Three dimensional dissipative diffeomorphisms with homoclinic tangencies

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Abstract

Given a two-parameter family of three-dimensional diffeomorphism $\{f_{a,b}\}_{a,b}$, with a dissipative (but not sectionally dissipative) saddle fixed point, we prove that the existence of a certain type of homoclinic tangency of the invariant manifolds, implies that there exists a return map $f^n_{a,b}$, near the homoclinic orbit, for values of the parameter near such tangency and for n greater enough, such that after a change of variables and reparametrization depending on n, this return map tends to a simple quadratic map. This implies the existence of strange attractors and infinitely many sinks as in other known cases. Moreover, there appear attracting invariant circles, implying the existence of quasiperiodic behaviour near the homoclinic tangency.

1 Introduction and statement of results

Let $\{f_a\}_{a\in V}$ be a smooth family of diffeomorphisms in an n-dimensional smooth manifold M. This means that $a \in V \subset \mathbb{R}^m$ is a parameter, V is an open subset and that there exists a smooth map $f : M \times V \mapsto M$ such that $f_a = f(\cdot, a)$ and each $f_a : M \mapsto M$ is a diffeomorphism.

Let us suppose that for $a = a_0$, f_{a_0} has a dissipative saddle fixed point $p_0 \in M$. This means that $f_{a_0}(p_0) = p_0$, $Df_{a_0}(p_0)$ has all the eigenvalues with absolute value different from 1, at least one with absolute value greater than 1, and $|\det Df_{a_0}(p_0)| < 1$. Then, associated to this point, there exist the stable and unstable invariant manifolds

$$W^s(f_{a_0}, p_0) = \{q \in M \;:\; \lim_{n\to\infty} f^n(q) = p_0\}, \quad W^u(f_{a_0}, p_0) = W^s(f^{-1}_{a_0}, p_0),$$

by the stable manifold theorem (see e. g. Palis and de Melo [10]). All these objects (the fixed point and the invariant manifolds) are persistent by perturbations, so there exist for values of the parameter sufficiently near to a_0. It is known that transversal intersections of these manifolds (transversal homoclinic orbits) produce hyperbolic behaviour of the orbits near the homoclinic one.

The existence of homoclinic bifurcations, or nontransverse homoclinic orbits, become important when going beyond the hyperbolic theory. In particular, it is related to the existence of bifurcations of periodic points, appearance of periodic attractors and strange attractors. However, it is far from being completely understood. The way to face the problem of proving the existence of these nonhyperbolic phenomena, is to reduce it to the study of perturbations of simple model families of maps. This is made in the following way:

Consider a (nontransverse) homoclinic point q of the map f_{a_0}. Then, for values of the parameter near a_0, for points r near q, and $n \in \mathbb{N}$ larger enough (depending on the distance between q and r and between a and a_0) $f_a^n(r)$ is near q. This allows to define a return map near the homoclinic point. There are some known cases in which this return map has a simple limit (in a suitable way) when n tends to infinity:

- When $n = 2$, $m = 1$ and there exists a generically unfolding quadratic homoclinic tangency, the limit map is the logistic map. As a consequence it can be proven that there exists a nonzero Lebesgue measure set A, in the parameter space, such that if $a \in A$ the map f_a exhibits a strange attractor (see Mora and Viana [8]). The nontransverse homoclinic orbit is in the closure of the parameter values for which there exist attracting periodic orbits. There also appear other phenomena, as the coexistence of infinitely many attractors (see e. g. Newhouse [9] and Palis and Takens [11]). Moreover, it is possible to get some information about the shape of the basin of attraction of the attracting periodic orbits which a appear near the homoclinic tangency (see Tatjer and Simó [15]).

- When n is arbitrary, $m = 1$, $\dim W^u(p_0) = 1$, there exists a generically unfolding quadratic homoclinic tangency and p_0 is sectionally dissipative (all the products of two different eigenvalues are less than 1 in absolute value), then again the limit map is, essentially, the logistic map. There are similar results concerning the coexistence of infinitely many sinks (see Palis and Viana [12]) and strange attractors (see Viana [16]).

- When $n = 2$, $m = 2$ and the nontransverse homoclinic tangency is cubic, then the limit map is a one dimensional cubic map. In this case it is possible to prove the existence of cusp and codimension two flip bifurcations associated to this bifurcation (see Tatjer [14]).

- When n is arbitrary, $m = 1$ and there is a hyperbolic fixed point of index $1 \leq s \leq n-2$ such that its stable and unstable invariant manifolds have an orbit of quadratic tangency unfolding generically of codimension greater or equal than 2, for some value of the parameter, then in general the phenomenon of existence of infinitely many sinks or strange attractor does not appear. However, it is possible to prove similar results if we substitute sinks by periodic orbits of index $s + 1$ and strange attractors by strange saddle sets. See Romero [13].

- There are other situations in the multidimensional case ($n > 2$) in which Gonchenko et al. [5, 6] found the phenomenon of coexistence of infinitely many attractors, near a diffeomorphism having a quadratic homoclinic tangency. We note that only in the second paper there are some details of the proofs in specific cases.

Our definition of strange attractor is that of Viana [16].

The goal of this paper is to obtain other cases in dimension $n = 3$ for which there exist limit return maps. This means to delete the condition of sectional dissipation. As in general, a one-parameter family of diffeomorphisms without this condition, does not have such limit map, we must add one more parameter ($m = 2$) and a new condition over the homoclinic tangency. This new type of homoclinic tangency will be called *generalized homoclinic tangency*, and will be defined in section 2. The existence of these limit return maps allow us to find some phenomena associated to this new type of tangencies.

The role that the saddle-node bifurcation plays in the two dimensional case will be played in the three dimensional case by the Bogdanov-Takens (BT) bifurcation. We recall that for such bifurcation of periodic points, the corresponding spectrum has one double eigenvalue equal to 1, with associated eigenspace of dimension 1. See section 2 for more details. In particular, we shall find invariant circles, and therefore, quasiperiodic behaviour.

Before giving our main results we will introduce a linearization assumption and a definition concerning limit return maps associated with homoclinic tangencies.

Linearization assumption

Let $\{f_a\}_{a\in V}$ be a family of diffeomorphisms $f_a : M \mapsto M$, M being an 3-dimensional smooth manifold, and $V \subset \mathbb{R}^k$ an open set. Suppose that $a_0 \in V$ and f_{a_0} has a hyperbolic fixed point p_0 with real eigenvalues λ_i^0, $i = 1, 2, 3$ such that $|\lambda_1^0| < |\lambda_2^0| < 1$ and $|\lambda_3^0| > 1$ or $|\lambda_1^0| < 1$, $|\lambda_3^0| > |\lambda_2^0| > 1$. We denote by $p = p(a)$ the corresponding fixed point for f_a and a near a_0 such that $p(a_0) = p_0$, and by λ_i, $i = 1, 2, 3$ its eigenvalues. We change, if necessary, V to have that $p(a)$ exists for all $a \in V$. We say that f_a satisfies the linearization assumption with respect to the fixed point $p(a)$ if there exist an open set $U \subset \mathbb{R}^3$ and a smooth map $\mathbf{x} : U \times V \subset \mathbb{R}^3 \times \mathbb{R}^k \mapsto M$ such that:

1. $\Lambda_a(U) \subset U$, where $\Lambda_a(t_1, t_2, t_3) = (\lambda_1 t_1, \lambda_2 t_2, \lambda_3 t_3)$, for $(t_1, t_2, t_3) \in \mathbb{R}^3$.
2. U is a neighbourhood of $\{(t_1, t_2, t_3) \ : \ (t_1^2 + t_2^2)t_3 = 0\}$, if $|\lambda_1^0| < |\lambda_2^0| < 1$ and $|\lambda_3^0| > 1$, or a neighbourhood of $\{(t_1, t_2, t_3) \ : \ (t_2^2 + t_3^2)t_1 = 0\}$, in the other case.
3. $\mathbf{x}(0, 0, 0, a) = p(a)$, for all $a \in V$.
4. $\mathbf{x}(\lambda_1 t_1, \lambda_2 t_2, \lambda_3 t_3, a) = f_a(\mathbf{x}(t_1, t_2, t_3, a))$, for $(t_1, t_2, t_3) \in U$.

We note that if the family is linearizable the invariant manifolds have natural parametrizations. For example, if $|\lambda_1| < |\lambda_2| < 1 < |\lambda_3|$ then:

$$W^s(p(a)) = \{q \in M \ : \ q = \mathbf{x}(t_1, t_2, 0) \text{ for some } (t_1, t_2) \in \mathbb{R}^2\},$$

$$W^u(p(a)) = \{q \in M \ : \ q = \mathbf{x}(0, 0, t_3) \text{ for some } t_3 \in \mathbb{R}\}.$$

The linearization assumption is a generic condition. Indeed, we only need to have a smooth conjugacy of the map near the fixed point to its linear part. See, e. g. Banyaga et al. [1]. Then we use the maps f_a and f_a^{-1} to build up the map $\mathbf{x}$. Moreover, we can use, for all $a \in V$, the inverse of $\mathbf{x}(\cdot, a)$ in some neighbourhood of the fixed point $p(a)$, as a local chart of M. In these coordinates we have that $g_a^n(t_1, t_2, t_3) = \mathbf{x}_a^{-1}\mathbf{x}(\lambda_1^n t_1, \lambda_2^n t_2, \lambda_3^n t_3, a)$, where $\mathbf{x}_a(t_1, t_2, t_3) = \mathbf{x}(t_1, t_2, t_3, a)$, $g_a^n = \mathbf{x}_a^{-1} \circ f_a^n \circ \mathbf{x}_a$, and (t_1, t_2, t_3) and $\mathbf{x}(t_1, t_2, t_3, a)$ are in a small neighbourhood of $(0, 0, 0)$.

In order to avoid the use of coordinates, it is not restrictive to suppose that $M = \mathbb{R}^3$, $p(a) = (0, 0, 0)$, for all $a \in V$, and, if the linearization assumption holds, there exists a map

$\mathbf{x}$ as before for which $\mathbf{x}_a$ is the identity map in a neighbourhood of the origin. Moreover, $f_a^n(t_1,t_2,t_3) = \mathbf{x}(\lambda_1^n t_1, \lambda_2^n t_2, \lambda_3^n t_3, a)$, for (t_1,t_2,t_3) in a small neighbourhood of the origin and for all $a \in V$.

Definition 1.1 *Let $\{f_a\}_{a\in V}$ be a smooth family of diffeomorphisms in some m-dimensional manifold M, depending on a parameter $a \in V \subset \mathbb{R}^k$, V being an open subset. Suppose that for $a = a_0$ there exists a homoclinic orbit O_0 of some dissipative fixed point. We say the the family $\{f_a\}_{a\in V}$ has a limit return map associated to the homoclinic orbit O_0, in the C^l topology, if there exist a point of the orbit q, and a natural number N, such that for any positive integer $n \geq N$ there exist reparametrizations $a = M_n(\tilde{a})$ of the a variable and $\tilde{a}$-dependent coordinate transformations $x = \Psi_{n,\tilde{a}}(\tilde{x})$ satisfying the following properties:*

1. *For each compact set K in the $(\tilde{a},\tilde{x})$ space the images of K under the maps*
$$(\tilde{a},\tilde{x}) \mapsto (M_n(\tilde{a}), \Psi_{n,\tilde{a}}(\tilde{x}))$$
converge , for $n \to \infty$, in the (a,x) space, to (a_0,q).
2. *The domains of the maps*
$$(\tilde{a},\tilde{x}) \mapsto (\tilde{a}, (\Psi_{n,\tilde{a}}^{-1} \circ f_{M_n(\tilde{a})}^n \circ \Psi_{n,\tilde{a}}))$$
converge, for $n \to \infty$, to all of $\mathbb{R}^{m+k}$, and the maps converge, in the C^l topology to some map of the form
$$(\tilde{a},\tilde{x}) \mapsto (\tilde{a}, \tilde{f}_{\tilde{a}}(\tilde{x})).$$

In such case, the map $\tilde{f}_{\tilde{a}}$ will be called a limit return map *associated to the homoclinic orbit O_0.*

In the definition, a and x are used for the parameter and variable of the initial family f_a.

For a two-parameter family of three dimensional maps having a special type of codimension-two homoclinic tangency, that we call generalized homoclinic tangency, unfolding generically, we will prove the following theorems, that describe several behaviours depending on the dimensions of the invariant manifolds of the fixed point. We note that in none of the cases the fixed point is sectionally dissipative.

Theorem 1 *Let $\{f_{a,b}\}$ be a two-parameter family of three dimensional diffeomorphisms, having a hyperbolic dissipative fixed point p_0 for $(a,b) = (0,0)$ with eigenvalues λ_1, λ_2 and λ_3. Let suppose that $|\lambda_1| < 1$, $|\lambda_2| < 1$, $|\lambda_3| > 1$, $|\lambda_1\lambda_3| < 1$ and $|\lambda_2\lambda_3| > 1$. Moreover, the invariant manifolds of p_0 have a generalized homoclinic tangency which unfolds generically and the family $f_{a,b}$ satisfies the linearization assumption. Then:*

1. *For n large enough there are values of the parameter (a_n,b_n) for which the map $f_{a,b}$ undergoes a generic Bogdanov-Takens bifurcation. Moreover $(a_n,b_n) \to (0,0)$ when $n \to \infty$.*
2. *There exists a limit return map $\tilde{f}_{\tilde{a},\tilde{b}} : \mathbb{R}^3 \mapsto \mathbb{R}^3$ associated to the generalized homoclinic tangency, such that*
$$\tilde{f}_{\tilde{a},\tilde{b}}(\tilde{x},\tilde{y},\tilde{z}) = (\tilde{z}, \tilde{b}\tilde{z}, \tilde{a} + \tilde{y} + \tilde{z}^2),$$

or

$$\tilde f_{\tilde a,\tilde b}(\tilde x,\tilde y,\tilde z)=(0,\tilde z,\tilde a+\tilde b\tilde y+\tilde z^2),$$

depending on the type of generalized homoclinic tangency (I or II resp., see section 2).

3. *$(a,b)=(0,0)$ is in the closure of parameter values for which there exist differentiable invariant circles. Depending on the geometry of the tangency we can add attracting or saddle (repelling inside the centre manifold).*

4. *$(a,b)=(0,0)$ is in the closure of the values of the parameters for which there are attracting periodic orbits.*

5. *There is a positive Lebesgue measure set E of parameter values arbitrarily near $(0,0)$ such that, for $(a,b)\in E$ the diffeomorphism $f_{a,b}$ exhibits a strange attractor near the orbit of tangency.*

6. *There are open sets $U\subset\mathbb{R}^2$, arbitrarily near $(0,0)$ such that for a generic $(a,b)\in U$ the map $f_{a,b}$ has infinitely many sinks.*

Remark 1.2 *The two quadratic maps that appear in the previous theorem are conjugate if $\tilde b\neq 0$. When $\tilde b=0$ they are not conjugate but if we restrict the first map to the plane $\tilde y=0$ and the second to $\tilde x=0$, we obtain the same two-dimensional map.*

Theorem 2 *Let $\{f_{a,b}\}_{a,b}$ be a two-parameter family of three dimensional diffeomorphisms having a dissipative fixed point p_0 for $(a,b)=(0,0)$, with eigenvalues λ_1, λ_2, λ_3 such that: $|\lambda_1|<1$, $|\lambda_3|>|\lambda_2|>1$. Suppose that the fixed point has a generalized homoclinic tangency which unfolds generically, and the family $f_{a,b}$ satisfies the linearization assumption. Then:*

1. *For n large enough there exists (a_n,b_n) for which the map f_{a_n,b_n} undergoes a generic BT bifurcation of period n. Moreover $(a_n,b_n)\to(0,0)$ when $n\to\infty$.*

2. *$(a,b)=(0,0)$ is in the closure of parameter values for which there are differentiable attracting invariant circles.*

3. *$(a,b)=(0,0)$ is in the closure of parameter values for which there are attracting periodic orbits.*

4. *There is a positive Lebesgue measure set E of parameter values arbitrarily near $(0,0)$ such that for $(a,b)\in E$ the diffeomorphism $f_{a,b}$ exhibits a strange attractor.*

5. *There are open sets $U\subset\mathbb{R}^2$, arbitrarily near $(0,0)$ such that for a generic $(a,b)\in U$ the map $f_{a,b}$ has infinitely many sinks.*

6. *There exists a limit return map associated to the generalized homoclinic tangency, $\tilde f_{\tilde a,\tilde b}:\mathbb{R}^3\mapsto\mathbb{R}^3$, such that*

$$\tilde f_{\tilde a,\tilde b}(\tilde x,\tilde y,\tilde z)=(0,\tilde a+\tilde b\tilde y+\tilde z^2,\tilde y).$$

Finally, as it is easy to observe, the limit return map of theorem 1 is, essentially the two-dimensional Hénon map. As it is known, all the nontrivial quadratic maps in $\mathbb{R}^2$ with constant jacobian are conjugate to this map. In the next theorem we prove that the limit return map of theorem 2 is also representative of all the quadratic maps in $\mathbb{R}^2$ whose double iterate is also quadratic. Moreover, we find the simplest quadratic map in $\mathbb{R}^3$ that can be taken as a model to study chaotic behaviour.

Theorem 3 *1. Let $f : \mathbb{R}^2 \mapsto \mathbb{R}^2$ a quadratic map such that $f \circ f$ is also quadratic. Suppose that f does not have any invariant linear foliation. Then it is linearly conjugate to the map $(x, y) \mapsto (a + bx - y^2, x)$ for some real numbers a and b.*

2. Let $f : \mathbb{R}^3 \mapsto \mathbb{R}^3$ a quadratic map such that $f \circ f$ is also quadratic and having constant jacobian. Suppose that f does not have any invariant linear foliation. Then it is linearly conjugate to the map $(x, y, z) \mapsto (a+bx+z-y^2, x, cy)$, for some real numbers a, b and c.

It is important to stress that it is easy to find numerically strange attractors that are topologically two-dimensional, for the quadratic maps of Theorem 3. We conjecture that these strange attractor really exist, and for a set in the (a, b) parameter plane with positive Lebesgue measure, for the first map, and the same for the second but for c small enough. Thinking of the scheme of the proof of existence of strange attractors in the known cases, it does not seem very risky to suppose that the same result can be true for a two-dimensional family of dissipative diffeomorphisms as the one of Theorem 2, and for values of the parameters near the corresponding to the generalized homoclinic tangency. In any case the existence of this type of tangency seems to produce a very rich and, probably, new phenomena.

Remark 1.3 *According to Gonchenko et al. [5, 6], the maps given in theorem 3 can be seen as approximations of return maps near a (non-generalized) quadratic homoclinic tangency when some of the eigenvalues of the initial fixed point are not real. For the second three-dimensional quadratic map the dimension of the space where the initial diffeomeorphism is defined, has to be at least four. However, no detailed proofs of these statements are provided in the cited papers.*

The paper is organized as follows. In section 2 we introduce the definition of generalized homoclinic tangency and introduce the Bogdanov-Takens bifurcations, giving the conditions of generic unfolding. In section 3 we obtain bifurcations associated to the existence of homoclinic tangencies in the one-parameter case, and with the generalized homoclinic tangency in the two-parameter case, and other related results. In section 4 we prove the existence of the limit return maps and theorems 1 and 2. Finally, in section 5 we prove theorem 3.

Through the paper we use the concept of smooth map. We mean that the considered map is sufficiently differentiable.

