THE NODAL SURFACE OF THE SECOND EIGENFUNCTION OF THE LAPLACIAN IN $\mathbf{R}^D$ CAN BE CLOSED

SØREN FOURNAIS

Abstract. We construct a set in $\mathbf{R}^D$ with the property that the nodal surface of the second eigenfunction of the Dirichlet Laplacian is closed, i.e. does not touch the boundary of the domain. The construction is explicit in all dimensions $D \geq 2$ and we obtain explicit control of the connectivity of the domain.

Contents

1. Introduction

It is a famous conjecture by Payne [Pay67] that the nodal surface of the 2nd eigenfunction of the Dirichlet Laplacian on a bounded, connected domain Ω in $\mathbf{R}^D$ touches the boundary $\partial\Omega$. For convex domains in $\mathbf{R}^2$ this conjecture was proved by Melas [Mel92]. However, M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof and N. Nadirashvili [HOHON98] constructed a non-simply connected counterexample to the general conjecture in $\mathbf{R}^2$. The 2-dimensional example in [HOHON98] relies heavily on choosing a very symmetric domain and using symmetry arguments. In higher dimensions it is not possible to choose similar, very symmetric domains. The obstruction being that there are only a finite number of regular polyhedra in any dimension greater than or equal to 3 (in 3 dimensions these are the platonic solids). Below we will look at almost the same domain as in the above mentioned paper. Because of the lack of symmetries the argument from [HOHON98] cannot be applied. We shall use an alternative, and in a way more direct, argument to reach the desired conclusion.

1.1. Generalities.

For a bounded connected domain Ω (with sufficiently regular boundary) we will look at the Laplace operator with Dirichlet conditions at the boundary. This defines a

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positive, self adjoint operator $-\Delta_\Omega$ with domain $W_0^{1,2}(\Omega)$ (see [GT83] for notation) and purely discrete spectrum. We denote the eigenvalues (eigenfunctions) by $\{\lambda_j(\Omega)\}_{j=1}^\infty$ ($\{u_j(\Omega)\}_{j=1}^\infty$), so

$$\begin{aligned} -\Delta_\Omega u_j(\Omega) &= \lambda_j(\Omega) u_j(\Omega) \text{ in } \Omega \\ u_j(\Omega) &\equiv 0 \text{ on } \partial\Omega, \end{aligned}$$

and the eigenvalues are ordered according to size: $0 < \lambda_1(\Omega) < \lambda_2(\Omega) \leq \lambda_3(\Omega) \leq ...$ (It is a general result that $\lambda_1(\Omega)$ is simple and strictly positive). We may take the eigenfunctions to be real and orthonormal: $\langle u_j(\Omega), u_k(\Omega)\rangle = \delta_{j,k}$, where δ is the Kronecker delta. Since the eigenfunctions are real, and the first can be chosen positive, the second eigenfunction $u_2(\Omega)$ has to take both positive and negative values. According to Courant's Nodal Domains Theorem Ω splits into exactly two connected open sets Ω_+, Ω_- such that $u_2 > 0$ on Ω_+, $u_2 < 0$ on Ω_- and $\overline{\Omega} = \overline{\Omega}_+ \cup \overline{\Omega}_-$.
It is now natural to study the geometry of the nodal set $\mathcal{N}(u_2)$, where

$$\mathcal{N}(u_2) = \overline{\{x \in \Omega | u_2(\Omega)(x) = 0\}}.$$

Generically (see [Uhl72]), this is a manifold of codimension 1, and one may ask wether it always touches the boundary of the domain, i.e. wether

$$\mathcal{N}(u_2) \cap \partial\Omega \neq \emptyset$$

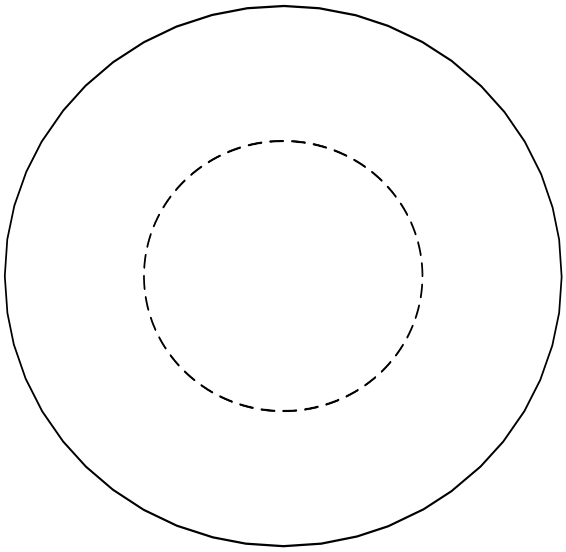
always. This is the above mentioned conjecture by Payne [Pay67].

1.2. **The domain.** We choose $0 < R_1 < R_2$ such that

$$\lambda_1(B(R_1)) < \lambda_1(B(R_2) \setminus \overline{B(R_1)}) < \lambda_2(B(R_1)).$$

Furthermore we let $N \in \mathbf{N}$, and let $\{x_1, ..., x_N\} \subset \{x \in \mathbf{R}^D \big| |x| = R_1\}$. Then we let $\epsilon \in \mathbf{R}_+ \cup \{0\}$ and define

$$\Omega_\epsilon = (B(R_2) \setminus \overline{B(R_1)}) \cup \left(\cup_{j=1}^N B(x_j, \epsilon)\right) \cup B(R_1).$$



A typical domain

As a measure of the distance between the points we introduce:

$$\delta = \inf\left\{\delta > 0 \middle| \{x \in \mathbf{R}^D \big| |x| = R_1\} \subset \cup_{j=1}^N B(x_j, \delta)\right\}.$$

This measures the maximal distance between neighboring points. Let us also introduce the simpler:

$$\rho = \min_{j \neq k}\{\mathrm{dist}_{\mathbb{S}^{N-1}}(x_j, x_k)\},$$

where $\mathrm{dist}_{\mathbb{S}^{N-1}}$ is geodesic distance on the sphere. We will always assume that $\rho > 0$ and that $\epsilon < \rho/2$. To make sure that the holes are evenly distributed, we assume $\delta/\rho \leq c_0$, where c_0 is some constant which will be assumed fixed throughout the paper.

