Smooth normalization of reversible vector fields

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Abstract

The paper deals with the problem of reduction of a smooth reversible vector field near a hyperbolic singularity to a resonant polynomial normal form via a smooth symmetric transformation. The authors show that the Sternberg-Chen theory remain true in that specific context.

Introduction

It is known [1, 2, 3, 4] that given a hyperbolic linear vector field A and a positive integer k one can find a number K, $K \geq k$, such that each two C^K vector fields having a singularity at the origin, and having the same linear part A and the same K-jet at the origin are locally C^k conjugate. By virtue of the formal theory of normal forms due to Poincare, this gives in particular that each C^K vector field having a hyperbolic singularity at the origin with the linear part A is locally C^k equivalent to a resonant polynomial vector field of order less than K.

In the last years much attention has been paid to the problem of normalization of vector fields with extra structures. This means that when considering a specific category of vector fields and a related group of transformations, respecting this category, it is natural to ask whether it is possible to reduce a given vector field to a normal form via such a transformation. For instance, in [5] it is shown that a smooth symplectic (volume preserving, contact) vector field can be normalized by a smooth canonical coordinate change. In the recent work of Bonckaert [6] the problem of symmetric normalization is solved: a smooth enough symmetric vector field admits a smooth symmetric reduction to the resonant polynomial normal form. A similar problem of normalization of a reversible vector field is posed there.

Problem. *Given an linear involution R and a C^K vector field ξ having a hyperbolic equilibrium at the origin such that $R_*\xi = -\xi$, is it basically true that ξ admits a reduction to a resonant polynomial normal form via a C^k transformation commuting with R?*

Gaeta [7] has shown that the related formal problem is solved positively. Namely, given a formal R-reversible vector field then the Poincaré normal form is R-reversible and the formal conjugacy map commutes with R. In this paper we prove that a smooth hyperbolic R-reversible vector field is finitely determined with respect to the group of R-symmetric C^k local diffeomorphisms. Together with the result of Gaeta, this solves the problem of smooth symmetric normalization of a reversible vector field.

Basic notions and definitions

Given a diffeomorphism H of d-dimensional linear space E and a vector field ξ on E, by $H_*\xi$ denote the vector field

$$H_*\xi(z) \equiv DH(H^{-1}(z))\xi(H^{-1}(z)) \qquad (z \in E).$$

Let $R : E \to E$ be a linear involution, *i.e.*, $R^2 = I$. A vector field ξ is said to be *R-reversible* if

$$R_*\xi = -\xi.$$

Two vector fields ξ and η are called *R-equivalent* if there is a diffeomorphism $H : E \to E$ such that $H \circ R = R \circ H$ and

$$H_*\xi = \eta.$$

Remark 1. Observe that the notions of R-reversibility is *jetwise* : every finite jet of an R-reversible vector field is itself R-reversible.

Consider a linear hyperbolic R-reversible vector field A. Fix a coordinate system bringing A to the block-diagonal form

$$A = \operatorname{diag}(A_1, \ldots, A_p)$$

in such a way that all the eigenvalues of the block A_i have one and the same real part θ_i $(i = 1, \ldots, p)$, and $\theta_i \neq \theta_j$ for $i \neq j$. This gives an A-invariant splitting $E = E_1 \oplus \ldots \oplus E_p$ of the phase space.

Since A is R-reversible and R is invertible, one can easily show that after some renumbering of invariant subspaces, A takes the form

$$A = \operatorname{diag}(A_1, \ldots, A_n, -A_1, \ldots, -A_n) \equiv \operatorname{diag}(B, -B) \tag{1}$$

with respect to the splitting $E = X_1 \oplus \ldots \oplus X_n \oplus Y_1 \oplus \ldots \oplus Y_n$, where $\dim X_i = \dim Y_i = m_i$, $\theta_i > 0$ $(i = 1, \ldots, n)$. Suppose the numbers θ_i are odered by

$$0 < \theta_1 < \ldots < \theta_n.$$

One can easily prove the following algebraic

Lemma 1. *If R is an involution and A is a linear hyperbolic R-reversible vector field then in adopted coordinates A takes the form* (1) *and R can be written by*

$$R = \begin{pmatrix} & 0 & & I_{m_1} & & \\ & & & & \ddots & \\ & & & & & I_{m_n} \\ I_{m_1} & & & & & \\ & \ddots & & & 0 & \\ & & I_{m_n} & & & \end{pmatrix} = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}, \tag{2}$$

where I_p denotes the identity of $\mathbf{R}^p$.