2 Unfoldings of homoclinic tangencies and bifurcations

We have divided this section in four parts. In the first part, we study the generic unfolding of a quadratic tangency between a curve and a surface in a three-dimensional space. We use this to define a homoclinic quadratic tangency unfolding generically. In the second we define what we call generalized transversality between a curve and a surface. In this case it is supposed that in the surface we have a foliation. Then we define homoclinic transversality. It is necessary in order to give the behaviour of the generalized (quadratic) homoclinic tangency that is defined in the fourth section. We give in all the cases the conditions of generic unfolding that are necessary for the proof of our main theorems. Finally, in the last part we give the conditions of existence of a generic Bogdanov-Takens bifurcation.

2.1 Unfolding of quadratic tangencies

In this section we want to define generic unfolding of the tangency of a curve and a surface in $\mathbb{R}^3$, when these objects depend on one real parameter.

First we will give a general definition:

Definition 2.1 *Let $\{C_a\}_{a\in V}$, be a smooth family of smooth regular curves $C_a \subset \mathbb{R}^3$, and $\{S_a\}_{a\in V}$ be a smooth family of smooth regular surfaces $S_a \subset \mathbb{R}^3$ depending on a parameter $a \in V \subset \mathbb{R}$, V being an open subset, such that, for $a = a_0$, C_a and S_a intersect in a point $p_0 \in \mathbb{R}^3$. We say that C_a and S_a have a quadratic tangency in p_0 (or a contact of order 1) which unfolds generically with a in $a = a_0$ iff there exists some smooth change of variables, such that in the new variables $p_0 = (0,0,0)$, $S_a = \{(x,y,z) \in U \ : \ z = 0\}$, where U is a neighbourhood of $0 \in \mathbb{R}^3$ and C_a is represented by the parametrized curve $\gamma(t,a) = (x(t,a), y(t,a), z(t,a))$, and the following properties hold:*

1. $z(0, a_0) = 0$,
2. $D_1 z(0, a_0) = 0$,
3. $D_{11} z(0, a_0) \neq 0$ *and*
4. $D_2 z(0, a_0) \neq 0$.

It is easy to see that this definition does not depend on the used change to obtain that S_a is represented by the equation $z = 0$.

The manifolds C_a and S_a can be represented in the following way: $S_a = \{(x,y,z) \in U \ : \ f(x,y,z,a) = 0\}$, where U is an open neighbourhood of $p_0 = (x_0, y_0, z_0)$, and $C_a = \{\gamma(t,a) \ : \ t \in I \subset \mathbb{R}\}$, where I is an open interval, and $f(p_0, a_0) = 0$, and $\gamma(0, a_0) = p_0$. Then, it is easy to prove the following proposition:

Proposition 2.2 *Suppose that C_a and S_a have a quadratic tangency in p_0 unfolding generically in $a = a_0$, and f and γ are as before. Then:*

1. $Df_0(p_0)D_1\gamma(0, a_0) = 0$,
2. $D^2 f_0(p_0)(D_1\gamma(0, a_0), D_1\gamma(0, a_0)) + Df_0(p_0)D_{11}\gamma(0, a_0) \neq 0$,
3. $Df_0(p_0)D_2\gamma(0, a_0) + D_2 f(p_0, a_0) \neq 0$,

where $f_0 = f(\cdot, \cdot, \cdot, a_0)$.

The next proposition gives the behaviour of quadratic tangencies which unfold generically.

Proposition 2.3 *Let C_a and S_a be as in the beginning of the section, and $p_0 \in C_{a_0} \cap S_{a_0}$. If C_a and S_a has a quadratic tangency in p_0 which unfolds generically with a in $a = a_0$ then there are an open neighbourhood I_0 of $0 \in \mathbb{R}$, and a smooth map $a = a(t) \in \mathbb{R}$ such that:*

1. $a(0) = a_0$, $\dot{a}(0) = 0$ *and* $\ddot{a}(0) \neq 0$.
2. $\gamma(t, a(t)) \in S_{a(t)}$ *for all* $t \in I_0$.

We will consider one special case we are interested in.

We can perform a smooth change of variables and parameters, in order to have $\gamma(t,a) = (t,0,0)$. Moreover, we suppose that the surface S_a has the following parametrization:

$$\mathbf{x}(u,v,a) = (X(u,v,a), Y(u,v,a), Z(u,v,a)).$$

As $p_0 = (x_0, 0, 0) \in S_{a_0}$, there exists $(u_0, v_0) \in \mathbb{R}^2$ such that $p_0 = \mathbf{x}(u_0, v_0, a_0)$. We want to obtain simple necessary conditions of quadratic tangency, in terms of the maps Y and Z.

Proposition 2.4 *If the families of curves γ and surfaces S_a have a quadratic tangency in $(x_0, 0, 0)$ which unfolds generically with a in a_0 then:*

1. $Y(u_0, v_0, a_0) = Z(u_0, v_0, a_0) = 0$.
2. *Let* $g(u,v,a) = D_1Y(u,v,a)D_2Z(u,v,a) - D_2Y(u,v,a)D_1Z(u,v,a)$. *Then* $g(u_0, v_0, a_0) = 0$.
3. *Let* $g_0 = g(\cdot,\cdot,a_0)$, $Y_0 = Y(\cdot,\cdot,a_0)$ *and* $Z_0 = Z(\cdot,\cdot,a_0)$. *Then*

$$\mathit{rank}\,(Dg_0(u_0,v_0), DY_0(u_0,v_0), DZ_0(u_0,v_0)) = 2.$$

4. $\mathit{rank}\,(DY(u_0,v_0,a_0), DZ(u_0,v_0,a_0)) = 2$.

Proof:
Consider a map $G = (G_1, G_2, G_3)$, defined in an open neighbourhood of $(u_0, v_0, 0, a_0) \in \mathbb{R}^4$ and with range in $\mathbb{R}^3$, and such that $G(u,v,0,a) = \mathbf{x}(u,v,a)$. We denote $G_a := G(\cdot,\cdot,\cdot,\cdot,a)$. As S_a is a regular surface, we can define G in such a way $G(u,v,0,a) = \mathbf{x}(u,v,a)$, and G_a is a diffeomorphism for each a. Then G can be used as a change of variables to be in the conditions of definition 2.1. Hence, we have, by items 1. and 2. of the definition, that if there is a quadratic tangency then items 1. and 2. of the proposition hold. Moreover, as 2. holds, we can take $G(u,v,w,a) = (X(u,v,a), Y(u,v,a) + g_2(w), Z(u,v,a) + g_3(w))$, where $g_2(w) = w$ and $g_3(w) = 0$ or $g_2(w) = 0$ and $g_3(w) = w$. Using this latter change it is easy to deduce the rest of the proposition from items 3 and 4 of the definition. □

In order to define quadratic homoclinic tangencies, we add some generic conditions. First we begin with the following definition:

Definition 2.5 *Let f a three-dimensional diffeomorphism, having a hyperbolic fixed point p. We denote by λ_i, $i = 1,2,3$ the eigenvalues of $Df(p)$, that we suppose real and different in absolute value. Suppose that one of the invariant manifolds that we denote W has dimension 2. Let $\tilde{W}$ denote the strong stable or unstable manifold contained in W. We say that $q \in W$ is in general position if $q \notin \tilde{W}$ and there exists some point $\tilde{q}$ in the orbit of q, and some smooth conjugacy of f to $\Lambda(t_1,t_2,t_3) = (\lambda_1 t_1, \lambda_2 t_2, \lambda_3 t_3)$ such that the image of q_0 for this conjugacy is not in any of the coordinate axes.*

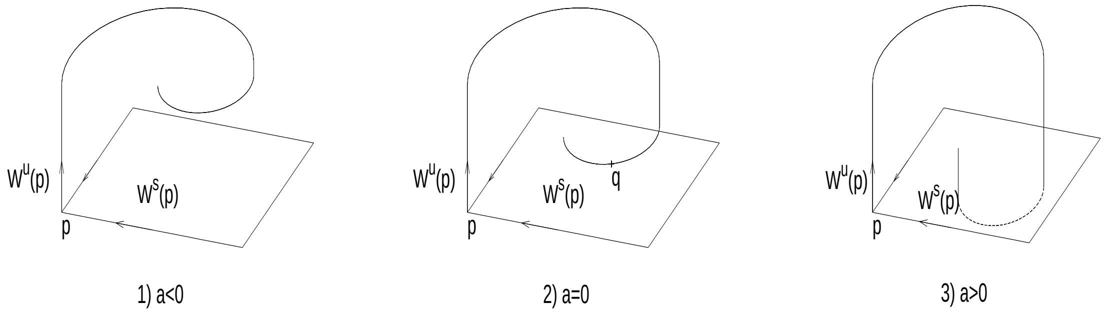

Figure 1: Quadratic homoclinic tangency. 1) No intersection of the invariant manifolds. 2) A tangencial intersection. 3) Two transversal intersections.

Remark 2.6 *We note that if, for example,* $|\lambda_1| < |\lambda_2| < 1$ *in the preceding definition, and* $|\lambda_1\lambda_2^{-n}| < 1$ *there exists a nontrivial* C^n *diffeomorphism* G *which commutes with* Λ *(that is* $G\circ\Lambda = \Lambda\circ G$*). Indeed, we have to distinguish two cases: a)* $\lambda_1 > 0$*. It is enough to perform the following change of coordinates to the linearized map:* $(x, y, z) \mapsto (x + C|y|^{\beta}, y, z)$*, where* C *is a constant different from zero and* $\beta = \log|\lambda_1|/\log|\lambda_2|$*. b) If* $\lambda_1 < 0$ *then we consider the change:* $(x, y, z) \mapsto (x + C\sin[\pi(\log|\lambda_2|)^{-1}\log|y|]|y|^{\beta}, y, z)$*. This means that in this case, if the point* q *is not in the strong stable manifold, the second condition of the definition can be always achieved if we work with* C^n *diffeomorphisms.*

Then we are in conditions to give the definition of quadratic homoclinic tangency.

Definition 2.7 *Let* $\{f_a\}_{a\in V}$ *be a family of three-dimensional diffeomorphism, having a hyperbolic fixed point* $p = p(a)$*, for all* $a \in V$*. Suppose that the eigenvalues of* $Df_a(p(a))$ *that we denote by* λ_i*,* $i = 1, 2, 3$*, are all real and different in absolute value:* $|\lambda_1| < |\lambda_2| < |\lambda_3|$*. Moreover* $|\lambda_1| < 1$ *and* $|\lambda_3| > 1$*. Then*

1. *If* $|\lambda_2| < 1$*, we say that* $\{f_a\}_{a\in V}$ *has a quadratic homoclinic tangency in a point* q *unfolding generically in* $a = a_0$ *if the manifolds* $W^u(p(a))$ *and* $W^s(p(a))$ *have a quadratic tangency in* q *which unfolds generically in* $a = a_0$*, and such that* q *is in general position.*
2. *If* $|\lambda_2| > 1$*, we say that* $\{f_a\}_{a\in V}$ *has a quadratic homoclinic tangency in a point* q *unfolding generically in* $a = a_0$ *if* $\{f_a^{-1}\}_{a\in V}$ *has a quadratic homoclinic tangency unfolding generically in* $a = a_0$*.*

A typical quadratic homoclinic tangency is shown in figure 1.

In the last part of this section, we will give some adapted coordinates and parameters for families of diffeomorphisms having a quadratic homoclinic tangency.

Proposition 2.8 *Let* $\{f_a\}_{a\in V}$*,* V *being an open neighbourhood of* a_0*, be a 1-parameter family of three-dimensional diffeomorphisms having a hyperbolic fixed point* $p = p(a)$*, with real eigenvalues* λ_i*,* $i = 1, 2, 3$ *such that* $|\lambda_1| < |\lambda_2| < 1 < |\lambda_3|$*. We assume that this* *family has a quadratic homoclinic tangency in a point q unfolding generically with a in $a = a_0$, and that it is smoothly linearizable near the fixed point p. Then, via a change of variables and a reparametrization if necessary, we can suppose that this family satisfies the following: There exists a smooth map $\mathbf{x} = \mathbf{x}(t_1, t_2, t_3, a)$, defined in a neighbourhood of $\{(t_1, t_2, t_3) \in \mathbb{R}^3 \ : \ (t_1^2 + t_2^2)t_3 = 0\} \times \{0\}$ such that*

1. *$\mathbf{x}(\cdot, \cdot, \cdot, a)$ is the identity in a neighbourhood U of $\{(t_1, t_2, 0) \ : \ t_1^2 + t_2^2 \leq 3\}$, for all $a \in V$, and U does not depend on a.*
2. *$f_a(t_1, t_2, t_3) = \mathbf{x}(\lambda_1 t_1, \lambda_2 t_2, \lambda_3 t_3, a)$, for all $(t_1, t_2, t_3) \in U$ and $a \in V$.*
3. *If $\mathbf{x} = (x, y, z)$ then*
$$x(0,0,1,a) = 1, \quad y(0,0,1,a) = 1, \quad z(0,0,1,a) = a, \quad D_3 z(0,0,1,a) = 0,$$
for all $a \in V$ and $D_{33} z(0,0,1,0) \neq 0$.
4. *The point of homoclinic tangency can be taken as $q = (1,1,0) = \mathbf{x}(0,0,1,0)$.*

Proof:

In the first place, we can suppose that $p(a) = 0$, for all $a \in V$ and $a_0 = 0$, by performing a translation in the variables and the parameter.

As a consequence of the smooth linearization assumption, it is possible to perform a change of variables to have a map $\mathbf{x}$ which satisfy the properties 1 and 2. By the definition of quadratic homoclinic tangency, there exists a real number $\bar{t}_3 \neq 0$ such that $z(0,0,\bar{t}_3,0) = D_3 z(0,0,\bar{t}_3,0) = 0$, $D_4 z(0,0,\bar{t}_3,0) \neq 0$ and $D_{33} z(0,0,\bar{t}_3,0) \neq 0$. Moreover, as the homclinic point is in general position, $x(0,0,\bar{t}_3,0) \neq 0$ and $y(0,0,\bar{t}_3,0) \neq 0$.

Let $\tilde{t}_3(a)$ be a map defined in a neighbourhood of zero such that for all a in this neighbourhood
$$D_3 z(0,0,\tilde{t}_3(a),a) = 0 \text{ and } \tilde{t}_3(0) = \bar{t}_3.$$
Such a map exists because $D_3 z(0,0,\bar{t}_3,0) = 0$, $D_{33} z(0,0,\bar{t}_3,0) \neq 0$ and we can apply the implicit function theorem. Let $g_a = \Lambda_{H(a)}^{-1} \circ f_{H(a)} \circ \Lambda_{H(a)}$, where
$$\Lambda_a(t_1, t_2, t_3) = (x(0,0,\tilde{t}_3(a),a)t_1, y(0,0,\tilde{t}_3(a),a)t_2, \tilde{t}_3(a)t_3),$$
and H is the inverse of the map G such that $G(a) = z(0,0,\tilde{t}_3(a),a)/\tilde{t}_3(a)$, and $G(0) = 0$, defined in a neighbourhood of $0 \in \mathbb{R}$. The map H exists because the first three conditions of non-degenerate homoclinic tangency which unfolds generically, permit us to apply the inverse function theorem. Then the family g_a is smoothly conjugated to f_a and has the same representation. If we call again this new map f_a, we can represent, as before,
$$f_a^n(t_1, t_2, t_3) = \mathbf{x}(\lambda_1^n t_1, \lambda_2^n t_2, \lambda_3^n t_3, a),$$
with $\mathbf{x}$ satisfying all the properties we wanted. □

Proposition 2.9 *Let $\{f_a\}_{a \in V}$ a 1-parameter family of three-dimensional diffeomorphisms having a hyperbolic fixed point $p = p(a)$, with real eigenvalues λ_i, $i = 1,2,3$ such that $|\lambda_1| < 1 < |\lambda_2| < |\lambda_3|$. We assume that this family has a quadratic homoclinic tangency in a point q unfolding generically with a in $a = a_0$, and that it is smoothly linearizable near the fixed point p. Then, via a change of variables and a reparametrization if necessary, we can suppose that this family satisfies the following: There exists a smooth map $\mathbf{x} = \mathbf{x}(t_1, t_2, t_3, a)$, defined in a neighbourhood of $\{(t_1, t_2, t_3) \in \mathbb{R}^3 \ : \ (t_2^2 + t_3^2)t_1 = 0\} \times \{0\}$ such that*

1. *$\mathbf{x}(\cdot,\cdot,\cdot,a)$ is the identity in a neighbourhood U of $\{(t_1,0,0)\ :\ |t_1|\leq 3\}$, for all $a\in V$, and U does not depend on a.*
2. *$f_a^n(t_1,t_2,t_3)=\mathbf{x}(\lambda_1^n t_1,\lambda_2^n t_2,\lambda_3^n t_3,a)$, for all $(t_1,t_2,t_3)\in U$ and $a\in V$.*
3. *The point of homoclinic tangency can be taken as $q=(1,0,0)$, for $a=0$. Moreover, if $\mathbf{x}=(x,y,z)$ and*

$$g(t_2,t_3,a)=D_2y(0,t_2,t_3,a)D_3z(0,t_2,t_3,a)-D_3y(0,t_2,t_3,a)D_2z(0,t_2,t_3,a),$$

then one of the following possibilities holds:

(a) $x(0,1,1,a)=1$, $y(0,1,1,a)=0$, $z(0,1,1,a)=a$, $g(1,1,a)=0$, for all $a\in V$ and

$$D_1g(1,1,0)D_3y(0,1,1,0)-D_2g(1,1,0)D_2y(0,1,1,0)\neq 0.$$

(b) $x(0,1,1,a)=1$, $y(0,1,1,a)=a$, $z(0,1,1,a)=0$, $g(0,1,1,a)=0$, for all $a\in V$ and

$$D_1g(1,1,0)D_3z(0,1,1,0)-D_2g(1,1,0)D_2z(0,1,1,0)\neq 0.$$

Proof:

As before, we can suppose that there exists a map $\mathbf{x}$ satisfying 1 and 2, and that $a_0=0$. By the definition of quadratic homoclinic point, there exists $\bar{t}_2$ and $\bar{t}_3$ such that $\mathbf{x}(0,\bar{t}_2,\bar{t}_3,0)$ is a homoclinic point. Moreover $\bar{t}_2\neq 0$, $\bar{t}_3\neq 0$, $x(0,\bar{t}_2,\bar{t}_3,0)\neq 0$ and $y(0,\bar{t}_2,\bar{t}_3,0)=z(0,\bar{t}_2,\bar{t}_3,0)=0$.

Now we suppose that the homoclinc tangency unfolds generically. Applying proposition 2.4 we have the following additional conditions on the map $\mathbf{x}$: $g(\bar{t}_2,\bar{t}_3,0)=0$,

$$\operatorname{rank}\begin{pmatrix} D_1g & D_2g\\ D_2y & D_3y\\ D_2z & D_3z\end{pmatrix}=2,$$

and

$$\operatorname{rank}\begin{pmatrix} D_2y & D_3y & D_4y\\ D_2z & D_3z & D_4z\end{pmatrix}=2,$$

where the derivatives of y and z are evaluated in $(0,\bar{t}_2,\bar{t}_3,0)$ and the derivatives of g in $(\bar{t}_2,\bar{t}_3,0)$. We distinguish two cases:

a) $D_1gD_3y-D_2gD_2y\neq 0$, where the derivatives are evaluated in the same points as before.

Taking into account that $g(\bar{t}_2,\bar{t}_3,0)=0$, $y(0,\bar{t}_2,\bar{t}_3,0)=0$ we can apply the Implicit Function Theorem to the system:

$$\begin{aligned} g(t_2,t_3,a) &= 0\\ y(0,t_2,t_3,a) &= 0\end{aligned}$$

Then there exist maps $\tilde{t}_2(a)$, $\tilde{t}_3(a)$ such that $\tilde{t}_i(0)=\bar{t}_i$, for $i=1,2$, and satisfy the previous equations for all a small enough.