We will simplify notation by writing $\lambda_{j,\epsilon}$ and $u_{j,\epsilon}$ instead of the heavier $\lambda_j(\Omega_\epsilon)$ and $u_j(\Omega_\epsilon)$

Notice that $\Omega_0 = (B(R_2) \setminus \overline{B(R_1)}) \cup B(R_1)$, is the union of two disjoint domains and that the Dirichlet Laplacian on this set is explicitly solvable in terms of Bessel functions.

In [Sto95] it was proved that $-\Delta_\epsilon$ converges to $-\Delta_0$ in strong resolvent sense as $\epsilon \searrow 0$. Thus in particular

$$\lambda_{j,\epsilon} \nearrow \lambda_{j,0}.$$

From this we see that when ϵ is sufficiently small, then $\lambda_{2,\epsilon}$ is a simple eigenvalue. We choose $u_{1,0}$ and $u_{2,0}$ to be positive functions.

Define $\mathcal{N}(u_{2,\epsilon}) = \overline{\{x \in \Omega_\epsilon \big| u_{2,\epsilon}(x) = 0\}}$, then the result of this paper is:

Theorem 1.1. *There exists a δ_0 such that if $\delta \leq \delta_0$ then*

$$\mathcal{N}(u_{2,\epsilon}) \cap \partial\Omega_\epsilon = \emptyset,$$

for all ϵ sufficiently small.

Remark 1.2. Thus, the theorem says, that if we cut many, small holes, and they are almost evenly distributed over the sphere, then the nodal surface will be closed.

In section 4 we get an explicit upper bound on the minimal number of holes necessary which is $\approx 10^9$.

We use [Bas95] and [PS78] as standard references for results on stochastic processes. In those books references to the original articles can be found.

2. Preliminary Estimates

The most important result in this section is the following estimate:

Lemma 2.1. *$\exists C > 0$ independent of ϵ, δ such that*

$$|\langle u_{j,\epsilon}, u_{2,0}\rangle| \le C \frac{\lambda_{j,\epsilon}^{1+D/4}}{|\lambda_{j,\epsilon} - \lambda_{2,0}|} N\epsilon^D + O(\epsilon^{D+1}),$$

when ϵ is sufficiently small (dep. on δ). Here C can be chosen as:

$$C = (\partial_r u_{2,0}|_{r=R_1})\sigma_{D-1} e^{1+\frac{1}{8\pi}} C_0,$$

where C_0 is the constant given in Lemma 2.6 below, and $\sigma_{D-1} = \mathrm{vol}_{\mathbf{R}^{D-1}}(B(1))$.

Remark 2.2. $\delta^{-(D-1)}$ is proportional to the number of holes, so N in the above Lemma can be changed to $\delta^{-(D-1)}$, up to a change in the constant C.

The proof of Lemma 2.1 is given in the rest of this section as a series of lemmas.

Lemma 2.3.

$$\langle u_{j,\epsilon}, u_{2,0}\rangle = -\frac{\partial_r u_{2,0}|_{r=R_1}}{\lambda_{j,\epsilon} - \lambda_{2,0}} \int_{S_\epsilon} u_{j,\epsilon}(y) d\sigma(y),$$

where $S_\epsilon = \{x \in \Omega_\epsilon \big| |x| = R_1\}$ and σ is surface measure on the sphere $\{|x| = R_1\}$.

Proof. This is, in fact, just Green's identity:

$$\begin{aligned} \lambda_{j,\epsilon}\langle u_{j,\epsilon}, u_{2,0}\rangle &= \langle -\Delta_\epsilon u_{j,\epsilon}, u_{2,0}\rangle \\ &= \lambda_{2,0}\langle u_{j,\epsilon}, u_{2,0}\rangle - \int_{S_\epsilon} u_{j,\epsilon}(y)\partial_r u_{2,0}(y) d\sigma(y). \end{aligned}$$

Now we use that $u_{2,0}$ is rotationally symmetric to reach the conclusion. □

Thus we need to estimate $\int_{S_\epsilon} u_{j,\epsilon}(y) d\sigma(y)$.

Notice the following argument:

Lemma 2.4. *$\forall y \in \Omega_\epsilon$ we have:*

$$|u_{j,\epsilon}(y)| \le \|u_{j,\epsilon}\|_\infty e \lambda_{j,\epsilon} \mathbb{E}^y[\tau_\epsilon],$$

where τ_ϵ is the exit time of Brownian motion from Ω_ϵ i.e.

$$\tau_\epsilon = \inf\{t > 0 | W_t \notin \Omega_\epsilon\},$$

where W_t is D-dimensional Brownian motion and $\mathbb{E}$ denotes the expectation.

Proof.

$$\begin{aligned} u_{j,\epsilon}(y) &= e^{t\lambda_{j,\epsilon}}(e^{t\Delta_\epsilon} u_{j,\epsilon})(y) \\ &= e^{t\lambda_{j,\epsilon}} \mathbb{E}^y[u_{j,\epsilon}(W_t), t < \tau_\epsilon] \\ &\le e^{t\lambda_{j,\epsilon}} \|u_{j,\epsilon}\|_\infty \mathbb{E}^y[1 < \tau_\epsilon/t] \\ &\le e^{t\lambda_{j,\epsilon}} \|u_{j,\epsilon}\|_\infty \mathbb{E}^y[\tau_\epsilon]/t. \end{aligned}$$

Now we put $t = \lambda_{j,\epsilon}^{-1}$ to get the lemma. □

Lemma 2.5.

$$\|u_{j,\epsilon}\|_\infty \leq e^{1/(8\pi)}(\lambda_{j,\epsilon})^{D/4}.$$

Proof. This was proved in [Dav89][p. 63]. I am grateful to T. Hoffmann-Ostenhof for pointing my attention to this reference. □

Thus we will prove a bound $\mathbb{E}^y[\tau_\epsilon] = O(\epsilon)$ when $y \in S_\epsilon$. This will finish the proof of Lemma 2.1.

