to follow a new approach which seems vaguely related to the well known two–block estimates of hydrodynamical limit for interacting particle systems with a conservation law (see [LY]) and which avoids completely the proof of a CLT. Roughly speaking we proceed as follows. By means of Fourier transform, for any function f with support Δ, we write

$$\nu_\Lambda^\tau(f) - \mu_\Lambda^\tau(f) = \frac{\int_{-T}^{T} dt\, \mu_\Lambda^\tau\big(e^{i\frac{t}{\sigma}(N_\Lambda(\eta)-N)}, f\big)}{2T\mu_\Lambda^\tau(N_\Lambda = N)}$$

where $T = \frac{\pi}{\lambda\sigma}$, $\sigma^2 = \mu_\Lambda^\tau(N_\Lambda, N_\Lambda)$ and λ is such that λN_Λ takes integer values. Then, using DLR equations, elementary Taylor expansions of only part of the characteristic function of N_Λ and a simple gaussian bound on the characteristic function of N_Λ, we prove the correct bound

$$|\mu_\Lambda^\tau(f) - \nu_\Lambda^\tau(f)| \le C\|f\|_\infty \frac{|\Delta|}{|\Lambda|}$$

for any f with support Δ and such that f is "almost orthogonal" to the r.v. N_Λ (*i.e.* the μ_Λ^τ–covariance between f and N_Λ is almost zero). Next we decompose any function f with support Δ into a part which is "almost orthogonal" to N_Λ plus something proportional to N_Δ, simply by adding and subtracting a suitable const. $\times N_\Delta$. Thus, one is left with the problem of comparing the canonical and grand canonical expectations of N_Δ, or, by additivity, of a single contribution h_x. It is at this stage that we exploit in a crucial way the identity $\sum_{x\in\Lambda}(\mu_\Lambda^\tau(h_x) - \nu_\Lambda^\tau(h_x)) = 0$. Let $D_x = \mu_\Lambda^\tau(h_x) - \nu_\Lambda^\tau(h_x)$. Then, thanks to the first part, for arbitrary $x, y \in \Lambda$,

$$|D_x - \alpha_{xy} D_y| \le O\left(\frac{1}{|\Lambda|}\right)$$

with $\alpha_{xy} = \frac{\mathrm{Cov}(h_y, N_\Lambda)}{\mathrm{Cov}(h_x, N_\Lambda)}$ (if necessary we have to replace the single sites by small cubes around them in such a way that the resulting covariances become positive). For most sites x, y, namely all those sites which are not too close to the boundary $\partial\Lambda$, $\alpha_{xy} \approx 1$ since $\mathrm{Cov}(h_x, N_\Lambda) \approx \mathrm{Cov}(h_y, N_\Lambda)$ because of the mixing property of μ_Λ^τ. Then the identity $\sum_x D_x = 0$ immediately allows us to conclude that

$$|D_y| \le O(1)\Big|\sum_x \alpha_{xy}\Big|^{-1} = O\left(\frac{1}{|\Lambda|}\right)$$

and the result follows.

It is important to observe that the assumption of having lattice distribution for the r.v. $h(\eta(x))$ is rather crucial in order to be able to use discrete Fourier transform.

We conclude this short introduction with a brief survey of the paper.

- In section 2 we give the basic setting.

- In section 3 we prove several simple bounds on various kind of covariances and on the characteristic function of N_Λ.
- In section 4 we prove the first main result on the variational distance between the marginals on a subste Δ of μ^τ_Λ and ν^τ_Λ.
- In section 5 we extend the result of section 4 to a multicanonical case, namely when several r.v N_{Λ_i} are fixed, equal to their average values, in the different atoms $\{\Lambda_i\}_{i=1}^k$ of a partition of Λ.
- In section 6 we extend the results of section 5 to the random bond dilute Ising model in the so called Griffiths phase. The interest here is to show how our method works when the basic mixing condition which is behind the results of sections 4,5 doesn't work. For this model it seems to us the methods based on complete analiticity may fail.
- Finally in section 7 we discuss several application of our results.

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2. Notation

2.1 The lattice and the configuration space

The Lattice. We consider the d dimensional lattice $\mathbb{Z}^d$ with sites $x = \{x_1, \dots, x_d\}$ and norm $|x| = \max_{i\in\{1,\dots,d\}} |x_i|$. The associated distance function is denoted by $d(\cdot,\cdot)$. By Q_l we denote the cube of all $x = (x_1, \dots, x_d) \in \mathbb{Z}^d$ such that $x_i \in \{0, \dots, l-1\}$. If $x \in \mathbb{Z}^d$, $Q_l(x)$ stands for $Q_l + x$. We also let B_l be the ball of radius l centered at the origin, *i.e.* $B_l = Q_{2l+1}((-l, \dots, -l))$.
If Λ is a finite subset of $\mathbb{Z}^d$ we write $\Lambda \subset\subset \mathbb{Z}^d$. The cardinality of Λ is denoted by $|\Lambda|$. $\mathbb{F}$ is the set of all nonempty finite subsets of $\mathbb{Z}^d$.
We finally define the exterior *n–boundary* as $\partial_n^+\Lambda = \{x \in \Lambda^c : d(x,\Lambda) \le n\}$ and the interior *n–boundary* as $\partial_n^-\Lambda = \{x \in \Lambda : d(x,\Lambda^c) \le n\}$. Given $r \in \mathbb{Z}_+$, we say that a subset V of $\mathbb{Z}^d$ is *r-connected* if for any two sites $,y$ in V there exists $\{x^1, \dots, x^n\} \subset V$ such that $x^1 = y$, $x^n = z$ and $|x^{i+1} - x^i| \le r$ for $i = 2, \dots, n$.

Regular sets. A finite subset Λ of $\mathbb{Z}^d$ is said to be a *l–regular*, $l \in \mathbb{Z}_+$, if there exists $x \in \mathbb{Z}^d$ such that Λ is the union of a finite number of cubes $Q_l(x^i + x)$ where $x^i \in l\mathbb{Z}^d$. We denote the class of all such sets by $\mathbb{F}_l$. Notice that any set is 1–regular *i.e.* $\mathbb{F}_{l=1} = \mathbb{F}$.

The configuration space. Our *configuration space* is $\Omega = S^{\mathbb{Z}^d}$, where S is a finite space, or $\Omega_V = S^V$ for some $V \subset \mathbb{Z}^d$. The configuration space Ω is endowed with the product topology and $\mathcal{F}$ will denote the σ–algebra generated by cylinders. Given $\eta \in \Omega$ and $\Lambda \subset \mathbb{Z}^d$ we denote by η_Λ the natural projection over Ω_Λ. If U, V are disjoint, $\eta_U\tau_V$ is the configuration on $U\cup V$ which is equal to η on U and τ on V.

Local functions. If f is a function on Ω, the *support* of f, denoted by Δ_f, is the smallest subset of $\mathbb{Z}^d$ such that $f(\sigma)$ depends only on σ_{Δ_f}. The *l–support* of a function, $l\in\mathbb{Z}_+$, is the smallest l–regular set V such that $\Delta_f\subset V$. f is called *local* if Δ_f is finite. By $\|f\|_\infty$ we mean the supremum norm of f and by $\|\nabla_x f\|_\infty$ we mean the supremum over all configurations η,τ that agree outside x of $|f(\eta)-f(\tau)|$.

Generalized h–charge. Given a real function $h\,:\,S[0,1]$ we define the h–charge in the set $V\subset\mathbb{Z}^d$ for the configuration η as $N_V(\eta)=\sum_{x\in V}h_x(\eta)$ where $h_x(\eta)=h(\eta(x))$.

2.2 Gibbs Measures

Definition 2.1. *A finite range, translation–invariant potential is a collection of real, local functions $\{\Phi_\Lambda\}$, $\Lambda\in\mathbb{F}$, $|\Lambda|\geq 2$, on Ω with the following properties*

(1) $\Phi_\Lambda=\Phi_{\Lambda+x}$ for all $\Lambda\in\mathbb{F}$ and all $x\in\mathbb{Z}^d$

(2) For each Λ the support of Φ_Λ coincides with Λ

(3) There exists $r>0$ such that $\Phi_\Lambda=0$ if $\operatorname{diam}\Lambda>r$. r is called the range of the interaction.

(4) $\|\Phi\|:=\sum_{\Lambda\ni 0}\|\Phi_\Lambda\|_\infty\ <\ \infty$

Given a *potential* or *interaction* Φ with the above four properties and $V\in\mathbb{F}$, we define the Hamiltonian $H_V^\Phi:\Omega\mapsto\mathbb{R}$ by

$$H_V^\Phi(\sigma)=-\sum_{\Lambda:\,\Lambda\cap V\neq\emptyset}\Phi_\Lambda(\sigma)$$

For $\sigma,\tau\in\Omega$ we also let $H_V^{\Phi,\tau}(\sigma)=H_V^\Phi(\sigma_V\tau_{V^c})$ and τ is called the *boundary condition.* For each $V\in\mathbb{F}$, $\tau\in\Omega$ the (finite volume) conditional Gibbs measure on $(\Omega,\mathcal{F})$, are given by

$$d\mu_V^{\Phi,\tau}(\sigma)=\begin{cases}\left(Z_V^{\Phi,\tau}\right)^{-1}\exp[\,-H_V^{\Phi,\tau}(\sigma)\,]\prod_{x\in\Lambda}d\mu_0^{\{x\}}\big(\sigma(x)\big) & \text{if }\sigma(x)=\tau(x)\text{ for all }x\in V^c\\ 0 & \text{otherwise.}\end{cases}\tag{2.1}$$

where $Z_V^{\Phi,\tau}$ is the proper normalization factor called partition function and for each $x\in\mathbb{Z}^d$ $\mu_0^{\{x\}}(\cdot)$ is some apriori positive probability measure on S. Notice that in (2.1) we have adsorbed in the potential Φ the standard inverse temperature factor β in front of the Hamiltonian while we have put into the apriori measure $\mu_0^{\{x\}}(\cdot)$ the usual one–body part of the interaction. In most notation we will drop the superscript Φ if that does not generate confusion.

Given a measurable bounded function f on Ω, $\mu_V(f)$ denotes the *function* $\sigma\mapsto\mu_V^\sigma(f)$ where $\mu_V^\sigma(f)$ is just the average of f w.r.t. μ_V^σ. Analogously, for any event X, $\mu_V^\tau(X):=\mu_V^\tau(\mathbb{I}_X)$, where $\mathbb{I}_X$ is the characteristic function of X. $\mu_V^\tau(f,g)$ stands for the covariance or *truncated correlation* (with respect to μ_V^τ) of f and g. The set of measures (2.1) satisfies the DLR compatibility conditions

$$\mu_\Lambda^\tau(\mu_V(X))=\mu_\Lambda^\tau(X)\qquad\forall\,X\in\mathcal{F}\qquad\forall\,V\subset\Lambda\subset\subset\mathbb{Z}^d\tag{2.2}$$

Definition 2.2. *A probability measure* μ *on* $(\Omega, \mathcal{F})$ *is called a Gibbs measure for* Φ *if*

$$\mu(\mu_V(X)) = \mu(X) \qquad \forall\, X \in \mathcal{F} \qquad \forall\, V \in \mathbb{F} \tag{2.3}$$

Remark. In the above definition we could have replaced the $\sigma-$algebra $\mathcal{F}$ with $\mathcal{F}_V$ (see section 2.3.2 in [EFS]).

3. Some preliminary results for Gibbs measures satisfying a strong mixing condition

In this section we derive some simple bounds for finite volume Gibbs measures satisfying the following "strong mixing assumption" (see [M2] for more details).

In order to formulate our condition let us fix positive numbers C, m, l and an arbitrary set $\Lambda \in \mathbb{F}_l$, $\Lambda = \cup_{i\in I} Q_l(x^i + x)$ with $x^i \in l\mathbb{Z}^d$.

Definition of property $SMT(\Lambda, C, m, l)$**.** *For any pair of bounded local functions f and g*

$$\sup_\tau |\mu^\tau_\Lambda(f,g)| \le C \sup_\tau \mu^\tau_\Lambda(|f|)\mu^\tau_{\Lambda\setminus V_f}(|g|) \sum_{x\in\partial^-_r V_f} \sum_{y\in\partial^-_r V_g} e^{-m|x-y|}$$

provided that $d(V_f, V_g) \ge l$. Here, if $\Lambda = \cup_{i\in I} Q_l(x^i + x)$ with $x^i \in lZ^d$, V_f (V_g) is the union of those cubes $Q_l(x^i + x)$, $i \in I$, that intersect the support of f (g).

3.1 Bounds on various covariances.

Let now $\Lambda \in \mathbb{F}_l$, $\Lambda = \cup_{i\in I} Q_l(x^i + x)$, and let $I_1 \dots I_k$ be a partition of I. Set $\Lambda_i = \cup_{j\in I_i} Q_l(x^j + x)$, $i = 1 \dots k$. We assume that the apriori probability measure $\mu_0^{\{x\}}$ is the same in each atom of the partition and we denote by $\mu_0^{(i)}$ the apriori measure in the i^{th}–atom. By assuming property $SMT(V, C, m, l)$ for any finite set $V \in \mathbb{F}_l$ we first derive some simple bounds on covariances of the form $\mu(f, N_{\Lambda_i})$, $\mu(N_{\Lambda_i}, N_{\Lambda_i})$, $\mu(f, N_{\Lambda_i}, N_{\Lambda_i})$ or $\mu(N_{\Lambda_i}, N_{\Lambda_k}, N_{\Lambda_j})$, where $\mu(\cdot) = \mu^\tau_\Lambda(\cdot)$, f is a local function, $\Delta := V_f$ is as in the definition of property SMT, and $N_{\Lambda_i}(\eta) := \sum_{x\in\Lambda_i} h_x$ for some function $h : S \mapsto [0,1]$ with a unique global minimum (where h is equal to zero) and maximum (where h is equal to one). A typical example in the case of lattice gas variables, $S = \{0,1\}$, is $h(\eta) = \eta$. Such bounds will be essential in the proof of the equivalence of ensembles.

Proposition 3.1. *Assume $SMT(V, C, m, l)$ for any l–regular finite set V. Fix $\Lambda \in \mathbb{F}_l$ and a partition of Λ into l–regular sets $\Lambda_1, \dots, \Lambda_k$, together with a local function f as above. Let $\rho_i = \frac{\mu(N_{\Lambda_i})}{|\Lambda_i|}$, $i = 1 \dots k$. Without loss of generality we assume that $\rho_i \le \frac{1}{2}$, $i = 1 \dots k$. Then for any l large enough there exists a constant A depending only on $C, m, \|\Phi\|, l, k$ such that*

$$\begin{aligned}
&a)\quad |\mu(f, N_{\Lambda_i})| \le A\|f\|_\infty \min\{\sqrt{\rho_i|\Lambda_i|}, \rho_i|\Lambda_i|\}\\
&b)\quad |\mu(N_{\Lambda_i}, N_{\Lambda_i})| \le A\rho_i\,|\Lambda_i|\\
&c)\quad \mu(N_{\Lambda_i}, N_{\Lambda_i}) \ge A^{-1}\rho_i\,|\Lambda_i| \qquad \textit{(here mixing is not needed)}\\
&d)\quad |\mu(N_{\Lambda_i}, N_{\Lambda_j})| \le A\rho_i\rho_j \Big\{|\partial^-_r\Lambda_i||\partial^-_r\Lambda_j|\Big\}^{1/2}\\
&e)\quad |\mu(\bar N^2_{\Lambda_i}, \bar N^2_{\Lambda_j})| \le A\,\rho_i\rho_j|\Lambda_i||\Lambda_j| \\
\textit{if } k=1 &\\
&f)\quad |\mu(f, N_\Lambda)| \le A\|f\|_\infty \min\{\sqrt{\rho|\Delta|},\, \rho|\Delta|\}\\
&g)\quad |\mu(f, \bar N^2_\Lambda)| \le A\|f\|_\infty\,\rho|\Delta|\\
&h)\quad |\mu(N_\Delta, \bar N^2_\Lambda)| \le A\rho|\Delta|\\
&i)\quad \mu(\bar N^4_\Lambda) \le A\max\{\rho|\Lambda|, (\rho|\Lambda|)^2\}
\end{aligned}\tag{3.1}$$

where $\rho := \rho_1$ *and* $\bar N_\Lambda := N_\Lambda - \mu^\tau_\Lambda(N_\Lambda)$.

Proof.
a) We can either write

$$|\mu(f, N_{\Lambda_i})| \le \|f\|_\infty \, \mu\big(N_{\Lambda_i}, N_{\Lambda_i}\big)^{1/2} \le A \|f\|_\infty \sqrt{\rho_i |\Lambda_i|}$$

or

$$|\mu(f, N_{\Lambda_i})| = |\mu\big(\mu_{\Lambda\setminus\Lambda_i}(f), N_{\Lambda_i}\big)| \le \mu\big(N_{\Lambda_i}, N_{\Lambda_i}\big)^{1/2} \mu\big(\mu_{\Lambda\setminus\Lambda_i}(f), \mu_{\Lambda\setminus\Lambda_i}(f)\big)^{1/2} \tag{3.2}$$

In order to conclude it is enough now to use *b)* below together with the bound

$$\mu\big(\mu_{\Lambda\setminus\Lambda_i}(f), \mu_{\Lambda\setminus\Lambda_i}(f)\big) \le \|f\|^2_\infty \, \mu\big(\exists x \in \Lambda_i : \; h_x(\eta) \ne 0\big) \le C \|f\|^2_\infty \, \mu(N_{\Lambda_i}) \le C \|f\|^2_\infty \, \rho_i |\Lambda_i| \tag{3.3}$$

b). First of all we observe that

$$\sup_{x \in \Lambda_i} \sup_{V \subset \Lambda} \sup_\tau \mu^\tau_V(h_x, h_x) \le C \rho_i \tag{3.4}$$

for a suitable constant C. In order to prove it we notice that, for any $x, y \in \Lambda_i$

$$\sup_{V \subset \Lambda} \sup_\tau \mu^\tau_V(h_x, h_x) \le \sup_{V \subset \Lambda} \sup_\tau \mu^\tau_V(h_x) \le e^{2\|\Phi\|} \mu(h_y) \tag{3.5}$$

Therefore

$$\sup_{x \in \Lambda_i} \mu(h_x, h_x) \le e^{2\|\Phi\|} \frac{\mu(N_{\Lambda_i})}{|\Lambda_i|} = e^{2\|\Phi\|} \rho_i \tag{3.6}$$

and (3.4) follows. Next, for any x, y in Λ_i with $|x - y| \ge l$, we immediately get from the strong mixing assumption and (3.4), the bound

$$|\mu(h_y, h_x)| \le C \rho_i e^{-m|x-y|} \tag{3.7}$$

Thus the estimate *b)* follows at once.

c). Let $\bar\Lambda_i$ be the largest subset of Λ_i such that for each $x, y \in \bar\Lambda_i$ $|x - y| \ge r$ where r is the range of the interaction. Then we write

$$\begin{aligned} \mu(N_{\Lambda_i}, N_{\Lambda_i}) &\ge \mu\big(\mu^\eta_{\bar\Lambda_i}(N_{\bar\Lambda_i}, N_{\bar\Lambda_i})\big) \\ &\ge \min_{x \in \Lambda_i} \inf_\tau \mu^\tau_{\{x\}}(h_x, h_x) \, |\bar\Lambda_i| \end{aligned} \tag{3.8}$$

As in (3.6) one gets immediately that

$$\mu^\tau_{\{x\}}(h_x, h_x) \ge \frac{1}{2} e^{-2\|\Phi\|} \mu(h_y) \qquad \forall y \in \Lambda_i$$

i.e.