Put $X = X_1 \oplus \ldots \oplus X_n$, $Y = Y_1 \oplus \ldots \oplus Y_n$. In what follows, for a point $z \in E$ we use the notation

$$z \equiv (x, y) \equiv (x_1, \ldots, x_n, y_1, \ldots, y_n),$$

where $x \in X$, $y \in Y$; $x_i \in X_i$, $y_i \in Y_i$ $(i = 1, \ldots, n)$. In this notation,

$$R(x, y) = (y, x),$$

a vector field ξ is R-reversible *iff*

$$\xi(x,y) = F(x,y)\frac{\partial}{\partial x} - F(y,x)\frac{\partial}{\partial y},$$

and a mapping H is R-symmetric *iff*

$$H(x,y) = (h(x,y), h(y,x)).$$

In what follows, given a function $f(x_1,\ldots,x_n)$ by $D_i^p f$ (p — positive integer, $i \in \{1,\ldots,n\}$) we mean partial derivative of this function of order p with respect to the i-th variable.

Main result

Let K and k be positive integers, $1 \le k \le K$. A vector field $\zeta \in C^K$ is said to be *Q-flat at the origin*, $k \le Q \le K$, if

$$D^s\zeta(0) = 0 \qquad (s = 0,1,\ldots,Q).$$

Theorem. *Let ξ and η be R-reversible C^K vector fields, $\xi(0) = \eta(0) = 0$, $A \equiv D\xi(0) = D\eta(0)$ be hyperbolic, and the vector field $(\xi - \eta)$ be Q-flat at the origin. If*

$$Q > \frac{\theta_n}{\theta_1}(k+1), \qquad K \ge Q + k \tag{3}$$

then ξ and η are locally C^k R-conjugate.

Deformation method

Apply the deformation method [8, 5] to find a conjugacy. For $\varepsilon \in \mathbf{R}$ set $\xi_\varepsilon = \varepsilon\eta + (1-\varepsilon)\xi$. Put $\xi(z,\varepsilon) = \xi_\varepsilon(z)$. Define the vector field Ξ on $E \times \mathbf{R}$ by

$$\Xi(z,\varepsilon) = \xi(z,\varepsilon)\frac{\partial}{\partial z} + 0 \cdot \frac{\partial}{\partial \varepsilon}.$$

Consider a vector field Φ on $E \times \mathbf{R}$ of the form

$$\Phi(z,\varepsilon) = \varphi(z,\varepsilon)\frac{\partial}{\partial z} + 1 \cdot \frac{\partial}{\partial \varepsilon},$$

which commutes with Ξ, *i.e.*

$$[\Phi, \Xi] = 0. \tag{4}$$

Lemma 2. *Let Φ satisfy* (4) *and $P^t(z,\varepsilon)$ ($z \in E$, $\varepsilon \in \mathbf{R}$, $t \in \mathbf{R}$) denote the phase flow of Φ. Put $H(z) = \mathrm{pr}_1 \circ P^1(z,0)$. Then $H_*\xi = \eta$, i.e., ξ and η are conjugate by H.*

Proof. Let Q^t be the phase flow of the vector field Ξ. Then

$$P^t(x,\varepsilon) = (p^t(x,\varepsilon), \varepsilon + t), \qquad Q^t(x,\varepsilon) = (q^t(x,\varepsilon), \varepsilon).$$

Note that the phase flows of ξ and η are, respectively, $q_0^t(x) = q^t(x,0)$ and $q_1^t(x) = q^t(x,1)$. Since the phase flows P^t and Q^t commute then

$$Q^{t_2}(P^{t_1}(x,\varepsilon)) = (q^{t_2}(p^{t_1}(x,\varepsilon), \varepsilon + t_1), \varepsilon + t_1) = (p^{t_1}(q^{t_2}(x,\varepsilon),\varepsilon), \varepsilon + t_1) = P^{t_1}(Q^{t_2}(x,\varepsilon)),$$

or

$$q^{t_2}(p^{t_1}(x,\varepsilon),\varepsilon+t_1)=p^{t_1}(q^{t_2}(x,\varepsilon),\varepsilon).$$

For $\varepsilon=0$, $t_1=1$, $t_2=t$ this gives

$$q^t(p^1(x,0),1)=p^1(q^t(x,0),0).$$

For $H(x)=p^1(x,0)$ we get

$$q^t(H(x),1)=H(q^t(x,0))$$

or

$$q_1^t(H(x))=H(q_0^t(x))$$

that is H is a conjugacy between (1) and (2).

Observe that (4) is equivalent to

$$D_z\varphi(z,\varepsilon)\cdot\xi(z,\varepsilon)=D_z\xi(z,\varepsilon)\cdot\varphi(z,\varepsilon)+\eta(z)-\xi(z)\quad(z\in E),\tag{5}$$

which is called the *cohomology equation*. The corresponding characteristic system is

$$\dot\gamma=D_z\xi(z,\varepsilon)\cdot\gamma+\eta(z)-\xi(z),\ \ \dot z=\xi(z,\varepsilon)\quad(\gamma,z\in E;\ \varepsilon\in\mathbf{R}).\tag{6}$$

The solutions of (5) are nothing else than invariant sections $\gamma=\varphi(z,\varepsilon)$ of (6). Thus, the deformation method reduces the conjugacy problem to finding a smooth invariant section $\gamma=\varphi(z,\varepsilon)$ of the affine extension (6) and then to integrating the system $\dot z=\varphi(z,\varepsilon)$, $\dot\varepsilon=1$.