Lemma 2.6. *Let $W_t = (X_t^1, X_t^2, ..., X_t^D)$ be D-dimensional Brownian motion. Let*

$$\tau_\epsilon = \inf\{t > 0 | W_t \notin \Omega_\epsilon\}.$$

Then $\exists C_0 > 0$ such that $\forall \delta > 0$

$$\sup_{y \in S_\epsilon} \mathbb{E}^y[\tau_\epsilon] \leq C_0\epsilon + O(\epsilon),$$

when ϵ is sufficiently small (depending on δ). Here C_0 can be chosen as:

$$C_0 = \frac{(D-2)D}{\sqrt{D-1}} \frac{R_1^{-(D-1)}}{R_1^{-(D-2)} - R_2^{-(D-2)}}(R_2^2 - R_1^2)(1 + R_2/R_1),$$

for $D \geq 3$ and

$$C_0 = \frac{1}{2}(R_2^2 - R_1^2)R_1^{-1}\frac{1 + R_2/R_1}{\log(R_2/R_1)},$$

for $D = 2$.

Proof. Let $M_t = W_t^2 - Dt$, then M_t is a martingale since the different coordinates of W_t are independent 1-dimensional Brownian motions. Let $x \in S_\epsilon$, we may choose the coordinates so that $x = (R_1, 0, ..., 0)$. Now, by the martingale property,

$$\mathbb{E}^x[M_t] = \mathbb{E}^x[M_0] = R_1^2,$$

so

$$\begin{aligned} D\mathbb{E}^x[\tau_\epsilon] &= \mathbb{E}^x[W_{\tau_\epsilon}^2] - R_1^2 \\ &= R_1^2\mathbb{P}^x[|W_{\tau_\epsilon}| = R_1] + R_2^2\mathbb{P}^x[|W_{\tau_\epsilon}| = R_2] - R_1^2 \\ &= (R_2^2 - R_1^2)\mathbb{P}^x[|W_{\tau_\epsilon}| = R_2]. \end{aligned}$$

We will now prove that when ϵ is sufficiently small, then

$$\mathbb{P}^x[|W_{\tau_\epsilon}| = R_2] \leq C_1\epsilon + (\frac{1}{2} + O(\epsilon)) \sup_{y \in S_\epsilon} \mathbb{P}^y[|W_{\tau_\epsilon}| = R_2].$$

Since $x \in S_\epsilon$ was arbitrary, this proves the lemma, with $C_0 = 2\frac{R_2^2 - R_1^2}{D}C_1$.

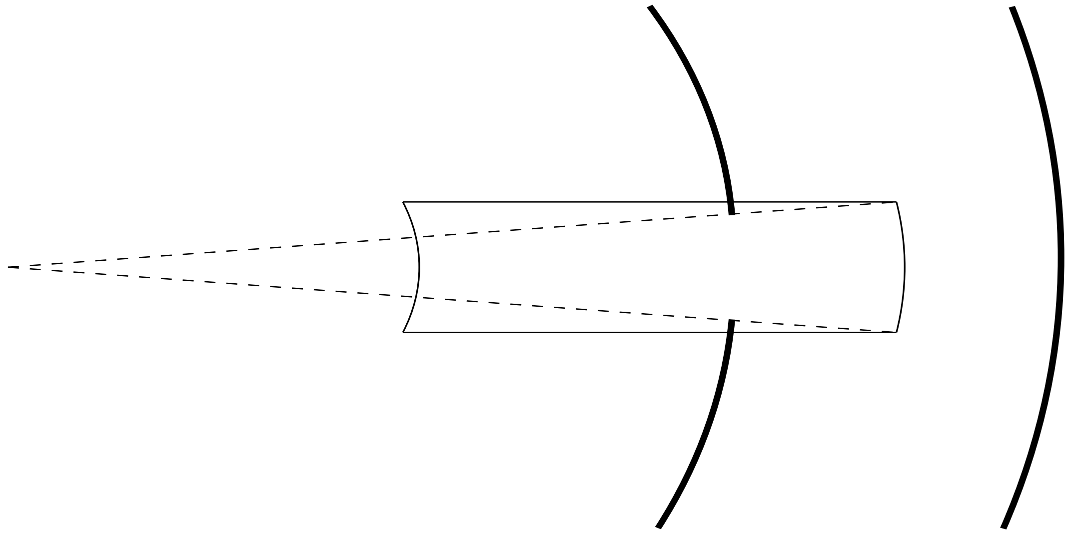
We introduce the following stopping times:

Let

$$k = \frac{\epsilon}{\sqrt{R_1^2 - \epsilon^2}} \frac{R_1 + R_2}{2} \approx \frac{R_1 + R_2}{2R_1}\epsilon,$$

and define

$$\tau_1^\epsilon = \inf\{t > 0 | \sqrt{\sum_{j=2}^D (X_t^j)^2} = k \text{ or } |X_t^1| = \frac{R_1 + R_2}{2} \text{ or } |X_t^1| = R_1/2\}.$$



The stopping time τ_1^ϵ

Define furthermore

$$\begin{aligned}\tau_1 &= \inf\{t>0 | |W_t| = R_1\},\\ \tau_2 &= \inf\{t>0 | |W_t| = R_2\}.\end{aligned}$$

Then

$$\begin{aligned}\mathbb{P}^x[|W_{\tau_\epsilon}| = R_2] &= \mathbb{P}^x[|W_{\tau_\epsilon}| = R_2 \wedge \tau_1^\epsilon < \tau^\epsilon]\\ &\leq \mathbb{P}^x[|W_{\tau_\epsilon}| \circ \theta_{\tau_1^\epsilon} = R_2]\\ &= \mathbb{E}^x[\mathbb{P}^{W_{\tau_1^\epsilon}}[|W_{\tau_\epsilon}| = R_2]],\end{aligned}$$

where we used the strong Markov property of Brownian motion. We continue the calculation:

$$\begin{aligned}\mathbb{E}^x[\mathbb{P}^{W_{\tau_1^\epsilon}}[|W_{\tau_\epsilon}| = R_2]] &= \mathbb{E}^x[\mathbb{P}^{W_{\tau_1^\epsilon}}[|W_{\tau_\epsilon}| = R_2, \tau_1<\tau_2, W_{\tau_1}\in S_\epsilon]] + \mathbb{E}^x[\mathbb{P}^{W_{\tau_1^\epsilon}}[\tau_1>\tau_2]]\\ &\equiv a+b.\end{aligned}$$

Let us look at the first term a. Below we will once again use the strong Markov property of Brownian motion.