$$\mu^\tau_{\{x\}}(h_x, h_x) \ge C \rho_i$$

Thus the r.h.s of (3.8) is larger than $C\rho_i|\bar\Lambda_i| \ge C'\rho_i|\Lambda_i|$.

d). From the mixing assumption and the bound $\sup_{V,\tau}\mu_V^\tau(h_x) \le C\rho_i$ if $x\in\Lambda_i$ (see (3.4)...(3.6)), we get that for any $x\in\Lambda_i$ and $y\in\Lambda_j$

$$|\mu(h_x,h_y)| \le \begin{cases} C\rho_i\rho_j e^{-m|x-y|} & \text{if } |x-y|\ge l \\ C\rho_i\rho_j & \text{otherwise} \end{cases}$$

The above bound implies that

$$|\mu(N_{\Lambda_i},N_{\Lambda_j})| \le A\rho_i\rho_j\,|\partial_l^-\Lambda_i|$$

for a suitable constants A. Similarily with i and j interchanged. Thus the sought bound follows then by taking the geometric mean of the two estimates.

e). Let $\bar h_x(\eta) := h_x - \mu(h_x)$. Given $x_i\in\Lambda$, $i=1,\dots,4$, it is quite easy to check, using once more property $SMT(\Lambda,C,m,l)$ together with $\sup_\tau|\mu^\tau_{\{x\}}(h_x)| \le C\|h\|_\infty\rho_x$ where $\rho_x=\rho_i$ if $x\in\Lambda_i$, and $\mu(\prod_{i=1}^n h_{x_i}) \le \prod_{i=1}^n \sup_\tau \mu^\tau_{\{x_i\}}(h_{x_i})$ valid if all points $\{x_i\}_{i=1}^n$ are different, that

$$|\mu(\prod_{i=1}^4 \bar h_{x_i})| \le \begin{cases} C\|h\|_\infty^4 \Big\{\prod_i \rho_{x_i}\Big\}^{1/2} e^{-m'd(x_1,\dots,x_4)} & \text{if } d(x_1,\dots,x_4)\ge 4l \\ & \text{and all points are different} \\ \|h\|_\infty^4\rho_x & \text{if } x_1=x_2=x_3=x_4=x \\ C\|h\|_\infty^4\rho_x\rho_y & \text{if } x_1=x_2=x \text{ and } x_3=x_4=y \\ C\|h\|_\infty^4\rho_x\big\{\rho_{x_3}\rho_{x_4}\big\}^{1/2}e^{-m'd(x_1,\dots,x_4)} & \text{if } d(x_1,\dots,x_4)\ge 4l,\ x_1=x_2=x \\ & \text{and } x_3\ne x_4\ne x \end{cases} \tag{3.9}$$

for suitable positive constants C and $m'=m'(m)$, where $d(x_1,\dots,x_4)=\sum_{i=1}^4 d(x_i,\cup_{j\ne i}x_j)$. In particular, from (3.9) above, it follows that

$$\mu\big(\bar N^2_{\Lambda_i}\bar N^2_{\Lambda_j}\big) \le \sum_{x_1,x_2\in\Lambda_i}\ \sum_{x_3,x_4\in\Lambda_j} |\mu(\prod_{i=1}^4 \bar h_{x_i})| \le C'\rho_i\rho_j|\Lambda_i||\Lambda_j| \tag{3.10}$$

for some constant C'. $\square$

f). For $j=0,1\dots$, let Δ_j be the union of those cubes $Q_l(x_i+x)$, $x_i\in lZ^d$, that form Λ, whose distance from Δ is between jl and $(j+1)l$. Let also $N_j(\eta)=\sum_{x\in\Delta_j}h_x(\eta)$. Then we can write

$$|\mu(f,N_\Lambda)| \le 2\sum_{j\ge1}|\mu(f,N_j)| + |\mu(f,N_{\Delta_0})| \tag{3.11}$$

Property SMT together *a)* above and the bound (see e.g. (3.3))

$$\mu\big(f,f\big) \le C\|f\|_\infty^2 \min\Big\{\rho|\Delta|,1\Big\} \tag{3.12}$$

give

$$|\mu(f,N_j)| \le C\|f\|_\infty \min\{\sqrt{\rho|\Delta_j|},\, \rho\big[|\Delta||\Delta_j|\big]^{1/2}\}\, e^{-mjl} \qquad j\ge 1$$

Thus, using the trivial estimate $|\Delta_j| \le Cj^d|\Delta|$, we obtain

$$|\mu(f,N_\Lambda)| \le C'\|f\|_\infty \min\{\sqrt{\rho|\Delta|},\, \rho|\Delta|\} + \mu(f,f)^{1/2}\mu\big(N_{\Delta_0},N_{\Delta_0}\big)^{1/2} \le C''\|f\|_\infty\{\sqrt{\rho|\Delta|},\, \rho|\Delta|\} \tag{3.13}$$

g). We proceed as in *f)* and we write

$$|\,\mu(f,\bar N_\Lambda^2)\,| = |\,\mu(\bar f \bar N_\Lambda^2)\,| \le 2\sum_{j\le k}|\mu(\bar f\bar N_j\bar N_k)| \tag{3.14}$$

where $\bar f = f-\mu(f)$. Next we observe that, thanks to property $SMT(\Lambda,C,m,l)$, *b)* and *i)* above,

$$|\mu(\bar f\bar N_j\bar N_k)| \le \begin{cases} C'\|f\|_\infty\,\rho\sqrt{|\Delta_j||\Delta_k|}\,e^{-ml(k-j)} & \text{if } j\ge 1 \text{ and } k\ge 2j \\ C'\|f\|_\infty\,\rho\sqrt{|\Delta_j||\Delta_k|}\,e^{-mlj} & \text{if } j\ge 1 \text{ and } k< 2j \\ C'\|f\|_\infty\,\rho|\Delta_0| & \text{if } j=0 \text{ and } k\le 1 \end{cases} \tag{3.15}$$

If we now use the trivial bound $|\Delta_k| \le C(k+1)^d|\Delta|$ and plug (3.15) into the r.h.s of (3.14), we immediately get the sought result.

h). We write

$$|\mu(N_\Delta,\bar N_\Lambda^2)| \le \sum_{x\in\Delta}|\mu(h_x,\bar N_\Lambda^2)|$$

and we apply to each term $|\mu(h_x,\bar N_\Lambda^2)|$ the bound

$$|\mu(h_x,\bar N_\Lambda^2)| \le A\|h\|_\infty\rho$$

which follows from g) applied to $f=h_x$.

i). ¿From (3.9) above it follows that

$$\mu\big((\bar N_\Lambda)^4\big) \le \sum_{\{x_1,\dots,x_4\}\in\Lambda}|\mu(\prod_{i=1}^4\bar h_{x_i})| \le C''\max\{\rho|\Lambda|,\rho^2|\Lambda|^2\} \tag{3.16}$$

for another constant C''. Notice that the "new" term $\rho|\Lambda|$ that was absent in *e)* represent the contribution to the above sum when all the points are the equal. □

For future purposes it will be useful to have the following simple corollaries.

Corollary 3.2. *In the same setting of proposition 3.1 let* $\tilde{I}_i \subset I_i$, $i = 1, \dots, k$ *and let* $V_i = \cup_{j \in \tilde{I}_i} Q_l(x^j + x)$. *Then the bounds a), ..., i) hold with* N_{Λ_i} *replaced by* N_{V_i} *and* A *replaced by* $e^{4\|\Phi\|}A$.

Proof. It is enough to observe that

$$e^{-4\|\Phi\|}\rho_i|V_i| \le \mu(N_{V_i}) \le e^{-4\|\Phi\|}\rho_i|V_i| \qquad \forall\, i = 1, \dots, k$$

and use proposition 3.1 with the finer partition $\{V_i, \Lambda_i \setminus V_i\}_{i=1}^k$. □

3.2 Gaussian upper bound on characteristic functions

In the same setting of proposition 3.1 but *without* assuming property $SMT(V, C, m, l)$ for all l–regular sets V, we prove, following [DT], a gaussian upper bound on the characteristic function of the random variables N_{Λ_i}, $i = 1, \dots, k$.

Proposition 3.3. *Given* $T > 0$ *and positive numbers* $\alpha_0, \alpha_1, \dots, \alpha_k$ *assume that*

$$\mathrm{Var}_0^j\big(\cos(th(\eta))\big) + \mathrm{Var}_0^j\big(\sin(th(\eta))\big) \ge \alpha_j \mu_0^{(j)}(h)t^2, \quad j = 1, \dots, k \quad \forall\, t \in [-T, T]$$

where Var_0^j *denotes the variance w.r.t. the single spin measure* $\mu_0^{(j)}$. *Let* $\sigma_j^2 := \mu_\Lambda^\tau(N_{\Lambda_j}, N_{\Lambda_j})$ *and assume that* $\sigma_j^2 \le \alpha_j \rho_j |\Lambda_j|$, $j = 1, \dots, k$. *Let* $F_\Lambda^\tau(t_1, \dots, t_k) := \mu_\Lambda^\tau\Big(\exp\big(i\sum_j \frac{t_j}{\sigma_j}\bar{N}_{\Lambda_j}\big)\Big)$. *Then there exists a constant c such that*

$$\sup_\tau |F_\Lambda^\tau(t_1, \dots, t_i)| \le e^{-c\sum_{j=1}^i \alpha_j t_j^2}$$

for any $i \le k$, $t_j \in [-T\sigma_j, T\sigma_j]$, $j = 1, \dots, i$.

Proof. Consider the subset Λ_r of Λ which consists of all those points $x \in \Lambda$ such that $x \in (2r+1)\mathbb{Z}^d$. Notice that the Gibbs measure on Λ_r conditioned to $\Lambda \setminus \Lambda_r$ is a product measure over the sites in Λ_r. Then, using DLR and the fact that $|F_{\Lambda \setminus \Lambda_r}^\tau(t_1, \dots, t_i)| \le 1$, we get

$$\sup_\tau F_\Lambda^\tau(t_1, \dots, t_i) \le \prod_{j=1}^k g_j\big(t_j/\sigma_j\big)^{|\Lambda_r \cap \Lambda_j|}$$

where

$$g_j(t) = \sup_\tau |\sup_{x \in \Lambda_j} \mu_{\{x\}}^\tau(e^{ith(\eta(x))})|$$

Let us now estimate from above $g_j(t)$ for $t \in [-\pi, \pi]$.
We write

$$g_j(t) \le e^{\frac{1}{2}(g_j(t)^2 - 1)}$$

and we compute

$$\begin{aligned}
\frac{1}{2}[g_j(t)^2 - 1] &= -\inf_\tau \inf_{x \in \Lambda_j} \mathrm{Var}_{\mu_{\{x\}}^\tau}\big(\cos(th(\eta))\big) + \mathrm{Var}_{\mu_{\{x\}}^\tau}\big(\sin(th(\eta))\big) \\
&\le -e^{-4\|\Phi\|}\Big\{\mathrm{Var}_0^j\big(\cos(th(\eta))\big) + \mathrm{Var}_0^j\big(\sin(th(\eta))\big)\Big\} \\
&\le -e^{-4\|\Phi\|}\alpha_j \mu_0^{(j)}(h)t^2 \\
&\le -e^{-6\|\Phi\|}\alpha_j \rho_j t^2
\end{aligned}$$

because of $\mu_0^{(j)}(h) \geq e^{-2\|\Phi\|} \frac{\sum_{x\in\Lambda_j} \mu\big(h(\eta(x)\big)}{|\Lambda_j|} = e^{-2\|\Phi\|}\rho_j$.
In conclusion

$$\sup_\tau F_\Lambda^\tau(t_1,\dots,t_i) \leq \exp\big(-e^{-6\|\Phi\|}\sum_j \alpha_j \frac{\rho_j}{\sigma_j^2}|\Lambda_r\cap\Lambda_j|t_j^2\big) \leq \exp\big(-c\sum_j \alpha_j t_j^2\big)$$

for a suitable numerical constant c, because of *b)* of proposition 3.1. □

Remark 1. Clearly the hypothesis $\sigma_j^2 \leq \alpha_0\rho_j|\Lambda_j|,\ j=1,\dots,k$ is satisfied for a suitable constant α_0 if we assume SMT for all l–regular volumes. However, when discussing the random dilute Ising model, we will see that the above assumption is satisfied with high probability even if property SMT is not, at least with constants C, m uniform in the volume.

Remark 2. It is important to understand the limitation on the function h and on the single spin measure $\mu_0^{(j)}$ put by the new hypothesis of proposition 3.3. Let us start with the simple, but physically relevant, case of $S=\{0,1\}$ and $h(\eta)=\eta$. Here the assumption of the proposition is valid with $T=\pi$ and constants $\{\alpha_j\}_{j=1}^k$ independent of ρ_j by the following argument. Let us denote by $p_j = \inf_\tau \inf_{x\in\Lambda_j} \mu^\tau_{\{x\}}(\eta(x)=1)\big(1-\mu^\tau_{\{x\}}(\eta(x)=1)\big)$ and observe that (see e.g the proof of *i)* of proposition 3.1) $p_j \geq C\rho_j$ for a suitable constant C since we are assuming $\rho_j \leq \frac{1}{2}$. Then we compute

$$\begin{aligned}\inf_\tau \inf_{x\in\Lambda_j} \mathrm{Var}_{\mu^\tau_{\{x\}}}\big(\cos(th(\eta)\big) + \mathrm{Var}_0^j\big(\sin(th(\eta)\big) &\geq p_j \sin(t/2)^2\\ &\geq p_j\big(\frac{1}{2}\wedge\frac{1}{8}t^2\big)\\ &\geq c_0\rho_j t^2 \qquad \forall\, t\in[-\pi,\pi]\end{aligned}$$

for a suitable numerical constant c_0.
When the cardinality of the single spin space is greater than two some complication may arise and the constants α_j may depend on the densities ρ_j. As an example let us consider the case $S=\{0,2,3\}$, $h(\eta)=\frac{1}{3}\eta$ and $\mu_0^{(j)}(\eta)$ proportional to $e^{-\lambda\eta}$, $\lambda>0$. It is immediate to check that, for $\rho_j \ll 1$, $c_1\rho_j \leq e^{-2\lambda} \leq c_2\rho_j$ for suitable constants c_1 and c_2. Take now $t^* = \frac{3\pi}{2}$ and compute

$$\mathrm{Var}_0^j\big(\cos(t^*h(\eta)\big) + \mathrm{Var}_0^j\big(\sin(t^*h(\eta)\big) = 2\mu_0^{(j)}(\eta=3)\mu_0^{(j)}(\eta\neq 3) \leq c\rho_j^{3/2}$$

Thus in this case if we want to take e.g. $T=t^*$ then necessarily we must take the constants α_j proportional to $\rho_j^{1/2}$ and the gaussian bound on the characteristic function $|F_\Lambda^\tau(t_1,\dots,t_k)|$ for $t_i\in[-t^*,t^*]$ $i=1,\dots,k$ is valid but with a constant in front of t_j^2 proportional to $\rho_j^{1/2}$. In some sense if $\rho_j \ll 1$ then the absolute value of the characteristic function $\mu^\tau_{\Lambda_j}\big(e^{i\frac{t_j}{\sigma_j}N_j}\big)$ looks more like the superposition of two gaussians, one centered at $t_j=0$ and the other at $t_j=t^*$. In other words one should be able to prove a gaussian bound in the smallest between the distance of t_j from zero or from t^*. Although such an analysis is feasible, we decided not to carry over it in order not to burden too much the reader.

3.3 A lower bound on the integral of characteristic functions

Here we prove a lower bound on the integral of the characteristic function $F^{\tau}_{\Lambda}(t_1,\dots,t_k)$ that will be essential in order to estimate from below the probability that $N_{\Lambda_j}=\mu(N_{\Lambda_j})$, $j=1,\dots,k$.

Proposition 3.4. *In the same assumptions of proposition 3.3 there exists positive constants $c=c(\Phi,d)$ and A_0 such that, for any $A\ge\max\{A_0,\prod\frac{1}{\alpha_i}\sum_j\frac{1}{\alpha_j}\}$*

$$\int_{-A}^{A}dt_1\dots\int_{-A}^{A}dt_k\,Re\,F(t_1,\dots,t_k)\ge c \tag{3.17}$$

provided that $\sigma_j\ge A^{d+4+k}$, $j=1,\dots,k$.

Proof. Given an arbitrary element Λ_j of the partition $\Lambda_1,\dots,\Lambda_k$ of the l–regular set Λ and a large integer A, let V_j be an l–regular subset of Λ_j with the properties that $\frac{|\Lambda_j\setminus V_j|}{V_j}\le A^{-1}$, $V_j=\cup_{\alpha=1}^{n_j}C_\alpha$ where each C_α is l–regular, $\mathrm{diam}(C_\alpha)\le Al$, $\mathrm{dist}(C_\alpha,\Lambda\setminus\Lambda_j)\ge 2(r+1)$, $\mathrm{dist}(C_\alpha,V_j\setminus C_\alpha)\ge 2(r+1)$. It is easy to convince oneself that a choice of V_j with the above properties always exists. We set $V=\cup_jV_j$. Notice that the Gibbs measure on V conditioned to $\Lambda\setminus V$ is a product measure over the sets V_j, $j=1,\dots,k$. Then we write (see [DT])

$$\begin{aligned}F\big(t_1,\dots,t_k\big)&=\mu\Big(\prod_j e^{i\frac{t_j}{\sigma_j}\left[\bar N_{\Lambda_j\setminus V_j}(\eta)+\mu^\eta_{V_j}(\bar N_{V_j})\right]}F^\eta_j(t_j)\Big)\\&=\mu\Big(\prod_j F^\eta_j(t_j)\Big)+\mu\Big(\prod_j F^\eta_j(t_j)\big[\,e^{i\sum_j\frac{t_j}{\sigma_j}\bar N_{\Lambda\setminus V_j}(\eta)}e^{i\sum_j\frac{t_j}{\sigma_j}\mu^\eta_{V_j}\left(\bar N_{V_j}\right)}-1\big]\Big)\end{aligned}\tag{3.18}$$

where $F^\eta_j(t_j):=\mu^\eta_{V_j}\big(e^{i\frac{t_j}{\sigma_j}\bar N_{V_j}}\big)$. Let us consider separately the two terms in the r.h.s. of (3.18). Since each measure $\mu^\eta_{V_j}(\cdot)$ is a product measure over the sets $\{C_\alpha\}$ that form V_j, the characteristic function $F^\eta_j(t_j)$ is equal to

$$F^\eta_j(t_j)=\prod_\alpha\mu^\eta_{C_\alpha}\big(e^{i\frac{t_j}{\sigma_j}\bar N_{C_\alpha}}\big)$$

Then we write

$$\mu^\eta_{C_\alpha}\big(e^{i\frac{t_j}{\sigma_j}\bar N_{C_\alpha}}\big)=1-\frac{t_j^2}{2\sigma_j^2}\mu^\eta_{C_\alpha}(N_{C_\alpha},N_{C_\alpha})+R_\alpha$$

where

$$|R_\alpha|\le\frac{1}{6}\,\frac{|t_j|^3}{\sigma_j^3}\,\mu^\eta_{C_\alpha}(|\bar N_{C_\alpha}|^3)\le C\frac{|t_j|^3}{\sigma_j^3}A^d\,\mu^\eta_{C_\alpha}(N_{C_\alpha},N_{C_\alpha})$$