Moreover, $D_2 z D\tilde{t}_2(0) + D_3 z D\tilde{t}_3(0) + D_4 z \neq 0$ where the derivatives of z are evaluated at $(0, \bar{t}_2, \bar{t}_3, 0)$. Indeed, suppose that it is not true. If we compute the derivative of $y(0, \tilde{t}_2(a), \tilde{t}_3(a), a) = 0$ with respect to a, we obtain the system:

$$\begin{aligned} D_2 y D\tilde{t}_2 + D_3 y D\tilde{t}_3 &= -D_4 y \\ D_2 z D\tilde{t}_2 + D_3 z D\tilde{t}_3 &= -D_4 z \end{aligned}$$

But this system is incompatible by the definition of quadratic homoclinic tangency which unfolds generically. Therefore, the assertion is true.

Let $g_a = \Lambda_a^{-1} \circ f_a \circ \Lambda_a$, where

$$\Lambda_a(t_1, t_2, t_3) = (x(0, \tilde{t}_2(a), \tilde{t}_3(a), a)t_1, \tilde{t}_2(a)t_2, \tilde{t}_3(a)t_3),$$

If we rename g_a again as f_a then $f_a(t_1, t_2, t_3) = \mathbf{x}(\lambda_1 t_1, \lambda_2 t_2, \lambda_3 t_3, a)$, where $x(0, 1, 1, a) = 1$, $y(0, 1, 1, a) = 0$, $g(1, 1, a) = 0$ and $D_4 z(0, 1, 1, a) \neq 0$. So, if we consider a new parameter $\bar{a} = z(0, 1, 1, a)$ and renaming again the new parameter as a we obtain a map $\mathbf{x}$ which satisfies items 1, 2, 3(a).

b) $D_1 g D_3 z - D_2 g D_2 z \neq 0$, where the derivatives are evaluated in the same points as before.

In this case we can proceed in a similar way. The differences are that first the maps $\tilde{t}_2$ and $\tilde{t}_3$ satisfy the equations:

$$\begin{aligned} g(t_2, t_3, a) &= 0 \\ z(0, t_2, t_3, a) &= 0 \end{aligned}$$

and in the final step we take $\bar{a} = y(0, 1, 1, a)$ as the new parameter. Then, we obtain a map $\mathbf{x}$ which satisfies 1, 2, 3(b). $\square$

2.2 Generalized homoclinic transversality

The goal of this section is to introduce a new type of transversality between invariant manifolds of a fixed point. First, we introduce some objects that we shall need for the definition.

Given a diffeomorphism $f : M \mapsto M$, where M is a 3-dimensional smooth manifold, with a fixed point p, such that the eigenvalues λ_i, $i = 1, 2, 3$, of $Df(p)$ satisfies $|\lambda_1| < |\lambda_2| < 1 < |\lambda_3|$ then there exists a foliation $\mathcal{F}^{ss}$, called the strong stable invariant foliation, defined in $W^s(p)$ such that it is characterized in the following way:

Given a smooth manifold $N \subset M$ we denote

$$L_i(N) := \{(x, L) \,|\, x \in N,\ L \text{ is a } i\text{-dimensional linear subspace of } T_x M \text{ and } T_x N \cap L \neq \{0\}\}.$$

Consider the diffeomorphism $f_* : L_1(W^s(p)) \mapsto L_1(W^s(p))$, where f_* is the diffeomorphism induced by Df on $L_1(W^s(p))$. Let E^{ss} be the linear subspace tangent to $W^{ss}(p)$ in p. Then (p, E^{ss}) is a hyperbolic fixed point of f_* with eigenvalues λ_1, λ_2 and $\lambda_2\lambda_1^{-1}$. We observe that $|\lambda_1| < |\lambda_2| < 1 < |\lambda_2\lambda_1^{-1}|$. Hence, there exist a two-dimensional stable invariant manifold $W^s(p, E^{ss})$. The foliation generated by this vector bundle is the strong stable invariant foliation $\mathcal{F}^{ss}(p)$.

Now we characterize a family of planes associated to the unstable invariant manifold. We suppose as before, that M is a smooth three-dimensional manifold.

Proposition 2.10 *Let $f : M \longrightarrow M$ be a smooth diffeomorphism such that there is a hyperbolic fixed point p with eigenvalues λ_1, λ_2 and λ_3 satisfying: $|\lambda_1| < |\lambda_2| < 1$ and $|\lambda_3| > 1$. Let f_* be the diffeomorphism induced on $L_2(W^u(p))$ by the derivative of f. Then the point $(p, E^{cu}) \in L_2(W^u(p))$, where E^{cu} is the invariant linear subspace associated to the eigenvalues λ_2 and λ_3, is a saddle fixed point of f_* with eigenvalues λ_3 and $\lambda_1\lambda_2^{-1}$.*

Proof:
As this can be considered as a local problem, we can suppose that $M = \mathbb{R}^3$. We can suppose that $p = 0$ and $Df(0) = \mathrm{diag}\,(\lambda_1, \lambda_2, \lambda_3)$. Let $\mathbf{x}(t)$ a parametrization of $W^u(p)$ such that $\mathbf{x}(0) = 0$ and $f(\mathbf{x}(t)) = \mathbf{x}(\lambda_3 t)$. We define coordinates of $L(W^u(p))$ in a neighbourhood U of $(0, E^{cu})$ given by the map $\psi : U \longrightarrow \mathbb{R}^2$, such that $\psi(q, L) = (t, n_2)$, where $\mathbf{x}(t) = q$ and $n = (n_1, n_2, n_3)$ is the unit normal vector associated to L, such that $n_1 > 0$. An easy computation shows that the point (p, E^{cu}), represented as $(0, 0) \in \mathbb{R}^2$ in these coordinates, is the wanted hyperbolic fixed point. □

The unstable invariant manifold $W^u(p, E^{cu})$ will be called the *unstable invariant tangent vector bundle* associated to E^{cu}. By definition the planes of the vector bundle are tangent to $W^u(p)$.

We note that if $|\lambda_1| < 1 < |\lambda_2| < |\lambda_3|$, we can define, in an analogous way, the strong unstable invariant foliation $\mathcal{F}^{uu}(p)$ and the stable invariant tangent vector bundle $W^s(p, E^{cs})$, associated to E^{cs}.

It is interesting to observe that if f satisfies the linearization assumption and, therefore, we have $f(\mathbf{x}(t_1, t_2, t_3)) = \mathbf{x}(\lambda_1 t_1, \lambda_2 t_2, \lambda_3 t_3)$, with $\mathbf{x}$ as in the introduction and we suppose that $|\lambda_1| < |\lambda_2| < 1 < |\lambda_3|$ then the leaves of the strong stable foliation of the origin $0 \in \mathbb{R}^3$ are parametrized by the maps $\mathbf{x}(t, \bar{t}_2, 0)$ with t a real parameter small enough and $\bar{t}_2$ a small constant. Moreover the unstable tangent vector bundle is generated in any point $\mathbf{x}(0, 0, t_3)$ of $W^u(0)$ by the vectors $D_2\mathbf{x}(0, 0, t_3)$ and $D_3\mathbf{x}(0, 0, t_3)$. In the case $|\lambda_1| < 1 < |\lambda_2| < |\lambda_3|$, the leaves of $\mathcal{F}^{uu}$ are parametrized by $\mathbf{x}(0, \bar{t}_2, t)$ and the stable vector bundle is generated in a point $\mathbf{x}(t_1, 0, 0)$ by the vectors $D_1\mathbf{x}(t_1, 0, 0)$ and $D_2\mathbf{x}(t_1, 0, 0)$.

The next step is to define generalized transversality between a one-parameter family of foliated surfaces and a family of one-parameter vector bundles with a curve as a zero section and planes as fibers. First we recall the local definition of foliation:

Let $S \subset \mathbb{R}^3$ a surface and $\Phi = \{\mathcal{L}_\alpha\}_{\alpha \in A}$, A being a set of indices, a partition of S into disjoint sets called *leaves*. The partition Φ is called a *foliation* if there is a parametrization $\mathbf{x} : I \times J \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$ of S such that $\mathcal{L}_\alpha = \mathbf{x}(I \times \{c_\alpha\})$, where I and J are open intervals, $S = \mathbf{x}(I \times J)$, and c_α is a constant for each $\alpha \in A$. We call the pair (S, Φ) a *foliated surface*.

On one hand, we consider a family of foliated surfaces $\{(S_a, \Phi_a)\}_{a \in V}$, $V \subset \mathbb{R}$ being an open set. We can always suppose that the family of surfaces is implicitly defined by the equation $f(x, y, z, a) = 0$, On the other hand, we consider a family of vector bundles whose zero section is a curve $\gamma = \gamma(t,a)$ and with fibers that are tangent planes to the curve. This can be represented by a family of vectors $v(t,a)$, such that $v(t,a) \in T_{\gamma(t,a)}\mathbb{R}^3$, that are not colinear to the tangent vector of the curve. That is we have the family $\{(\gamma_a, v_a)\}_{a\in V}$, where $\gamma_a(t) = \gamma(t,a)$ and $v_a(t) = v(t,a)$.

Definition 2.11 *We say that $\{(S_a,\Phi_a)\}_{a\in V}$ and $\{(\gamma_a,v_a)\}_{a\in V}$ have a generalized transversal intersection in a point $p\in\mathbb{R}^3$ for $a=a_0\in V$ which unfolds generically with a if there exist coordinates (x,y,z) for which, the leaves of the foliation are the straight lines $z=0$, $y=c$ and, if we write $\gamma(t,a)=(x(t,a),y(t,a),z(t,a))$, $v=(v_1,v_2,v_3)$ then:*

1. *$p=\gamma(0,a_0)$,*
2. *$z(0,a_0)=0$,*
3. *$D_1z(0,a_0)\neq 0$,*
4. *If $T(t,a)=v_2(t,a)D_1z(t,a)-v_3(t,a)D_1y(t,a)=0$ then $T(0,a_0)=0$.*
5. *$D_1z(0,a_0)D_2T(0,a_0)-D_2z(0,a_0)D_1T(0,a_0)\neq 0$.*

The next proposition gives the behaviour of the generalized transversality:

Proposition 2.12 *Suppose that $\{(S_a,\Phi_a)\}_{a\in V}$ and $\{(\gamma_a,v_a)\}_{a\in V}$ have a generalized transversal intersection in a point $p=\gamma(t_0,a_0)\in\mathbb{R}^3$ for $a=a_0\in V$ which unfolds generically with a. Then there exists a map $t=t(a)$ defined in a neighbourhood of t_0 such that*

1. *S_a has a transversal intersection with γ_a in $\gamma(t(a),a)$.*
2. *The plane determined by $D_1\gamma(t(a),a)$ and $v(t(a),a)$ is transversal to the leave of (S_a,Φ_a) passing through $\gamma(t(a),a)$, except if $a=a_0$.*

Proof:

First we consider coordinates like the ones of the previous definition. The map $t=t(a)$ exists because $D_1z(0,a_0)\neq 0$ and $z(0,a_0)=0$. So, we can apply the implicit function theorem to the equation $z(t,a)=0$, in order to get $t(a)$. This proves item 1.

To prove item 2 we use the second condition of existence of the generalized transversal point. As $z(t(a),a)=0$ then $\dot{t}(a_0)=-D_2z(0,a_0)/D_1z(0,a_0)$. Then by condition 2 of the definition we have that $T(t(a),a)$ is strictly monotone and $T(t(a_0),a_0)=0$. This proves item 2. □

We can give now the definition of generalized homoclinic transversality:

Definition 2.13 *Let $\{f_a\}_{a\in V}$, $V\subset\mathbb{R}$ being an open set, be a smooth 1-parameter family of three dimensional diffeomorphisms having a hyperbolic fixed point $p=p(a)$, for $a\in V$, with real eigenvalues λ_i, $i=1,2,3$. such that $|\lambda_1|<|\lambda_2|<|\lambda_3|$ and $|\lambda_1|<1<|\lambda_3|$. Then:*

1. *If $|\lambda_2|<1$ we say that $\{f_a\}_{a\in V}$ has a generalized transversal homoclinic point in $q\in\mathbb{R}^3$ unfolding generically with a in $a=a_0\in V$, if the family of foliated surfaces $\{(W^s(p(a)),\mathcal{F}^{ss}(p(a)))\}_{a\in V}$ and the family of invariant tangent vector bundles $\{(W^u(p(a)),E^{cu}(p(a)))\}_{a\in V}$ have a generalized transversal point in q, which unfolds generically with a in $a=a_0$. Moreover $q\in W^s(p(a_0))$ is in general position.*

2. *If $|\lambda_2| > 1$ we say that $\{f_a\}_{a\in V}$ has a generalized transversal homoclinic point in $q \in \mathbb{R}^3$ unfolding generically with a in $a = a_0 \in V$, if $\{f_a^{-1}\}_{a\in V}$ has a generalized transversal homoclinic point in q for $a = a_0$.*

In the next proposition we give adapted coordinates for such type of transversality.

Proposition 2.14 *Let $\{f_a\}_{a\in V}$, V being an open neighbourhood of a_0, a 1-parameter family of three-dimensional diffeomorphisms having a hyperbolic fixed point $p = p(a)$, with real eigenvalues λ_i, $i = 1, 2, 3$ such that $|\lambda_1| < |\lambda_2| < 1 < |\lambda_3|$. We assume that this family has a generalized transversal homoclinic point q unfolding generically with a in $a = a_0$, and that it is smoothly linearizable near the fixed point p. Then, via a change of variables and a reparametrization if necessary, we can suppose that this family satisfies the following: There exists a smooth map $\mathbf{x} = \mathbf{x}(t_1, t_2, t_3, a)$, defined in a neighbourhood of $\{(t_1, t_2, t_3) \in \mathbb{R}^3 \ : \ (t_1^2 + t_2^2)t_3 = 0\} \times \{0\}$ such that*

1. *$\mathbf{x}(\cdot, \cdot, \cdot, a)$ is the identity in a neighbourhood U of $\{(t_1, t_2, 0) \ : \ t_1^2 + t_2^2 \leq 3\}$, for all $a \in V$, and U does not depend on a.*
2. *$f_a^n(t_1, t_2, t_3) = \mathbf{x}(\lambda_1^n t_1, \lambda_2^n t_2, \lambda_3^n t_3, a)$, for all $(t_1, t_2, t_3) \in U$ and $a \in V$.*
3. *If $\mathbf{x} = (x, y, z)$ then*
$$x(0, 0, 1, a) = 1, \quad y(0, 0, 1, a) = 1, \quad z(0, 0, 1, a) = 0,$$
and
$$D_2 y(0, 0, 1, a) D_3 z(0, 0, 1, a) - D_2 z(0, 0, 1, a) D_3 y(0, 0, 1, a) = a,$$
for all $a \in V$ and $D_3 z(0, 0, 1, 0) \neq 0$.
4. *The point q can be taken as $q = (1, 1, 0) = \mathbf{x}(0, 0, 1, 0)$.*

In figure 2 we have a typical example of a generalized homoclinic transversality.

2.3 Generalized homoclinic tangencies

Now we want to introduce new types of intersection of invariant manifolds of a fixed point, that we shall call generalized homoclinic tangency, that can be seen as a degeneracy of a homoclinic tangency and a generalized transversal intersection of invariant manifolds.

As before we introduce first this concept with respect to a family of surfaces with a smooth foliation and a vector bundle with a curve as a zero section and tangent planes as fibers. We shall give two types of this tangency. As before we consider a family of foliated surfaces $\{(S_{a,b}, \Phi_{a,b})\}_{(a,b)\in V}$, with associated parametrization $\mathbf{x}$, and a family of tangent vector bundles $\{(\gamma_{a,b}, v_{a,b})\}_{(a,b)\in V}$ where $V \subset \mathbb{R}^2$ is an open neighbourhood of (a_0, b_0), and $\gamma_{a,b}(t) = \gamma(t, a, b)$ is a regular parametrized curve and $v_{a,b}(t) = v(t, a, b) \in T_{\gamma(t,a,b)}\mathbb{R}^3$. Let $t_0 \in \mathbb{R}$ such that $p_0 = \gamma(t_0, a_0, b_0) \in S_{a_0,b_0}$.

Definition 2.15 *We say that $\{(S_{a,b}, \Phi_{a,b})\}_{(a,b)\in V}$ and $\{(\gamma_{a,b}, v_{a,b})\}_{(a,b)\in V}$ have a generalized tangency of type I in p_0 which unfolds generically with (a, b) in (a_0, b_0) if there exist some coordinate system (x, y, z) for which $S_{a,b}$ is the plane $z = 0$ and the leaves of the foliation are the straight lines $z = 0$, $y = c$, such that if we write $\gamma(t, a, b) = (x(t, a, b), y(t, a, b), z(t, a, b))$ then*

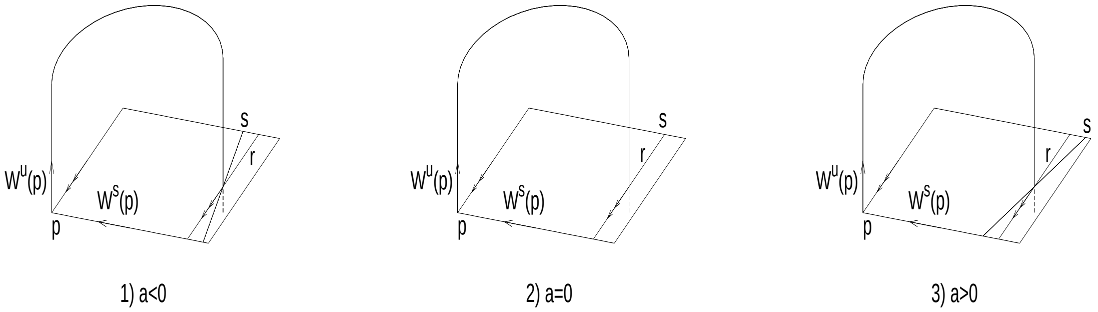

Figure 2: Generalized transversality. The line from p with two arrows represents the strong stable manifold $W^{ss}(p)$, the line r is a leave of the strong stable foliation of $W^s(p)$ and the line s represents the intersection of $W^s(p)$ with the fiber of the unstable invariant vector bundle passing through the point of intersection of the invariant manifolds. In 1) and 3) r and s are different and in 2) (corresponding to the generalized homoclinic transversality) they coincide. Observe that in all the cases the invariant manifolds have transversal intersection.

1. $z(t_0, a_0, b_0) = 0$, $D_1 z(t_0, a_0, b_0) = 0$, $D_1 y(t_0, a_0, b_0) = 0$,
2. $D_{11} z(t_0, a_0, b_0) \neq 0$, $v_3(t_0, a_0, b_0) \neq 0$,
3. $$\begin{vmatrix} 0 & D_2 z(t_0, a_0, b_0) & D_3 z(t_0, a_0, b_0) \\ D_{11} z(t_0, a_0, b_0) & D_{12} z(t_0, a_0, b_0) & D_{13} z(t_0, a_0, b_0) \\ D_{11} y(t_0, a_0, b_0) & D_{12} y(t_0, a_0, b_0) & D_{13} y(t_0, a_0, b_0) \end{vmatrix} \neq 0,$$

   and
4. $D_{11} y(t_0, a_0, b_0) v_3(t_0, a_0, b_0) - D_{11} z(t_0, a_0, b_0) v_2(t_0, a_0, b_0) \neq 0$, *where* $v = (v_1, v_2, v_3)$.