$$\begin{aligned}\mathbb{P}^{W_{\tau_1^\epsilon}}[|W_{\tau_\epsilon}| = R_2, \tau_1<\tau_2, W_{\tau_1}\in S_\epsilon] &= \mathbb{P}^{W_{\tau_1^\epsilon}}[|W_{\tau_\epsilon}|\circ\theta_{\tau_1} = R_2, \tau_1<\tau_2, W_{\tau_1}\in S_\epsilon]\\ &= \mathbb{E}^{W_{\tau_1^\epsilon}}[1_{\{\tau_1<\tau_2\}}1_{\{W_{\tau_1}\in S_\epsilon\}}\mathbb{P}^{W_{\tau_1}}[|W_{\tau_\epsilon}| = R_2]]\\ &\leq (\sup_{y\in S_\epsilon}\mathbb{P}^y[|W_{\tau_\epsilon}| = R_2])\mathbb{E}^{W_{\tau_1^\epsilon}}[1_{\{\tau_1<\tau_2\}}1_{\{W_{\tau_1}\in S_\epsilon\}}]\\ &\leq (\frac{1}{2}+O(\epsilon))(\sup_{y\in S_\epsilon}\mathbb{P}^y[|W_{\tau_\epsilon}| = R_2]),\end{aligned}$$

when ϵ is sufficiently small (dep. on δ). Here we used that due to symmetry the chance of "falling back" into the hole we came from, is $\leq 1/2$, and the probability of falling into another hole is $O(\epsilon)$ as $\epsilon\to 0$ (this follows from [PS78][Thm. 3.1. p. 102]).

Thus we only need an estimate of order ϵ of the term b. This is easily accomplished: First for $D\geq 3$. Since $|W_t|^{-(D-2)}$ is a martingale and leaves $[R_2^{-(D-2)}, R_1^{-(D-2)}]$ with probability one, we get ([Bas95][Cor. 4.10 p.33])

$$\mathbb{P}^{W_{\tau_1^\epsilon}}[\tau_1>\tau_2] = \frac{R_1^{-(D-2)} - |W_{\tau_1^\epsilon}|^{-(D-2)}}{R_1^{-(D-2)} - R_2^{-(D-2)}}1_{\{|W_{\tau_1^\epsilon}|>R_1\}}.$$

Therefore, by a first order Taylor expansion,

$$\begin{aligned} b &\leq c\mathbb{E}^x[|X^1_{\tau_1^\epsilon} - R_1|] \\ &\leq c\sqrt{\mathbb{E}^x[|X^1_{\tau_1^\epsilon} - R_1|^2]} \\ &= c\mathbb{E}^x[\sqrt{\tau_1^\epsilon}], \end{aligned}$$

by the Jensen inequality, since $(X_t^1)^2 - t$ is a martingale. Here c can be chosen (up to errors of higher order in ϵ) as

$$c = \frac{(D-2)R_1^{-(D-1)}}{R_1^{-(D-2)} - R_2^{-(D-2)}}.$$

Now $\tau_1^\epsilon \leq \tilde{\tau}_1^\epsilon$, where $\tilde{\tau}_1^\epsilon = \inf\{t > 0 | \sqrt{\sum_{j=2}^D (X_t^j)^2} = k\}$. Remember that $x = (R_1, 0, .., 0)$ By scaling

$$\mathbb{E}^x[\tilde{\tau}_1^\epsilon] = k^2 \mathbb{E}^x[\tau],$$

where $\tau = \inf\{t > 0 | \sqrt{\sum_{j=2}^D (X_t^j)^2} = 1\}$.

For $D = 2$ we have to use $\log |W_t|$ instead of $|W_t|^{-(D-2)}$.

$$\begin{aligned} \mathbb{P}^{W_{\tau_1^\epsilon}}[\tau_1 > \tau_2] &= \frac{\log |W_{\tau_1^\epsilon}| - \log R_1}{\log(R_2/R_1)} 1_{\{|W_{\tau_1^\epsilon}| > R_1\}} \\ &\leq R_1^{-1} \frac{\big||W_{\tau_1^\epsilon}| - R_1\big|}{\log(R_2/R_1)} 1_{\{|W_{\tau_1^\epsilon}| > R_1\}} \\ &\leq R_1^{-1} \frac{\Big||X^1_{\tau_1^\epsilon}| - R_1\Big|}{\log(R_2/R_1)} 1_{\{|W_{\tau_1^\epsilon}| > R_1\}} + O(\epsilon^2). \end{aligned}$$

Now, $\sum_{j=2}^D (X_t^j)^2 - (D-1)t$ is a martingale, thus

$$\mathbb{E}^x[\tau] = \frac{1}{D-1} \mathbb{E}^x[\sum_{j=2}^D (X_\tau^j)^2] = \frac{1}{D-1},$$

and we get

$$C_1 = \frac{D-2}{\sqrt{D-1}} \frac{R_1^{-(D-1)}}{R_1^{-(D-2)} - R_2^{-(D-2)}} \frac{1 + R_2/R_1}{2},$$

for $D \geq 3$, and

$$C_1 = \frac{R_1^{-1}}{\log(R_2/R_1)} \frac{1 + R_2/R_1}{2},$$

for $D = 2$. □

3. Proof of the main theorem

Here we will prove the following, more precise, version of the main theorem. Let us first fix the relative magnitude of the various parameters: Since $\rho \approx \delta$, $N \approx \delta^{-(D-1)}$ and we can express everything in terms of δ.

Theorem 3.1. $\exists \delta_0 > 0\ \forall \delta < \delta_0\ \exists \epsilon > 0$ *such that* $u_{2,\epsilon}(x) \geq 0\ \forall x$ *with* $|x| = R_1 - \delta$.

Remark 3.2. The following argument shows that Theorem 3.1 implies Theorem 1.1:
Let $R_0^\epsilon < R_1$ be chosen such that
$$\lambda_1(B(R_0^\epsilon)) = \lambda_{2,\epsilon}.$$
Then $R_0^\epsilon \searrow R_0$, where $\lambda_1(B(R_0)) = \lambda_{2,0}$. Suppose now that $u_{2,\epsilon}(x) \geq 0 \; \forall |x| \leq R_1 - \delta$. For δ and ϵ small enough this implies that $u_{2,\epsilon}(x)$ is positive on an open set containing $B(R_0)$ and therefore, by Courant's Nodal Domains Theorem, that $\lambda_{2,\epsilon} > \lambda_{2,0}$. This is a contradiction, thus $u_{2,\epsilon}$ takes negative values inside the sphere of radius $R_1 - \delta$ and the nodal surface gets trapped.