Notice that, thanks to *b)* and *c)* of proposition 3.1 and the fact that e.g. $\frac{|V_j|}{|\Lambda_j|}\ge\frac{1}{2}$,

$$\frac{\sum_\alpha\mu^\eta_{C_\alpha}\big(N_{C_\alpha},N_{C_\alpha}\big)}{\sigma_j^2}\le c \tag{3.19}$$

for a suitable constant c independent of η. Thus, in particular, for all $t_j \in [-A, A]$

$$\sum_\alpha |R_\alpha| \le C' \frac{|t_j|^3}{\sigma_j^3} A^d\, \sigma_j^2 \le C'' A^{-(1+k)}$$

because it is assumed that $\sigma_j \ge A^{d+4+k}$. Thus, for A large enough,

$$|\,\mu\Big(\prod_j F_j^\eta(t_j)\Big) - \mu\Big(\prod_j \prod_\alpha [1 - \frac{t_j^2}{2\sigma_j^2} \mu_{C_\alpha}^\eta (N_{C_\alpha}, N_{C_\alpha})]\Big)\,| \le 3 \sum_j \sum_\alpha |R_\alpha| \le 3C'' A^{-(1+k)}$$

In conclusion, for any A large enough, we have shown that

$$\begin{aligned} &\int_{-A}^{A} dt_1 \ldots \int_{-A}^{A} dt_k \,\mathrm{Re}\, \mu\Big(\prod_j F_j^\eta(t_j)\Big) \ge \\ &\qquad \ge \int_{-A}^{A} dt_1 \ldots \int_{-A}^{A} dt_k \,\mu\Big(\prod_{j,\alpha} [1 - \frac{t_j^2}{2\sigma_j^2} \mu_{C_\alpha}^\eta (N_{C_\alpha}, N_{C_\alpha})]\Big) \;- 6\,C'' A^{-1} \\ &\qquad \ge \int_{-A}^{A} dt_1 \ldots \int_{-A}^{A} dt_k \, e^{-c\sum_j t_j^2} \;- 6\,C'' A^{-1} \ge c_0 \end{aligned} \tag{3.20}$$

for some $c_0 > 0$. Above we have used the simple bound

$$\frac{t_j^2}{2\sigma_j^2} \mu_{C_\alpha}^\eta (N_{C_\alpha}, N_{C_\alpha}) \le C\rho |C_\alpha| A^{-2(d+4+k)+2} \le C' A^{-(d+6+2k)}$$

in order to estimate from below $1 - \frac{t_j^2}{2\sigma_j^2} \mu_{C_\alpha}^\eta (N_{C_\alpha}, N_{C_\alpha})$ in terms of a negative exponential.
In order to complete the proof of (3.17) we need to bound the integrals over $t_1, \ldots, t_k$ of the second term in the r.h.s. of (3.18). We first Taylor expand up to zero order each factor $e^{i\sum_j \frac{t_j}{\sigma_j} \bar N_{\Lambda \setminus V_j}(\eta)}$ and $e^{i\sum_j \frac{t_j}{\sigma_j} \mu_{V_j}^\eta (\bar N_{V_j})}$ to get

$$e^{i\sum_j \frac{t_j}{\sigma_j} \bar N_{\Lambda \setminus V_j}(\eta)} = 1 + \beta_1\,, \qquad |\beta_1| \le \sum_j \frac{|t_j|}{\sigma_j} |\bar N_{\Lambda_j \setminus V_j}(\eta)|$$

$$e^{i\sum_j \frac{t_j}{\sigma_j} \mu_{V_j}^\eta (\bar N_{V_j})} = 1 + \beta_2\,, \qquad |\beta_2| \le \sum_j \frac{|t_j|}{\sigma_j} |\mu_{V_j}^\eta (\bar N_{V_j})|$$

and then we write

$$|\mu\Big(\prod_j F_j^\eta(t_j) \big[e^{i\sum_j \frac{t_j}{\sigma_j} \bar N_{\Lambda \setminus V_j}(\eta)} e^{i\sum_j \frac{t_j}{\sigma_j} \mu_{V_j}^\eta \left(N_{V_j} - \mu(N_{V_j})\right)} - 1 \big]\Big)| \le e^{-c\sum_j \alpha_j t_j^2} \mu\big(|\beta_1| + |\beta_2|\big) \tag{3.21}$$

where we have used the gaussian upper bound proved in proposition 3.3 applied to each $F^{\eta}_j(t_j)$. The goal is now to show that both terms in the r.h.s. of (3.21), $\mu(|\beta_1|)$ and $\mu(|\beta_2|)$, are bounded from above by $\sum_j |t_j| A^{-\frac{1}{2}}$ for A large enough. For the first one we get

$$\begin{aligned}\mu(|\beta_1|) &\le \sum_j \frac{|t_j|}{\sigma_j}\mu(|\bar N_{\Lambda_j\setminus V_j}|)\\ &\le C\sum_j \frac{|t_j|}{\sigma_j}\left(\rho_j\,|\Lambda_j\setminus V_j|\right)^{\frac{1}{2}}\\ &\le C'\sum_j |t_j|\,A^{-1/2}\end{aligned}\tag{3.22}$$

because $\frac{|\Lambda_j\setminus V_j|}{V_j}\le A^{-1}$ and $\sigma_j^2\ge C''|V_j|\rho_j$. Similarily

$$\mu_{\Lambda_j}(|\beta_2|)\le \sum_j \frac{|t_j|}{\sigma_j}\mu_{\Lambda_j}\left(\mu^{\eta}_{V_j}(N_{V_j}),\mu^{\eta}_{V_j}(N_{V_j})\right)^{1/2}\le C\sum_j \frac{|t_j|}{\sigma_j}\left\{\,|\Lambda_j\setminus V_j|\right\}^{\frac{1}{2}}\le C\sum_j |t_j|\,A^{-\frac{1}{2}}\tag{3.23}$$

for a suitable constant C. Above we have used the Poincare' inequality for μ with "heat bath" jump rates (see e.g [M1] and references therein):

$$\mu_{\Lambda_j}(f,f)\le C_1\sum_{x\in\Lambda_j}\mu_{\Lambda_j}\left(\mu_{\{x\}}(f,f)\right)\tag{3.24}$$

with $f(\eta):=\mu^{\eta}_{V_j}(N_{V_j})$ and $C_1=C_1(\|\Phi\|,d,r,C,m)$ a suitable constant, together with the simple bounds

$$\mu^{\eta}_{\{x\}}(f,f)\le C'\|\nabla_x f\|^2_\infty\rho_j(1-\rho_j)$$

and

$$\|\nabla_x f\|_\infty\le\begin{cases}0 & \text{if } x\notin\Lambda_j\setminus V_j\\ C'' & \text{if } x\in\Lambda_j\setminus V_j\end{cases}\tag{3.25}$$

where the latter easily follows from our mixing assumption and the fact that the interaction is of finite range r. In conclusion, if we now combine (3.23) and (3.22) we have shown that

$$|\text{ second term in the r.h.s. of (3.21) }|\le C\sum_j |t_j|e^{-c\sum_j\alpha_j t_j^2}A^{-\frac{1}{2}}\tag{3.26}$$

Such a bound together with (3.20) implies that

$$\begin{aligned}\int_{-A}^{A}dt_1\ldots\int_{-A}^{A}dt_k\,\mathrm{Re}\,F(t_1,\ldots,t_k)&\ge c_0-C\,A^{-\frac{1}{2}}\int_{-A}^{A}dt_1\ldots\int_{-A}^{A}dt_k C\sum_j|t_j|e^{-c\sum_j\alpha_j t_j^2}\\ &\ge c_0-C\,A^{-\frac{1}{2}}\sum_j\alpha_j^{-\frac{1}{2}}\prod_i\alpha_i^{-\frac{1}{2}}\frac{c_0}{2}\end{aligned}$$

provided that A was chosen larger than $\max\{A_0,\prod\frac{1}{\alpha_i}\sum_j\frac{1}{\alpha_j}\}$ for a suitable constant A_0 independent of the α_j's and of the densities ρ_j. □

Remark. Exactly the same result hold for any choice of the atoms $\Lambda_{i_1},\ldots,\Lambda_{i_j}$, $j\le k$.

4. Finite volume equivalence between canonical and grand canonical Gibbs measures

In this section we assume that the apriori measures $\mu_0^{\{x\}}$ are all equal to a given positive probability measure μ_0 on the state space S. We then fix a positive number α and take the generalized $h-$charge $h : S \mapsto [0,1]$ such that

Assumption H(α)**.**
i) The values of h are of the form λj, j a non negative integer equal to $0, 1, j_2, \dots, j_{max}$, with $\lambda = j_{max}^{-1}$.
ii) $\mu_0(h) \le \frac{1}{2}$.
iii) $\mathrm{Var}_0\big(\cos(th(\eta))\big) + \mathrm{Var}_0\big(\sin(th(\eta))\big) \ge \alpha\mu_0(h)t^2$ *for all* $t \in [-\frac{\pi}{\lambda}, \frac{\pi}{\lambda}]$.
iv) $\mu_0(h = \lambda) \ge \alpha\mu_0(h)$.

Remark. As already remarked the above assumption is satisfied for lattice gases, $S = \{0, 1\}$ and $h(\eta) = \eta$.

Next we fix a l–regular set Λ, a possibile value N of the h–charge in Λ, a boundary condition τ outside Λ and we assume that the grand canonical Gibbs measure $\mu := \mu^\tau_\Lambda$ is such that $\mu(N_\Lambda(\eta)) = N$. We also set $\nu(\cdot) := \mu\big(\cdot \,|\, N_\Lambda(\eta) = N\big)$ and $\rho = \frac{N}{|\Lambda|}$. An elementary estimate together with *ii)* of assumption $H(\alpha)$ shows that necessarily $\rho \le 1 - \frac{\lambda}{2}e^{-2\|\Phi\|}$. Then we prove the following

Theorem 4.1. *Assume condition $SMT(V, C, m, l)$ for all sets $V \in \mathbb{F}_l$. Then, for any l large enough independent of ρ and $\epsilon \in (0, 1)$, there exist constants $C' = C'(C, m, \|\Phi\|, \alpha, l, \epsilon)$ and $v = v(C, m, \|\Phi\|, \alpha, l, \epsilon)$ such that for all l–regular sets Λ with $|\Lambda| \ge v$ and all local functions f with l–support Δ satisfying $|\Delta| \le |\Lambda|^{1-4\epsilon}$*

$$|\nu(f) - \mu(f)| \le C'\|f\|_\infty \frac{|\Delta|}{|\Lambda|}$$

Proof. Let χ_N be the characteristic function of the event $N_\Lambda(\eta) = N$. Then we write

$$\nu(f) - \mu(f) = \frac{\mu(\chi_N, f)}{\mu(N_\Lambda = N)} \tag{4.1}$$

Next, using Fourier transform, we express χ_N as

$$\chi_N(\eta) = \frac{1}{2T}\int_{-T}^{T} dt\, e^{i\frac{t}{\sigma}(N_\Lambda(\eta) - N)} \tag{4.2}$$

where $T = \frac{\pi\sigma}{\lambda}$, so that (4.1) becomes

$$\nu(f) - \mu(f) = \frac{\int_{-T}^{T} dt\, \mu\big(e^{i\frac{t}{\sigma}(N_\Lambda(\eta) - N)}, f\big)}{\int_{-T}^{T} dt\, F(t)} \tag{4.3}$$

where $\sigma = \mu^\tau_\Lambda(N_\Lambda, N_\Lambda)$ and $F(t) = \mu\big(e^{i\frac{t}{\sigma}(N_\Lambda(\eta) - N)}\big)$.
As a first step we show that the denominator is bounded away from zero uniformly in the density ρ and in the volume Λ.

Lemma 4.2. *In the same assumptions of theorem 4.1 there exist constants* $C'' = C''(C, m, \|\Phi\|, \alpha, l)$ *and* $v = v(C, m, \|\Phi\|, \alpha, l, \epsilon$ *such that for all* $l-$*regular sets* Λ *with* $|\Lambda| \geq v$

$$\int_{-T}^{T} dt\, F(t) \geq C''$$

Proof. We fix a number $A \geq \max\{A_0, \alpha^{-2}\}$ where A_0 is the constant appearing in proposition 3.4 and we distinguish between two cases:
(a). $\sigma \geq A^{d+5}$
(b). $\sigma \leq A^{d+5}$
In case (a) we split the integral into two regions as follows

$$\int_{-\frac{\pi\sigma}{\lambda}}^{\frac{\pi\sigma}{\lambda}} dt\, F(t) = \int_{-\frac{\pi\sigma}{\lambda}}^{\frac{\pi\sigma}{\lambda}} dt\, Re\, F(t) = \int_{-A}^{A} dt\, Re\, F(t) + \int_{A \leq |t| \leq \frac{\pi\sigma}{\lambda}} dt\, Re\, F(t) \tag{4.4}$$

The first integral is larger than some fixed constant $c(\alpha, \|\Phi\|, d, r)$ because of proposition 3.4 while the absolute value of the second one is smaller than $e^{-c' A^2}$ because of the gaussian bound on $|F(t)|$ given in proposition 3.3. Thus the whole integral is larger than e.g. $c(\alpha, \|\Phi\|, d, r)/2$ if A is large enough.
It is important to observe that in this case we did not use the rather special assumption that the first non zero value of the function h is λ (see assumption $H(\alpha)$).
In case (b), namely extremely low particle density ρ if $|\Lambda|^{\frac{1}{2}}$ is large compared to A^{d+5} since $A^{d+5} \geq \sigma \geq c\sqrt{N}$ because of (c) of proposition 3.1, we first bound from below the integral by

$$\int_{-\frac{\pi\sigma}{\lambda}}^{\frac{\pi\sigma}{\lambda}} dt\, F(t) = \frac{2\pi\sigma}{\lambda} \mu(N_\Lambda = N) \geq c' \mu(N_\Lambda = N)$$

because of assumption $H(\alpha)$. Next we estimate $\mu(N_\Lambda = N)$ as follows. Let η^* be the unique element of S such that $h(\eta^*) = 0$. Then we write

$$\mu(N_\Lambda = N) \geq \sum_{\substack{V \subset \Lambda \\ |V| = n}} \mu\Big(h_x = \lambda\ \forall x \in V\, ;\, \eta(y) = \eta^*\ \forall\, y \notin V\Big) \tag{4.5}$$

where $n = \frac{N}{\lambda}$. Then, using DLR, each term in the r.h.s. of (4.5) can be bounded from below by

$$\begin{aligned} e^{-2n\|\Phi\|} \mu_0(h(\eta) = \lambda)^n\, (1 - e^{2\|\Phi\|} \mu_0(h \neq 0))^{|\Lambda| - n} &\geq C(N) \mu_0(h)^n (1 - \lambda^{-1} e^{2\|\Phi\|} \mu_0(h))^{|\Lambda| - n} \\ &\geq C'(N) p^n (1 - p)^{|\Lambda| - n} \end{aligned}$$

where $p = \lambda^{-1} e^{2\|\Phi\|} \mu_0(h) \leq \lambda^{-1} e^{4\|\Phi\|} \rho \ll 1$ if $|\Lambda|$ is large enough. Thus the r.h.s of (4.5) is bounded from below by a constant depending on N, $\|\Phi\|$ and the result follows. $\square$

Next we analyze the numerator of (4.3) and we begin by proving an upper bound of the right order for a rather special class of local functions that, roughly speaking, have almost zero covariance

with the h-charge N_Λ. Later on, using the conservation law $N_\Lambda = N$, we will extend our result to general functions.
Let for simplicity $\Lambda = \cup_{i=1}^n Q_l(x^i)$, $x^i \in l\mathbb{Z}^d$, let g be a local function and V_g be its l–support inside Λ (see definition of property SMT). Let also $V_g' = \cup_{i\in I} Q_l(x^i)$, $I \subset \{1,\dots,n\}$ be an arbitrary collection of those l–cubes that form Λ such that $|V_g'| = |V_g|$. We then set $\Delta = V_g \cup V_g'$. Given $\epsilon \ll 1$ we next define, for any positive large M,

$$\bar\Delta = \begin{cases} \{x \in \Lambda :\ d(x,\Delta) \le M \log|\Lambda|\,\} & \text{if } \rho \ge |\Lambda|^{-\epsilon} \\ \{x \in \Lambda :\ d(x,\Delta) \le M\,\} & \text{if } \rho \le |\Lambda|^{-\epsilon} \end{cases}$$

and $f = g - \alpha_g N_{V_g'}$ with

$$\alpha_g = \frac{\mu(g, N_{\bar\Delta})}{\mu(N_{V_g'}, N_{\bar\Delta})}$$

Notice that α_g is well defined, namely the denominator is positive, because of *c)* and *d)* of corollary 3.2, and moreover it satisfies the bound

$$|\alpha_g| \le c\|g\|_\infty \frac{\min\{\sqrt{\rho|\Delta|}, \rho|\Delta|\}}{\rho|\Delta|} \tag{4.6}$$

for a suitable constant c again because of corollary 3.2, provided that l is large enough independent of Λ. Then we have the following

Lemma 4.3. *In the same assumptions of theorem 4.1 there exists a constant $c = c(C, m, \|\Phi\|, \alpha, l)$ such that*

$$\int_{-T}^{T} dt\, |\, \mu\big(e^{i\frac{t}{\sigma}(N_\Lambda(\eta)-N)}, f\big)\, | \le c\|g\|_\infty \frac{|\Delta|}{|\Lambda|}$$

Proof. Let $A = \partial^+\bar\Delta$ and $B = \Lambda \setminus (\bar\Delta \cup A)$ and let

$$\begin{aligned} G_t(\eta) &:= \mu_B^\eta\big(e^{i\frac{t}{\sigma}N_B}\big) \\ H_t(\eta) &:= e^{i\frac{t}{\sigma}N_A(\eta)} \\ K_t(\eta) &:= e^{-i\frac{t}{\sigma}\mu_{\bar\Delta}^\eta(N_{\bar\Delta})} \end{aligned}$$

Then, thanks to the Markov property, we can write

$$\begin{aligned} \mu(f, e^{i\frac{t}{\sigma}N_\Lambda}) &= \mu\big(H_t\, G_t\, K_t\, \mu_{\bar\Delta}^\eta(f, e^{i\frac{t}{\sigma}\bar N_{\bar\Delta}})\big) + \mu\Big(H_t\, G_t\, K_t\, \mu_{\bar\Delta}^\eta\big(e^{i\frac{t}{\sigma}\bar N_{\bar\Delta}}\big)[\mu_{\bar\Delta}^\eta(f) - \mu(f)]\Big) \\ &= \mu\big(H_t\, G_t\, K_t\, \mu_{\bar\Delta}^\eta(f, e^{i\frac{t}{\sigma}\bar N_{\bar\Delta}})\big) + R_t \end{aligned}$$

where the bar on top of the h–charge means that we have substracted the mean w.r.t $\mu_{\bar\Delta}^\eta$ and

$$|R_t| \le \sup_{\eta,\eta'} |\mu_{\bar\Delta}^{\eta'}(f) - \mu_{\bar\Delta}^\eta(f)| \tag{4.7}$$

Thus we get that

$$|\mu(f, e^{i\frac{t}{\sigma}N_\Lambda})| \le \|G_t\|_\infty \left\{ \|\mu^\eta_{\bar\Delta}(f, e^{i\frac{t}{\sigma}\bar N_{\bar\Delta}}))\|_\infty + C\|f\|_\infty |R_t| \right\}$$

Let us examine the factor $\|G_t\|_\infty$. Thanks to proposition 3.3 and the fact that $|\bar\Delta| \ll \frac{1}{2}|\Lambda|$ we have a gaussian bound

$$\|G_t\|_\infty \le e^{-ct^2} \tag{4.8}$$

for some constant $c = c(\alpha, \lambda, \|\Phi\|)$.
As far as the term R_t is concerned we have the bound

$$|R_t| \le \|g\|_\infty \, |\Lambda|^{-2} \tag{4.9}$$

provided that the constant M appearing in the definition of $\bar\Delta$ was chosen large enough. If the density ρ is larger than $|\Lambda|^{-\epsilon}$ (4.9) follows immediately from the strong mixing assumption and the definition of $\bar\Delta$. If instead the density ρ is smaller than $|\Lambda|^{-\epsilon}$ then $\sup_\tau \sup_x \mu^\tau_{\{x\}}(\eta(x) \neq \eta^*) \le e^{2\|\Phi\|}\rho \ll 1$, where η^* is the unique element of S such that $h(\eta^*) = 0$. Thus one can apply Dobrushin's uniqueness criterium to get that the rate of decay of covariances, the constant m in the definition of property $SMT(\Lambda, C, m, l)$, is proportional to $-\log\rho$. The bound (4.9) then follows immediately.
We can now bound from above the key term $\|\mu^\eta_{\bar\Delta}(f, e^{i\frac{t}{\sigma}\bar N_{\bar\Delta}})\|_\infty$. We distinguish two cases: $\rho \ge |\Lambda|^{-\epsilon}$ or $\rho \le |\Lambda|^{-\epsilon}$.