The next proposition gives some geometric properties of this type of tangency:

Proposition 2.16 *Let* $(S_{a,b}, \Phi_{a,b})$ *be a family of foliated surfaces, with associated parametrization* $\mathbf{x}$ *and a tangent vector bundle as before, determined by* $(\gamma(t, a, b), v(t, a, b))$. *If there is a generalized quadratic tangency of type I between the family of foliated surfaces and the family of vector bundles in a point* $p = \gamma(t_0, a_0, b_0)$ *unfolding generically with* (a, b) *in* (a_0, b_0) *then there exist two maps* $(t_1(s), a_1(s), b_1(s))$ *and* $(t_2(s), a_2(s), b_2(s))$ *such that:*

1. $t_i(0) = t_0$, $(a_i(0), b_i(0)) = (a_0, b_0)$, *for* $i = 1, 2$.
2. $(a_i(s), b_i(s))$ *are regular curves for* $i = 1, 2$.
3. $S_{a_1(s), b_1(s)}$ *has a quadratic tangency with the curve* $\gamma(t, a_1(s), b_1(s))$ *in* $p_1(s) = \gamma(t_1(s), a_1(s), b_1(s))$, *for* s *small enough.*

4. *$S_{a_2(s),b_2(s)}$ has a generalized transversal intersection with $\gamma(t,a_2(s),b_2(s))$ in $p_2(s)=\gamma(t_2(s),a_2(s),b_2(s))$, for $s\neq 0$ small enough. If $s=0$ only the transversality of the curve and the surface does not remain.*
5. *Let $\mathcal{L}_{\alpha(s),a_1(s),b_1(s)}$ be the leaf of $\Phi_{a_1(s),b_1(s)}$ which passes through $p_1(s)$. Then the curves $\mathcal{L}_{\alpha(s),a_1(s),b_1(s)}$ and $\gamma(t,a_2(s),b_2(s))$ are tangent for $s=0$ and non-tangent for $s\neq 0$ small enough.*
6. *There is a quadratic tangency between $(a_1(s),b_1(s))$ and $(a_2(s),b_2(s))$ in (a_0,b_0).*

Proof:

First we perform a change of coordinates to be in the conditions of definition 2.15. Then we have that the foliated surface $S_{a,b}$ has associated parametrization $\mathbf{x}(u,v,a,b)=(u,v,0)$. Let suppose that $\gamma(t,a,b)=(x(t,a,b),y(t,a,b),z(t,a,b))$. By means of a change of the parameters a,b and a reparametrization of t we can suppose that: $(a_0,b_0)=(0,0)$, $t_0=1$, $D_1z(1,a,b)=0$, $z(1,a,b)=a$ and $D_1y(1,a,b)=b$, for all a,b small enough.

In these coordinates and parameters we have that $(t_1(s),a_1(s),b_1(s))=(1,0,s)$. To obtain the curve $(t_2(s),a_2(s),b_2(s))$ we have to solve the system of equations:

$$\begin{aligned} z(t,a,b) &= 0 \\ v_2(t,a,b)D_1z(t,a,b)-v_3(t,a,b)D_1y(t,a,b) &= 0 \end{aligned}$$

that have a solution $(t,a,b)=(1,0,0)$. By the conditions of the definition we can apply the implicit function theorem to obtain the desired map and, moreover, we can choose as parameter $s=a$ or $s=b$. Then the proof of the proposition is a simple check. □

Definition 2.17 *We say that $\{(S_{a,b},\Phi_{a,b})\}_{(a,b)\in V}$ and $\{(\gamma_{a,b},v_{a,b})\}_{(a,b)\in V}$ have a generalized tangency of type II in p_0 which unfolds generically with (a,b) in (a_0,b_0) if there exist some coordinate system (x,y,z) for which $S_{a,b}$ is the plane $z=0$ and the leaves of the foliation are the straight lines $z=0$, $y=c$, such that if we write $\gamma(t,a,b)=(x(t,a,b),y(t,a,b),z(t,a,b))$ then*

1. $z(t_0,a_0,b_0)=0$, $D_1z(t_0,a_0,b_0)=0$, $v_3(t_0,a_0,b_0)=0$,
2. $D_{11}z(t_0,a_0,b_0)\neq 0$, $D_1y(t_0,a_0,b_0)\neq 0$,
3. $$\begin{vmatrix} 0 & D_2z(t_0,a_0,b_0) & D_3z(t_0,a_0,b_0) \\ D_{11}z(t_0,a_0,b_0) & D_{12}z(t_0,a_0,b_0) & D_{13}z(t_0,a_0,b_0) \\ D_1v_3(t_0,a_0,b_0) & D_2v_3(t_0,a_0,b_0) & D_3v_3(t_0,a_0,b_0) \end{vmatrix}\neq 0$$

   and
4. $D_1y(t_0,a_0,b_0)D_1v_3(t_0,a_0,b_0)-D_{11}z(t_0,a_0,b_0)v_2(t_0,a_0,b_0)\neq 0$.

Both definitions do not depend on changes of coordinates and parameters, and in the chosen vector v to define the vector bundle.

As before we give some geometric properties of this tangency:

Proposition 2.18 *Let $(S_{a,b},\Phi_{a,b})$ be a family of foliated surfaces, with associated parametrization* $\mathbf{x}$ *and a tangent vector bundle as before, determined by $(\gamma(t,a,b),v(t,a,b))$. If there is a generalized quadratic tangency of type II between the* *family of foliated surfaces and the family of vector bundles in a point* $p = \gamma(t_0, a_0, b_0)$ *unfolding generically with* (a, b) *in* (a_0, b_0) *then there exist two maps* $(t_1(s), a_1(s), b_1(s))$ *and* $(t_2(s), a_2(s), b_2(s))$ *such that:*

1. $t_i(0) = t_0$, $(a_i(0), b_i(0)) = (a_0, b_0)$, *for* $i = 1, 2$.
2. $(a_i(s), b_i(s))$ *are regular curves for* $i = 1, 2$.
3. $S_{a_1(s),b_1(s)}$ *has a quadratic tangency with the curve* $\gamma(t, a_1(s), b_1(s))$ *in* $p_1(s) = \gamma(t_1(s), a_1(s), b_1(s))$, *for* s *small enough.*
4. $S_{a_2(s),b_2(s)}$ *has a generalized transversal intersection with* $\gamma(t, a_2(s), b_2(s))$ *in* $p_2(s) = \gamma(t_2(s), a_2(s), b_2(s))$, *for* $s \neq 0$ *small enough. If* $s = 0$ *neither the transversality of the curve and the surface nor the transversality of the fiber passing through* $p_2(s)$ *with the surface remain.*
5. *Let* $\mathcal{L}_{\alpha(s),a_1(s),b_1(s)}$ *be the leaf of* $\Phi_{a_1(s),b_1(s)}$ *which passes through* $p_1(s)$. *Then the curves* $\mathcal{L}_{\alpha(s),a_1(s),b_1(s)}$ *and* $\gamma(t, a_2(s), b_2(s))$ *are not tangent for* s *small enough.*
6. *There is a quadratic tangency between* $(a_1(s), b_1(s))$ *and* $(a_2(s), b_2(s))$ *in* (a_0, b_0).

Proof:

As before, we perform a change of coordinates to be in the conditions of definition 2.17, and use the same notation as in the previous proposition. By means of a change of the parameters a, b and a reparametrization of t we can suppose that: $(a_0, b_0) = (0, 0)$, $t_0 = 1$, $D_1 z(1, a, b) = 0$, $z(1, a, b) = a$, $v_3(1, a, b) = b$, for all (a, b) near $(0, 0)$.

The rest is similar to the previous proposition. $\square$

Now, we can define generalized homoclinic tangencies:

Definition 2.19 *Let* $\{f_{a,b}\}_{(a,b)\in V}$, $V \subset \mathbb{R}^2$ *being an open set, be a family of three-dimensional diffeomorphism, having a hyperbolic fixed point* $p = p(a, b)$, *for all* $(a, b) \in V$. *Suppose that the eigenvalues of* $Df_{a,b}(p(a, b))$ *that we denote by* λ_i, $i = 1, 2, 3$, *are all real and different in absolute value:* $|\lambda_1| < |\lambda_2| < |\lambda_3|$. *Moreover* $|\lambda_1| < 1$ *and* $|\lambda_3| > 1$. *We distinguish several cases.*

1. *Let* $|\lambda_2| < 1$, *and consider* $W^s(p(a, b))$ *as a manifold foliated by the strong stable foliation* $\mathcal{F}^{ss}(p(a, b))$. *We say that* $\{f_{a,b}\}_{(a,b)\in V}$ *has a generalized quadratic homoclinic tangency of type I (resp. type II) in a point* q *unfolding generically in* $(a, b) = (a_0, b_0)$ *if the families of vector bundles* $(W^u(p(a, b)), W^u(p, E^{cu}))$ *and of foliated surfaces* $(W^s(p(a, b)), \mathcal{F}^{ss})$ *have a generalized quadratic tangency of type I (resp. type II) in* q *which unfolds generically in* $(a, b) = (a_0, b_0)$, *and, moreover,* q *is in general position.*
2. *If* $|\lambda_2| > 1$, *we say that* $\{f_{a,b}\}_{(a,b)\in V}$ *has a generalized quadratic homoclinic tangency of type I (respectively type II) in a point* q *unfolding generically in* $(a, b) = (a_0, b_0)$ *if* $\{f^{-1}_{a,b}\}_{(a,b)\in V}$ *has a generalized quadratic homoclinic tangency of type I (resp. type II) unfolding generically in* $(a, b) = (a_0, b_0)$.

In the last part we give adapted coordinates and parameters for families having generalized homoclinic tangencies.

Proposition 2.20 *Let* $\{f_{a,b}\}_{(a,b)\in V}$, V *being an open neighbourhood of* (a_0, b_0), *a 2-parameter family of three-dimensional diffeomorphisms having a hyperbolic fixed point*

$p = p(a,b)$, with real eigenvalues λ_i, $i = 1,2,3$ such that $|\lambda_1| < |\lambda_2| < 1 < |\lambda_3|$. We assume that this family has a generalized quadratic homoclinic tangency in a point q unfolding generically with (a,b) in $(a,b) = (a_0,b_0)$, and that it is smoothly linearizable near the fixed point p. Then, via a change of variables and a reparametrization if necessary, we can suppose that this family satisfies the following: There exists a smooth map $\mathbf{x} = \mathbf{x}(t_1,t_2,t_3,a,b)$, defined in a neighbourhood of $\{(t_1,t_2,t_3) \in \mathbb{R}^3 \,:\, (t_1^2+t_2^2)t_3 = 0\} \times \{(0,0)\}$ such that

1. *$\mathbf{x}(\cdot,\cdot,\cdot,a,b)$ is the identity in a neighbourhood U of $\{(t_1,t_2,0) \,:\, t_1^2+t_2^2 \leq 3\}$, for all $(a,b) \in V$, and U does not depend on a,b.*
2. *$f_{a,b}^n(t_1,t_2,t_3) = \mathbf{x}(\lambda_1^n t_1, \lambda_2^n t_2, \lambda_3^n t_3, a, b)$, for all $(t_1,t_2,t_3) \in U$ and $(a,b) \in V$.*
3. *If $\mathbf{x} = (x,y,z)$ then*
   (a) *If the generalized homoclinic tangency is of type I:*
   $$x(0,0,1,a,b) = 1, \quad y(0,0,1,a,b) = 1, \quad z(0,0,1,a,b) = a,$$
   $$D_3z(0,0,1,a,b) = 0, \quad D_3y(0,0,1,a,b) = b,$$
   for all $a \in V$ and $D_{33}z(0,0,1,0,0) \neq 0$, $D_2z(0,0,1,0,0) \neq 0$, and
   $$\begin{vmatrix} D_{33}y(0,0,1,0,0) & D_{33}z(0,0,1,0,0) \\ D_2y(0,0,1,0,0) & D_2z(0,0,1,0,0) \end{vmatrix} \neq 0.$$
   (b) *If the generalized homoclinic tangency is of type II:*
   $$x(0,0,1,a,b) = 1, \quad y(0,0,1,a,b) = 1, \quad z(0,0,1,a,b) = a,$$
   $$D_3z(0,0,1,a,b) = 0, \quad D_2z(0,0,1,a,b) = b,$$
   for all $a \in V$ and $D_{33}z(0,0,1,0,0) \neq 0$, $D_3y(0,0,1,0,0) \neq 0$, and
   $$\begin{vmatrix} D_{23}z(0,0,1,0,0) & D_{33}z(0,0,1,0,0) \\ D_2y(0,0,1,0,0) & D_3y(0,0,1,0,0) \end{vmatrix} \neq 0.$$
4. *The tangent homoclinic point can be taken as $q = (1,1,0) = \mathbf{x}(0,0,1,0,0)$.*

Proof:

In the first place, we can suppose that $p(a,b) = 0$, for all $a \in V$ and $(a_0,b_0) = (0,0)$. Moreover, we can assume that $f_{a,b}(t_1,t_2,t_3) = \mathbf{x}(\lambda_1 t_1, \lambda_2 t_2, \lambda_3 t_3, a, b)$, where $\mathbf{x}$ satisfies items 1 and 2.

We distinguish the generalized tangencies of type I and II:

a) Type I:

By definition 2.19 we have that there exists a real number $\bar{t}_3 \neq 0$ such that $z(0,0,\bar{t}_3,0,0) = D_3z(0,0,\bar{t}_3,0,0) = 0$ and $D_{33}z(0,0,\bar{t}_3,0,0) \neq 0$, $x(0,0,\bar{t}_3,0,0) \neq 0$ and $y(0,0,\bar{t}_3,0,0) \neq 0$.

Let $\tilde{t}_3(a,b)$ be a map defined in a neighbourhood of $(0,0)$ such that $\tilde{t}_3(0,0) = \bar{t}_3$ and $D_3z(0,0,\tilde{t}_3(a,b),a,b) = 0$. By means of this map, we can assume (via a change of variables) that the map $\mathbf{x}$ satisfies: $x(0,0,1,a,b) = y(0,0,1,a,b) = 1$, $D_3z(0,0,1,a,b) = 0$. Now, the point of homoclinic tangency is $q = (1,1,0) = \mathbf{x}(0,0,1,0,0)$. Moreover, by the definition, we have:

$D_2y(0,0,1,0,0)=0$, $D_{33}z(0,0,1,0,0)\neq 0$, $D_2z(0,0,1,0,0)\neq 0$ and

$$\left|\begin{array}{cc} D_4z(0,0,1,0,0) & D_5z(0,0,1,0,0) \\ D_{34}y(0,0,1,0,0) & D_{35}y(0,0,1,0,0)\end{array}\right|\neq 0.$$

These conditions are deduced from definition 2.15.

Finally, we can define new parameters, that are well defined by the last condition:

$$\begin{array}{rcl}\bar{a} &=& z(0,0,1,a,b)\\ \bar{b} &=& D_3y(0,0,1,a,b)\end{array}$$

If we rename these new parameters as (a,b) and consider the new family, we obtain a map $\mathbf{x}$ satisfying the conditions 1, 2, 3(a).

b) Type II

As before we can suppose that $f_{a,b}(t_1,t_2,t_3)=\mathbf{x}(\lambda_1 t_1,\lambda_2 t_2,\lambda_3 t_3,a,b)$, where $\mathbf{x}$ satisfies items 1 and 2. Moreover,

$$x(0,0,1,a,b)=y(0,0,1,a,b)=1,\quad D_3z(0,0,1,a,b)=0,$$

and the point of homoclinic tangency is $q=(1,1,0)=\mathbf{x}(0,0,1,0,0)$. From definition 2.19 one has:

$D_2z(0,0,1,0,0)=0$, $D_{33}z(0,0,1,0,0)\neq 0$, $D_3y(0,0,1,0,0)\neq 0$ and

$$\left|\begin{array}{cc} D_4z(0,0,1,0,0) & D_5z(0,0,1,0,0) \\ D_{24}z(0,0,1,0,0) & D_{25}z(0,0,1,0,0)\end{array}\right|\neq 0.$$

This last conditions follows from definition 2.17, taking into account that in its notation we have to substitute $\gamma(t,a,b)$ by $\mathbf{x}(0,0,t,a,b)$, and $v(t,a,b)$ by $D_2z(0,0,t,a,b)$, and $f(x,y,z,a,b)=z$.

Finally we perform the change of parameters:

$$\begin{array}{rcl}\bar{a} &=& z(0,0,1,a,b)\\ \bar{b} &=& D_2z(0,0,1,a,b)\end{array}$$

Renaming again the new parameters as (a,b) the new family owns a map $\mathbf{x}$ satisfying items 1, 2, 3(b), as we wanted. □

Proposition 2.21 *Let $\{f_{a,b}\}_{(a,b)\in V}$, V being an open neighbourhood of (a_0,b_0), a 2-parameter family of three-dimensional diffeomorphisms having a hyperbolic fixed point $p=p(a,b)$, with real eigenvalue s λ_i, $i=1,2,3$ such that $|\lambda_1|<1<|\lambda_2|<|\lambda_3|$. We assume that this family has a generalized quadratic homoclinic tangency in a point q unfolding generically with (a,b) in $(a,b)=(a_0,b_0)$, and that it is smoothly linearizable near the fixed point p. Then, via a change of variables and a reparametrization if necessary, we can suppose that this family satisfies the following: There exists a smooth map $\mathbf{x}=\mathbf{x}(t_1,t_2,t_3,a,b)$, defined in a neighbourhood of $\{(t_1,t_2,t_3)\in\mathbb{R}^3\ :\ (t_2^2+t_3^2)t_1=0\}\times\{(0,0)\}$ such that*

1. *$\mathbf{x}(\cdot,\cdot,\cdot,a,b)$ is the identity in a neighbourhood U of $\{(t_1,0,0)\ :\ |t_1|\leq 3\}$, for all $(a,b)\in V$, and U does not depend on a,b.*

2. $f^n_{a,b}(t_1,t_2,t_3) = \mathbf{x}(\lambda_1^n t_1, \lambda_2^n t_2, \lambda_3^n t_3, a, b)$, *for all* $(t_1,t_2,t_3) \in U$ *and* $(a,b) \in V$.
3. *If* $\mathbf{x} = (x,y,z)$ *then*
   (a) *If the generalized homoclinic tangency is of type I:*

$$x(0,0,1,a,b) = 1, \quad y(0,0,1,a,b) = a,$$

$$z(0,0,1,a,b) = 0, \quad D_3 z(0,0,1,a,b) = b,$$

$$D_2 y(0,1,1,a,b) D_3 z(0,1,1,a,b) - D_3 y(0,1,1,a,b) D_2 z(0,1,1,a,b) = 0,$$

   for all $(a,b) \in V$, *and* $D_3 y(0,1,1,0,0) = 0$, $D_2 z(0,1,1,0,0) \neq 0$, *and*

$$D_2 y D_{33} z - D_{33} y D_2 z \neq 0,$$

   where all the derivatives are evaluated at $(0,1,1,0,0)$.

   (b) *If the generalized homoclinic tangency is of type II:*

$$x(0,0,1,a,b) = 1, \quad y(0,0,1,a,b) = 1,$$

$$z(0,0,1,a,b) = a, \quad D_3 z(0,0,1,a,b) = b,$$

$$D_2 y(0,1,1,a,b) D_3 z(0,1,1,a,b) - D_3 y(0,1,1,a,b) D_2 z(0,1,1,a,b) = 0,$$

   for all $(a,b) \in V$ *and*

$$D_2 z(0,1,1,0,0) = D_3 z(0,1,1,0,0) = 0, \quad D_3 y(0,1,1,0,0) \neq 0,$$

   and

$$(D_2 y)^2 D_{33} z - 2 D_2 y D_3 y D_{23} z + (D_3 y)^2 D_{22} z \neq 0,$$

   where all the derivatives are evaluated at $(0,1,1,0,0)$.
4. *The tangent homoclinic point can be taken as* $q = (1,0,0) = \mathbf{x}(0,1,1,0,0)$.