The strategy of the proof is the following:
We look at
$$(-\Delta_\epsilon)^{-(n+1)} u_{2,0}(x),$$
where $|x| = R_1 - \delta$. On one hand, we can express this, using the spectral theorem, as a sum of terms of the form
$$\frac{1}{\lambda_{j,\epsilon}^{n+1}} \langle u_{j,\epsilon}, u_{2,0} \rangle u_{j,\epsilon}(x),$$
where we expect the term with $j = 2$ to be the most important one. If $u_{2,\epsilon}(x) \leq 0$ we get
$$(-\Delta_\epsilon)^{-(n+1)} u_{2,0}(x) \leq \left| \sum_{j \neq 2} \frac{1}{\lambda_{j,\epsilon}^{n+1}} \langle u_{j,\epsilon}, u_{2,0} \rangle u_{j,\epsilon}(x) \right|,$$
Which will be small in a suitable sense, i.e. $O(\epsilon^D)$.
On the other hand, we can estimate
$$(-\Delta_\epsilon)^{-(n+1)} u_{2,0}(x),$$
from below, rather explicitely, using Brownian motion techniques. This gives a lower bound, which is also of order ϵ^D. By keeping track of how the constants in both bounds depend on δ, we get a contradiction, for small δ, if $u_{2,\epsilon}(x) \leq 0$. Notice, that it is essential for the lower bound that $u_{2,0}$ is a positive function.

Now we will give the details:

Lemma 3.3. *Let $y \in S_\epsilon \equiv \{x \in \Omega_\epsilon \big| |x| = R_1\}$, and let $d = \operatorname{dist}(y, \partial\Omega_\epsilon) > 0$. Then $\exists c, C > 0$ independent of ϵ, δ such that*
$$\frac{1}{\lambda_{2,\epsilon}^{n+1}} \langle u_{2,\epsilon}, u_{2,0} \rangle u_{2,\epsilon}(y) \geq cd - C\epsilon^2 - C\epsilon^{D+1}\delta^{-1}.$$
Here c can be chosen as $c = \frac{1}{\lambda_{2,0}^{n+1}} \frac{1}{8} \partial_r u_{2,0}|_{r=R_1}$

Proof. Let F be a box around y with sidelength d with two sides at right angles to the vector from y to the origin. Let F_1 be the side of F with the largest distance to the origin. We define the stopping time τ_F as the exit time from F i.e.
$$\tau_F(\omega) = \inf\{t \geq 0 \big| W_t(\omega) \notin F\}.$$
Let also
$$\begin{aligned} \tau_0(\omega) &= \inf\{t \geq 0 \big| W_t(\omega) \notin \Omega_0\} \\ \tau_\epsilon(\omega) &= \inf\{t \geq 0 \big| W_t(\omega) \notin \Omega_\epsilon\}. \end{aligned}$$
Notice that $\tau_F \leq \tau_0 \leq \tau_\epsilon$. We look at the iterated resolvent $(-\Delta_\epsilon)^{-(n+1)}$ for a sufficiently big n:
$$\left[(-\Delta_\epsilon)^{-(n+1)} u_{2,0}\right](y) = \sum_{j=1}^{\infty} \frac{1}{\lambda_{j,\epsilon}^{n+1}} \langle u_{2,0}, u_{j,\epsilon} \rangle u_{j,\epsilon}(y)$$

From Section 2 we get:

$$\left| \sum_{j \neq 2} \frac{1}{\lambda_{j,\epsilon}^{n+1}} \langle u_{j,\epsilon}, u_{2,0} \rangle u_{j,\epsilon}(y) \right| \leq C\epsilon^{D+1} N.$$

We apply the following elementary formula to the resolvent:

$$\int_0^\infty t^n e^{-t\lambda} dt = \frac{1}{c_n} \frac{1}{\lambda^{1+n}},$$

where c_n is a normalisation. Below, we will repeatedly use the fact that $u_{2,0} \geq 0$. We will write τ instead of τ_ϵ.

$$\begin{aligned}
\left[(-\Delta_\epsilon)^{-(n+1)} u_{2,0}\right](y) &= c_n \int_0^\infty t^n (e^{t\Delta_\epsilon} u_{2,0})(y) dt \\
&= c_n \mathbb{E}^y [\int_0^\infty t^n u_{2,0}(W_{t\wedge\tau}) dt] \\
&\geq c_n \mathbb{E}^y [\int_{\tau_F}^\infty t^n u_{2,0}(W_{t\wedge\tau}) dt] \\
&= c_n \mathbb{E}^y [\int_0^\infty (t+\tau_F)^n u_{2,0}(W_{(t+\tau_F)\wedge\tau}) dt] \\
&\geq c_n \int_0^\infty t^n \mathbb{E}^y [u_{2,0}(W_{(t+\tau_F)\wedge\tau})] dt \\
&= c_n \int_0^\infty t^n \mathbb{E}^y [\mathbb{E}^y [u_{2,0}(W_{(t+\tau_F)\wedge\tau}) | \mathcal{F}_{\tau_F}]] dt \\
&= c_n \int_0^\infty t^n \mathbb{E}^y [\mathbb{E}^{W_{\tau_F}} [u_{2,0}(W_{t\wedge\tau})]] dt \\
&\geq c_n \mathbb{E}^y [\mathbb{E}^{W_{\tau_F}} [\int_0^{\tau_0} t^n u_{2,0}(W_{t\wedge\tau}) dt]] \\
&= \mathbb{E}^y [(-\Delta_0)^{-(n+1)} u_{2,0}(W_{\tau_F})] \\
&= \frac{1}{\lambda_{2,0}^{n+1}} \mathbb{E}^y [u_{2,0}(W_{\tau_F})] \\
&\geq \frac{1}{\lambda_{2,0}^{n+1}} \frac{1}{4} \min\{u_{2,0}(z) | z \in F_1\} \\
&\geq \frac{1}{\lambda_{2,0}^{n+1}} \frac{1}{4} \frac{d}{2} \partial_r u_{2,0}|_{r=R_1} + O(\epsilon^2).
\end{aligned}$$