Case $\rho \ge |\Lambda|^{-\epsilon}$. Here we use the Taylor expansion up to second order

$$e^{ix} = 1 + ix - \frac{x^2}{2} + \gamma(x)$$

with $|\gamma(x)| \le |x|^3$, to write (remember that $f = g - \alpha_g N_{V'_g}$)

$$\begin{aligned} \|\mu^\eta_{\bar\Delta}(f, e^{i\frac{t}{\sigma}\bar N_{\bar\Delta}})\|_\infty \le{}& \frac{|t|}{\sigma}\big|\mu^\eta_{\bar\Delta}(g, N_{\bar\Delta}) - \alpha_g \mu^\eta_{\bar\Delta}(N_{V'_g}, N_{\bar\Delta})\big| + \\ & \frac{t^2}{2\sigma^2}\Big[\|\mu^\eta_{\bar\Delta}(g, \bar N^2_{\bar\Delta})\|_\infty + |\alpha_g|\,\|\mu^\eta_{\bar\Delta}(\bar N_{V'_g}, \bar N^2_{\bar\Delta})\|_\infty\Big] + \delta_3 \end{aligned} \tag{4.10}$$

where

$$\delta_3 = \frac{|t|^3}{\sigma^3}\Big[\,\|g\|_\infty\,\|\mu^\eta_{\bar\Delta}\big(|\bar N_{\bar\Delta}|\big)^3\,)\|_\infty + |\alpha_g|\,\|\mu_{\bar\Delta}\big(|\bar N_{V'_g}||\bar N_{\bar\Delta}|\big)^3\,)\|_\infty$$

In order to bound from above the linear term in (4.10) we observe that, thanks to property SMT, for any $x \in \bar\Delta$

$$|\mu^\tau_{\bar\Delta}(g, h_x) - \mu(g, h_x)| \le \begin{cases} \|g\|_\infty\,\|h\|_\infty C e^{-md(x,\Delta)} & \text{if } d(x,\Delta) \ge \frac{1}{2}M\log|\Lambda| \\ \|g\|_\infty\,\|h\|_\infty |\bar\Delta| C e^{-\frac{m}{2}M\log|\Lambda|} & \text{if } d(x,\Delta) \le \frac{1}{2}M\log|\Lambda| \end{cases}$$

so that

$$|\mu^\tau_{\bar\Delta}(g, h_x) - \mu(g, h_x)| \le c\|g\|_\infty\,|\Lambda|^{-3} \tag{4.11}$$

provided that the constant M was chosen large enough. An analogous result holds when $\rho \le |\Lambda|^{-\epsilon}$ (see the proof of (4.9)). The above rough bound is enough to conclude, by the very definition of the constant α_g, that the linear term in (4.10) is smaller than

$$\frac{|t|}{\sigma}\left|\mu^{\eta}_{\bar\Delta}(g,N_{\bar\Delta})-\alpha_g\mu^{\eta}_{\bar\Delta}(N_{V'_g},N_{\bar\Delta})\right| \le \frac{|t|}{\sigma}c\|g\|_\infty\,|\Lambda|^{-1}\le tc'\|g\|_\infty\,|\Lambda|^{-1} \tag{4.12)}$$

because $\sigma^2 \ge c''\rho|\Lambda| = c''N$.
If we use *c), g)* and *h)* of corollary 3.2 together with the bound (4.6) we immediately conclude that the second order term in (4.10) gives a contribution smaller than

$$\frac{t^2}{\sigma^2}\,c\|g\|_\infty\,\rho|\Delta| \le t^2c'\|g\|_\infty\,\frac{|\Delta|}{|\Lambda|} \tag{4.13}$$

Finally the error term δ_3, thanks again to corollary 3.2 together with the Schwartz inequality, (4.6) and the assumption $\rho \ge |\Lambda|^{-\epsilon}$, is smaller than

$$\begin{aligned}\delta_3 &\le \frac{|t|^3}{|\Lambda|^{3/2(1-\epsilon)}}c\mu^{\eta}_{\bar\Delta}(|\bar N_{\bar\Delta}|^4)^{3/4}\Big(\|g\|_\infty+|\alpha_g|\mu^{\eta}_{\bar\Delta}(|\bar N_{V'_g}|^4)^{1/4}\Big)\\ &\le c'\frac{|t|^3}{|\Lambda|^{3/2(1-\epsilon)}}\|g\|_\infty\,\max\{\rho|\bar\Delta|,(\rho|\bar\Delta|)^2\}^{3/4}\\ &\le c''\frac{|t|^3}{|\Lambda|^{3/2(1-\epsilon)}}\|g\|_\infty\,|\bar\Delta|^{3/2}\end{aligned} \tag{4.14}$$

By putting together (4.12), (4.13) and (4.14) we conclude that for not too small values of the density

$$\|\mu^{\eta}_{\bar\Delta}(f,e^{i\frac{t}{\sigma}\bar N_{\bar\Delta}})\|_\infty \le c(|t|+t^2+|t|^3)\|g\|_\infty\,\frac{|\Delta|}{|\Lambda|} \tag{4.15}$$

for a suitable constant c.

Case $\rho \le |\Lambda|^{-\epsilon}$. Here we Taylor expand up to first order

$$e^{ix} = 1 + ix + \gamma(x)$$

with $|\gamma(x)| \le |x|^2$, to write

$$\begin{aligned}\|\mu^{\eta}_{\bar\Delta}(f,e^{i\frac{t}{\sigma}\bar N_{\bar\Delta}})\|_\infty &\le \\ &\le \frac{|t|}{\sigma}\left|\mu^{\eta}_{\bar\Delta}(g,N_{\bar\Delta})-\alpha_g\mu^{\eta}_{\bar\Delta}(N_{V'_g},N_{\bar\Delta})\right|+\delta_2\end{aligned}$$

with

$$\begin{aligned}\delta_2 &\le \frac{t^2}{2\sigma^2}\Big[\|g\|_\infty\,\|\mu^{\eta}_{\bar\Delta}(\bar N^2_{\bar\Delta})\|_\infty+|\alpha_g|\,\|\mu^{\eta}_{\bar\Delta}(|\bar N_{V'_g}|\bar N^2_{\bar\Delta})\|_\infty\Big]\\ &\le t^2\Big\{\,c\|g\|_\infty\,\frac{|\Delta|}{|\Lambda|}+|\alpha_g|\,\|\mu^{\eta}_{\bar\Delta}(\bar N^4_{\bar\Delta})\|^{1/2}_\infty\|\mu^{\eta}_{\bar\Delta}(\bar N^2_{V'_g})\|^{1/2}_\infty\,\Big\}\\ &\le t^2c'\|g\|_\infty\,\frac{|\Delta|}{|\Lambda|}\end{aligned} \tag{4.16}$$

because of corollary 3.2, (4.6) and the fact that now $|\bar{\Delta}|$ is proportional to $|\Delta|$. Since the linear term has already been estimated in (4.12) we conclude that also for extremely low density

$$\|\mu^{\eta}_{\bar{\Delta}}(f, e^{i\frac{t}{\sigma}\bar{N}_{\bar{\Delta}}})\|_\infty \leq c(|t| + t^2)\|g\|_\infty \frac{|\Delta|}{|\Lambda|} \tag{4.17}$$

for a suitable constant c.
The bounds (4.8), (4.9), (4.15) and (4.17) finally prove the lemma. $\square$

We are now in a position to complete the proof of the theorem.
Using lemma 4.2 and lemma 4.3 it follows that for local functions f of the form $f = g - \alpha_g N_{V'_g}$ with g an arbitrary local function with l–support inside Λ equal to Δ,

$$|\nu(f) - \mu(f)| \leq C\|g\|_\infty \frac{|\Delta|}{|\Lambda|}$$

which implies, using (4.6),

$$|\nu(g) - \mu(g)| \leq C'\|g\|_\infty \Big(\frac{|\Delta|}{|\Lambda|} + |\nu(N_{V'_g}) - \mu(N_{V'_g})| \Big)$$

To complete the proof one has to prove the result for the special functions $N_i := N_{Q_l(x^i)}$, $x^i \in l\mathbb{Z}^d \cap \Lambda$.
Let $\Delta_{ij} = Q_l(x^i) \cup Q_l(x^j)$ and consider the function $N_i - \alpha_{ij}N_j$ where $\alpha_{ij} = \frac{\mu(N_{\bar{\Delta}_{ij}}, N_i)}{\mu(N_{\bar{\Delta}_{ij}}, N_j)}$. Notice that, thanks to corollary 3.2, there exists a positive constant α_0 independent of the density such that

$$\alpha_0 \leq \alpha_{ij}$$

Then, thanks to lemma 4.2 and lemma 4.3, we know that

$$\nu(N_i) - \mu(N_i) - \alpha_{ij}[\nu(N_j) - \mu(N_j)] = \epsilon_{ij}$$

with $\sup_{i,j} |\epsilon_{ij}| \leq \frac{C(l)}{|\Lambda|}$. Because of the conservation law $\sum_i [\nu(N_i) - \mu(N_i)] = 0$, we get that

$$-\Big\{\sum_i \alpha_{ij}\Big\}[\nu(N_j) - \mu(N_j)] = \sum_i \epsilon_{ij}$$

i.e.

$$[\nu(N_j) - \mu(N_j)] = -\Big\{\sum_i \epsilon_{ij}\Big\}\Big\{\sum_i \alpha_{ij}\Big\}^{-1}$$

Thus

$$|\nu(N_j) - \mu(N_j)| \leq \frac{C}{|\Lambda|}$$

for a suitable constant C and the theorem is proved. $\square$

Remark. As it should be clear from the proof, the condition $|\Delta| \le |\Lambda|^{1-4\epsilon}$ can be relaxed to $|\bar\Delta| = o(|\Lambda|)$ if $\rho \ge \rho_0 > 0$ uniformly in $|\Lambda|$. In particular, for "normal" values of the density, the result of the theorem is valid for all cubic regions Δ with $|\Delta| = o(|\Lambda|)$.

Remark. Exactly the same result can be obtained if instead of comparing the finite volume canonical Gibbs measure with its grand canonical counterpart, we take the marginal on $\mathcal{F}_\Lambda$ of the infinite volume Gibbs state conditioned to $N_\Lambda = N$ and compare it to the marginal on $\mathcal{F}_\Lambda$ of the unconditioned one. That can be useful in some application (see e.g. [CZ]).

5. Equivalence between multi–canonical and multi–grand canonical Gibbs measures.

In this section we extend the result of theorem 4.1 to the case in which the reference measure μ_0 is different in the atoms $\Lambda_1, \dots, \Lambda_k$ of a partition of an l–regular set Λ. More precisely our setting is as follows.

We fix $\alpha, \delta, \epsilon \in (0,1)$, a function $h : S \mapsto [0,1]$ and a set $\Lambda \in \mathbb{F}_l$, $\Lambda = \cup_{i \in I} Q_l(x^i + x)$ such that $|\partial^-_{l-1}\Lambda| \le |\Lambda|^{\frac{d-1+\epsilon}{d}}$. Let $I_1 \dots I_k$ be a partition of I such that, if $\Lambda_i = \cup_{j \in I_i} Q_l(x^j + x)$, then $0 < \delta \le \frac{|\Lambda_i|}{|\Lambda|}$ and $|\partial^-_{l-1}\Lambda_i| \le |\Lambda|^{\frac{d-1+\epsilon}{d}}$ for all $i = 1, \dots, k$.

We assume that the apriori probability measure $\mu_0^{\{x\}}$ is the same in each atom of the partition and equal to $\mu_0^{(i)}$. Let then $\mathbf{N}_\Lambda = \{N_{\Lambda_i}\}_{i=1}^k$, let $\mathbf{N} = \{N_i\}_{i=1}^k$ be a set of possible values of the h–charge in each atom Λ_i and let $\rho_i = \frac{N_i}{|\Lambda_i|}$. Without loss of generality we assume $\rho_i \le \frac{1}{2}$, $i = 1, \dots, k$. Given a boundary condition τ, we also assume that $\mu^\tau_\Lambda(\mathbf{N}_\Lambda) = \mathbf{N}$. We then set $\mu := \mu^\tau_\Lambda$ and $\nu := \mu^\tau_\Lambda(\cdot | \mathbf{N}_\Lambda = \mathbf{N})$. Next, given $M > 0$ and $\Delta \subset \Lambda$, we say that Δ is good if either

a) Δ together with its $M \log |\Lambda|$ neighborhood is entirely contained in some atom Λ_i with $\rho_i \ge |\Lambda|^{-\epsilon}$

or

b) Δ together with its M neighborhood is entirely contained in some atom Λ_i with $\rho_i \le |\Lambda|^{-\epsilon}$.

A set is bad if it is not good. For good sets $\Delta \subset \Lambda_i$, $i = 1, \dots, k$, we define

$$\bar\Delta = \begin{cases} \{x \in \Lambda : \ d(x, \Delta) \le M \log |\Lambda| \} & \text{if } \rho_i \ge |\Lambda|^{-\epsilon} \\ \{x \in \Lambda : \ d(x, \Delta) \le M \} & \text{if } \rho_i \le |\Lambda|^{-\epsilon} \end{cases}$$

while for bad sets Δ

$$\bar\Delta = \{x \in \Lambda : \ d(x, \Delta) \le M \log |\Lambda| \}$$

With these notation our result reads as follows

Theorem 5.1. *Assume condition $SMT(V, C, m, l)$ for all sets $V \in \mathbb{F}_l$. Assume condition $H(\alpha)$ for each single spin measures $\mu_0^{(i)}$, $i = 1, \dots, k$. Then, for any l, M large enough and ϵ small enough independent of $\{\rho_i\}_{i=1}^k$, there exist constants $C' = C'(C, m, \|\Phi\|, \alpha, l, \delta, M, \epsilon)$, $v = v(C, m, \|\Phi\|, \alpha, l, M, \delta, \epsilon)$ such that if $|\Lambda| \ge v$ then, for all local functions f with l–support Δ satisfying $|\Delta| \le |\Lambda|^{1-4\epsilon}$*

$$|\nu(f) - \mu(f)| \le \begin{cases} C' \|f\|_\infty \frac{|\Delta|}{|\Lambda|} & \textit{if } \Delta \textit{ is good} \\ C' \|f\|_\infty \frac{|\Delta|}{|\Lambda|} \left(\frac{|\bar\Delta|}{|\Delta|} \right)^2 & \textit{if } \Delta \textit{ is bad} \end{cases}$$

Remark. It is worthwhile to notice that if either all densities are bounded away from zero uniformly in the volume Λ or if they are all smaller than $|\Lambda|^{-\epsilon}$ then the error would be always $\frac{|\Delta|}{|\Lambda|}$ for good or bad sets.

Proof. We proceed as in the proof of theorem 4.1. Let χ_i be the characteristic function of the event $N_{\Lambda_i}(\eta) = N_i$. Then we write

$$\nu(f) - \mu(f) = \frac{\mu(\prod_i \chi_i, f)}{\mu(\prod_i \chi_i)} \tag{5.1}$$

Next, using Fourier transform, we express χ_i as

$$\chi_i(\eta) = \frac{1}{2T_i} \int_{-T_i}^{T_i} dt\, e^{i\frac{t}{\sigma_i}(N_{\Lambda_i}(\eta) - N_i)} \tag{5.2}$$

where $T_i = \frac{\pi\sigma_i}{\lambda}$ and $\sigma_i = \mu_\Lambda^\tau(N_{\Lambda_i}, N_{\Lambda_i})$. Thus (5.1) becomes

$$\nu(f) - \mu(f) = \frac{\int_{-T_1}^{T_1} dt_1 \ldots \int_{-T_k}^{T_k} dt_k\, \mu\big(e^{i\sum_i \frac{t_i}{\sigma_i}(N_{\Lambda_i}(\eta) - N_i)}, f\big)}{\int_{-T_1}^{T_1} dt_1 \ldots \int_{-T_k}^{T_k} dt_k\; F(t_1, \ldots, t_k)} \tag{5.3}$$

where $F(t_1, \ldots, t_k) = \mu\big(e^{i\sum_i \frac{t_i}{\sigma_i}(N_{\Lambda_i}(\eta) - N_i)}\big)$.
We begin by proving the analogous of lemma 4.2.

Lemma 5.2. *In the same assumptions of theorem 5.1 there exists a constant $C'' = C''(C, m, \|\Phi\|, \alpha, l)$ such that*

$$\int_{-T_1}^{T_1} dt_1 \, \ldots \int_{-T_k}^{T_k} dt_k\, F(t_1, \ldots, t_k) \geq C''$$

Proof. We proceed as in lemma 4.2. We fix a number $A \geq \max\{A_0, \alpha^{-2}\}$ where A_0 is the constant appearing in proposition 3.4 and we assume, for simplicity that $\sigma_i \geq A^{d+4+k}$ for $i = 1, \ldots, j$ while $\sigma_i \leq A^{d+4+k}$ for $i = j+1, \ldots, k$, $j \leq k$. Notice that $N_i^{1/2} \leq cA^{d+4+k}$, $i = j+1, \ldots k$ because of *(c)* of proposition 3.1. Then, remembering the identity (4.2) and using the DLR equations, we can bound from below the multiple integral in question by

$$\begin{aligned}
&\int_{-T_1}^{T_1} dt_1 \, \ldots \int_{-T_k}^{T_k} dt_k\, F(t_1, \ldots, t_k) \geq \\
&\int_{-T_1}^{T_1} dt_1 \, \ldots \int_{-T_j}^{T_j} dt_j\, F(t_1, \ldots, t_j) \min_\tau \prod_{i=j+1}^{k} 2T_i\, \mu^\tau_{\Lambda_{j+1}\cup\ldots\Lambda_k}(N_{\Lambda_{j+1}} = N_{j+1}, \ldots, N_{\Lambda_k} = N_k)
\end{aligned} \tag{5.4}$$

The factor $\min_\tau \prod_{i=j+1}^k 2T_i \mu^\tau_{\Lambda_{j+1}\cup\ldots\Lambda_k}(N_{\Lambda_{j+1}} = N_{j+1}, \ldots, N_{\Lambda_k} = N_k)$ can be bounded from below by $C = C(N_{j+1}, \ldots, N_k)$ exactly as in the second part of the proof of lemma 4.2. The remaining multiple integral in (5.4) is bounded from below as in the first part of the proof of lemma 4.2 by a constant independent of the volumes $\Lambda_1, \ldots \Lambda_j$ thanks to propositions 3.3, 3.4. $\square$

Next we turn to the analysis of the numerator of (5.3) and, as in the proof of theorem 4.1, we begin by proving an upper bound of the right order for a rather special class of local functions.
Let g be a local function with l–support inside Λ equal to V_g with $|V_g| \le |\Lambda|^{1-4\epsilon}$. We have to distinguish between two different cases.