Proof:
As before, we can suppose that $f_{a,b}(t_1,t_2,t_3) = \mathbf{x}(\lambda_1 t_1, \lambda_2 t_2, \lambda_3 t_3, a, b)$, where $\mathbf{x}$ satisfies items 1 and 2, and that $(a_0, b_0) = (0,0)$, $p(a,b) = (0,0)$.

We distinguish between the types I and II:

a) Type I:
By definition, there exist real numbers $\bar{t}_2 \neq 0$ and $\bar{t}_3 \neq 0$, such that $x(0,\bar{t}_2,\bar{t}_3,0,0) \neq 0$, $y(0,\bar{t}_2,\bar{t}_3,0,0) = 0$, $z(0,\bar{t}_2,\bar{t}_3,0,0) = 0$. This is the homoclinic point. The other conditions of definition 2.19, except the one concerning the generic unfolding, give us the following properties of the map $\mathbf{x}$: $D_3 y = D_3 z = 0$, $D_2 z \neq 0$, $D_2 y D_{33} z - D_2 z D_{33} y \neq 0$, where all the derivatives are evaluated at $(0,\bar{t}_2,\bar{t}_3,0,0)$.
Then we consider maps $\tilde{t}_2 = \tilde{t}_2(a,b)$, $\tilde{t}_3 = \tilde{t}_3(a,b)$, such that $\tilde{t}_2(0,0) = \bar{t}_2$, $\tilde{t}_3(0,0) = \bar{t}_3$, and satisfy, for (a,b) near $(0,0)$, the following system of equations:

$$\begin{aligned} D_2 y(0,\tilde{t}_2,\tilde{t}_3,a,b) D_3 z(0,\tilde{t}_2,\tilde{t}_3,a,b) - D_2 z(0,\tilde{t}_2,\tilde{t}_3,a,b) D_3 y(0,\tilde{t}_2,\tilde{t}_3,a,b) &= 0 \\ z(0,\tilde{t}_2,\tilde{t}_3,a,b) &= 0 \end{aligned}$$

These maps exist because $D_2 y D_{33} z - D_2 z D_{33} y \neq 0$, where the derivatives are evaluated at $(0, \bar{t}_2, \bar{t}_3, 0, 0)$, and we can apply the implicit function theorem.

The existence of the maps $\tilde{t}_2$ and $\tilde{t}_3$, allows as to perform a linear change of variables, such that if we call the new maps after the change as before, we have that the point of homoclinic tangency is $(1,0,0) = \mathbf{x}(0,1,1,0,0)$, and

$$x(0,1,1,a,b) = 1, \quad z(0,1,1,a,b) = 0,$$

$$D_2 y(0,1,1,a,b) D_3 z(0,1,1,a,b) - D_2 z(0,1,1,a,b) D_3 y(0,1,1,a,b) = 0,$$

for all (a,b) in a neighbourhood of $(0,0)$. Moreover, as before: $D_2 z \neq 0$, $D_3 y = D_3 z = 0$, and $D_2 y D_{33} z - D_2 z D_{33} y \neq 0$ where all the derivatives are evaluated at $(0,1,1,0,0)$.

Now, we only need to perform reparametrization, to obtain a map $\mathbf{x}$ satisfying items 1, 2, 3(a) and 4. In order to do this we have to compute the last condition of definition 2.19 (generic unfolding):

Let $G_{a,b}$ denote the inverse map of $\mathbf{x}_{a,b} := \mathbf{x}(\cdot,\cdot,\cdot,a,b)$, in a neighbourhood of $(1,0,0)$, for all (a,b) in a neighbourhood of $(0,0)$, and $G(s_1,s_2,s_3,a,b) := G_{a,b}(s_1,s_2,s_3)$. Let $g_{a,b} = G_{a,b} \circ f_{a,b} \circ \mathbf{x}_{a,b}$. Then $g_{a,b}$ has a generalized quadatic homoclinic tangency at the point $(0,1,1)$ of type I between a the foliated surface $S_{a,b}$ and the curve $\gamma(t) = G_{a,b}(t,0,0)$, such that $\gamma(1) = (0,1,1)$. The foliated surface can be represented by means of the parametrization $\tilde{\mathbf{x}}(u,v,a,b) = (0,v,u)$. Using definition 2.15, it is easy to see that the condition 3 of generic unfolding for $g_{a,b}$ is, if $G = (G_1, G_2, G_3)$:

$$\begin{vmatrix} 0 & D_4 G_1 & D_5 G_1 \\ D_{11} G_1 & D_{14} G_1 & D_{15} G_1 \\ D_{11} G_2 & D_{14} G_2 & D_{15} G_2 \end{vmatrix} \neq 0,$$

where the derivatives are taken in $(0,1,1,0,0)$. Computing all these derivatives, taking into account the properties of $\mathbf{x}$, we obtain that this is equivalent to:

$$D_{34} z(0,1,1,0,0) D_5 y(0,1,1,0,0) - D_{35} z(0,1,1,0,0) D_4 y(0,1,1,0,0) \neq 0.$$

Finally, we consider the following reparametrization:

$$\begin{aligned} \bar{a} &= y(0,1,1,a,b) \\ \bar{b} &= D_3 z(0,1,1,a,b) \end{aligned}$$

It is well-defined by the last condition. Moreover, if we rename the new parameters and the new map as before, we obtain that $f_{a,b}$ satisfies items 1, 2, 3(a) and 4, as we wanted.

b) Type II

As before there exist $\bar{t}_2 \neq 0$ and $\bar{t}_3 \neq 0$ such that $x(0,\bar{t}_2,\bar{t}_3,0,0) \neq 0$, $y(0,\bar{t}_2,\bar{t}_3,0,0) = 0$, $z(0,\bar{t}_2,\bar{t}_3,0,0) = 0$. Computing the other conditions of existence of a generalized homoclinc tangency (but the generic unfolding), we have: $D_2 y D_3 z - D_2 z D_3 y = 0$, $D_2 z = D_3 z = 0$, $D_3 y \neq 0$, and

$$D_{22} z (D_3 y)^2 + D_{33} z (D_2 y)^2 - 2 D_2 y D_3 y D_{23} z \neq 0, \tag{1}$$

where all the derivatives are evaluated at $(0,\bar{t}_2,\bar{t}_3,0,0)$.

Then we consider maps $\tilde{t}_2=\tilde{t}_2(a,b)$, $\tilde{t}_3=\tilde{t}_3(a,b)$, such that $\tilde{t}_2(0,0)=\bar{t}_2$, $\tilde{t}_3(0,0)=\bar{t}_3$, and satisfy, for (a,b) near $(0,0)$, the following system of equations:

$$\begin{aligned} D_2y(0,\tilde{t}_2,\tilde{t}_3,a,b)D_3z(0,\tilde{t}_2,\tilde{t}_3,a,b)-D_2z(0,\tilde{t}_2,\tilde{t}_3,a,b)D_3y(0,\tilde{t}_2,\tilde{t}_3,a,b) &= 0\\ y(0,\tilde{t}_2,\tilde{t}_3,a,b) &= 0\end{aligned}$$

These maps exist because equation (1) allows to apply the implicit function theorem.

In a similar way as before, we can perform a linear change of parameters to obtain a map (that we call again $\mathbf{x}$) satisfying the following properties:

The point of tangent homoclinic tangency is $q=(1,0,0)=\mathbf{x}(0,1,1,0,0)$, and

$$x(0,1,1,a,b)=1,\quad y(0,1,1,a,b)=0,$$

$$D_2y(0,1,1,a,b)D_3z(0,1,1,a,b)-D_2z(0,1,1,a,b)D_3y(0,1,1,a,b)=0,$$

for all (a,b) in a neighbourhood of $(0,0)$. Moreover, as before: $D_2z=D_3z=0$, $D_3y\neq 0$ and $D_{22}z(D_3y)^2+D_{33}z(D_2y)^2-2D_2yD_3yD_{23}z\neq 0$, where all the derivatives are evaluated at $(0,1,1,0,0)$.

Now, we consider the third condition of definition 2.15. We proceed as in the type I, using $\mathbf{x}_{a,b}$ as a conjugacy and obtain that it is equivalent to

$$D_{34}z(0,1,1,0,0)D_5z(0,1,1,0,0)-D_{35}z(0,1,1,0,0)D_4z(0,1,1,0,0)\neq 0.$$

Now, we consider the following change of parameters:

$$\begin{aligned}\bar{a} &= z(0,1,1,a,b)\\ \bar{b} &= D_3z(0,1,1,a,b)\end{aligned}$$

If we rename the new parameters and maps as before, we obtain a map $\mathbf{x}$ satisfying items 1, 2, 3(b) and 4.

This finishes the proof of the proposition. $\square$

In figures 3, 4 and 5 we have a typical example of a generalized homoclinic tangency.

In figure 3 we have drawn the plane of parameters, and marked several lines (t_1, t_2, r_1, r_2), a point (O) and regions (I, II, III, IV). The lines t_1 and t_2 corresponds to a homoclinic tangency, and the lines r_1 and r_2 to a generalized homoclinic transversality. In region IV there is not intersection between the invariant manifolds.

In figures 4 and 5 we have several possibilities of the behaviour of the invariant manifolds, depending on the values of the parameters considered. The meaning of the letters on the pictures are the following: p is the fixed point, q is a point of homoclinic tangency, S, S_1 and S_2 are leaves of the strong stable foliation of $W^s(p)$, L, L_1, L_2 are the intersection of a fiber of the unstable vector bundle with $W^s(p)$ (that is the plane $z=0$, in coordinates (x,y,z)) In the case of generalized homoclinic tangency, S and L coincides, and in the case of generalized transversality, S_i, L_i (for $i=1$ or 2, depending on the case) coincides. More arrows means more attraction. Finally, all the dashed curves are below the plane $z=0$.

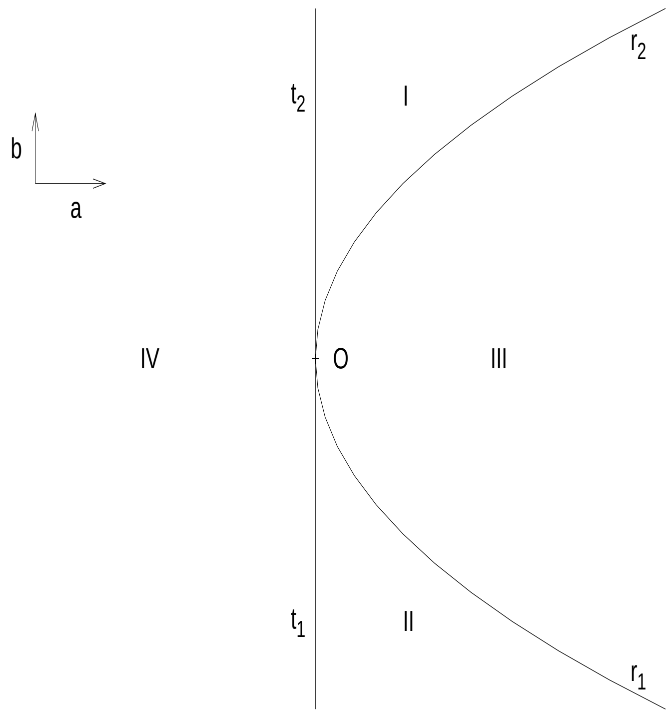

Figure 3: Generalized homoclinic tangency (type I). Parameter plane

2.4 The Bogdanov-Takens bifurcation

Let $\{f_{a,b}\}_{a,b\in V}$, $V \subset \mathbb{R}^2$ being an open neighbourhood of (a_0, b_0), be a smooth 2-parameter 2-dimensional family of diffeomorphisms, having a fixed point p_0, with two eigenvalues equal to 1 and $Df_{a_0,b_0}(p_0) \neq \mathrm{Id}$. The objective of this section is to give explicit conditions on the family to have a Bogdanov-Takens family of diffeomorphisms unfolding generically, as defined in [3].

Definition 2.22 *We say that $\{f_{a,b}\}_{(a,b)\in V}$ has a Bogdanov-Takens bifurcation in p_0 which unfolds generically with (a, b) in (a_0, b_0), if via an smooth change of variables and a reparametrization we can obtain the family of maps:*

$$\begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} u+v \\ v+\mu+u^2+v(\nu+\tilde{\gamma}u)+\tilde{c}v^2+\tilde{r}_3(u,v) \end{pmatrix}, \tag{2}$$

where p_0 is transformed in $(0,0)$, $\mu = \mu(a, b)$ and $\nu = \nu(a, b)$ are the new parameters such that $(\mu(a_0, b_0), \nu(a_0, b_0)) = (0, 0)$ and $|\frac{\partial(\mu,\nu)}{\partial(a,b)}|$ is different from zero when $(a, b) = (a_0, b_0)$, $\tilde{\gamma}$ and $\tilde{c}$ are functions of (a, b) such that $\tilde{\gamma}(a_0, b_0) \neq 2$, and $\tilde{r}_3 = \tilde{r}_3(u, v, a, b)$ is a smooth function whose Taylor expansion around $(u, v) = (0, 0)$, for each fixed (a, b), begins at degree 3.

In the following we include some properties related to the B-T bifurcation. For a proof and deeper properties see Broer et al. [3].

1. The map (2) has a saddle-node bifurcation in $(0, 0)$ for $\mu = 0$ and all ν small.

2. The map (2) has a Hopf bifurcation in $(u, 0)$ for

$$(\mu, \nu) = (-u^2 + O_3, (2 - \tilde{\gamma})u + O_2),$$

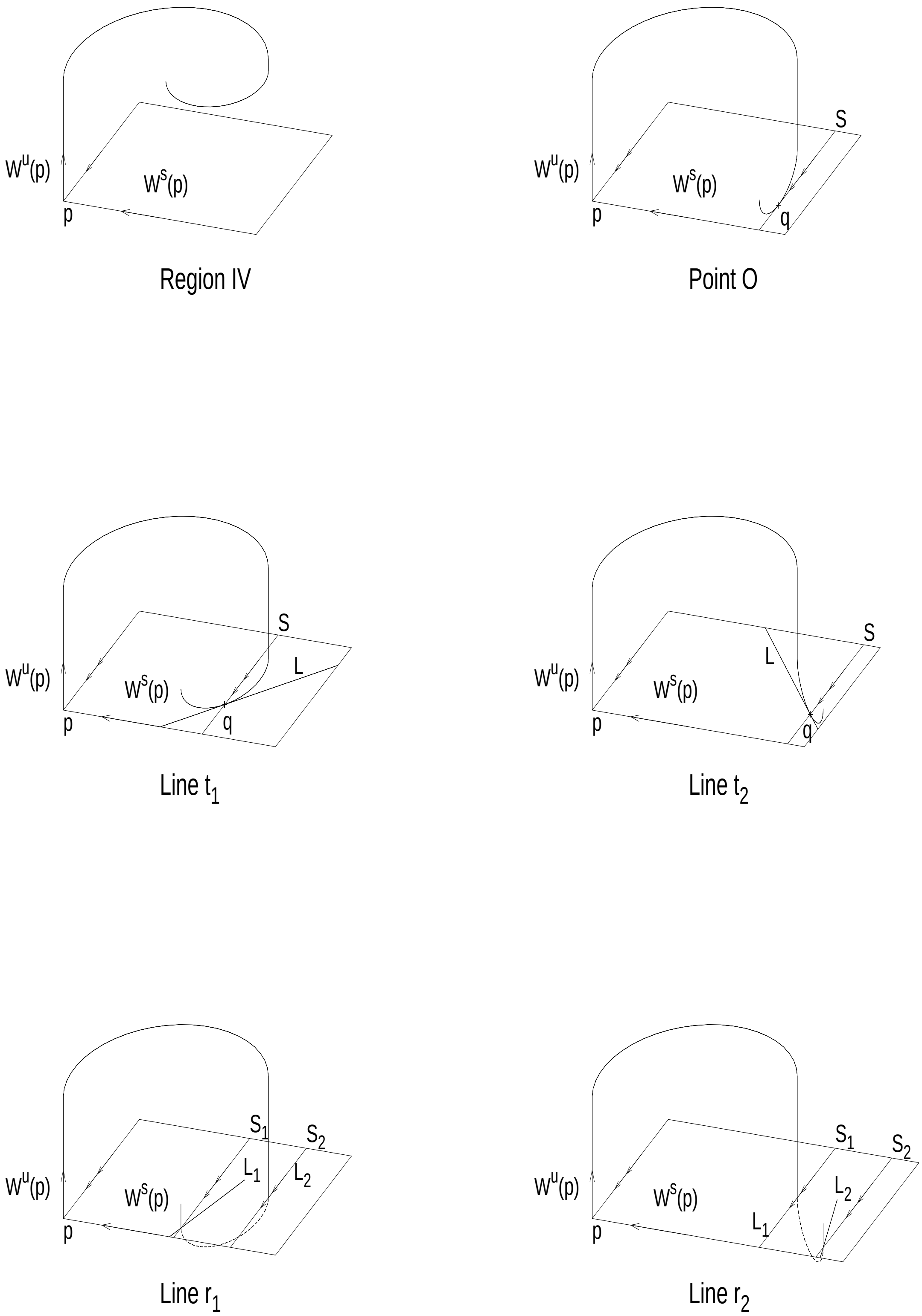

Figure 4: Generalized homoclinic tangency (type I). Phase space (see explanations on the text).

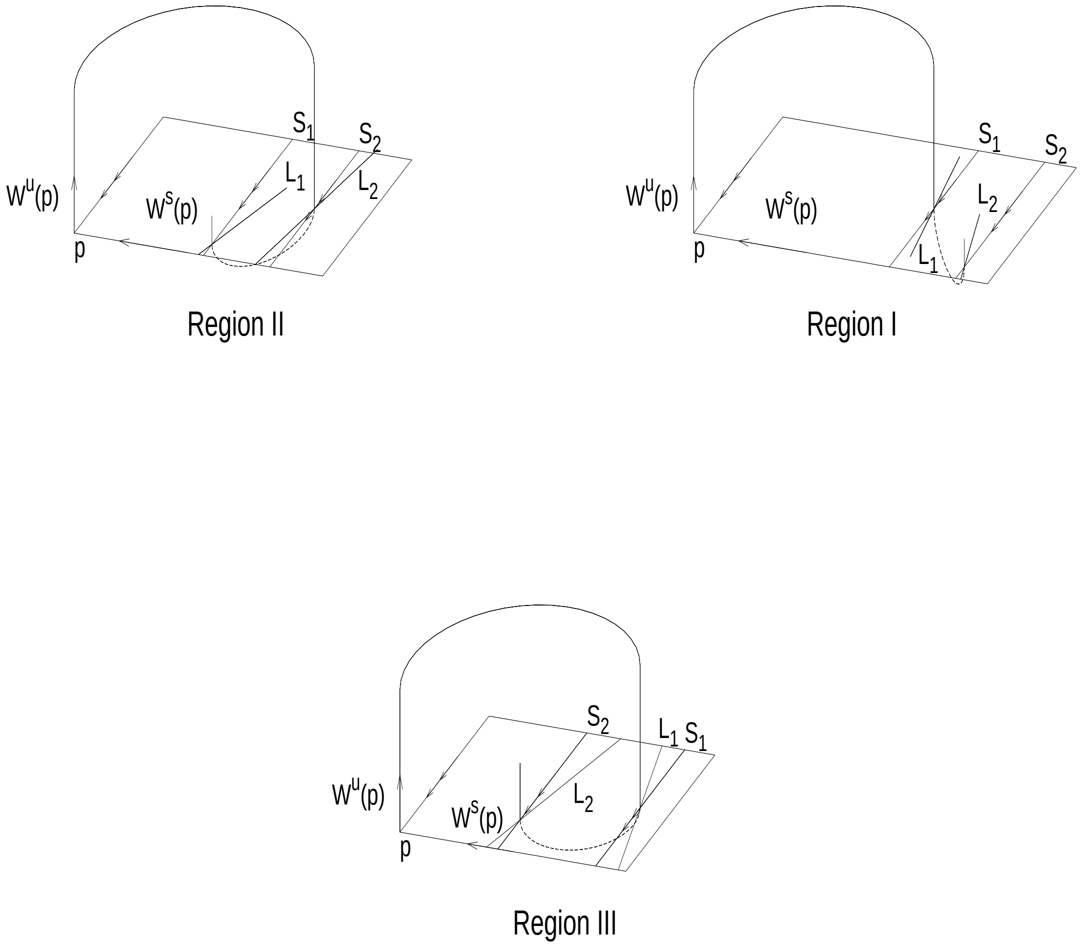

Figure 5: Generalized homoclinic tangency (type I). Continuation of the previous figure.

for all $u < 0$ small enough. Here O_i means order i in u. As a consequence, the curves of saddle-node and Hopf bifurcations have a quadratic contact in the origin.