□

Remark 3.4. Let us look at one of the holes

$$H = \{|y| = R_1\} \cap B(x_k, \epsilon).$$

Let $d(y) = \operatorname{dist}(y, \partial\Omega_\epsilon)$, and let $d\sigma$ be <u>normalised</u> surface measure on $\{|y| = R_1\}$, then

$$\begin{aligned}
\int_H d(y)\, d\sigma(y) &\geq \int_{\{y \in H | d(y) \geq \epsilon/2\}} \epsilon/2 \, d\sigma(y) \\
&= \epsilon/2(\epsilon/2)^{D-1} \frac{\sigma_{D-1}}{R_1^{D-1} \tilde{\sigma}_D} + O(\epsilon^{D+1}) \\
&= 2^{-D} \epsilon^D \frac{\sigma_{D-1}}{R_1^{D-1} \tilde{\sigma}_D} + O(\epsilon^{D+1}),
\end{aligned}$$

where $\sigma_m = \mathrm{vol}_{\mathbf{R}^m}(B(1))$ and $\tilde{\sigma}_D$ is the surface measure of $\{|y|=1\}$ in $\mathbf{R}^D$

Lemma 3.5. *Let $x \in \Omega_\epsilon$, $|x| = R_1 - \delta$. Then there exists a positive constant c_1 such that*

$$\left[(-\Delta_\epsilon)^{-(n+1)} u_{2,0}\right](x) \geq c_1 \epsilon^D \delta^{-(D-1)} + O(\epsilon^{D+1}).$$

Here c_1 can be chosen as

$$c_1 = 2^{-(2D+3)} \frac{\sigma_{D-1}}{\tilde{\sigma}_D} R_1^{-1} (2R_1 - \delta) \frac{1}{\lambda_{2,0}^{n+1}} (\partial_r u_{2,0}|_{r=R_1}).$$

Proof. The same calculus as above, again using the strong Markov property of Brownian motion, proves that

$$\left[(-\Delta_\epsilon)^{-(n+1)} u_{2,0}\right](x) \geq c_n \int_0^\infty t^n \mathbb{E}^x \left[\mathbb{E}^{W_{\tau_B}}[u_{2,0}(W_{t\wedge\tau})]\right] dt,$$

where

$$\tau_B(\omega) = \inf\{t \geq 0 \big| |W_t(\omega)| \geq R_1\},$$

is the exit time from the ball of radius R_1. Using the integral formula and the spectral theorem, the right hand side is equal to

$$\mathbb{E}^x \left[\sum_{j=1}^\infty \frac{1}{\lambda_{j,\epsilon}^{n+1}} \langle u_{2,0}, u_{j,\epsilon}\rangle u_{j,\epsilon}(W_{\tau_B}) \right].$$

Now, the exit distribution of the ball for Brownian motion started at x is explicitly known ([Bas95][p.92]) so we get

$$\begin{aligned}
\left[(-\Delta_\epsilon)^{-(n+1)} u_{2,0}\right](x) &\geq \sum_{j=1}^\infty \frac{1}{\lambda_{j,\epsilon}^{n+1}} \langle u_{2,0}, u_{j,\epsilon}\rangle \int_{S_\epsilon} K(x,y) u_{j,\epsilon}(y) d\sigma(y) \\
&= \frac{1}{\lambda_{2,\epsilon}^{n+1}} \langle u_{2,0}, u_{2,\epsilon}\rangle \int_{S_\epsilon} K(x,y) u_{2,\epsilon}(y) d\sigma(y) + O(\epsilon^{D+1}).
\end{aligned}$$

where $d\sigma$ is <u>normalised</u> surface measure on S_ϵ and

$$K(x,y) = R_1^{D-2} \frac{R_1^2 - |x|^2}{|y-x|^D}.$$

If we just include the hole nearest to x in the integral, we get:

$$\begin{aligned}
&\int_{S_\epsilon} K(x,y) \frac{1}{\lambda_{2,\epsilon}^{n+1}} \langle u_{2,0}, u_{2,\epsilon}\rangle u_{2,\epsilon}(y) d\sigma(y) \\
\geq\ & \left(R_1^{D-2} \frac{2R_1 - \delta}{(\delta+\delta)^D} \delta\right) \left(2^{-D} \epsilon^D \frac{\sigma_{D-1}}{R_1^{D-1} \tilde{\sigma}_D}\right) \left(\frac{1}{\lambda_{2,0}^{n+1}} \frac{1}{8} \partial_r u_{2,0}|_{r=R_1}\right) + O(\epsilon^{D+1}).
\end{aligned}$$

Thus c_1 can be chosen as

$$c_1 = 2^{-(2D+3)} \frac{\sigma_{D-1}}{\tilde{\sigma}_D} R_1^{-1} (2R_1 - \delta) \frac{1}{\lambda_{2,0}^{n+1}} (\partial_r u_{2,0}|_{r=R_1}).$$

□

Now, we can prove Theorem 3.1:

Proof. If $|x| = R_1 - \delta$ and $\delta \gg \epsilon$, then

$$\begin{aligned}
|u_{j,\epsilon}(x)| &\leq \|u_{j,\epsilon}\|_\infty e \lambda_{j,\epsilon} \mathbb{E}^x[\tau_\epsilon] \\
&= \|u_{j,\epsilon}\|_\infty e \lambda_{j,\epsilon} o_\delta(1),
\end{aligned}$$

where $o_\delta(1)$ tends to zero as δ gets small[1]. Therefore

$$\left|\sum_{j\neq 2}\frac{1}{\lambda_{j,\epsilon}^{n+1}}\langle u_{j,\epsilon},u_{2,0}\rangle u_{j,\epsilon}(x)\right|\leq \epsilon^D\delta^{-(D-1)}o_\delta(1).$$

On the other hand,

$$\left[(-\Delta_\epsilon)^{-(n+1)}u_{2,0}\right](x) \quad\geq\quad c_1\epsilon^D\delta^{-(D-1)}+O(\epsilon^{D+1}).$$

Choosing now ϵ,δ sufficiently small, we get a contradiction if $u_{2,\epsilon}\leq 0$. $\square$