Case 1. The set V_g is a good subset of Λ_i .
In this case, following the proof of theorem 4.1, we define

$$f = g - \alpha_g N_{V_g'} \quad , \quad \alpha_g = \frac{\mu(g, N_{\bar\Delta})}{\mu(N_{V_g'}, N_{\bar\Delta})} \tag{5.5}$$

where $\Delta := \Delta_g = V_g \cup V_g'$ and V_g' is an arbitrary sub–collection of the l–cubes that form Λ_i such that V_g' is also good and $|V_g'| = |V_g|$. Since, by assumption, $|\bar\Delta| \ll |\Lambda_i|$, we can then repeat step by step the proof of lemma 4.3 to get that

$$\int_{-T_1}^{T_1} dt_1 \ldots \int_{-T_k}^{T_k} dt_k \, |\mu\big(e^{i\sum_i \frac{t_i}{\sigma_i}(N_{\Lambda_i}(\eta)-N_i)}, f\big)| \le c\|g\|_\infty \frac{|\Delta|}{|\Lambda_i|} \le c\delta^{-1}\|g\|_\infty \frac{|\Delta|}{|\Lambda|} \tag{5.6}$$

Case 2. The set V_g is bad.
Let $V_{g,i}'$ be an arbitrary sub–collection of the l–cubes that form Λ_i such that $V_{g,i}'$ is good and $|V_{g,i}'| = |V_g|$. Notice that such a set always exists (actually there are plenty of them) if $|\Lambda|$ is large enough because of the regularity assumption $|\partial_{l-1}^- \Lambda_i| \le |\Lambda|^{\frac{d-1+\epsilon}{d}}$ for all $i = 1, \ldots, k$ and the bound $|V_g| \le |\Lambda|^{1-4\epsilon}$. Then we set $\Delta := \Delta_g = \cup_i \{V_{g,i}' \cup V_g\}$, $\bar\Delta_i = \bar\Delta \cap \Lambda_i$ and

$$f = g - \sum_i \alpha_i N_{V_{g,i}'} \quad , \quad \alpha_i = \frac{\mu(g, N_{\bar\Delta_i})}{\mu(N_{V_{g,i}'}, N_{\bar\Delta_i})} \qquad i = 1, \ldots, k \tag{5.7}$$

Notice that, as in the previous section, the constants α_i are well defined and, thanks to proposition 3.1 and its corollary, satisfy

$$|\alpha_i| \le c\|g\|_\infty \frac{|\bar\Delta_i|}{|V_{g,i}'|} \min\Big\{1, \frac{1}{(\rho_i|\bar\Delta_i|)^{\frac12}}\Big\} \qquad i = 1, \ldots, k \tag{5.8}$$

if l was taken large enough.
We then proceed as in *case 1* but now we Taylor expand the relevant term $\mu_{\bar\Delta}^\tau\big(f, \exp(i\sum_{j=1}^k \frac{t_j}{\sigma_j} N_{\bar\Delta_j})\big)$ only up to first order in $\sum_{j=1}^k \frac{t_j}{\sigma_j} N_{\bar\Delta_j}$ as in the case of very low density of theorem 4.1, irrespectively of the values of the densities $\rho_1, \ldots, \rho_k$. The absolute value of the linear term can be bounded as follows

$$\begin{aligned}
&\sum_j \frac{|t_j|}{\sigma_j} |\mu_{\bar\Delta}^\tau\big(g, N_{\bar\Delta_j}\big)\big) - \sum_i \alpha_i \mu_{\bar\Delta}^\tau\big(N_{V_{g,i}'}, N_{\bar\Delta_j}\big)\big)| \le \\
&\sum_j \frac{|t_j|}{\sigma_j} |\mu_{\bar\Delta}^\tau\big(g, N_{\bar\Delta_j}\big)\big) - \alpha_j \mu_{\bar\Delta}^\tau\big(N_{V_{g,i}'}, N_{\bar\Delta_j}\big)\big)| + \sum_j \frac{|t_j|}{\sigma_j} \sum_{i\ne j} |\alpha_i||\mu_{\bar\Delta}^\tau\big(N_{V_{g,i}'}, N_{\bar\Delta_j}\big)\big)|
\end{aligned} \tag{5.9}$$

The first sum in the r.h.s. of (5.9) can be made smaller than $\|g\|_\infty |\Lambda|^{-1}$ by taking M large enough, because of our definition of the constants α_j, $j=1,\dots,k$, and the strong mixing property exactly as in section 3. The second double sum is also smaller than $\|g\|_\infty |\Lambda|^{-1}$ if M is large enough because $d(V'_{g,i}, \bar\Delta_j) \ge M \log|\Lambda|$ for $i \neq j$ since $V'_{g,i}$ is good.
The error term in the Taylor expansion is bounded by

$$\sup_\tau \sum_j \frac{t_j^2}{\sigma_j^2}\Big(\mu^\tau_{\bar\Delta}\big(|g|\bar N^2_{\bar\Delta_j}\big) + \sum_i |\alpha_i||\mu^\tau_{\bar\Delta}\big(|\bar N_{V'_{g,i}}|\bar N^2_{\bar\Delta_j}\big)\Big)$$

where the bar over the h–charge means that we have subtracted the mean w.r.t. $\mu^\tau_{\bar\Delta}(\cdot)$. Each term $\frac{1}{\sigma_j^2}\,\mu^\tau_{\bar\Delta}\big(|g|\bar N^2_{\bar\Delta_j}\big)$ is bounded from above by $c\|g\|_\infty\,\frac{|\bar\Delta_j|}{|\Lambda_j|}$ for a suitable constant c while the term $\frac{1}{\sigma_j^2}\,\sum_i |\alpha_i||\mu^\tau_{\bar\Delta}\big(|\bar N_{V'_{g,i}}|\,\bar N^2_{\bar\Delta_j}\big)$ is bounded from above by

$$\frac{1}{\sigma_j^2}\,|\alpha_j|\,|\mu^\tau_{\bar\Delta}\big(|\bar N_{V'_{g,j}}|\bar N^2_{\bar\Delta_j}\big)+\frac{1}{\sigma_j^2}\sum_{i\neq j}|\alpha_i||\mu^\tau_{\bar\Delta}\big(|\bar N_{V'_{g,i}}|\,,\bar N^2_{\bar\Delta_j}\big)+\frac{1}{\sigma_j^2}\sum_i|\alpha_i||\mu^\tau_{\bar\Delta}\big(|\bar N_{V'_{g,i}}|\big)\,\mu^\tau_{\bar\Delta}\big(\bar N^2_{\bar\Delta_j}\big) \quad (5.10)$$

In turn the first term in (5.10), using (5.8) and *b)* of proposition 3.1, is bounded from above by $c\|g\|_\infty\,\frac{|\bar\Delta_j|}{|V'_{g,j}|}\,\frac{|\bar\Delta_j|}{|\Lambda_j|}$ for a suitable constant c exactly as in (4.16). The second term is smaller than e.g. $c\|g\|_\infty\,\frac{1}{|\Lambda|}$ if M was taken large enough because $d(V'_{g,i},\bar\Delta_j)\ge M\log|\Lambda|$ if $i\neq j$ since $V'_{g,i}$ is a good set. The third term is again bounded from above by $c\|g\|_\infty\,\sup_i\frac{|\bar\Delta_i|}{|V'_{g,i}|}\,\frac{|\bar\Delta_j|}{|\Lambda_j|}$ for a suitable constant c because of (5.8) and *b)* of proposition 3.1.
If we now put together (5.9), (5.10) and the Gaussian upper bound of proposition 3.3 we conclude that

$$\int_{-T_1}^{T_1} dt_1 \dots \int_{-T_k}^{T_k} dt_k\,|\mu\big(e^{i\sum_i \frac{t_i}{\sigma_i}(N_{\Lambda_i}(\eta)-N_i)}, f\big)| \le c\sum_i \|g\|_\infty\,\frac{|\bar\Delta_i|}{|V'_{g,i}|}\,\frac{|\bar\Delta_i|}{|\Lambda_i|} \le c\delta^{-1}\|g\|_\infty\,\frac{|\bar\Delta|}{|\Delta|}\,\frac{|\bar\Delta|}{|\Lambda|} \quad (5.11)$$

In conclusion, thanks to (5.6), (5.11) together with lemma 5.2, given a local function g with $|V_g|\le |\Lambda|^{1-4\epsilon}$ we have proved that

$$|\nu(f_g)-\mu(f_g)| \le \begin{cases} C\|g\|_\infty\,\frac{|\Delta|}{|\Lambda|} & \text{if } V_g \text{ is good} \\ C\|g\|_\infty\,\Big(\frac{|\bar\Delta|}{|\Delta|}\Big)^2\,\frac{|\Delta|}{|\Lambda|} & \text{if } V_g \text{ is bad}\end{cases} \quad (5.12)$$

where f_g is defined either as (5.5) or as in (5.7) depending whether V_g is good or bad.
As in section 3, (5.12) together with (4.6), (5.8) and the definition of f implies that

$$|\nu(g)-\mu(g)| \le C'\|g\|_\infty\,|V_g|\,\Big(\frac{1}{|\Lambda|} + \sup_{\substack{i\subset I \\ \mathrm{Q_l(x^i)\ is\ good}}} |\nu(N_{Q_l(x^i)})-\mu(N_{Q_l(x^i)})|\Big)$$

or

$$|\nu(g)-\mu(g)| \le C'\|g\|_\infty\,|V_g|\,\Big(\frac{|\bar\Delta|}{|\Delta|}\Big)^2\,\Big(\frac{1}{|\Lambda|} + \sup_{\substack{i\subset I \\ \mathrm{Q_l(x^i)\ is\ good}}} |\nu(N_{Q_l(x^i)})-\mu(N_{Q_l(x^i)})|\Big)$$

depending whether V_g is good or bad.
To complete the proof one has to prove the result for the special functions $N_{Q_l(x^i)}$, $i \in I$, with $Q_l(x^i)$ good.
For this purpose let $Q_l(x^{i_1}), \dots, Q_l(x^{i_k})$ be a collection of good l–cubes, one for each atom Λ_i. Given $i \in I$ let $f = N_{Q_l(x^i)} - \sum_{j=1}^k \alpha_{ij} N_{Q_l(x^{i_j})}$ where

$$\alpha_{ij} = \begin{cases} 0 & \text{if } Q_l(x^i) \text{ is good and } i \notin I_j \\ \frac{\mu(N_{Q_l(x^i)}, N_{\bar\Delta})}{\mu(N_{Q_l(x^{i_j})}, N_{\bar\Delta})} & \text{if } Q_l(x^i) \text{ is good in } \Lambda_j, \text{ with } \Delta = Q_l(x^i) \cup Q_l(x^{i_j}) \\ \frac{\mu(N_{Q_l(x^i)}, N_{\bar\Delta_j})}{\mu(N_{Q_l(x^{i_j})}, N_{\bar\Delta_j})} & \text{if } Q_l(x^i) \text{ is bad with } \Delta = Q_l(x^i) \cup Q_l(x^{i_1}) \ldots \cup Q_l(x^{i_k}) \end{cases} \tag{5.13}$$

Notice that, using (5.8) and *b),c),d)* of proposition 3.1, the following simple bounds hold

$$\begin{aligned} &\alpha_{ij} \geq \alpha_0 \quad \text{if } i \in I_{i_j} \\ &|\alpha_{ij}| \leq \alpha_\infty (\log|\Lambda|)^{2d} \quad \text{otherwise} \end{aligned} \tag{5.14}$$

for suitable positive constants α_0, α_∞.
Let now $D_i := \nu(N_{Q_l(x^i)}) - \mu(N_{Q_l(x^i)})$. Then (5.12) gives

$$|D_i - \sum_j \alpha_{ij} D_{i_j}| := |\epsilon_i| \leq \begin{cases} \frac{c}{|\Lambda|} & \text{if } Q_l(x^i) \text{ is good} \\ \frac{c(\log(|\Lambda|))^{2d}}{|\Lambda|} & \text{if } Q_l(x^i) \text{ is bad} \end{cases} \tag{5.15}$$

for a suitable constant c.
We use at this point the k conservation laws $\sum_{i \in I_j} D_i = 0$, $j = 1, \dots, k$, to write

$$A\vec{D} = -\vec{\mathcal{E}} \tag{5.16}$$

where

$$A_{j'j} = \sum_{i \in I_{j'}} \alpha_{ij}, \quad \vec{D} = \big(D_{i_1}, \dots, D_{i_k} \big), \quad \vec{\mathcal{E}} = \big(\sum_{i \in I_{i_1}} \epsilon_i, \dots, \sum_{i \in I_{i_k}} \epsilon_i \big)$$

Notice that (5.14) immediately implies that the diagonal matrix elements A_{jj} satisfy the bounds

$$\begin{aligned} A_{jj} &\geq \alpha_0 \#\{i \in I_j : Q_l(x^i) \text{ is good}\} - \alpha_\infty (\log|\Lambda|)^{2d} \#\{i \in I_j : Q_l(x^i) \text{ is bad}\} \\ &\geq \alpha_0 l^{-d} |\Lambda_j| \{1 - \alpha_\infty M^d \frac{|\partial_r^- \Lambda_j| (\log|\Lambda|)^{3d}}{|\Lambda_j|}\} \\ &\geq c|\Lambda| \end{aligned}$$

for Λ large enough, while the off–diagonal terms obey

$$|A_{j'j}| \leq M^d l^{-d} \alpha_\infty \sum_{i=1}^k |\partial_r^- \Lambda_i| (\log|\Lambda|)^d \qquad j' \neq j$$

Moreover, because of (5.15),

$$|\mathcal{E}_j| \le c_1 + \frac{c(\log(|\Lambda|))^d}{|\Lambda|} \#\{i \in I_j : Q_l(x^i) \text{ is bad}\} \le \epsilon_\infty$$

for a suitable constant ϵ_∞.
With the above estimates at our disposal we can finally easily invert (5.16) to get that

$$\|\vec{D}\| \le \frac{c}{|\Lambda|}$$

for a suitable constant c. The theorem is proved. □

6. Extension to the bond dilute Ising model.

In this section we discuss the equivalence of multicanonical and multigrand canonical ensembles for a system where the assumption of *translation invariance* is removed and property $SMT(V, C, m, l)$ does no longer hold for all sets $V \in \mathbb{F}_l$. The simplest example of such a system is the (bond) dilute Ising ferromagnet in the Griffiths phase which can be described as follows. For each $\eta \in S^{\mathbb{Z}^d}$, $S = \{0, 1\}$, and $V \in \mathbb{F}$ the Hamiltonian is given by

$$H_V^J(\eta) = -\sum_{[x,y]\in V} J_{xy}(2\eta(x) - 1)(2\eta(y) - 1) + \text{ boundary term}$$

where $[x, y]$ denotes a generic pair of nearest neighbors sites (a bond in $\mathbb{Z}^d$) and $[x, y] \in V$ if both x and y belong to V. The couplings $\{J_{xy}\}$ are assumed to be i.i.d. random variables taking only two values, $J_{xy} = 0$ and $J_{xy} = \beta$ with probability $1 - p$ and p respectively, independently for each bond. Here the parameter β is chosen larger than the critical value β_c for the pure Ising model in $\mathbb{Z}^d$. If $J_{xy} = 0$ the bond $[x, y]$ is said to be closed, while if $J_{xy} = \beta$ is said to be open.
Given $\delta \in (0, 1)$, we consider the dilute model in a volume $\Lambda \in \mathbb{F}$ satisfying $|\partial^-\Lambda|\phi(|\Lambda|)^4 \le |\Lambda|^{1-\delta}$, with $\phi(|\Lambda|)$ a positive function such that $\log(|\Lambda|) \le \phi(|\Lambda|) \le |\Lambda|^\delta$. We also consider a partition of Λ, $\Lambda_1, \dots, \Lambda_k$, such that $0 < \delta \le \frac{|\Lambda_i|}{|\Lambda|}$ and $|\partial^-\Lambda_i|\phi(|\Lambda|)^4 \le |\Lambda|^{1-\delta}$ for all $i = 1, \dots, k$. We assume that the apriori probability measure $\mu_0^{\{x\}}$ is the same in each atom of the partition and we take as h-charge the number of particles. To simplify the notation we set $\mu(\cdot) = \mu_\Lambda^\tau(\cdot)$. As before we set $\rho_i = \mu(N_{\Lambda_i})/|\Lambda_i|$ and we assume, without loss of generality, $\rho_i \le \frac{1}{2}$, $i = 1, \dots, k$.

Given a bond configuration in Λ (*i.e.* a configuration of the couplings $\{J_{xy}\}_{[x,y]\in\Lambda}$), we denote by C_x the cluster of the site $x \in \Lambda$, that is the set af all sites which are connected to x by a path of open bonds, and for each set $V \subseteq \Lambda$ and $n \in \mathbb{N}$ we define

$$V_i := V \cap \Lambda_i, \quad \bar{V} := \cup_{x\in V} C_x, \quad V^{(n)} := \sum_{x\in V} |C_x \cap V|^n \tag{6.1}$$

Notice that $C_x = \{x\}$ if all the bonds with one endpoint equal to x are closed and that $V^{(0)} = |V|$. Clearly the Gibbs measure μ factorizes over the clusters so that the covariance between observables localized in different clusters is zero but inside a given cluster the system is just a low temperature pure Ising model. In particular property $SMT(\Lambda, C, m, l)$ cannot hold unless the basic length scale l is taken larger than the largest among the diameters of the clusters C_x. Such a loss of uniformity in the decay of covariances will be replaced by some regularity assumptions on the bond configuration, which, roughly speaking, are typical of the non–percolative regime. In order to formulate our conditions we fix two positive numbers K_1, K_2 and we define

Assumption $H1(K_1)$**.**

$$\max_{x\in\Lambda} |C_x| \,\le K_1\, \phi(|\Lambda|)$$

Assumption $H2(K_2)$**.** *For* $n = 1, \dots, 4$

$$\Lambda_i^{(n)} \le K_2\, \Lambda_i^{(0)} \quad i = 1, \dots, k$$

Remark. It is not difficult to see that when $p < p_c$, p_c being the bond percolation threshold in $\mathbb{Z}^d$, one can take $\phi(|\Lambda|) = \log|\Lambda|$ and choose K_1 and K_2 such that $H1(K_1)$, $H2(K_2)$ hold with probability one for all large enough cubes centered at the origin.

6.1 Preliminary results

Here we give the analogous of propositions 3.1, 3.4 and of lemma 5.2 in the present case. We refer for the notation to section 3.

Proposition 6.1. *There exists a constant c, depending only on* β*, such that for any* $V \subseteq \Lambda$*, and any bounded local function f with support* $\Delta_f \subset \Lambda$

$$\begin{aligned}
&a)\quad |\mu(N_{C_x\cap V_i}, N_{C_x\cap V_j})| \le c\, \min\{\rho_i,\rho_j\}\, |C_x\cap V_i||C_x\cap V_j| \\
&b)\quad \mu(\bar N^2_{C_x\cap V_i}) \ge c^{-1}\,\rho_i |C_x\cap V_i| \\
&c)\quad \mu(\bar N^2_{V_i}) \le c\,\rho_i\, V_i^{(1)} \\
&d)\quad \mu(\bar N^2_{V_i}) \ge c^{-1}\,\rho_i\,|V_i| \\
&e)\quad |\mu(f, N_{V_i})| \le c\,\|f\|_\infty\, \min\{\,\rho_i|\bar\Delta_f\cap V_i|,\, \Big(\rho_i\big(\bar\Delta_f\cap V_i\big)^{(1)}\Big)^{\frac12}\,\} \\
&f)\quad \mu(\bar N^4_{V_i}) \le c\,\max\{\,(\rho_i V_i^{(1)})^2,\,(\rho_i V_i^{(3)})\,\}
\end{aligned} \tag{6.2}$$

Proof. It follows immediately from the factorization property of the measure μ over the clusters and from the fact that there exists a positive constant $c = c(\beta)$ such that $c^{-1}\rho_i \le \mu(\eta_x, \eta_x) \le c\rho_i$ $\forall\, x \in \Lambda_i$. $\square$

Proposition 6.2. *Assume* $H2(K_2)$*. Then there exists a positive constant A such that*

$$\int_{-A}^{A} dt_1 \dots \int_{-A}^{A} dt_k\, Re\, F(t_1, \dots, t_k) \ge c \tag{6.3}$$

provided that $\sigma_j \ge A^{4+k}$, $j = 1, \dots, k$.