3. There is a function $\mu_1 = \mu_1(\nu)$ such that if $\mu_H = \mu_H(\nu)$ represents the curve corresponding to Hopf bifurcation in the (μ, ν) plane ($\nu \leq 0$ if $\tilde{\gamma} < 2$ and $\nu \geq 0$ if $\tilde{\gamma} > 2$) then there exists $\mu_0 < 0$ such that: a) If $\tilde{\gamma} < 2$ then there exists a $\nu_0 < 0$ such that the map (2) has an attracting C^1 invariant curve for any (ν, μ) with $\mu_1(\nu) < \mu < \mu_H(\nu)$ and $0 > \nu > \nu_0$. b) If $\tilde{\gamma} > 2$ then there exists a $\nu_0 > 0$ such that the map (2) has a repelling C^1 invariant curve for any (ν, μ) with $\mu_1(\nu) < \mu < \mu_H(\nu)$ and $\nu_0 > \nu > 0$.

   Moreover the maps μ_1 and μ_H has a quadratic contact in the origin.

4. Let $\{f_{a,b}\}_{a,b \in V}$, $V \subset \mathbb{R}^2$ being an open neighbourhood of (a_0, b_0), be a smooth 2-parameter family of 2-dimensional diffeomorphisms. Suppose that the map f_{a_0,b_0} has a fixed point p_0 with two eigenvalues equal to 1 and that there exists a 2×2 invertible matrix M such that

$$M^{-1} D f_{a_0,b_0}(p_0) M = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

   Let $v_i = M e_i$ and $w_i^t = e_i^t M^{-1}$, where $i = 1, 2$ and e_1, e_2 is the canonical basis of $\mathbb{R}^2$. If

$$w_2^t D^2 f_{a_0,b_0}(p_0)(v_1, v_1) \neq 0$$

   and

$$w_2^t D^2 f_{a_0,b_0}(p_0)(v_1, v_2) + w_1^t D^2 f_{a_0,b_0}(p_0)(v_1, v_1) - w_2^t D^2 f_{a_0,b_0}(p_0)(v_1, v_1) \neq 0,$$

   then, by means of a smooth change of variables it is possible to transform our family to one like (2). Moreover, if after the transformation we have that $|\frac{\partial(\mu,\nu)}{\partial(a,b)}|$ is different from zero when $(a, b) = (a_0, b_0)$, then the initial family has a Bogdanov-Takens bifurcation in (a_0, b_0) which unfolds generically.

3 Bifurcations of periodic points related to homoclinic tangencies

In this section, we will find bifurcations of periodic points near a homoclinic quadratic tangency, a generalized transversal homoclinic point (one-parameter case) and generalized quadratic homoclinic tangency (two-parameter case). We will consider a family of diffeomorphisms $f_c : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, where $c = a \in \mathbb{R}$ in the one-parameter case, and $c = (a, b) \in \mathbb{R}^2$ in the two-parameter case. We suppose, as before, that $\{f_c\}_c$ has a dissipative hyperbolic fixed point in $p = 0 \in \mathbb{R}^3$, for c near the origin. We call the eigenvalues of $Df_c(p)$ as λ_1, λ_2 and λ_3, where $\lambda_i = \lambda_i(c)$, $i = 1, 2, 3$, $|\lambda_1| < |\lambda_2| < |\lambda_3|$, and $|\lambda_1 \lambda_2 \lambda_3| < 1$. Moreover we introduce the following notation:

If $A = (a_{i,j})_{1 \leq i,j \leq 3}$ is a 3×3 matrix then $d_1(A) = \operatorname{trace} A$, $d_2(A) = \det A_1 + \det A_2 + \det A_3$, and $d_3(A) = \det A$, where A_i is the 2×2 matrix obtained by removing the row and the column i of the matrix A.

3.1 Quadratic tangency

In this case $c = a \in \mathbb{R}$. Assume that f_a has a quadratic tangency of its invariant manifolds in $0 \in \mathbb{R}^3$ which unfolds generically with a in $a = 0 \in \mathbb{R}$. We shall distinguish between the cases $\dim W^u(p) = 1$ or 2.

3.1.1 Case 1: $|\lambda_1| < |\lambda_2| < 1$, $|\lambda_3| > 1$.

In this case we know, using proposition 2.8, that we can write, after a change of variables and parameter:

$$f_a^n(t_1, t_2, t_3) = \mathbf{x}(\lambda_1^n t_1, \lambda_2^n t_2, \lambda_3^n t_3, a),$$

The map $\mathbf{x}$ satisfies:

$$x(0,0,1,a) = 1,\ y(0,0,1,a) = 1,\ z(0,0,1,a) = a,\ D_3 z(0,0,1,a) = 0, \tag{3}$$

for a in a neighbourhood of $0 \in \mathbb{R}$, and $D_{33}z(0,0,1,0) \neq 0$. Moreover, $\mathbf{x}(\cdot, a)$ is the identity in a neighbourhood of the disk $\{(t_1, t_2, 0) \ :\ t_1^2 + t_2^2 \leq 3\}$.

Now we want to find values of the parameter for which f_a^n has a saddle-node bifurcation. Let $(t_1, t_2, \lambda_3^{-n} t_3)$ be a saddle-node periodic point of f_a^n. Then:

$$f_a^n(t_1, t_2, \lambda_3^{-n} t_3) = (t_1, t_2, \lambda_3^{-n} t_3),$$

and if $A = Df_a^n(t_1, t_2, \lambda_3^{-n} t_3) = D\mathbf{x}(\lambda_1^n t_1, \lambda_2^n t_2, t_3, a) \circ \Lambda_n$, where Λ_n is the diagonal matrix $\Lambda_n = \operatorname{diag}(\lambda_1^n, \lambda_2^n, \lambda_3^n)$, then the condition of the existence of a saddle-node bifurcation is

$$1 - d_1(A) + d_2(A) - d_3(A) = 0.$$

If we write explicitly the equations, the last equation multiplied by λ_3^{-n}, and we put $s_1 = \lambda_1^n$, $s_2 = \lambda_2^n$ and $s_3 = \lambda_3^{-n}$, we have the following system of equations $\mathbf{F}(s_1, s_2, s_3, t_1, t_2, t_3, a) = 0$, where $\mathbf{F} = (F_1, F_2, F_3, F_4)$ and $F_1 = x(s_1 t_1, s_2 t_2, t_3, a) - t_1$, $F_2 = y(s_1 t_1, s_2 t_2, t_3, a) - t_2$, $F_3 = z(s_1 t_1, s_2 t_2, t_3, a) - s_3 t_3$, and $F_4 = s_3 - s_3 d_1(A) + s_3 d_2(A) - s_3 d_3(A)$, where

$$s_3 d_1(A) = s_1 s_3 D_1 x + s_2 s_3 D_2 y + D_3 z$$

$$s_3 d_2(A) = s_1 s_2 s_3 \begin{vmatrix} D_1 x & D_2 x \\ D_1 y & D_2 y \end{vmatrix} + s_1 \begin{vmatrix} D_1 x & D_3 x \\ D_1 z & D_3 z \end{vmatrix} + s_2 \begin{vmatrix} D_2 y & D_3 y \\ D_2 z & D_3 z \end{vmatrix}$$

$$s_3 d_3 = s_1 s_2 \det D\mathbf{x}(s_1 t_1, s_2 t_2, t_3, a),$$

and all the derivatives of x, y, z are evaluated at $(s_1 t_1, s_2 t_2, t_3, a)$. Moreover, we get $\mathbf{F}(0,0,0,1,1,1,0) = 0$, applying (3), that is, due to the existence of a homoclinic bifurcation. Finally, one sees that $\det D\mathbf{G}(1,1,1,0) = -D_{33}z(0,0,1,0) \neq 0$, where $\mathbf{G}(t_1, t_2, t_3, a) = \mathbf{F}(0,0,0,t_1,t_2,t_3,a)$, because the tangency is nondegenerate. Then we deduce, by the Implicit Function Theorem, the existence of n-period points p_n^+ for values of the parameter $a = a_n^+$ such that the eigenvalues of $Df_{a_n^+}^n(p_n^+)$ are 1, α_n and β_n and $|\alpha_n \beta_n| < 1$.

We note that, moreover,

$$\alpha_n + \beta_n + \alpha_n \beta_n = d_2(Df_{a_n^+}^n(p_n^+)) = \lambda_2^n \lambda_3^n (D_2 y D_3 z - D_2 z D_3 y + o(1)),$$

where all the derivatives are evaluated in $(0,0,1,0)$. This means that if in the case $|\lambda_2\lambda_3| > 1$ we want to have only one eigenvalue equal to 1 in absolute value, it is enough to impose that $D_2z(0,0,1,0) \neq 0$ and $D_3y(0,0,1,0) \neq 0$. This means that the homoclinic tangency is not generalized.

We can obtain also the existence of flip bifurcations in the same way. We have only to substitute the first equation by $1 + d_1 + d_2 + d_3 = 0$ and proceed as before.

Then we have proven the following proposition:

Proposition 3.1 *Let $\{f_a\}_{a\in V}$ be a smooth family of 3-dimensional diffeomorphisms $f_a : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, where V is an open neighbourhood of $a_0 \in \mathbb{R}$. Suppose that*

1. *For $a = a_0$, f_{a_0} has a dissipative saddle fixed point p_0, with eigenvalues $|\lambda_1| < |\lambda_2| < 1$ and $|\lambda_3| > 1$, such that it undergoes a nondegenerate homoclinic tangency in q_0 which unfolds generically with the parameter a in $a = a_0$.*
2. *If $|\lambda_2\lambda_3| > 1$ then the quadratic tangency is not a generalized quadratic tangency.*
3. *The family of diffeomorphisms is locally smoothly conjugate to its linear part near the corresponding fixed point $p = p(a)$, such that $p(a_0) = p_0$, if $|a - a_0|$ is smooth enough.*

Then, for n large enough there exist parameters a_n^+ and a_n^- such that

1. *$f_{a_n^+}$ has an n-periodic saddle-node point p_n^+.*
2. *$f_{a_n^-}$ has an n-periodic flip point p_n^-.*
3. *$\lim_{n\to\infty} a_n^+ = \lim_{n\to\infty} a_n^- = a_0$.*
4. *$\lim_{n\to\infty} p_n^+ = \lim_{n\to\infty} p_n^- = q_0$.*

Remark 3.2 *The index of stability of these bifurcations (that is the number of eigenvalues with absolute value less than one) depends on certain relations between the eigenvalues of the initial fixed point. The existence of periodic attractors only happens in the sectionally dissipative case.*

3.1.2 Case 2: $|\lambda_1| < 1$, $|\lambda_3| > |\lambda_2| > 1$.

For this case, using proposition 2.9, we can write, after a change of variables and reparametrization:

$$f_a^n(t_1,t_2,t_3) = \mathbf{x}(\lambda_1^n t_1, \lambda_2^n t_2, \lambda_3^n t_3, a),$$

where $\mathbf{x}$ is a smooth map, $\mathbf{x}(\cdot, a)$ is the identity in a neighbourhood of $\{(t_1,0,0) \ : \ |t_1| \leq 3\}$, and one of the two possibilities holds:

1. $x(0,1,1,a) = 1$, $y(0,1,1,a) = 0$, $z(0,1,1,a) = a$,

$$D_2y(0,1,1,a)D_3z(0,1,1,a) - D_2z(0,1,1,a)D_3y(0,1,1,a) = 0$$

and

$$D_1g(1,1,0)D_3y(0,1,1,0) - D_2g(1,1,0)D_2y(0,1,1,0) \neq 0.$$

2. $x(0,1,1,a) = 1$, $y(0,1,1,a) = a$, $z(0,1,1,a) = 0$,

$$D_2 y(0,1,1,a) D_3 z(0,1,1,a) - D_2 z(0,1,1,a) D_3 y(0,1,1,a) = 0$$

and

$$D_1 g(1,1,0) D_3 z(0,1,1,0) - D_2 g(1,1,0) D_2 z(0,1,1,0) \neq 0.$$

Here $g(t_2, t_3, a) = D_2 y D_3 z - D_2 z D_3 y$, where all the derivatives are evaluated at $(0, t_2, t_3, a)$.

In order to see if there are saddle-node bifurcations of period n we have to solve a similar system that we presented in the first case. The only difference i that in the present case we take as a periodic point: $(t_1, \lambda_2^{-n} t_2, \lambda_3^{-n} t_3)$, which is near to $(1,0,0)$. It is easy to see that in both cases the system has solution, by applying the Implicit Function Theorem. The condition to have no other eigenvalue equal to 1 in absolute value is, in this case: $D_3 z(0,1,1,0,0) \neq 0$ which means that the homoclinic tangency is not generalized.

It is possible to prove in a similar way that there are flip bifurcations.

In the other way there are also periodic points with the product of two of their eigenvalues equal to 1. To see this, first it is easy to see that the last condition in the previous case has to be changed by

$$1 + (d_1(A) - d_3(A)) d_3(A) - d_2(A) = 0.$$

By using the variables $s_1 = \lambda_1^n \lambda_3^n$, $s_2 = \lambda_2^{-n}$, $s_3 = \lambda_2^n \lambda_3^{-n}$, we can proceed as before, but multiplying the last equation by $\lambda_2^{-n} \lambda_3^{-n}$, and applying the implicit function theorem.

We collect these results in the following proposition:

Proposition 3.3 *Let $\{f_a\}_{a \in V}$ be a smooth family of 3-dimensional diffeomorphisms $f_a : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, where V is an open neighbourhood of $a_0 \in \mathbb{R}$. Suppose that*

1. *For $a = a_0$, f_{a_0} has a dissipative saddle fixed point p_0, which eigenvalues $|\lambda_1| < 1$ and $|\lambda_3| > |\lambda_2| > 1$, such that it undergoes a nondegenerate homoclinic tangency in q_0 which unfolds generically with the parameter a. Moreover this homoclinic tangency is not generalized.*
2. *The family of diffeomorphisms is locally smoothly conjugate to its linear part near the corresponding fixed point $p = p(a)$, such that $p(a_0) = p_0$, if $|a - a_0|$ is smooth enough.*

Then, for n large enough there exist parameters a_n^+, a_n^- and a_n^c such that

1. *$f_{a_n^+}$ has an n-periodic saddle-node point p_n^+.*
2. *$f_{a_n^-}$ has an n-periodic flip point p_n^-.*
3. *$f_{a_n^c}$ has an n-periodic point p_n^c with the product of two of the eigenvalues equal to 1.*
4. *$\lim_{n\to\infty} a_n^+ = \lim_{n\to\infty} a_n^- = \lim_{n\to\infty} a_n^c = a_0$.*
5. *$\lim_{n\to\infty} p_n^+ = \lim_{n\to\infty} p_n^- = \lim_{n\to\infty} p_n^c = q_0$.*

Moreover, the saddle-node and the flip have index of stability 1.

3.2 Generalized transversality

In this case we want to find periodic points with hyperbolic points having the product of two of the eigenvalues equal to 1. In order to get such situation we have to assume that $|\lambda_1| < |\lambda_2| < 1 < |\lambda_3|$, $|\lambda_2\lambda_3| > 1$, $|\lambda_1\lambda_3| < 1$. We note that in fact the periodic points we find are hyperbolic, but are important in the context of the generalized homoclinic tangency, because are the candidates to undergo a Hopf bifurcation, and a Bogdanov-Takens bifurcation.

We use proposition 2.14, so we can write

$$f_a^n(t_1, t_2, t_3) = \mathbf{x}(\lambda_1^n t_1, \lambda_2^n t_2, \lambda_3^n t_3),$$

where

$$x(0,0,1,a) = 1, \quad y(0,0,1,a) = 1, \quad z(0,0,1,a) = 0,$$

and

$$D_2 y(0,0,1,a) D_3 z(0,0,1,a) - D_2 z(0,0,1,a) D_3 y(0,0,1,a) = a,$$

for all $a \in V$ and $D_3 z(0,0,1,0) \neq 0$. Moreover the point of generalized homoclinic transversality is $(1,0,0) = \mathbf{x}(0,0,1,0)$.

As before, we suppose that the n-periodic point is $(t_1, t_2, \lambda_3^{-n} t_3)$. Let $A = Df_a^n(t_1, t_2, \lambda_3^{-n} t_3)$. The condition of existence of two eigenvalues with product equal to one is:

$$1 + (d_1(A) - d_3(A)) d_3(A) - d_2(A) = 0.$$

If we write the equations of existence of a n-periodic point with the product of two eigenvalues equal to one, with the last equation multiplied by $\lambda_2^{-n}\lambda_3^{-n}$ and we use the variables $s_1 = \lambda_2^{-n}\lambda_3^{-n}$, $s_2 = \lambda_2^n$ and $s_3 = \lambda_1^n \lambda_3^n$ then we have a system of equations $\mathbf{F}(s_1, s_2, s_3, t_1, t_2, t_3, a) = 0$, where $\mathbf{F} = (F_1, F_2, F_3, F_4)$ and

$$\begin{aligned}
F_1 &= x(s_1 s_2 s_3 t_1, s_2 t_2, t_3, a) - t_1, \\
F_2 &= y(s_1 s_2 s_3 t_1, s_2 t_2, t_3, a) - t_2, \\
F_3 &= z(s_1 s_2 s_3 t_1, s_2 t_2, t_3, a) - s_1 s_2 t_3, \\
F_4 &= D_2 y D_3 z - D_3 y D_2 z + O(|s_1, s_2, s_3|),
\end{aligned}$$

where, the derivatives are evaluated at $(s_1 s_2 s_3 t_1, s_2 t_2, t_3, a)$ in the last equation. Moreover $\mathbf{F}(0,0,0,1,1,1,0) = 0$. Using the implicit function theorem it is immediate to see that there exist n-periodic points p_n^c for values of the parameter $a = a_n^c$ such that the eigenvalues of $Df_{a_n^c}(p_n^c)$, α_n, β_n, γ_n satisfy: $\alpha_n \beta_n = 1$, $|\alpha_n \beta_n \gamma_n| < 1$ and all are real and different from 1 or -1.

We collect these results in the following proposition.

Proposition 3.4 *Let $\{f_a\}_{a \in V}$ be a smooth family of 3-dimensional diffeomorphisms $f_a : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, where V is an open neighbourhood of $a_0 \in \mathbb{R}$. Suppose that*

1. *For $a = a_0$, f_{a_0} has a dissipative saddle fixed point p_0, with eigenvalues satisfying $|\lambda_1| < |\lambda_2| < 1$, $|\lambda_3| > 1$ and $|\lambda_2\lambda_3| > 1$, such that it undergoes a nondegenerate generalized homoclinic transversality in q_0 which unfolds generically with the parameter a.*

2. *The family of diffeomorphisms is locally smoothly conjugate to its linear part near the corresponding fixed point $p = p(a)$, such that $p(a_0) = p_0$, if $|a - a_0|$ is smooth enough.*

Then, for n large enough there exist parameters a_n^c such that

1. *$f_{a_n^c}$ has an n-periodic point p_n^c with the product of two eigenvalues equal to 1.*
2. $\lim_{n\to\infty} a_n^c = a_0$.
3. $\lim_{n\to\infty} p_n^c = q_0$.