Proof of the Theorem.

Rewrite ξ and η by

$$\begin{aligned}\xi(x,y)&=(Bx+p(x,y))\frac{\partial}{\partial x}-(By+p(y,x))\frac{\partial}{\partial y}, &(7)\\ \eta(x,y)&=(Bx+p(x,y)+q(x,y))\frac{\partial}{\partial x}-(By+p(y,x)+q(y,x))\frac{\partial}{\partial y}, &(8)\end{aligned}$$

where p and q are of class C^K, and q is Q-flat at the origin. For every ε, put $g(x,y)\equiv g_\varepsilon(x,y)=p(x,y)+\varepsilon q(x,y)$. Then the characteristic system (6) takes the following form

$$\begin{aligned}\dot\alpha&=[B+D_1g(x,y)]\alpha+D_2g(x,y)\beta+q(x,y),\\ \dot\beta&=-D_2g(y,x)\alpha-[B+D_1g(y,x)]\beta-q(y,x),\\ \dot x&=Bx+g(x,y),\\ \dot y&=-(By+g(y,x)).\end{aligned}\tag{9}$$

Denote $\mathcal{R}=\mathrm{diag}(R,R)$. Note that the differential system (9) is $\mathcal{R}$-reversible. Let us show that there is a C^k $\mathcal{R}$-symmetric manifold $(\alpha,\beta)=(h(x,y),h(y,x))$ which is invariant under (9).

Suppose a manifold $(\alpha,\beta)=\Phi(x,y)\equiv(\varphi(x,y),\psi(x,y))$ is invariant under (9). Then

$$\begin{aligned}{}[B+D_1g(x,y)]\varphi(x,y)+D_2g(x,y)\psi(x,y)+q(x,y)&=D_1\varphi(x,y)(Bx+g(x,y))-D_2\varphi(x,y)(By+g(y,x)),\\ -D_2g(y,x)\varphi(x,y)-[B+D_1g(y,x)]\psi(x,y)-q(y,x)&=D_1\psi(x,y)(Bx+g(x,y))-D_2\psi(x,y)(By+g(y,x)).\end{aligned}\tag{10}$$

Replacing x by y and *vice versa* in (10) we get

$$\begin{aligned}
[B + D_1 g(y,x)]\varphi(y,x) + D_2 g(y,x)\psi(y,x) + q(y,x) &= D_1\varphi(y,x)(By + g(y,x)) - D_2\varphi(y,x)(Bx + g(x,y)), \\
-D_2 g(x,y)\varphi(y,x) - [B + D_1 g(x,y)]\psi(y,x) - q(x,y) &= D_1\psi(y,x)(By + g(y,x)) - D_2\psi(y,x)(Bx + g(x,y)).
\end{aligned} \tag{11}$$

It can be easily checked that (11) gives the invariance of the manifold $(\alpha,\beta) = \Psi(x,y) \equiv (\psi(y,x), \varphi(y,x))$ under (9). Put

$$\mathcal{H}(x,y) = \frac{1}{2}(\Phi(x,y) + \Psi(x,y)). \tag{12}$$

By linearity of (10) and (11), the manifold $(\alpha,\beta) = \mathcal{H}(x,y) = (\chi_1(x,y), \chi_2(x,y))$ is invariant under (9) as well.

Since q is Q-flat at the origin and $K \geq Q + k$ then it follows from [9, Lemma 3] that

$$q(x,y) = a(x,y)x^Q + b(x,y)y^Q,$$

where $a, b \in C^k$. Then by Special Smooth Section Theorem [5, Theorem 10.1], there is a C^k smooth manifold $(\alpha,\beta) = \Phi(x,y)$ invariant under (9). Then the manifold (12) constructed above is C^k smooth and invariant as well. It remains to observe that

$$\chi_2(y,x) = \frac{1}{2}(\psi(y,x) + \phi(x,y)) = \chi_1(x,y),$$

i.e. the manifold $(\alpha,\beta) = \mathcal{H}(x,y)$ is $\mathcal{R}$-symmetric. By integrating the $\mathcal{R}$-symmetric system

$$\dot{x} = \chi_1(x,y,\varepsilon), \quad \dot{y} = \chi_1(y,x,\varepsilon), \quad \dot{\varepsilon} = 1$$

we get an $\mathcal{R}$-symmetric C^k conjugacy.

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