Proof. Thanks to the factorization property of μ and to proposition 6.1 together with $H2(K_2)$, the proof becomes analogous to that of proposition 3.4.

Corollary 6.3. *Assume $H2(K_2)$. Then there exists a constant $C' = C'(\beta, K_2)$ such that*

$$\int_{-\pi\sigma_1}^{\pi\sigma_1} dt_1 \ldots \int_{-\pi\sigma_k}^{\pi\sigma_k} dt_k\, F(t_1, \ldots, t_k) \geq C'$$

Proof. If we use proposition 6.2 instead of proposition 3.4 we can then follow step by step the proof of lemma 5.2. □

6.2 Equivalence between multi–canonical and multi–grand canonical Gibbs measures.

Let $\mathbf{N}_\Lambda = \{N_{\Lambda_i}\}_{i=1}^k$ and let $\mathbf{N} = \{N_i\}_{i=1}^k$ be a set of possible values of the number of particles in each atom Λ_i. We assume that $\mu(\mathbf{N}_\Lambda) = \mathbf{N}$ and we set $\nu := \mu(\cdot|\mathbf{N}_\Lambda = \mathbf{N})$. We say that a set $\Delta \subset \Lambda$ is *good* in Λ_i if $\bar\Delta \cap \Lambda_j = \emptyset$ for $j \neq i$ where $\bar\Delta$ is defined in (6.1). The following is the main result of this section.

Theorem 6.4. *In the above setting assume $H1(K_1)$ and $H2(K_2)$. Then there exists constants $C = C(K_1, K_2, \delta, k, \beta)$ and $v = v(K_1, K_2, \delta, k, \beta)$ such that, if $|\Lambda| \geq v$, for all bounded local functions f with support $\Delta_f \subset \Lambda$ satisfying $\bar\Delta_f^{(3)} \leq |\Lambda|$ if Δ_f is good in one of the atoms of the partition or $|\Delta_f|\, \phi(|\Lambda|)^4 \ll |\Lambda|$ if Δ_f is bad*

$$|\nu(f) - \mu(f)| \leq C\, \|f\|_\infty \begin{cases} \frac{\bar\Delta_f^{(3)}}{|\Lambda|} & \text{if } \Delta_f \text{ is good} \\ \frac{|\Delta_f|\, \phi(|\Lambda|)^4}{|\Lambda|} & \text{otherwise} \end{cases}$$

where $\bar\Delta_f^{(3)}$ is defined in (6.1).

Proof. The proof is similar to the proof of theorem 5.1 but not completely equivalent.
The starting point is again (5.3). Because of *d)* of proposition 6.1

$$\sigma_i^2 = \mu(N_{\Lambda_i}, N_{\Lambda_i}) \geq c'\, \rho_i\, |\Lambda_i| \geq c\, \rho_i\, |\Lambda| \qquad i = 1, \ldots, k \tag{6.4}$$

so that the denominator in (5.3) is greater than some fixed constant thanks to corollary 6.3.
Next we analyze the numerator, and we begin by proving an upper bound when f takes the special form

$$f = g - \sum_{i=1}^k \alpha_i N_{\bar\Delta_i} \tag{6.5}$$

where g is some bounded local function with support Δ_g, $\{\Delta_j\}_{j=1}^k$ is an arbitrary family of sets such that Δ_j is good in Λ_j and $|\Delta_j| = |\Delta_g|$ for all $j = 1, \ldots, k$ and the constants α_i are given by

$$\alpha_j = \frac{\mu(g, N_{\bar\Delta_{gj}})}{\mu(N_{\bar\Delta_j}, N_{\bar\Delta_j})} = \frac{\mu_{\bar\Delta_f}(g, N_{\bar\Delta_{gj}})}{\mu_{\bar\Delta_f}(N_{\bar\Delta_j}, N_{\bar\Delta_j})} \qquad j = 1, \ldots, k \tag{6.6}$$

The support of f is $\Delta_f = \Delta_g \cup (\cup_{j=1}^k \Delta_i)$.
Notice that $\{\alpha_j\}_{j=1}^k$ are well defined and, because of *c)*, *d)* and *e)* of proposition 6.1, they satisfy the bound

$$|\alpha_j| \le c\,\|g\|_\infty \,\min\{\frac{|\bar\Delta_{gj}|}{|\bar\Delta_j|}, \frac{(\rho_j \bar\Delta_{gj}^{(1)})^{\frac12}}{\rho_j\,|\bar\Delta_j|}\} \tag{6.7}$$

for a suitable constant c. Moreover, if Δ_g is good in e.g. Λ_m, then $\alpha_i = 0$ for $i \neq m$.

Lemma 6.5. *Assume $H1(K_1)$ and $H2(K_2)$. Then for the particular function f defined above there exists a positive constant $c = c(\beta, k, \delta, K_1, K_2)$ such that*

$$\int_{-\pi\sigma_1}^{\pi\sigma_1} dt_1 \ldots \int_{-\pi\sigma_k}^{\pi\sigma_k} dt_k\, |\mu\big(e^{i\sum_{j=1}^k \frac{t_j}{\sigma_j}\bar N_{\Lambda_j}}, f\big)| \le$$
$$c\frac{\|g\|_\infty}{|\Lambda|}\begin{cases} \frac{(\bar\Delta_m^{(1)})^{\frac12}\left[(\bar\Delta_g^{(1)})^{\frac12}(\bar\Delta_f^{(1)}) + \bar\Delta_g^{(0)}(\bar\Delta_f^{(3)})^{\frac12}\right]}{\bar\Delta_m^{(0)}} & \textit{if } \Delta_g \textit{ is good in } \Lambda_m \\ |\Delta_g|\,\phi(|\Lambda|)^3\left\{|\Delta_g|, \phi(|\Lambda|)\right\} & \textit{otherwise}\end{cases}$$

Proof. If Δ_g is a good set in Λ_m then $\alpha_j = 0$ for all $j \neq m$ and $\Delta_f = \Delta_g \cup \Delta_m$ is also good in Λ_m. Using the factorization of the measure over the clusters we can then write

$$|\mu\big(e^{i\sum_{j=1}^k \frac{t_j}{\sigma_i}\bar N_{\Lambda_j}}, f\big)| = |\mu\big(e^{i\sum_{j\neq i}^k \frac{t_j}{\sigma_j}\bar N_{\Lambda_j} + \frac{t_m}{\sigma_m}\bar N_{\Lambda_m\setminus\bar\Delta_f}}\big)|\,|\mu_{\bar\Delta_f}\big(e^{i\frac{t_m}{\sigma_m}\bar N_{\bar\Delta_f}}, f\big)| \tag{6.8}$$

The first factor in the righthand side of (6.8) can be bounded by proposition 3.3

$$|\mu\big(e^{i\sum_{j\neq i}^k \frac{t_j}{\sigma_j}\bar N_{\Lambda_j} + \frac{t_m}{\sigma_m}\bar N_{\Lambda_m\setminus\bar\Delta_f}}\big)| \le e^{-c\sum_{j=1}^k t_j^2} \tag{6.9}$$

Exactly as in the proof of theorem 5.1 we can Taylor expand the second factor in the r.h.s. of (6.8) up to the first order which, due to the definition of α_m, vanishes. Because of *c) d)* and *f)* of proposition 6.1 and (6.7) we then obtain

$$\begin{aligned}&\left|\mu_{\bar\Delta_f}(f, e^{i\frac{t_m}{\sigma_m}\bar N_{\bar\Delta_f}})\right| \le \\ &c\,\|g\|_\infty\, t_m^2\,\frac{(\bar\Delta_m^{(1)})^{\frac12}}{\bar\Delta_m^{(0)}}\Big[\frac{(\bar\Delta_g^{(1)})^{\frac12}(\bar\Delta_f^{(1)}) + \bar\Delta_g^{(0)}(\bar\Delta_f^{(3)})^{\frac12}}{|\Lambda|}\Big]\end{aligned} \tag{6.10}$$

If we finally combine together (6.9) and (6.10) we immediately get the first part the lemma.
Consider now the case of a bad Δ_g. We need to bound from above

$$|\mu\big(e^{i\sum_{j=1}^k \frac{t_j}{\sigma_j}\bar N_{\Lambda_j}}, f\big)| = |\mu\big(e^{i\sum_{j=1}^k \frac{t_j}{\sigma_j}\bar N_{\Lambda_j\setminus\bar\Delta_{fj}}}\big)|\,|\mu_{\bar\Delta_f}\big(e^{i\sum_{j=1}^k \frac{t_j}{\sigma_i}\bar N_{\bar\Delta_{fj}}}, f\big)| \tag{6.11}$$

The first factor in the righthand side of (6.11) is again bounded from above as in (6.9) using proposition 3.3. As far as the second factor is concerned, we again Taylor expand up to the first order, which, due to the definition of $\{\alpha_i\}_{i=1}^k$, gives a vanishing term. By points *c) d)* and *f)* of proposition 6.1, (6.7) and $H1(K_1)$ we get

$$|\mu\big(e^{i\sum_{j=1}^k \frac{t_j}{\sigma_j}\bar N_{\Delta_{fj}}}, f\big)| \le c\,\|g\|_\infty\,(\sum_{j=1}^k t_j^2)\,\frac{|\Delta_g|\,\phi(|\Lambda|)^3}{|\Lambda|}\min\{\,|\Delta_g|,\phi(|\Lambda|)\,\}$$

Thanks to the gaussian bound on the first factor we can perform the integral over $\{t_j\}_{j=1}^k$ and also the second part of the lemma follows. □

Thanks to lemma 6.3 and corollary 6.3, we have that for the special functions defined in (6.5)

$$|\nu(f)-\mu(f)| \le c\,\frac{\|g\|_\infty}{|\Lambda|}\begin{cases}\frac{(\bar\Delta_m^{(1)})^{\frac12}\left[(\bar\Delta_g^{(1)})^{\frac12}(\bar\Delta_f^{(1)})+\bar\Delta_g^{(0)}(\bar\Delta_f^{(3)})^{\frac12}\right]}{\bar\Delta_m^{(0)}} & \text{if } \Delta_g \text{ is good in } \Lambda_m\\ |\Delta_g|\,\phi(|\Lambda|)^3\,\min\Big\{|\Delta_g|,\phi(|\Lambda|)\Big\} & \text{otherwise}\end{cases} \tag{6.12}$$

Therefore, by the very definition of f (see (6.5)), we can bound from above the quantity $|\nu(g)-\mu(g)|$ by

$$|\nu(g)-\mu(g)| \le |\nu(f)-\mu(f)| + \sum_{i=1}^k |\alpha_j||\nu(N_{\bar\Delta_j})-\mu(N_{\bar\Delta_j})| \tag{6.13}$$

If Δ_g is good, say in Λ_m, we can choose the corresponding function f as $f=g-\alpha_m N_{\Delta_g}$ so that, using (6.12) and (6.7)

$$|\nu(g)-\mu(g)| \le c\,\|g\|_\infty\,(\frac{\bar\Delta_g^{(3)}}{|\Lambda|} + |\nu(N_{\bar\Delta_g})-\mu(N_{\bar\Delta_g})|) \tag{6.14}$$

If Δ_g is bad (6.12) together with (6.13) give

$$|\nu(g)-\mu(g)| \le c\,\|g\|_\infty\,|\Delta_g|\,\phi(|\Lambda|)^3\,\min\Big\{|\Delta_g|\wedge\phi(|\Lambda|)\Big\} + \sum_{i=1}^k|\alpha_j||\nu(N_{\bar\Delta_j})-\mu(N_{\bar\Delta_j})| \tag{6.15}$$

Thus, in order to complete the proof of the theorem we must bound a generic term $|\nu(N_{\bar\Delta})-\mu(N_{\bar\Delta})|$, when Δ is good.
For this purpose, let $g=\eta_x$, $f=\eta_x-\sum_{i=1}^k \alpha_i N_{C_{y_i}}$, $C_{y_i}\subset\Lambda_i$ for any $i=1,\dots,k$, with

$$\alpha_i=\alpha_i(x)=\mu(\eta_x, N_{C_x\cap\Lambda_i})/\mu(N_{C_{y_i}},N_{C_{y_i}})$$

Define $\hat D_x=\nu(\eta_x)-\mu(\eta_x)$ and $D_x=\nu(N_{C_x})-\mu(N_{C_x})$. Notice that for any set Δ

$$|\nu(N_{\bar\Delta})-\mu(N_{\bar\Delta})| \le \sum_{x\in\Delta}|C_x|^{-1}\,|D_x|$$

We then have that

$$\epsilon_x := \nu(f)-\mu(f) = \hat D_x - \sum_{i=1}^k \alpha_i D_{y_i} \tag{6.16}$$

satisfies the bound

$$|\epsilon_x| \le \frac{c}{|\Lambda|} \begin{cases} |C_x|(|C_x|^2 + |C_{y_m}|^2) & \text{if } x \text{ is good in } |\Lambda_m| \\ \phi(|\Lambda|)^3 & \text{otherwise} \end{cases} \tag{6.17}$$

If we now sum over $x \in \Lambda_j$ and we use the conservation laws $\sum_{x\in\Lambda_j} \hat{D}_x = 0$ for all $j = 1, \ldots, k$ we get

$$A\vec{D} = -\vec{\mathcal{E}}$$

where A is the matrix whose elements are given by

$$A_{ji} = \sum_{x\in\Lambda_j} \alpha_i(x)$$

and $\vec{D}$, $\vec{\mathcal{E}}$ are the vectors

$$\vec{D} = (D_{y_1}, \ldots, D_{y_k}) \ , \qquad \vec{\mathcal{E}} = (\sum_{x\in\Lambda_1} \epsilon_x, \ldots, \sum_{x\in\Lambda_k} \epsilon_x)$$

In order to bound from above the components of the vector $\vec{D}$ we need to bound the inverse matrix A^{-1}.
As in the strong mixing case (see section 5), thanks to $H1(K_1)$, $H2(K_2)$ and the fact that $\phi(|\Lambda|)^4|\partial^-\Lambda_i| \le |\Lambda|^{1-\delta}$, the diagonal terms are largely dominant w.r.t the off–diagonal ones. We have in fact

$$\begin{aligned} A_{jj} &\ge c\,\frac{|\Lambda|}{|C_{y_j}|^2} \ge c\,\frac{|\Lambda|}{\phi(|\Lambda|)^2} \\ |A_{ji}| &\le c \sum_{x \text{ is bad in } \Lambda_j} |C_x \cap \Lambda_i| \le c\,|\partial^-\Lambda_i|\phi(|\Lambda|)^2 \end{aligned} \tag{6.18}$$

so that $\sup_{i\neq j} \frac{|A_{ji}|}{A_{jj}} \le c|\Lambda|^{-\delta}$. Moreover, using (6.17)

$$\sum_{x\in\Lambda_j} |\epsilon_x| \le \sum_{x \text{ is bad in } \Lambda_j} |\epsilon_x| + \sum_{x \text{ is bad in } \Lambda_j} |\epsilon_x| \ \le c\,|C_{y_j}|^2 \tag{6.19}$$

Thus, from (6.18) and (6.19) we easily get

$$|D_{y_j}| \le c\,\frac{\sum_{x\in\Lambda_j} |\epsilon_x|}{|A_{jj}|} \le c\,\frac{|C_{y_j}|^4}{|\Lambda|} \ , \qquad \forall\ j = 1, \ldots, k \tag{6.20}$$

In particular, due to the arbitrariness of the good sites $y_1, \ldots, y_k$, for any good set Δ

$$|\nu(N_{\bar{\Delta}}) - \mu(N_{\bar{\Delta}})| \le c\,\frac{\bar{\Delta}^{(3)}}{|\Lambda|} \tag{6.21}$$

The result of the theorem now follows by (6.14) together with (6.21) if Δ_g is a good set in Λ_m, and by (6.7), (6.15) and (6.21) otherwise. □

Corollary 6.6. *In the same setting of theorem 6.4 consider $B \subset \Lambda$ such that $\bar{B}^{(3)} \le |\Lambda|$ if B is good and $|B|\,\phi(|\Lambda|)^4 \le |\Lambda|$ otherwise. Let τ_B be an arbitrary spin configuration in B and let*

$$g(\eta) := \begin{cases} 1 & \text{if } \eta(x) = \tau_B(x) \quad \forall x \in B \\ 0 & \text{otherwise} \end{cases}$$

Then there exists a constant $c = c(\beta, K_1, K_2, k, \delta)$ such that

$$\frac{|\nu(g) - \mu(g)|}{\mu(g)} \le c \begin{cases} \frac{\max\{(\bar{B}^{(0)}\bar{B}^{(1)}), \bar{B}^{(3)}\}}{\rho_m |\Lambda|} & \text{if } B \text{ is good in } \Lambda_m \\ \frac{|B|\phi(|\Lambda|)^3 \max\{|B|, \phi(|\Lambda|)\}}{\min_i \rho_i \, |\Lambda|} & \text{otherwise} \end{cases}$$

Proof. If we use the r.h.s. of (6.7) with the simple bound

$$|\alpha_j| \le c\,\mu(\eta_B)\, \frac{|\bar{B}_j|}{\rho_j |\bar{\Delta}_j|}$$

then the proof becomes equal to that of theorem 6.4. $\square$

7. Applications for lattice gases

Here we discuss some simple consequences of our results for lattice gases, namely lattice models with single spin space $S = \{0, 1\}$. In this case the h-charge is given by $h(\eta) = \eta$ so that $N_\Lambda(\eta)$ counts the number of particles in the set Λ.