3.3 Generalized quadratic tangency

Now, we deal with a family $\{f_{a,b}\}_{(a,b)}$. Again we use the same notation than before, and distinguish between two cases, depending on the dimension of the stable invariant manifold of the fixed point.

3.3.1 Case 1: $|\lambda_1| < |\lambda_2| < 1$ and $|\lambda_2\lambda_3| > 1$

In this case, we use the change of variables and reparametrization of proposition 2.20. Hence, we have:

$$f_{a,b}^n(t_1, t_2, t_3) = \mathbf{x}(\lambda_1^n t_1, \lambda_2^n t_2, \lambda_3^n t_3, a, b),$$

where $\mathbf{x}$ is smooth and $\mathbf{x}(\cdot, a, b)$ is the identity in a neighbourhood of $\{(t_1, t_2, 0) \ : \ t_1^2 + t_2^2 \leq 3\}$.

We begin with the tangency of type I. This means that the map $\mathbf{x}$ satisfies:

$$x(0,0,1,a,b) = 1, \quad y(0,0,1,a,b) = 1, \quad z(0,0,1,a,b) = a,$$

$$D_3 z(0,0,1,a,b) = 0, \quad D_3 y(0,0,1,a,b) = b,$$

for all $a \in V$ and $D_{33}z(0,0,1,0,0) \neq 0$, $D_2 z(0,0,1,0,0) \neq 0$, and

$$\begin{vmatrix} D_{33}y(0,0,1,0,0) & D_{33}z(0,0,1,0,0) \\ D_2 y(0,0,1,0,0) & D_2 z(0,0,1,0,0) \end{vmatrix} \neq 0.$$

Now, we want to obtain periodic points which undergo a Bogdanov-Takens bifurcation. Let $s_1 = \lambda_2^{-n}\lambda_3^{-n}$, $s_2 = \lambda_2^n$ and $s_3 = \lambda_1^n \lambda_2^{-n}$. Suppose that $p_n = (t_1, t_2, \lambda_3^{-n} t_3) = (t_1, t_2, s_1 s_2 t_3)$ is a periodic point. Then it satisfies:

$$\begin{aligned} x(s_2 s_3 t_1, s_2 t_2, t_3) - t_1 &= 0 \\ y(s_2 s_3 t_1, s_2 t_2, t_3) - t_2 &= 0 \\ z(s_2 s_3 t_1, s_2 t_2, t_3) - s_1 s_2 t_3 &= 0 \end{aligned}$$

In order to have two eigenvalues equal to 1 if $A = Df_{a,b}^n(p_n)$, one has to impose: $1 - d_1(A) + d_2(A) - d_3(A) = 0$ and $1 - d_2(A) + 2d_3(A) = 0$. If we multiply the first equation by λ_3^{-n} and the second by $\lambda_2^{-n}\lambda_3^{-n}$, the equations take the form: $F(s_1, s_2, s_3, t_1, t_2, t_3, a, b) = 0$, where $F = (F_1, F_2)$ and

$$F_1(0,0,0,t_1,t_2,t_3,a,b) = -D_3 z(0,0,t_3,a,b),$$

and

$$F_2(0,0,0,t_1,t_2,t_3,a,b) = D_2yD_3z - D_2zD_3y,$$

where the derivatives are evaluated in $(0,0,t_3,a,b)$. Jointly with the first three equations we have a system that we can write as

$$\tilde{F}(s_1,s_2,s_3,t_1,t_2,t_3,a,b) = 0. \tag{4}$$

As $\tilde{F}(0,0,0,1,1,1,0,0) = 0$ and

$$\det D\tilde{F}(0,0,0,1,1,1,0,0) = -D_{33}z(0,0,1,0,0)D_2z(0,0,1,0,0) \neq 0,$$

by the conditions of existence of a generalized homoclinic tangency, we get the existence of periodic points having two eigenvalues equal to 1, as a consequence of the Implicit Function Theorem.

We can proceed similarly to obtain periodic points with eigenvalues $1,-1$ and with $-1,-1$, and the same result for the other type of generalized homoclinic tangency. We can summarize these results in the following proposition:

Proposition 3.5 *Let $\{f_{a,b}\}_{(a,b)\in V}$ a smooth two-parameter family of three dimensional diffeomorphisms $f_{a,b} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, where V is an open neighbourhood of $(a_0,b_0) \in \mathbb{R}^2$. Suppose that:*

1. *For $(a,b) = (a_0,b_0)$, f_{a_0,b_0} has a dissipative saddle fixed point p_0 with eigenvalues λ_1, λ_2 and λ_3 which satisfy: $|\lambda_1| < |\lambda_2| < 1$, $|\lambda_3| > 1$ and $|\lambda_2\lambda_3| > 1$.*
2. *The family $f_{a,b}$ has a nondegenerate generalized homoclinic tangency of type I or II in q_0 for $(a,b) = (a_0,b_0)$.*
3. *The family of diffeomorphisms is locally smoothly conjugate to its linear part near the corresponding fixed point $p = p(a,b)$, such that $p(a_0,b_0) = p_0$.*

Then, for n large enough there exist (a_n^{++}, b_n^{++}), (a_n^{+-}, b_n^{+-}) and (a_n^{--}, b_n^{--}), such that

1. *$f_{a_n^{++},b_n^{++}}$ has an n-periodic point p_n^{++} with 1 as a double eigenvalue.*
2. *$f_{a_n^{+-},b_n^{+-}}$ has an n-periodic point p_n^{+-} with one eigenvalue equal to 1 and other equal to -1.*
3. *$f_{a_n^{--},b_n^{--}}$ has an n-periodic point p_n^{--} with two eigenvalues equal to -1.*
4. $\lim_{n\to\infty} a_n^{++} = \lim_{n\to\infty} a_n^{+-} = \lim_{n\to\infty} a_n^{--} = a_0$.
5. $\lim_{n\to\infty} b_n^{++} = \lim_{n\to\infty} b_n^{+-} = \lim_{n\to\infty} b_n^{--} = b_0$.
6. $\lim_{n\to\infty} p_n^{\pm\pm} = q_0$.

Remark 3.6 *We postpone the proof of the existence of Bogdanov-Takens bifurcations, unfolding generically, because we will use a limit return map to simplify the calculations. It is also interesting to see if the other bifurcations unfolds generically, but we only want to use the properties that can be deduced from the existence of a BT bifurcation, so we do not take in consideration this problem.*

3.3.2 Case 2: $|\lambda_1| < 1 < |\lambda_2| < |\lambda_3|$.

In first place, we can write, as before:

$$f^n_{a,b}(t_1, t_2, t_3) = \mathbf{x}(\lambda_1^n t_1, \lambda_2^n t_2, \lambda_3^n t_3, a, b),$$

where $\mathbf{x} = (x, y, z)$ is a smooth map. We suppose that the generalized homoclinic tangency is of type I (the other type being similar). We can suppose by proposition 2.21:

$$x(0,0,1,a,b) = 1, \quad y(0,0,1,a,b) = a,$$

$$z(0,0,1,a,b) = 0, \quad D_3z(0,0,1,a,b) = b,$$

$$D_2y(0,1,1,a,b)D_3z(0,1,1,a,b) - D_3y(0,1,1,a,b)D_2z(0,1,1,a,b) = 0,$$

for all $(a,b) \in V$, and $D_3y(0,1,1,0,0) = 0$, $D_2z(0,1,1,0,0) \neq 0$, and

$$D_2yD_{33}z - D_{33}yD_2z \neq 0,$$

where all the derivatives are evaluated at $(0,1,1,0,0)$.

Let $s_1 = \lambda_1^n\lambda_2^n$, $s_2 = \lambda_2^{-n}$ and $s_3 = \lambda_2^n\lambda_3^{-n}$. Using these variables and considering the conditions of existence of an n-periodic BT point $(t_1, \lambda_2^{-n}t_2, \lambda_3^{-n}t_3)$ we obtain a system of equations: $F(s_1,s_2,s_3,t_1,t_2,t_3,a,b) = 0 \in \mathbb{R}^5$, such that $F = (F_1,F_2,F_3,F_4,F_5)$ and

$$\begin{aligned}
F_1(0,0,0,t_1,t_2,t_3) &= x(0,t_2,t_3,a,b) - t_1,\\
F_2(0,0,0,t_1,t_2,t_3) &= y(0,t_2,t_3,a,b)\\
F_3(0,0,0,t_1,t_2,t_3) &= z(0,t_2,t_3,a,b)\\
F_4(0,0,0,t_1,t_2,t_3) &= D_2yD_3z - D_3yD_2z\\
F_5(0,0,0,t_1,t_2,t_3) &= -D_3z(0,t_2,t_3,a,b)
\end{aligned}$$

where in the fourth equation the derivatives are evaluated in $(0,t_2,t_3,a,b)$. Then, we have that $F(0,0,0,1,1,1,0,0) = 0$ and $\det DF(0,0,0,1,1,1,0,0) = -D_2z(D_2yD_{33}z - D_{33}yD_2z) \neq 0$, where the derivatives are evaluated at $(0,1,1,0,0)$. Then, again by the implicit function theorem, we can prove the existence of periodic points with 1 as a double eigenvalue.

Analogously, we can deduce the existence of periodic points with eigenvalues $1,-1$ and $-1,-1$. Hence we have the following proposition:

Proposition 3.7 *Let $\{f_{a,b}\}_{(a,b)\in V}$ a smooth two-parameter family of three dimensional diffeomorphisms $f_{a,b} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, where V is an open neighbourhood of $(a_0,b_0) \in \mathbb{R}^2$. Suppose that:*

1. *For $(a,b) = (a_0,b_0)$, f_{a_0,b_0} has a dissipative saddle fixed point p_0 with eigenvalues λ_1, λ_2 and λ_3 which satisfy: $|\lambda_1| < 1 < |\lambda_2| < |\lambda_3|$.*
2. *The family $f_{a,b}$ has a nondegenerate generalized homoclinic tangency of type I or II in a point q_0 for $(a,b) = (a_0,b_0)$.*
3. *The family of diffeomorphisms is locally smoothly conjugate to its linear part near the corresponding fixed point $p = p(a,b)$, such that $p(a_0,b_0) = p_0$.*

Then, for n large enough there exist (a_n^{++}, b_n^{++}), (a_n^{+-}, b_n^{+-}) and (a_n^{--}, b_n^{--}), such that

1. $f_{a_n^{++}, b_n^{++}}$ *has an n-periodic point p_n^{++} with 1 as a double eigenvalue.*
2. $f_{a_n^{+-}, b_n^{+-}}$ *has an n-periodic point p_n^{+-} with one eigenvalue equal to 1 and other equal to -1.*
3. $f_{a_n^{--}, b_n^{--}}$ *has an n-periodic point p_n-- with two eigenvalues equal to -1.*
4. $\lim_{n\to\infty} a_n^{++} = \lim_{n\to\infty} a_n^{+-} = \lim_{n\to\infty} a_n^{--} = a_0$.
5. $\lim_{n\to\infty} b_n^{++} = \lim_{n\to\infty} b_n^{+-} = \lim_{n\to\infty} b_n^{--} = b_0$.
6. $\lim_{n\to\infty} p_n^{\pm\pm} = q_0$.

4 Limit return maps

In this section we will prove two of the main theorems of the paper, concerning the existence of limit return maps. We do not want to deal with the sectional dissipative case, studied by Viana ([16]), so we have to consider the two parameter cases we have presented in the previous sections, that have a generalized homoclinic tangency. The changes to obtain these limit return maps are suggested by the values of the parameters corresponding to the several bifurcations we have obtained. We use the same notation as in the previous section.

4.1 Case 1: $|\lambda_1| < |\lambda_2| < 1 < |\lambda_3|$, $|\lambda_1\lambda_3| < 1$ and $|\lambda_2\lambda_3| > 1$.

In this section we suppose that there is a generalized homoclinic tangency and we have performed the changes of variables and reparametrization of proposition 2.20.

We will use a change of variables and parameters for the map $f_{a,b}^n$ near the generalized homoclinic tangency, depending on n, in order to obtain a limit return map. First, we need to have some information about the points and values of parameters corresponding to saddle-node and BT bifurcations. This is the content of the following lemma

Lemma 4.1 *There exist smooth maps $T_i = T_i(s_1, s_2, s_3, b)$, $u = u(s_1, s_2, s_3, b)$ and $v = v(s_2)$, defined for (s_1, s_2, s_3) in a neighbourhood of $(0,0,0,0) \in \mathbb{R}^4$, such that:*

1. *If we take $s_1 = \lambda_1^n$, $s_2 = \lambda_2^n$ and $s_3 = \lambda_3^{-n}$ then the map $f_{u,b}$ has a n-periodic saddle-node bifurcation in (T_1, T_2, T_3), that is:*

$$\begin{array}{rcl} x(\lambda_1^n T_1, \lambda_2^n T_2, T_3, u, b) - T_1 & = & 0 \\ y(\lambda_1^n T_1, \lambda_2^n T_2, T_3, u, b) - T_2 & = & 0 \\ z(\lambda_1^n T_1, \lambda_2^n T_2, T_3, u, b) - \lambda_3^{-n} T_3 & = & 0 \\ \lambda_3^{-n}(1 - d_1 + d_2 - d_3) & = & 0, \end{array} \tag{5}$$

   and $T_i(0,0,0,b) = 1$, $i = 1,2,3$, $u(0,0,0,b) = 0$, for all n large enough and b small enough, where $d_i = d_i(Df_{u,b}(T_1, T_2, \lambda_3^{-n} T_3))$.

2. *If we set $T_i = T_i(0, \lambda_2^n, 0, v(\lambda_2^n))$, $i = 1,2,3$, $u = u(0, \lambda_2^n, 0, v(\lambda_2^n))$ then $v(0) = 0$ and:*

(a) If the generalized homoclinic tangency is of type I:

$$\begin{array}{rcl} x(0,\lambda_2^nT_2,T_3,u,v(\lambda_2^n))-T_1 & = & 0\\ y(0,\lambda_2^nT_2,T_3,u,v(\lambda_2^n))-T_2 & = & 0\\ z(0,\lambda_2^nT_2,T_3,u,v(\lambda_2^n)) & = & 0\\ D_3z(0,\lambda_2^nT_2,T_3,u,v(\lambda_2^n)) & = & 0\\ D_3y(0,\lambda_2^nT_2,T_3,u,v(\lambda_2^n)) & = & 0 \end{array} \tag{6}$$

(b) If the generalized homoclinic tangency is of type II:

$$\begin{array}{rcl} x(0,\lambda_2^nT_2,T_3,u,v(\lambda_2^n))-T_1 & = & 0\\ y(0,\lambda_2^nT_2,T_3,u,v(\lambda_2^n))-T_2 & = & 0\\ z(0,\lambda_2^nT_2,T_3,u,v(\lambda_2^n)) & = & 0\\ D_3z(0,\lambda_2^nT_2,T_3,u,v(\lambda_2^n)) & = & 0\\ D_2z(0,\lambda_2^nT_2,T_3,u,v(\lambda_2^n)) & = & 0, \end{array} \tag{7}$$

This lemma is an immediate consequence of the properties of the map $\mathbf{x}$ and the implicit function theorem.

Remark 4.2 *The second system of equations (6) or (7) is deduced from the one corresponding to the Bogdanov-Takens bifurcation (4), that we gave in the previous section. We recall that we used variables $s_1=\lambda_2^{-n}\lambda_3^{-n}$, $s_2=\lambda_2^n$ and $s_3=\lambda_1^n\lambda_2^{-n}$. If in that system we put $s_1=s_3=0$ we obtain the equations (6) or (7), depending on the type of tangency.*

With the information of the previous lemma, we can perform a changes of variables and reparametrization for the map $f^n_{a,b}$, depending on the type of tangency. Denote $(\bar{x},\bar{y},\bar{z})$ the new variables and $(\bar{a},\bar{b})$ the new parameter.

Type I. We consider the change:

$$\begin{array}{rcl} t_1 & = & T_1(\lambda_1^n,\lambda_2^n,\lambda_3^{-n},b)+\lambda_3^{-n}\bar{x}\\ t_2 & = & T_2(\lambda_1^n,\lambda_2^n,\lambda_3^{-n},b)+\lambda_2^{-n}\lambda_3^{-2n}\bar{y}\\ t_3 & = & \lambda_3^{-n}T_3(\lambda_1^n,\lambda_2^n,\lambda_3^{-n},b)+\lambda_3^{-2n}\bar{z}\\ a & = & u(\lambda_1^n,\lambda_2^n,\lambda_3^{-n},b)+\lambda_3^{-2n}\bar{a}\\ b & = & v(\lambda_2^n)+\bar{b}\lambda_2^{-n}\lambda_3^{-n} \end{array} \tag{8}$$

We want to express $f^n_{a,b}$ in terms of $\bar{x},\bar{y},\bar{z},\bar{a},\bar{b}$. Let $(\bar{x},\bar{y},\bar{z},\bar{a},\bar{b})$ denote a point. The (t_1,t_2,t_3,a,b) variables of this point are given by (8). After applying $f^n_{a,b}$ to this point we get as (t_1,t_2,t_3) coordinates:

$$\begin{array}{rcl} t_1 & = & x(\lambda_1^nT_1+\lambda_1^n\lambda_3^{-n}\bar{x},\lambda_2^nT_2+\lambda_3^{-2n}\bar{y},T_3+\lambda_3^{-n}\bar{z},u+\lambda_3^{-2n}\bar{a},v+\bar{b}\lambda_2^{-n}\lambda_3^{-n})\\ t_2 & = & y(\lambda_1^nT_1+\lambda_1^n\lambda_3^{-n}\bar{x},\lambda_2^nT_2+\lambda_3^{-2n}\bar{y},T_3+\lambda_3^{-n}\bar{z},u+\lambda_3^{-2n}\bar{a},v+\bar{b}\lambda_2^{-n}\lambda_3^{-n})\\ t_3 & = & z(\lambda_1^nT_1+\lambda_1^n\lambda_3^{-n}\bar{x},\lambda_2^nT_2+\lambda_3^{-2n}\bar{y},T_3+\lambda_3^{-n}\bar{z},u+\lambda_3^{-2n}\bar{a},v+\bar{b}\lambda_2^{-n}\lambda_3^{-n}) \end{array}$$

where $T_i=T_i(\lambda_1^n,\lambda_2^n,\lambda_3^{-n},b)$, $i=1,2,3$, $u=u(\lambda_1^n,\lambda_2^n,\lambda_3^{-n},b)$, and $v=v(\lambda_2^n)$. Transforming this back to the $(\bar{x},\bar{y},\bar{z})$ coordinates, and denoting the values of these coordinates of the new point by $\bar{x}'$, $\bar{y}'$, $\bar{z}'$,

$$\begin{array}{rcl} \bar{x}' & = & (x-T_1(\lambda_1^n,\lambda_2^n,\lambda_3^{-n}))\lambda_3^n\\ \bar{y}' & = & (y-T_2(\lambda_1^n,\lambda_2^n,\lambda_3^{-n}))\lambda_2^n\lambda_3^{2n}\\ \bar{z}' & = & (z-\lambda_3^{-n}T_3(\lambda_1^n,\lambda_2^n,\lambda_3^{-n}))\lambda_3^{2n} \end{array}$$

where x, y, z are evaluated as in the previous equations.