4. An explicit estimate on the number of holes for $D=2$

Let us redo the proof of Theorem 3.1 a little more carefully in the case where $D=2$ and the holes are evenly placed i.e. $2\pi R_1 = N\rho = 2N\delta$

First we estimate

$$\left|\sum_{j\neq 2}\frac{1}{\lambda_{j,\epsilon}^{n+1}}\langle u_{j,\epsilon},u_{2,0}\rangle u_{j,\epsilon}(x)\right|.$$

For $|x|=R_1-\delta$, where $\delta\gg\epsilon$, we get

$$\begin{aligned}|u_{j,\epsilon}(x)| &\leq \|u_{j,\epsilon}(x)\|_\infty e\lambda_{j,\epsilon}\mathbb{E}^x[\tau_\epsilon]\\ &= \|u_{j,\epsilon}(x)\|_\infty e\lambda_{j,\epsilon}(\mathbb{E}^x[\tau_0]+O(\epsilon)).\end{aligned}$$

Now

$$\begin{aligned}|x|^2 &= \mathbb{E}^x[W_{\tau_0}-2\tau_0]\\ &= R_1^2-2\mathbb{E}^x[\tau_0],\end{aligned}$$

so

$$\mathbb{E}^x[\tau_0]=\frac{R_1^2-|x|^2}{2},$$

and therefore

$$\begin{aligned}|u_{j,\epsilon}(x)| &\leq \|u_{j,\epsilon}(x)\|_\infty e\lambda_{j,\epsilon}\frac{2R_1-\delta}{2}\delta\\ &\leq e^{1+\frac{1}{8\pi}}\frac{2R_1-\delta}{2}\lambda_{j,\epsilon}^{3/2}\delta,\end{aligned}$$

by Lemma 2.5. Therefore

$$\begin{aligned}\left|\sum_{j\neq 2}\frac{1}{\lambda_{j,\epsilon}^{n+1}}\langle u_{j,\epsilon},u_{2,0}\rangle u_{j,\epsilon}(x)\right| &\leq e^{1+\frac{1}{8\pi}}\frac{2R_1-\delta}{2}C\sum_{j\neq 2}\frac{1}{\lambda_{j,\epsilon}^{n+1}}\frac{\lambda_{j,\epsilon}^{3/2}}{|\lambda_{j,\epsilon}-\lambda_{2,0}|}\lambda_{j,\epsilon}^{3/2}N\delta\epsilon^2\\ &= e^{1+\frac{1}{8\pi}}\frac{2R_1-\delta}{2}C\sum_{j\neq 2}\frac{\lambda_{j,\epsilon}^{2-n}}{|\lambda_{j,\epsilon}-\lambda_{2,0}|}\pi R_1\epsilon^2,\end{aligned}$$

[1]In fact,

$$\begin{aligned}\mathbb{E}^x[\tau_\epsilon] &= \mathbb{E}^x[\tau_0]+O(\epsilon)\\ &= \frac{R_1^2-|x|^2}{D}+O(\epsilon)\\ &= \frac{2R_1-\delta}{D}\delta+O(\epsilon).\end{aligned}$$

This is proved in section 4.

where C is the constant from Lemma 2.1. Let us estimate $\frac{1}{|\lambda_{j,\epsilon}-\lambda_{2,0}|}$ by

$$\max\{\frac{1}{|\lambda_{1,\epsilon}-\lambda_{2,0}|},\frac{1}{|\lambda_{3,\epsilon}-\lambda_{2,0}|}\} = \max\{\frac{1}{|\lambda_{1,0}-\lambda_{2,0}|},\frac{1}{|\lambda_{3,0}-\lambda_{2,0}|}\} + o(1),$$

as $\epsilon \searrow 0$.

Let us fix R_2/R_1 in such a way that

$$\lambda_{2,0} = \lambda_1(B(R_2)\setminus\overline{B(R_1)}) = \frac{\lambda_{1,0}+\lambda_{3,0}}{2} = \frac{\lambda_1(B(R_1))+\lambda_2(B(R_1))}{2}.$$

Furthermore, we have to estimate

$$\sum_{j\neq 2}\lambda_{j,\epsilon}^{2-n}.$$

We may take $n=4$, and get

$$\begin{aligned}\sum_{j\neq 2}\lambda_{j,\epsilon}^{-2} &\leq \sum \lambda_j(B(R_2))^{-2}\\ &\leq \sum_j\left(4\pi\frac{1}{2}\frac{j}{\pi R_2^2}\right)^{-2},\end{aligned}$$

by an inequality proved in [LY83][Cor. 1]. Thus

$$\sum_{j\neq 2}\lambda_{j,\epsilon}^{-2} \leq \frac{R_2^4}{4}\sum_j j^{-2} = \frac{\pi^2 R_2^4}{24},$$

and we finally obtain

$$\begin{aligned}&\left|\sum_{j\neq 2}\frac{1}{\lambda_{j,\epsilon}^5}\langle u_{j,\epsilon},u_{2,0}\rangle u_{j,\epsilon}(x)\right|\\ \leq\;& e^{1+\frac{1}{8\pi}}\frac{\pi^3}{48}(2R_1-\delta)CR_1R_2^4\frac{2}{\lambda_{3,0}-\lambda_{1,0}}\epsilon^2+o(\epsilon^2)\end{aligned}$$

On the other hand we get from Lemma 3.5 that

$$\left[(-\Delta_\epsilon)^{-5}u_{2,0}\right](x) \geq \frac{1}{128}\frac{2R_1-\delta}{\pi R_1}\frac{1}{\lambda_{2,0}^5}(\partial_r u_{2,0}|_{r=R_1})\epsilon^2\delta^{-1}+O(\epsilon^3).$$

Thus we get a contradiction when

$$\delta \;<\; \frac{3}{8}\pi^{-4}(e^{1+\frac{1}{8\pi}})^{-2}\lambda_{2,0}^{-5}(\lambda_{2,0}-\lambda_{1,0})\frac{R_1^{-1}R_2^{-4}\log(R_2/R_1)}{(R_2^2-R_1^2)(1+R_2/R_1)}.$$

This is equivalent to:

$$N > (e^{1+\frac{1}{8\pi}})^2\frac{8\pi^5}{3}\frac{1+R_2/R_1}{\log(R_2/R_1)}R_1^2R_2^4(R_2^2-R_1^2)\frac{\lambda_{2,0}^5}{\lambda_{3,0}-\lambda_{1,0}}.$$