7.1 Equivalence of the canonical and grand canonical ensembles

We begin by discussing the thermodynamic limit of canonical and grand canonical Gibbs measures. Take the apriori measures $\mu_0^{\{x\}}$ all equal to a given positive, non trivial, probability measure μ_0 on the state space S and let Φ be a finite range, translation invariant, interaction such that property $SMT(V, C, m, l)$ holds for all sets $V \in \mathbb{F}_l$ and some finite integer l. Let μ be the unique Gibbs state corresponding to Φ and let $\rho = \lim_{L\to\infty} (2L+1)^{-d}\, \mu^\tau_{B_L}(N_{B_L})$. Clearly the limit exists independent of the boundary conditions because of the mixing hypothesis. Next we fix a large L multiple of l, a boundary condition τ and take $\Lambda = B_L$. Then, for l large enough and any $\epsilon \in (0, 1)$, there exists positive constants C' and L_0 independent of τ such that for all local functions f with l–support $\Delta \subset \Lambda$ satisfying $|\Delta| \le L^{d(1-\epsilon)}$

$$|\,\mu^\tau_\Lambda(f \,|\, N_\Lambda = \lfloor \rho |\Lambda| \rfloor) - \mu^\tau_\Lambda(f)\,| \le C' \|f\|_\infty\, |\Delta|^{\frac{1}{2}}\, L^{-1} \tag{7.1}$$

provided that $L \ge L_0$. Notice the new error term $|\Delta|^{\frac{1}{2}}\, L^{-1}$ which is much larger than the one in theorem 4.1. As we will see such an error comes fron the fact that in general the average of N_Λ w.r.t. μ^τ_Λ is shifted by $O(L^{d-1})$, *i.e.* a surface term, w.r.t to its thermodinamic value $\rho|\Lambda|$.
To prove (7.1) we let $N = \lfloor \rho|\Lambda| \rfloor$ and we denote by $\lambda(N)$ the unique value of λ such that

$$\mu^{\tau,\lambda}_\Lambda(N_\Lambda) := \frac{\mu^\tau_\Lambda\big(N_\Lambda e^{\lambda N_\Lambda}\big)}{\mu^\tau_\Lambda\big(e^{\lambda N_\Lambda}\big)} = N$$

The existence and uniqueness of $\lambda(N)$ follow immediately from the computation $\frac{d}{d\lambda}\mu_\Lambda^{\tau,\lambda}(N_\Lambda) = \mu_\Lambda^{\tau,\lambda}(N_\Lambda, N_\Lambda) > 0$ together with $\lim_{\lambda\to\infty}\mu_\Lambda^{\tau,\lambda}(N_\Lambda) = |\Lambda|$ and $\lim_{\lambda\to-\infty}\mu_\Lambda^{\tau,\lambda}(N_\Lambda) = 0$. If we let $N' = \mu_\Lambda^\tau(N_\Lambda)$, then $|N - N'| \le CL^{d-1}$ because of the strong mixing assumption and moreover, thanks to *c)* of proposition 3.1,

$$|N - N'| = |\int_0^{\lambda(N)} d\lambda' \, \mu_\Lambda^{\tau,\lambda'}(N_\Lambda, N_\Lambda)\,| \ge |\lambda(N)|\, A^{-1}\,(N \wedge N') \tag{7.2}$$

so that

$$|\lambda(N)| \le A\frac{|N - N'|}{N \wedge N'} \le A'L^{-1}$$

Next we observe that, for δ small enough depending on C, m, l, the new Gibbsian specifications $\mu_V^{\tau,\lambda}(\cdot)$ with $|\lambda| \le \delta$ share the property $SMT(V, 2C, m/2)$ for all l–regular sets V. For this purpose one can either use the original results on completely analytical Gibbsian fields [DS] or check condition (a) of theorem 3.3. of [M2] for μ^λ. Therefore, for L large enough, we can apply theorem 4.1 to the measure $\mu_\Lambda^{\tau,\lambda(N)}$ and to get

$$|\mu_\Lambda^{\tau,\lambda(N)}(f \mid N_\Lambda = N) - \mu_\Lambda^{\tau,\lambda(N)}(f)| \le C''\|f\|_\infty \frac{|\Delta|}{|\Lambda|} \tag{7.3}$$

for a suitable constant C'' provided that l was chosen large enough. Finally we compare $\mu_\Lambda^{\tau,\lambda(N)}(f)$ and $\mu_\Lambda^\tau(f)$. We get

$$\begin{aligned}
|\mu_\Lambda^{\tau,\lambda(N)}(f) - \mu_\Lambda^\tau(f)| &\le \int_0^{\lambda(N)} d\lambda' \,|\, \mu_\Lambda^{\tau,\lambda'}(f, N_\Lambda)\,| \\
&\le B\|f\|_\infty \sup_{\lambda'\in[0,\lambda(N)]} |\, \mu_\Lambda^{\tau,\lambda'}(f, N_\Lambda)\,||\lambda(N)| \\
&\le B'\|f\|_\infty\, |\Delta|^{\frac{1}{2}} L^{-1}
\end{aligned} \tag{7.4}$$

for suitable constants B, B'. Thus (7.1) follows.

7.2 Influence of the boundary condition for multicanonical Gibbs measures

Here we discuss the influence of the boundary conditions in a multicanonical Gibbs measure under some kind of *uniform strong mixing condition.* The main result is that multiple conditionings produce an effective screening of the influence of the boundary conditions which is much larger than that observed in a simple conditioning. Such a result is a keystep in the analysis of the block spin renormalization group transformation [BCO] and in the study of the relaxational properties of spin exchange dynamics in the high temperature phase [CM1].

The setting is exactly that of section 5 in the case $S = \{0, 1\}$ and $h(\eta) = \eta$ and we thus refer to that part for all the necessary notation. However, in order to present the result in the simplest possible way, we decided to restrict the geometrical setting as follows.

Fix $\delta \in (0, 1)$ and an integer $k \ge 2$ with $\delta k < 1$. Let $L_1, \ldots, L_k$ be large multiples of the basic length scale l, let $L = \sum_i L_i$ and assume that $L_j \ge \delta L$ for any j. We then choose one coordinate direction, e.g. the d direction, and we take $\Lambda = Q_L$, Λ_1 equal to the first slice of Λ orthogonal to d–direction of width L_1, *i.e.* $\Lambda_1 = \{x \in \Lambda \,:\, 0 \le x_d < L_1\}$, Λ_2 equal to the slice of Λ on top of Λ_1 of width L_2 and so on. Finally τ will denote the special boundary condition such that $\mu(N_{\Lambda_i}) := \mu^\tau_\Lambda(N_{\Lambda_i}) = N_i$, $i = 1, \dots, k$, $\nu^\tau := \nu^\tau_\Lambda$ will denote the conditional Gibbs measure $\mu^\tau_\Lambda\big(\cdot \,|\, N_{\Lambda_i} = N_i,\, i = 1, \dots, k\big)$ and Ω_τ the set of configurations τ' that coincide with τ in the half space $\{x \in \mathbb{Z}^d \,:\, x_d < L\}$.

Proposition 7.1. *Assume condition $SMT(V, C, m, l)$ for all sets $V \in \mathbb{F}_l$ uniformly for all possible choices of the apriori measures $\mu_0^{(i)}$, $i = 1, \dots, k$. Then, for any l, M large enough and ϵ small enough independent of $\{\rho_i\}_{i=1}^k$, there exist constants $C' = C'(C, m, \|\Phi\|, \alpha, l, \delta, M, \epsilon)$, $L_0 = L_0(C, m, \|\Phi\|, \alpha, l, M, \delta, \epsilon)$ such that if $L \ge L_0$, for all local functions f with l–support $\Delta \subset \Lambda_n$, $n \le k-1$, satisfying $|\Delta| \le |\Lambda|^{1-\epsilon}$*

$$\sup_{\tau' \in \Omega_\tau} |\nu^\tau(f) - \nu^{\tau'}(f)| \le C' \|f\|_\infty \le \begin{cases} |\Delta| \left[\frac{1}{|\Lambda|} + \frac{1}{L^{k-n+1}}\right] & \text{if } \Delta \text{ is good} \\ \frac{|\Delta|}{|\Lambda|}\left(\frac{|\bar\Delta|}{|\Delta|}\right)^2 + \max_{j=n,n\pm1} \left[\frac{|\bar\Delta \cap \Lambda_j|^{\frac12}}{L^{k-j+1}}\right] & \text{if } \Delta \text{ is bad} \end{cases}$$

Corollary 7.2. *In the same assumptions of proposition 7.1 assume that $\Delta \subset \partial_r^- \Lambda_n \cap \partial_r^+ \Lambda_{n-1}$. Then*

$$\sup_{\tau' \in \Omega_\tau} |\nu^\tau(f) - \nu^{\tau'}(f)| \le C' \mathrm{Osc}(f) \,\Big[\frac{(\log L)^2}{L} + \frac{(\log L)^{\frac12}}{L^{k+1-n-(d-1)/2}}\Big]$$

where $\mathrm{Osc}(f) = \sup_{\eta,\eta'} |f(\eta) - f(\eta')|$.

Proof. It follows at once from proposition 7.1 since $\bar\Delta \cap \Lambda_{n+1} = \emptyset$ for large enough L. $\square$

If we now use the above corollary as the first step of an inductive procedure in n using the Markov property of ν^τ, we get the following

Corollary 7.3. *In the same assumptions of corollary 7.2 let $\bar n = \lfloor |\frac{d-1}{2} - 1| \rfloor + 1$ and $\gamma(L) = C' \left[\frac{(\log L)^2}{L} + \frac{(\log L)^{\frac12}}{L^{\bar n+1-(d-1)/2}}\right]$, where C' is the constant appearing in corollary 7.2. Then*

$$\sup_{\tau' \in \Omega_\tau} |\nu^\tau(f) - \nu^{\tau'}(f)| \le \mathrm{Osc}(f)\, \gamma(L)^{\lfloor \frac{k-n+1}{\bar n+1} \rfloor}$$

Proof. By the very definition of $\bar n$ and $\gamma(L)$ the result is true for $n = k - \bar n$. Assume the result valid for $n_j = k - j(\bar n + 1) + 1$, $1 \le j < \frac{k}{\bar n+1}$, and let us prove it for n_{j+1}. Let $V_j = \cup_{i=1}^{n_j-1} \Lambda_i$, let f be such that its support lies in $\partial_r^- \Lambda_{n_{j+1}} \cap \partial_r^+ \Lambda_{n_{j+1}-1}$ and let f' be defined by $f'(\eta) = \mu^\eta_{V_j}(f \,|\, N_{\Lambda_i} = N_i,\, i = 1, \dots, n_j - 1)$. Clearly the support of f' coincides with $\partial_r^- \Lambda_{n_j} \cap \partial_r^+ V_j$ and moreover $\nu^\tau(f) = \nu^\tau(f')$. Therefore, by the inductive hypothesis,

$$\sup_{\tau' \in \Omega_\tau} |\nu^\tau(f) - \nu^{\tau'}(f)| = \sup_{\tau' \in \Omega_\tau} |\nu^\tau(f') - \nu^{\tau'}(f')| =\le \gamma(L)^j \,\mathrm{Osc}(f')$$

In order to bound $\mathrm{Osc}(f')$ we can use corollary 7.2 with Λ replaced by V_j and k replaced by $n_j - 1$ together with the definition of $\bar n$ to get $\mathrm{Osc}(f') \le \gamma(L)\,\mathrm{Osc}(f)$. Thus the result holds for n_{j+1}. Let now $n<k$ be given and let us write it as $n=k-j_0(\bar n+1)+1-m$ with $0\le m<\bar n+1$. Let f have support inside $\partial_r^-\Lambda_n\cap\partial_r^+\Lambda_{n-1}$ and let f' be as above with j replaced by j_0. Then, by the same argument as above, we get

$$\sup_{\tau'\in\Omega_\tau}|\nu^\tau(f)-\nu^{\tau'}(f)|\le\gamma(L)^j\,\mathrm{Osc}(f')\le\gamma(L)^j\,\mathrm{Osc}(f)$$

and the statement of the corollary follows. □

Remark. Notice that in two and three dimensions $\bar n=1$ and $\gamma(L)=O\left(\frac{1}{L}\right)$. Thus, in this case, we gain a factor $O\left(\frac{1}{L}\right)$ for each slice but one separating the support of the function and the region where the boundary condition is changed.

Proof of the proposition. Let us fix $\tau'\in\Omega_\tau$ and let $N_i'=\mu_\Lambda^{\tau'}(N_{\Lambda_i})$, $i=1,\dots,k$. Because of condition $SMT(V,C,m,l)$

$$|N_i'-N_i|\le\begin{cases}C'e^{-\frac{m}{2}\delta L} & \text{if } i\neq k\\ C'\rho_kL^{d-1} & \text{if } i=k\end{cases}\tag{7.5}$$

Let also $\underline{\lambda}=(\lambda_1,\dots,\lambda_k)$ be such that

$$\mu^{\tau',\underline{\lambda}}(N_{\Lambda_i}):=\frac{\mu_\Lambda^{\tau'}\left(N_{\Lambda_i}e^{\sum_i\lambda_iN_{\Lambda_i}}\right)}{\mu_\Lambda^{\tau'}\left(e^{\sum_i\lambda_iN_{\Lambda_i}}\right)}=N_i\tag{7.6}$$

Notice that the new measure $\mu^{\tau',\underline{\lambda}}$ is simply obtained by replacing the original single site measure $\mu_0^{(i)}$ with $\frac{e^{\lambda_i\eta}\mu_0^{(i)}(\eta)}{\mu_0^{(i)}(e^{\lambda_i\eta})}$, $i=1,\dots,k$. By assumption, $\mu^{\tau',\underline{\lambda}}$ enjoys the property $SMT(V,C,m,l)$ for all sets $V\in\mathbb{F}_l$ uniformly in $\underline{\lambda}$.
The existence of the tilting field $\underline{\lambda}$ is guaranteed if we observe that the Jacobian of the transformation $\underline{\lambda}\mapsto\mathbf{N}(\underline{\lambda})=\mu^{\tau',\underline{\lambda}}(\mathbf{N}_\Lambda)$ is always positive and that for all $i=1,\dots,k$ $\lim_{\lambda_i\to\infty(-\infty)}N_i(\underline{\lambda})=|\Lambda_i|\ (0)$ uniformly in λ_j, $j\neq i$ (see appendix A for more details).
We now write

$$\begin{aligned}|\nu^\tau(f)-\nu^{\tau'}(f)|\le&\,|\mu^\tau(f)-\nu^\tau(f)|+|\mu^{\tau',\underline{\lambda}}(f)-\nu^{\tau'}(f)|+\\&|\mu^{\tau',\underline{\lambda}}(f)-\mu^{\tau,\underline{\lambda}}(f)|+|\mu^{\tau,\underline{\lambda}}(f)-\mu^\tau(f)|\end{aligned}\tag{7.7}$$

The first two terms in the r.h.s. of (7.7) can be bounded from above using theorem 5.1. The third term, using the exponential decay of the covariances of $\mu^{\tau',\underline{\lambda}}$ and the fact that the distance between the support of f and the region where τ' differs from τ is larger than δL, is bounded from above by $C''\|f\|_\infty e^{-\frac{m}{2}\delta L}$. We finally analyze the more interesting fourth term.
Let, for any $s\in[0,1]$, $\underline{\lambda}(s)=s\underline{\lambda}$. Then we write

$$|\mu^{\tau,\underline{\lambda}}(f)-\mu^\tau(f)|=|\ <\int_0^1ds\,\mu^{\tau,\underline{\lambda}(s)}(\mathbf{N}_\Lambda,f),\,\underline{\lambda}>\ |\tag{7.8}$$

where $< \cdot, \cdot >$ denotes the scalar product in $\mathbb{R}^k$. Using proposition 3.1 and the exponential decay of the covariances,

$$| \int_0^1 ds\, \mu^{\tau,\underline{\lambda}(s)}(N_{\Lambda_j}, f) \,| \le \begin{cases} \|f\|_\infty \, e^{-\frac{m}{2}\delta L} & \text{if } j \neq n \\ A\|f\|_\infty \, |\Delta|^{\frac{1}{2}}, & \text{otherwise} \end{cases} \tag{7.9}$$

if the support Δ is good in Λ_n. If instead Δ is bad we can use corollary 3.2 applied to $V_j = \bar{\Delta} \cap \Lambda_j$ and get

$$| \int_0^1 ds\, \mu^{\tau,\underline{\lambda}(s)}(N_{\Lambda_j}, f) \,| \le \begin{cases} \|f\|_\infty \, e^{-\frac{m}{2}\delta L} & \text{if } j \neq n,\, n \pm 1 \\ A\|f\|_\infty \, |\bar{\Delta} \cap \Lambda_j|^{\frac{1}{2}} & \text{otherwise} \end{cases} \tag{7.10}$$

We need now an estimate on the external field $\underline{\lambda}$. By construction the following identity holds

$$\mathbf{N} - \mathbf{N}' = \mu_\Lambda^{\tau',\underline{\lambda}}(\mathbf{N}_\Lambda) - \mu_\Lambda^{\tau'}(\mathbf{N}_\Lambda) = \mathcal{K}\underline{\lambda}$$

where the matrix $\mathcal{K}$ is given by

$$\big(\mathcal{K}\big)_{ij} = \int_0^1 ds\, \mu^{\tau,\underline{\lambda}(s)}(N_{\Lambda_i}, N_{\Lambda_j})$$

Let, for $j = 1, \ldots, k$, $\rho_j(s) := |\Lambda_j|^{-1}\mu^{\tau,\underline{\lambda}}(s)\left(N_{\Lambda_j}\right)$. Exactly as in (7.9), thanks to *c), d)* of proposition 3.1 we have the bounds

$$\begin{aligned} |\mathcal{K}_{ij}| &\le A L^{d-1} \int_0^1 ds\, \rho_i(s)\rho_j(s) \qquad \text{if } j = i \pm 1 \\ \mathcal{K}_{ii} &\ge A^{-1} L^d \int_0^1 ds\, \rho_i(s) \\ |\mathcal{K}_{ij}| &\le e^{-\frac{m}{2}\delta L} \qquad \text{otherwise} \end{aligned} \tag{7.11}$$

Thus, for L large enough, we can invert $\mathcal{K}$ by Neumann series and get, thanks to (7.5)

$$|\lambda_j| \le C \frac{1}{L^{k-j+1}} \tag{7.12}$$

In conclusion the r.h.s. of (7.8) is bounded from above by

$$|\mu^{\tau,\underline{\lambda}}(f) - \mu^\tau(f)| \le A'\|f\|_\infty \, \frac{|\Delta|^{\frac{1}{2}}}{L^{k-n+1}} \tag{7.13}$$

if Δ is good in Λ_n, while

$$|\mu^{\tau,\underline{\lambda}}(f) - \mu^\tau(f)| \le A'\|f\|_\infty \max_{j=n,n\pm1} \frac{|\bar{\Delta} \cap \Lambda_j|^{\frac{1}{2}}}{L^{k-j+1}} \tag{7.14}$$

if not.
The final result follows by putting together the above mentioned bounds on the first three terms in the r.h.s. of (7.7) together with (7.13), (7.14). □

We conclude this part by stating the analogous result for the bond dilute Ising model discussed in section 6. We refer the reader to that section for all the necessary notation.

Proposition 7.4. *In the same geometrical setting of proposition 7.1 assume $H1(K_1)$, $H2(K_2)$ and $\delta \leq \frac{1}{5d}$. Then there exists constants $C = C(K_1, K_2, \delta, k, \beta)$ and $L_0 = L_0(K_1, K_2, \delta, k, \beta)$ such that if $L > L_0$, for all local functions f with support $\Delta \subset \Lambda_n$, $n \leq k-1$, satsfying $\bar{\Delta}^{(3)} \leq |\Lambda|$ if Δ is good or $|\Delta|\phi(|\Lambda|) \ll |\Lambda|$ if Δ is bad,*

$$\sup_{\tau' \in \Omega_\tau} |\nu^\tau(f) - \nu^{\tau'}(f)| \leq C' \|f\|_\infty \, \epsilon(\Delta, L)$$

where

$$\epsilon(\Delta, L) \leq \begin{cases} \frac{\bar{\Delta}^{(3)}}{|\Lambda|} + \left(\frac{\phi(|\Lambda|)}{L}\right)^{k-n+1} \left[\min\{\, (\bar{\Delta}^{(1)})^{\frac{1}{2}}, |\bar{\Delta}|\,\}\right] & \text{if } \Delta \text{ is good} \\ \frac{|\Delta|\phi(|\Lambda|)^4}{|\Lambda|} + & \\ \max_{j=n,n\pm1} \left[\min\{\, \left((\bar{\Delta} \cap \Lambda_j)^{(1)}\right)^{\frac{1}{2}}, |\bar{\Delta} \cap \Lambda_j|\,\} \left(\frac{\phi(|\Lambda|)}{L}\right)^{k-j+1}\right] & \text{if } \Delta \text{ is bad} \end{cases}$$

Proof. Using $H1(K_1)$ and proposition 6.1 instead of proposition 3.1 we can follow the same steps of the proof of proposition 7.1 and obtain the result. □

Corollary 7.5. *Under the same assumptions of proposition 7.4 assume that $\Delta \subset \partial^- \Lambda_n \cap \partial^+ \Lambda_{n-1}$. Then*

$$\sup_{\tau' \in \Omega_\tau} |\nu^\tau(f) - \nu^{\tau'}(f)| \leq C' \mathrm{Osc}(f) \left[\frac{\phi(|\Lambda|)^4}{L} + \frac{\phi(|\Lambda|)^{k+2-n}}{L^{k+1-n-(d-1)/2}}\right]$$

Proof. It follows at once from proposition 7.4.