If we take $R_1=1$ then we get from a table of Bessel functions:

$$\begin{aligned}\lambda_{1,0} &= 5.7831 = (2.4048)^2\\ \lambda_{3,0} &= 14.6819 = (3.8317)^2\end{aligned}$$

Thus, we get $\lambda_{2,0} = 10.2325$. The Maple computation in the appendix shows that $R_2 = 1.9762$, and we get

$$N > 15.843\times 10^9.$$

Appendix A. A numerical calculation

In this appendix we have a short Maple code which gives us a graph where R_2 can be read off.

```
> with(plots):
```

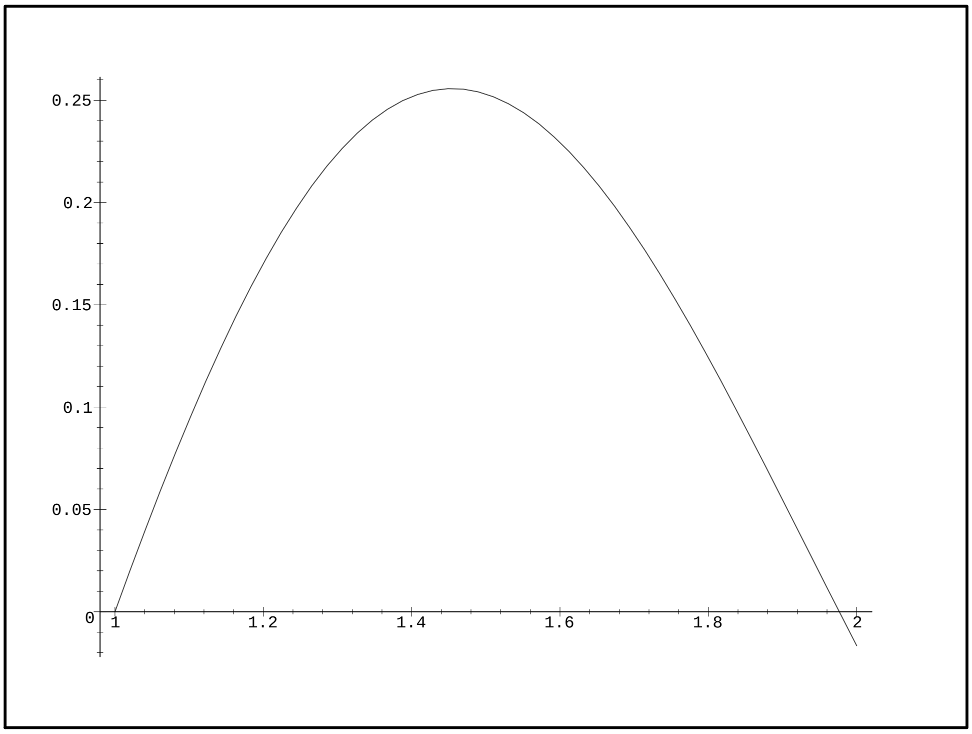
```
> e1:=diff(u(x),x$2)+diff(u(x),x)/x+10.2325*u(x ) = 0;
```

$$e1 := (\frac{\partial^2}{\partial x^2}\,\mathrm{u}(x)) + \frac{\frac{\partial}{\partial x}\,\mathrm{u}(x)}{x} + 10.2325\,\mathrm{u}(x) = 0$$

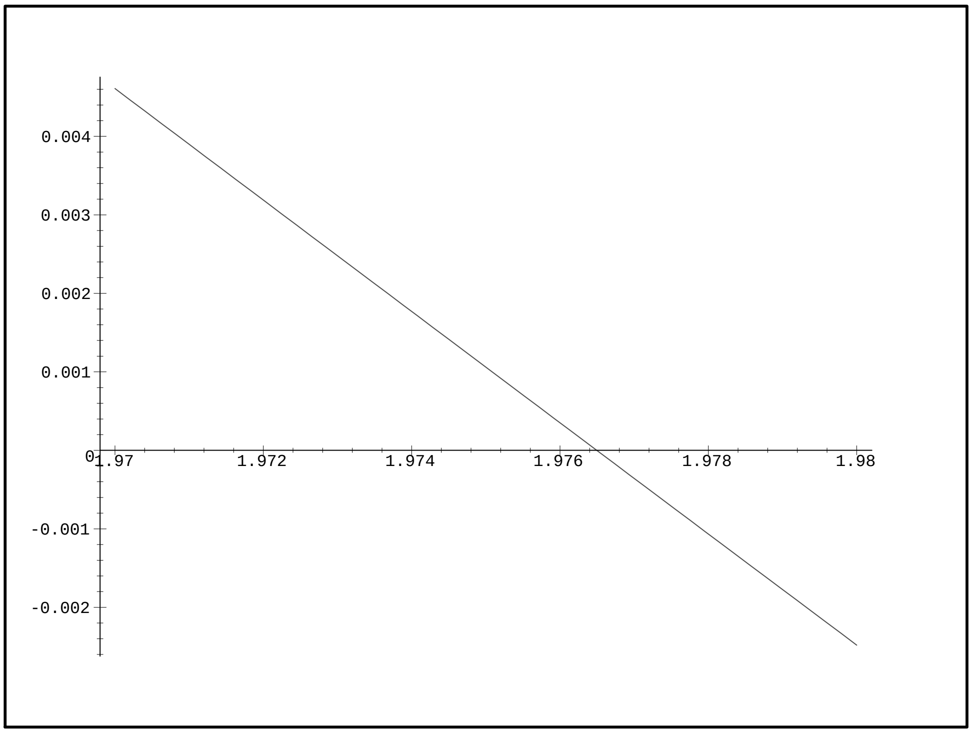
```
> p:=dsolve({e1,u(1)=0,D(u)(1)=1},u(x),type=n umeric);
```

$$p := \mathbf{proc}(rkf45\_x)\ \ldots\ \mathbf{end}$$

```
> odeplot(p,[x,u(x)],1..2);
```




```
> odeplot(p,[x,u(x)],1.97..1.98);
```



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E-mail address: `fournais@imf.au.dk`

Department of Mathematics, Ny Munkegade, 8000 Aarhus C, Denmark