7.3 Decay of covariances for finite volume multi–canonical Gibbs measures

Here we deduce decay bounds on covariances of finite volume multi–canonical measures. The setting and the notation are those of section 5 with $S = \{0, 1\}$ and $h(\eta) = \eta$.

Proposition 7.6. *Take l, Λ, f and g satisfying the hypothesis of theorem 4.1 and assume condition $SMT(V, C, m, l)$ for all sets $V \in \mathbb{F}_l$, uniformly in the densities $\rho_1, \ldots, \rho_k$. Assume moreover that the l–supports of f and g, Δ_f and Δ_g, are disjoint. Then*

$$|\,\nu(f,g)\,| \leq C' \, \|f\|_\infty \, \|g\|_\infty \, A_\Lambda(f,g) \begin{cases} \min\{\nu(|f|), \nu(|g|)\} & \text{if } \Delta_f \text{ and } \Delta_g \text{ are both good or bad} \\ \nu(|f|) & \text{if } \Delta_f \text{ is bad} \\ \nu(|g|) & \text{if } \Delta_g \text{ is bad} \end{cases} \tag{7.15}$$

where

$$A_\Lambda(f,g) = \begin{cases} |\Delta_f|\,|\Delta_g| \left(\frac{1}{|\Lambda|} + e^{-md(\Delta_f, \Delta_g)}\right) & \text{if } \Delta_f \text{ or } \Delta_g \text{ is good} \\ |\Delta_f|\,|\Delta_g| \left(\frac{1}{|\Lambda|}\left(\frac{|\bar{\Delta}_g|}{|\Delta_g|}\right)^2 \left(\frac{|\bar{\Delta}_f|}{|\Delta_f|}\right)^2 + e^{-md(\Delta_f, \Delta_g)}\right) & \text{if } \Delta_f \text{ and } \Delta_g \text{ are bad} \end{cases}$$

Proof. Let f, g be as in the theorem with zero mean w.r.t. the multi–canonical measure ν and let $V = \Lambda \backslash \Delta_f$. We partition V into k atoms $V_1, \ldots, V_k$, with $V_j = \Lambda_j$ if $j \neq i$ and $V_i = \Lambda_i \backslash \Delta_f$. Given a configuration $\tau \in \Omega_{\Delta_f}$, let $N'_j = N_j$ if $j \neq i$ and $N'_i = N_i - N_{\Delta_f}(\tau)$. Let finally $\underline{\lambda}(\tau) = (\lambda_1, \dots, \lambda_k)$ be such that $\mu_V^{\tau, \underline{\lambda}(\tau)}\left(N_{V_j}\right) = N'_j, \quad j = 1, \dots, k$. The existence of the tilting field $\underline{\lambda}$ follows at once by the argument given in the appendix. Moreover, exactly the same argument used in (7.5)...(7.12) together with the exponential decay of covariances of the measures $\mu_V^{\tau, \underline{\lambda}(\tau)}$ uniformly in τ, shows that

$$\sup_{\tau, \tau' \in \Omega_{\Delta_f}} |\, \mu_V^{\tau, \underline{\lambda}(\tau)}(g) - \mu_V^{\tau', \underline{\lambda}(\tau')}(g)\, | \leq$$

$$\leq C' \|g\|_\infty \, |\Delta_f| \, |\Delta_g| \, e^{-md(\Delta_f, \Delta_g)} + \begin{cases} C'' \|g\|_\infty \frac{|\Delta_f|\,|\Delta_g|}{|\Lambda|} & \text{if } \Delta_g \text{ is good} \\ C'' \|g\|_\infty \frac{|\Delta_f|\,|\bar{\Delta}_g|}{|\Lambda|} & \text{if } \Delta_g \text{ is bad} \end{cases} \tag{7.16}$$

since $|N'_i(\tau) - N_i| \leq |\Delta_f|$. Then we write

$$|\, \nu(f, g)\, | = |\, \nu\left(f \nu\left(g | \mathcal{F}_{\Delta_f}\right)\right)\, | \leq |\, \nu(|f|)\, | \sup_{\tau, \tau' \in \Omega_{\Delta_f}} |\, \nu(g|\tau) - \nu(g|\tau')\, | \tag{7.17}$$

In order to bound the second factor we simply apply theorem 5.1 to each term $\nu(g|\tau) = \mu_V^{\tau, \underline{\lambda}(\tau)}(g \,|\, \mathbf{N}_\Lambda = \mathbf{N}'(\tau))$ and use (7.16). $\square$

The following corollary is straightforward.

Corollary 7.7. *Under the same assumptions of proposition 7.6, take $k = 1$ and $f = \eta(x)$ and $g = \eta(y)$ then*

$$\nu(\eta(x), \eta(y)) \leq C' \rho(1 - \rho) \left(\frac{1}{\Lambda} + e^{-md(x,y)} \right)$$

Remark. It is worthwhile to mention that the above estimate cannot be improved in general as one can readily check for the simple case $k = 1$ and $\Phi = 0$ for which $\nu(\eta(x), \eta(y)) = -\frac{\rho(1-\rho)}{|\Lambda| - 1}$.

7.4 Mean field perturbation of Ising model

Here we briefly discuss the application of our results to mean field perturbations of the standard Ising model of the form

$$d\mu_\Lambda^{\beta, t} := Z^{-1} \exp\left(\frac{t}{2} \frac{M^2}{|\Lambda|} \right) d\mu_\Lambda^\beta$$

where $M := M_\Lambda = 2N_\Lambda - |\Lambda|$ is the usual (unnormalized) magnetization and μ_Λ^β denotes the Ising measure in Λ with *free* or *periodic* boundary conditions at inverse temperature β

$$\mu_\Lambda^\beta(\eta) = Z_\Lambda^{-1} \exp\bigl(\beta \sum_{n.n} (2\eta(x) - 1)(2\eta(y) - 1) \bigr)$$

Here $\sum_{n.n}$ means the sum over all nearest neighbor pairs in Λ.
Such an interesting problem was discussed in great detail in [CZ] by different methods. Here we

improve one of the result of [CZ] as follows. Let β be such that property $SMT(V,C,m,l)$ holds for some l and all sets $V \in \mathbb{F}_l$. In dimension $d=2$ this is true for all $\beta<\beta_c$, β_c being the critical value [MOS]. Then, for t small enough and Λ large enough,

$$|\,\mu_\Lambda^{\beta,t}(f)-\mu_\Lambda^{\beta}(f)\,| \le C\|f\|_\infty \frac{|\Delta|}{|\Lambda|} \tag{7.18}$$

for all f with support Δ smaller than $|\Lambda|^{1-\epsilon}$.

Remark 1. Essentially the limitation on the parameter t is of the form $t<\chi(\beta)^{-1}$ where $\chi(\beta)=\sum_{x\in\mathbb{Z}^d}\mu(\eta(0),\eta(x))$ is the magnetic susceptibility (see [CZ]).

Remark 2. The above estimate gives also the correct bound on the covariances of the perturbed measure,

$$|\,\mu_\Lambda^{\beta,t}(f,g)\,| \le C'\|f\|_\infty\,\|g\|_\infty\,\big(\frac{|\Delta_f|}{|\Lambda|}+\frac{|\Delta_g|}{|\Lambda|}+|\Delta_g|\,|\Delta_f|e^{-md(\Delta_f,\Delta_g)}\big) \tag{7.19}$$

as one can immediately check in the case $\beta=0$ and $f(\eta)=\eta(x)$, $g(\eta)=\eta(y)$, $x\neq y$.

In order to prove the bound (7.18) we proceed as follows. Let, for $m\in[-|\Lambda|,-|\Lambda|+2,\dots,|\Lambda|]$, $\mu^{(m)}$ be defined as

$$d\mu_\Lambda^{\beta,\lambda(m)} := Z_{\lambda(m)}^{-1}\exp\left(\lambda(m)N_\Lambda\right)d\mu_\Lambda^\beta$$

$\lambda(m)$ being the unique value of λ such that $\mu^\lambda(M)=m$. Because of the free b.c. $\lambda(m)=-\lambda(-m)$. Moreover, for any $m\le 0$, thanks to *c)* of proposition 3.1

$$|m| = |\int_0^{\lambda(m)} d\lambda'\,\mu_\Lambda^{\beta,\lambda'}(N_\Lambda,N_\Lambda)\,| \ge |\lambda(m)|\,A^{-1}\,\frac{(m+|\Lambda|)}{2}\wedge|\Lambda| \tag{7.20}$$

so that $|\lambda(m)|\le A\frac{|m|}{|\Lambda|}$ for all $|m|\le\frac{1}{2}|\Lambda|$.
For any small enough δ we now write

$$\begin{aligned}&|\,\mu_\Lambda^{\beta,t}(f)-\mu_\Lambda^{\beta}(f)\,|\le\\ &\sum_m \mu_\Lambda^{\beta,t}(M=m)\big|\,\mu_\Lambda^\beta(f\,|\,M=m)-\mu^{(m)}(f)\,\big|+\big|\sum_m\mu_\Lambda^{\beta,t}(M=m)[\,\mu_\Lambda^\beta(f)-\mu^{(m)}(f)\,]\,\big|\\ &\le C\|f\|_\infty\frac{|\Delta|}{|\Lambda|}+\|f\|_\infty\,\mu_\Lambda^{\beta,t}(|M|\ge\delta|\Lambda|)+\big|\sum_{\substack{m\\|m|\le\delta|\Lambda|}}\mu_\Lambda^{\beta,t}(M=m)[\,\mu_\Lambda^\beta(f)-\mu^{(m)}(f)\,]\,\big|\end{aligned} \tag{7.21}$$

where we have used theorem 4.1, the above bound on $\lambda(m)$ and the fact that the measure $\mu^{(m)}$ satisfies the usual mixing condition provided that $\lambda(m)$ is small enough.
The second term in the r.h.s. of (7.21) is exponentially small in $|\Lambda|$ provided that t is small enough by standard large deviations estimates (see [CZ] for a detailed discussion). We are left with the estimate of the third term in (7.21). We write

$$\mu^{\beta,\lambda(m)}(f)-\mu_\Lambda^\beta(f)=\int_0^{\lambda(m)}d\lambda'\,\mu^{\beta,\lambda'}(f,N_\Lambda)=\mu(f,N_\Lambda)\lambda(m)+F(\lambda(m))$$

with

$$|F(\lambda(m))| = |\int_0^{\lambda(m)} d\lambda' \int_0^{\lambda'} d\lambda''\, \mu^{\lambda''}(f, N_\Lambda, N_\Lambda)\,| \leq B_0 |\Delta| \lambda(m)^2 \qquad \forall\, |m| \leq \delta|\Lambda|$$

for a suitable constant B_0. Therefore

$$\begin{aligned}
\Big|\sum_{m:\,|m|\leq\delta|\Lambda|} \mu_\Lambda^{\beta,t}(M=m)\big[\mu^{(m)}(f) - \mu_\Lambda^{\beta}(f)\big]\,\Big| &\leq B_1 \|f\|_\infty\, |\Delta|^{\frac{1}{2}}\, \Big[\,\Big|\sum_{|m|\leq\delta|\Lambda|} \mu_\Lambda^{\beta,t}(M=m)\lambda(m)\,\Big| \\
&\qquad + \sum_{|m|\leq\delta|\Lambda|} \mu_\Lambda^{\beta,t}(M=m)\lambda^2(m)\,\Big] \\
&\leq B_2 \|f\|_\infty\, \mu_\Lambda^{\beta,t}(N_\Lambda, N_\Lambda)\, \frac{|\Delta|}{|\Lambda|^2} \\
&\leq B_3 \|f\|_\infty \frac{|\Delta|}{|\Lambda|}
\end{aligned} \tag{7.22)}$$

for suitable constants B_1, B_2, B_3. Above we have used the fact that

$$\sum_{|m|\leq\delta|\Lambda|} \mu_\Lambda^{\beta,t}(M=m)\lambda(m) = 0$$

due to the simmetry under global spin flip, together with the bound, valid for small t, $\mu_\Lambda^{\beta,t}(N_\Lambda, N_\Lambda) \leq C|\Lambda|$.
The desired estimate then follows by combining (7.22) and (7.21).

Appendix

Here we discuss in some more detail the existence of the tilting fields introduced in section 7.3.

Let $S = \{0,1\}$ and $h(\eta) = \eta$. Let $\Lambda = \cup_{i=1}^k \Lambda_i$, where the atoms $\Lambda_1, \dots, \Lambda_k$ are pairwise disjoint and such that there exists $x_1, \dots, x_k$, $x_i \in \Lambda_i \quad i = 1, \dots, k$, such that $\min_{i\neq j} d(x_i, x_j) \geq 2r$. Given $\underline{\lambda} = (\lambda_1, \dots, \lambda_k)$ let $d\mu_0^{(i)}(\eta) = \left(Z_0^{\lambda_i}\right)^{-1} \exp(\lambda_i \eta)\, d\mu_0(\eta)$, μ_0 being the fair measure on S and $Z_0^{\lambda_i}$ a normalization factor, and let $d\mu_\Lambda^{\tau,\underline{\lambda}}(\eta) := \left(Z_\Lambda^{\tau,\underline{\lambda}}\right)^{-1} \exp\left(-H_\Lambda^\tau(\eta)\right) \prod_{x\in\Lambda} d\mu_0^{\{x\}}(\eta(x))$, $\mu_0^{\{x\}} = \mu_0^{(i)}$ if $x \in \Lambda_i$, be the corresponding finite volume Gibbs measure. Let finally $\rho_i(\underline{\lambda}) := |\Lambda_i|^{-1} \mu_\Lambda^{\tau,\underline{\lambda}}(N_{\Lambda_i})$. Then, for any $N_i^* \in (0, |\Lambda_i|)$, $i = 1, \cdots, k$, there exists $\underline{\lambda}^*$ such that $\mu_\Lambda^{\tau,\underline{\lambda}^*}(N_{\Lambda_i}) = N_i^*, \quad i = 1, \cdots, k$.

Proof. Given $k' \leq k$, consider the Jacobian matrix $\mathbf{J}_{ij}^{(k')} = \frac{\partial}{\partial\lambda_i} \mu_\Lambda^{\tau,\underline{\lambda}}(N_{\Lambda_i}) = \mu_\Lambda^{\tau,\underline{\lambda}}\left(N_{\Lambda_i}, N_{\Lambda_j}\right)$, $i,j = 1, \dots, k'$. It is relatively simple to prove that such a matrix is positive definite. For each atom Λ_i fix in fact a site x_i as above. Let $\mathcal{F}_0$ be the σ–algebra generated by $\eta(x)$, $x \in \Lambda$, $x \neq \{x_1, \dots, x_k\}$. Then, by the formula for conditional covariance, we get

$$\begin{aligned} \mu_\Lambda^{\tau,\underline{\lambda}}\left(N_{\Lambda_i}, N_{\Lambda_j}\right) &= \mu_\Lambda^{\tau,\underline{\lambda}}\left(\mu_\Lambda^{\tau,\underline{\lambda}}(N_{\Lambda_i}|\mathcal{F}_0), \mu_\Lambda^{\tau,\underline{\lambda}}\left(N_{\Lambda_j}|\mathcal{F}_0\right)\right) + \mu_\Lambda^{\tau,\underline{\lambda}}\left(\mu_\Lambda^{\tau,\underline{\lambda}}\left(N_{\Lambda_i}, N_{\Lambda_j}|\mathcal{F}_0\right)\right) \\ &= \mu_\Lambda^{\tau,\underline{\lambda}}\left(\mu_\Lambda^{\tau,\underline{\lambda}}(N_{\Lambda_i}|\mathcal{F}_0), \mu_\Lambda^{\tau,\underline{\lambda}}\left(N_{\Lambda_j}|\mathcal{F}_0\right)\right) + \mu_\Lambda^{\tau,\underline{\lambda}}\left(\mu_\Lambda^{\tau,\underline{\lambda}}\left(\eta(x_i), \eta(x_j)|\mathcal{F}_0\right)\right) \end{aligned} \tag{7.23}$$

Let us denote for simplicity α_{ij} and β_{ij} the first and second term in the r.h.s. of (7.23). Since $d(x_i, x_j) \geq 2r$ then clearly $\beta_{ij} = 0$ unless $i = j$ and in that case $\beta_{ii} > 0$. Moreover the matrix (α_{ij}) is non negative since $\sum_{i,j} t_i t_j \alpha_{ij} = \mathrm{Var}_\Lambda^{\tau,\underline{\lambda}}\left(\sum_i t_i\, \mu_\Lambda^{\tau,\underline{\lambda}}(N_{\Lambda_i}|\mathcal{F}_0)\right) \geq 0$. Thus the Jacobian $\mathbf{J}^{(k')}$ is positive definite for any $k' \leq k$.

Assume now by induction that, given $N_1, \dots, N_k$ with $N_i \in (0, |\Lambda_i|)$ $\forall\, i = 1, \dots, k$ and $\lambda_{i+1}, \dots, \lambda_k$, we have been able to fix $\lambda_1, \dots, \lambda_i$ depending on $N_1, \dots, N_i, \lambda_{i+1}, \dots, \lambda_k$ in such a way that $\mu_\Lambda^{\tau,\underline{\lambda}}(N_{\Lambda_j}) = N_j$, $j = 1 \dots i$. When $i = 1$ this is trivial since $\mu_\Lambda^{\tau,\underline{\lambda}}(N_{\Lambda_1})$, as a function of only λ_1, is monotone increasing with $\lim_{\lambda_1\to+\infty} \mu_\Lambda^{\tau,\underline{\lambda}}(N_{\Lambda_1}) = |\Lambda_1|$ and $\lim_{\lambda_1\to-\infty} \mu_\Lambda^{\tau,\underline{\lambda}}(N_{\Lambda_1}) = 0$. Moreover, thanks to the DLR equations, the above limits are attained uniformly in $\lambda_2, \dots, \lambda_k$. In order to fix now the next field λ_{i+1} we compute

$$\frac{d}{d\lambda_{i+1}} \mu_\Lambda^{\tau,\underline{\lambda}}\left(N_{\Lambda_{i+1}}\right) = \frac{|\mathbf{J}^{(i+1)}|}{|\mathbf{J}^{(i)}|}$$

where $\mathbf{J}^{(i)} := \mathbf{J}^{(i)}(\lambda_{i+1}, \dots \lambda_k)$ is the Jacobian matrix defined above. Thus, thanks to the previous result, $\mu_\Lambda^{\tau,\underline{\lambda}}\left(N_{\Lambda_{i+1}}\right)$ is a monotone increasing function of λ_{i+1} with image $(0, |\Lambda_{i+1}|)$ and we can therefore find $\lambda_{i+1} = \lambda_{i+1}(N_{i+1}, \lambda_{i+2}, \dots, \lambda_k)$ such that $\mu_\Lambda^{\tau,\underline{\lambda}}\left(N_{\Lambda_{i+1}}\right) = N_{i+1}$. The induction step is complete. $\square$

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