In order to obtain a limit map, we use variables $s_1 = \lambda_2^{-n}\lambda_3^{-n}$, $s_2 = \lambda_2^n$ and $s_3 = \lambda_1^n\lambda_3^n$, and use the Taylor expansion of the map $\mathbf{x}(\cdot, b)$, about the point

$$(\lambda_1^n T_1, \lambda_2^n T_2, T_3, u) = (s_1 s_2 s_3 T_1, s_2 T_2, T_3, u),$$

where now we can write $T_i = T_i(s_1 s_2 s_3, s_2, s_1 s_2 i, b)$ and $u = u(s_1 s_2 s_3, s_2, s_1 s_2, b)$.

It is easy to see, using the system of equations (5), that

$$\begin{aligned} \bar{x}' &= D_3 x(0,0,1,0,0)\bar{z} + f_1 \\ \bar{y}' &= D_3 y(s_1 s_2 s_3 T_1, s_2 T_2, T_3, u, v(s_2) + \bar{b}s_1)\bar{z}s_1^{-1} + f_2 \\ \bar{z}' &= D_2 z(0,0,1,0,0)\bar{y} + D_3 z\, \bar{z} s_1^{-1} s_2^{-1} + \bar{a} + \frac{1}{2} D_{33} z(0,0,1,0,0)\bar{z}^2 + f_3 \end{aligned}$$

where $f_i = f_i(s_1, s_2, s_3, \bar{x}, \bar{y}, \bar{z}, \bar{a}, \bar{b})$ are smooth maps and $f_i(0,0,0,\bar{x},\bar{y},\bar{z},\bar{a},\bar{b}) = 0$, and $D_3 z = D_3 z(s_1 s_2 s_3 T_1, s_2 T_2, T_3, u, v(s_2) + \bar{b}s_1)$.

We know that $D_3 y(0, s_2 T_2, T_3, u, v(s_2)) = 0$, when $s_1 = 0$, by the fifth equation of (6). Hence, if we expand the map $Y(s_1) = D_3 y(s_1 s_2 s_3 T_1, s_2 T_2, T_3, u, v + \bar{b}s_1)$ about $s_1 = 0$, and using that $D_3 y(0,0,1,a,b) = b$, we obtain that

$$\bar{y}' = \bar{b}\bar{z} + f_2,$$

where f_2 is different from the previous one but has the same properties.

Finally, by the last equation of (5) we have:

$$(1 - s_2 D_2 y) D_3 z(s_1 s_2 s_3 T_1, s_2 T_2, T_3, u, v + \bar{b}s_1) = s_1 s_2 (1 + f_3) + D_3 y D_2 z\, s_2,$$

where $D_2 y$, $D_3 y$ and $D_2 z$ are evaluated in the same point as $D_3 z$.

As $D_3 y = (\bar{b} + \tilde{f}_2)s_1$, and $D_2 z(0,0,1,0,0) \neq 0$, the component $\bar{z}'$ satisfies:

$$\bar{z}' = \bar{a} + D_2 z\, \bar{y} + (1 - D_2 z \bar{b})\bar{z} + \frac{1}{2} D_{33} z \bar{z}^2 + f_3,$$

where $D_2 z$, $D_3 z$ and $D_{33} z$ are evaluated at $(0,0,1,0,0)$.

This means that in the type I generalized homoclinic tangency we obtain, after a change of variables and a reparametrization, that the limit return map is:

$$\begin{pmatrix} \bar{x}' \\ \bar{y}' \\ \bar{z}' \end{pmatrix} = \begin{pmatrix} \alpha \bar{z} \\ \bar{b}\bar{z} \\ \bar{a} + \beta \bar{y} + (1 - \beta \bar{b})\bar{z} + \gamma \bar{z}^2 \end{pmatrix} \tag{9}$$

where α, β and γ are non-zero constants.

Type II. We use the same notation as in the previous case. Now, we perform the change:

$$\begin{aligned} t_1 &= T_1(\lambda_1^n, \lambda_2^n, \lambda_3^{-n}, b) + \lambda_3^{-n}\bar{x} \\ t_2 &= T_2(\lambda_1^n, \lambda_2^n, \lambda_3^{-n}, b) + \lambda_3^{-n}\bar{y} \\ t_3 &= \lambda_3^{-n} T_3(\lambda_1^n, \lambda_2^n, \lambda_3^{-n}, b) + \lambda_3^{-2n}\bar{z} \\ a &= u(\lambda_1^n, \lambda_2^n, \lambda_3^{-n}, b) + \lambda_3^{-2n}\bar{a} \\ b &= v(\lambda_2^n) + \bar{b}\lambda_2^{-n}\lambda_3^{-n}. \end{aligned} \tag{10}$$

where T_i, $i = 1, 2, 3$, u and v satisfy the equations (5) and (7).

As before we want to express $f^n_{a,b}$ in the new coordinates and parameter. The (t_1, t_2, t_3, a, b) variables of a point $(\bar{x}, \bar{y}, \bar{z}, \bar{a}, \bar{b})$ are given by (10). After applying $F^n_{a,b}$ we get:

$$\begin{aligned} t_1 &= x(\lambda_1^n T_1 + \lambda_1^n \lambda_3^{-n}\bar{x}, \lambda_2^n T_2 + \lambda_2^n \lambda_3^{-n}\bar{y}, T_3 + \lambda_3^{-n}\bar{z}, u + \lambda_3^{-2n}\bar{a}, v + \bar{b}\lambda_2^{-n}\lambda_3^{-n}) \\ t_2 &= y(\lambda_1^n T_1 + \lambda_1^n \lambda_3^{-n}\bar{x}, \lambda_2^n T_2 + \lambda_2^n \lambda_3^{-n}\bar{y}, T_3 + \lambda_3^{-n}\bar{z}, u + \lambda_3^{-2n}\bar{a}, v + \bar{b}\lambda_2^{-n}\lambda_3^{-n}) \\ t_3 &= z(\lambda_1^n T_1 + \lambda_1^n \lambda_3^{-n}\bar{x}, \lambda_2^n T_2 + \lambda_2^n \lambda_3^{-n}\bar{y}, T_3 + \lambda_3^{-n}\bar{z}, u + \lambda_3^{-2n}\bar{a}, v + \bar{b}\lambda_2^{-n}\lambda_3^{-n}) \end{aligned}$$

As before, we expand the maps $x(\cdot, b)$, $y(\cdot, b)$, $z(\cdot, b)$ about $(s_1 s_2 s_3 T_1, s_2 T_2, T_3, u)$. Then we obtain:

$$\begin{aligned} \bar{x}' &= D_3 x(0, 0, 1, 0, 0)\bar{z} + f_1 \\ \bar{y}' &= D_3 y(0, 0, 1, 0, 0)\bar{z} + f_2 \\ \bar{z}' &= D_2 z\, \bar{y} s_1^{-1} + D_3 z\, \bar{z} s_1^{-1} s_2^{-1} + \bar{a} + \frac{1}{2} D_{33} z(0, 0, 1, 0, 0)\bar{z}^2 + f_3 \end{aligned}$$

where $D_2 z$ and $D_3 z$ are evaluated at $(s_1 s_2 s_3 T_1, s_2 T_2, T_3, u, v + \bar{b} s_1)$.

We know that $D_2 z(0, s_2 T_2, T_3, u, v(s_2)) = 0$, when $s_1 = 0$, by the fifth equation of (7). It is easy to see that if we expand the map

$$Z(s_1) = D_2 z(s_1 s_2 s_3 T_1, s_2 T_2, T_3, u, v + \bar{b} s_1),$$

about $s_1 = 0$, and using that $D_2 z(0, 0, 1, a, b) = b$, we obtain

$$\bar{z}' = \bar{b}\bar{y} + + D_3 z\, \bar{z} s_1^{-1} s_2^{-1} + \bar{a} + \frac{1}{2} D_{33} z(0, 0, 1, 0, 0)\bar{z}^2 + f_3.$$

Using, as in the preceding case, the last equation of (5), and taking into account the value of $D_2 z$ we find, finally:

$$\bar{z}' = \bar{a} + \bar{b}\bar{y} + (1 - \bar{b} D_3 y)\bar{z} + \frac{1}{2}(D_{33} z)\bar{z}^2 + f_3,$$

where $D_3 y$ and $D_{33} z$ are evaluated in $(0, 0, 1, 0, 0)$.

This means that in the type II a limit return map is:

$$\begin{pmatrix} \bar{x}' \\ \bar{y}' \\ \bar{z}' \end{pmatrix} = \begin{pmatrix} \alpha \bar{z} \\ \beta \bar{z} \\ \bar{a} + \bar{b}\bar{y} + (1 - \beta \bar{b})\bar{z} + \gamma \bar{z}^2 \end{pmatrix} \tag{11}$$

where β and γ are non-zero constants.

The two cases we have studied give limit maps that are, essentially like the two-dimensional Hénon map. Using simple linear changes of variables and reparametrization we can prove the following proposition:

Proposition 4.3 *Let $\{f_{a,b}\}$ a smooth two-parameter family of three dimensional diffeomorphisms having a hyperbolic dissipative fixed point p_0 for $(a, b) = (a_0, b_0)$, with eigenvalues λ_1, λ_2, λ_3 which satisfies: $|\lambda_1| < |\lambda_2| < 1$, $|\lambda_3| > 1$, $|\lambda_1 \lambda_3| < 1$ and $|\lambda_2 \lambda_3| > 1$.*

Suppose that f_{a_0,b_0} has a generalized homoclinic tangency at p_1 which unfolds generically, and with the homoclinic orbit in a general position. Then, after a change of variables and a reparametrization depending on n, the map $f^n_{a,b}$ for (a,b) near (a_0,b_0) and near p_1 has a limit when $n \to \infty$ which is

$$(\bar{x},\bar{y},\bar{z}) \mapsto (\bar{z},\bar{b}\bar{z},\bar{a}+\bar{y}+\bar{z}^2),$$

if the generalized homoclinic tangency is of type I and

$$(\bar{x},\bar{y},\bar{z}) \mapsto (0,\bar{z},\bar{a}+\bar{b}\bar{y}+\bar{z}^2),$$

if it is of type II.

Remark 4.4 *In fact the two limit return maps are conjugate, except in the case $\bar{b}=0$. But in this case these maps restricted to some suitable 1-dimensional manifold are conjugate to the logistic map.*

The next step is to obtain properties of the initial map, using the information we have of the return map. We first will prove that there are invariant circles near the tangency. In order to do this, we have to see that the Bogdanov-Takens bifurcation, that exists for all period sufficiently large, is generic and unfolds generically. There is a difficulty here. The limit map near the BT bifurcation is the conservative Hénon map. So we have to include some terms of higher order in the return map. We have the following proposition:

Proposition 4.5 *Let $\{f_{a,b}\}$ be a two-parameter family of three dimensional diffeomorphisms, having a hyperbolic dissipative fixed point p_0 for $(a,b)=(a_0,b_0)$ with eigenvalues λ_1, λ_2 and λ_3. Let suppose that $|\lambda_1|<1$, $|\lambda_2|<1$, $|\lambda_2|>1$, $|\lambda_1\lambda_3|<1$ and $|\lambda_2\lambda_3|>1$. Moreover, the invariant manifolds of p_0 have a generalized homoclinic tangency which unfolds generically. Then:*

1. *For n large enough there are values of the parameter (a_n^{++},b_n^{++}) for which the map $f^n_{a,b}$ undergoes a generic Bogdanov-Takens bifurcation. These values of the parameters are those of proposition 3.5.*
2. *$(a,b)=(a_0,b_0)$ is in the closure of parameter values for which there exist differentiable invariant circles. Depending on the geometry of the tangency and in the sign of λ_2, we can add attracting or saddle (repelling inside the centre manifold).*

Proof:
We shall distinguish between generalized tangencies of type I and II. It will be useful to consider as before the following variables $s_1=\lambda_2^{-n}\lambda_3^{-n}$, $s_2=\lambda_2^n$ and $s_3=\lambda_1^n\lambda_3^n$.

Type I:
In order to prove the proposition we will take the return map $f^n_{a,b}$ after the change of variables and reparametrization (8) and prove that it undergoes a generic Bogdanov-Takens bifurcation. We shall verify the conditions of item 4. in section 2.4. Then the statements of the proposition are consequence of the properties of this bifurcation (see item 3. in the same section).

It is not difficult to see that the return map has the following expression:

$$\begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} \mapsto \begin{pmatrix} D_3x\,\bar{z} + f_1\bar{z} + s_2O_1 \\ D_2y\,\bar{y}s_2 + \frac{1}{2}D_{33}y\,\bar{z}^2 s_2 + \bar{b}\bar{z} + f_2\bar{z} + s_2O_1 \\ \bar{a} + D_2z\,\bar{y} + (1 - D_2z\bar{b})\bar{z} + \frac{1}{2}D_{33}z\,\bar{z}^2 + f_3 \end{pmatrix}, \tag{12}$$

where all the derivatives are evaluated at $(0,0,1,0,0)$, f_i, $i = 1,2,3$, are smooth functions that are equal to zero when $s_1 = s_2 = s_3 = 0$ and depend on $s_1, s_2, s_3, \bar{a}, \bar{b}, \bar{x}, \bar{y}, \bar{z}$. Moreover, O_i means i-th order in (s_1, s_2, s_3), and f_3 satisfies $\partial^2 f_3/\partial\bar{x}^2 = s_2O_1$, $\partial^2 f_3/\partial\bar{x}\partial\bar{y} = s_2O_1$, $\partial^2 f_3/\partial\bar{x}\partial\bar{z} = s_2O_1$, $\partial^2 f_3/\partial\bar{y}^2 = s_2O_1$ and $\partial^2 f_3/\partial\bar{y}\partial\bar{z} = s_2O_1$.

On the other hand, the map (12) has a fixed point $(\bar{x},\bar{y},\bar{z}) = O_1$, for $\bar{a} = O_1$, $\bar{b} = -1/D_2z(0,0,1,0,0) + O_1$, with one eigenvalue less than 1 in absolute value and an eigenvalue equal to 1 with multiplicity two. We call these values of the parameters $\bar{a}_0$ and $\bar{b}_0$, respectively. Moreover, a centre manifold associated to the eigenvalue equal to 1 has the expression: $\bar{x} = \frac{D_3x}{\bar{b}}\bar{y} + O_1$, for the map (12), where the derivative is evaluated at $(0,0,1,0,0)$. Note that, as we saw before, D_3x and D_2z are not zero in $(0,0,1,0,0)$. The restriction of the map to the centre manifold, that we call $F_{\bar{a},\bar{b}}$ has the form:

$$\begin{pmatrix} \bar{y} \\ \bar{z} \end{pmatrix} \overset{F_{\bar{a},\bar{b}}}{\mapsto} \begin{pmatrix} \bar{b}\bar{z} + \tilde{f} \\ \bar{a} + D_2z\,\bar{y} + (1 - D_2z\,\bar{b})\bar{z} + \frac{1}{2}D_{33}z\,\bar{z}^2 + \tilde{g} \end{pmatrix},$$

where the derivatives are evaluated at $(0,0,1,0,0)$, $\tilde{f}$, $\tilde{g}$ are of first order in (s_1, s_2, s_3), and satisfy:

$$\frac{\partial\tilde{f}}{\partial\bar{y}} = D_2y\,s_2 + s_2O_1, \quad \frac{\partial^2\tilde{f}}{\partial\bar{y}^2} = s_2O_1, \quad \frac{\partial^2\tilde{f}}{\partial\bar{y}\partial\bar{z}} = s_2O_1, \quad \frac{\partial^2\tilde{f}}{\partial\bar{z}^2} = D_{33}y\,s_2 + s_2O_1,$$

$$\frac{\partial^2\tilde{g}}{\partial\bar{y}^2} = s_2O_1, \quad \frac{\partial^2\tilde{g}}{\partial\bar{y}\partial\bar{z}} = s_2O_1,$$

where the derivatives of y are evaluated at $(0,0,1,0,0)$. Moreover, the fixed point $\bar{p}_0 \in \mathbb{R}^2$ of $F_{\bar{a}_0,\bar{b}_0}$ with a double eigenvalue 1 satisfies $\bar{p}_0 = O_1$.

In order to prove that the Bogdanov-Takens bifurcation is generic we have to see that $w_2^t D^2F_{\bar{a}_0,\bar{b}_0}(\bar{p}_0)(v_1,v_1) \neq 0$ and

$$w_2^t D^2F_{\bar{a}_0,\bar{b}_0}(\bar{p}_0)(v_1,v_2) + w_1^t D^2F_{\bar{a}_0,\bar{b}_0}(\bar{p}_0)(v_1,v_1) - w_2^t D^2F_{\bar{a}_0,\bar{b}_0}(\bar{p}_0)(v_1,v_1) \neq 0,$$

where $w_i^t = e_i^t M^{-1}$, $v_i = Me_i$, $i = 1,2$, being e_1, e_2 the canonical basis of $\mathbb{R}^2$ and M the linear map that conjugates $DF_{\bar{a}_0,\bar{b}_0}(\bar{p}_0)$ to

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

We denote c_1 and c_2 the two real numbers that have to be different from zero.

If we denote $DF_{\bar{a}_0,\bar{b}_0}(\bar{p}_0) = (a_{ij})_{1\leq i,j\leq 2}$, as usual, it is easy to see that

$$v_1 = \begin{pmatrix} a_{11} - 1 \\ -(a_{11}-1)^2/a_{12} \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$w_1 = \begin{pmatrix} 0 \\ -a_{12}/(a_{11}-1)^2 \end{pmatrix}, \quad w_2 = \begin{pmatrix} 1 \\ a_{12}/(a_{11}-1) \end{pmatrix}.$$

Moreover $a_{11} = D_2y(0,0,1,0,0)s_2 + s_2O_1$, $a_{12} = -1/D_2z(0,0,1,0,0) + O_1$.

Carrying out some computations we finally see that the two conditions to have a BT bifurcation are $c_1 = D_{33}z(0,0,1,0,0)D_2z(0,0,1,0,0) + O_1 \neq 0$ and $c_2 = ((D_2yD_{33}z - D_{33}yD_2z)D_2z + O_1)s_2 \neq 0$, where the derivatives are evaluated at $(0,0,1,0,0)$. Using the conditions 3(a) of proposition 2.20, we see that c_1 and c_2 are different from zero if s_1, s_2, s_3 are small enough, that is n is large enough.

To see that the BT bifurcation unfolds generically, we have to compute the parameters μ and ν of the definition. It is easy to see that $\mu = \frac{1}{2}D_{33}z\bar{a} + O_1$, and $\nu = -D_2z\bar{b} - 1 + O_1$. As $|\frac{\partial(\mu,\nu)}{\partial(\bar{a},\bar{b})}| \neq 0$ for $(\bar{a},\bar{b}) = (\bar{a}_0,\bar{b}_0)$ and n large enough, we have finished the first part of the proposition.

Type II:

We follow the same steps as before. In this case te return map is:

$$\begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} \mapsto \begin{pmatrix} D_3x\,\bar{z} + f_1\bar{z} + s_2O_1 \\ D_2y\,\bar{y}s_2 + D_3y\,\bar{z} + f_2\bar{z} + s_2O_1 \\ \bar{a} + \bar{b}\bar{y} + (1 - D_3y\bar{b})\bar{z} + \frac{1}{2}D_{33}z\,\bar{z}^2 + D_{23}z\,\bar{y}\bar{z}s_2 + f_3 \end{pmatrix}, \tag{13}$$

where all the derivatives are evaluated at $(0,0,1,0,0)$, f_i, $i = 1,2,3$, are smooth functions that are equal to zero when $s_1 = s_2 = s_3 = 0$ and depend on $s_1, s_2, s_3, \bar{a}, \bar{b}, \bar{x}, \bar{y}, \bar{z}$. Moreover, O_i means i-th order in (s_1, s_2, s_3), and f_3 satisfies $\partial^2 f_3/\partial\bar{x}^2 = s_2O_1$, $\partial^2 f_3/\partial\bar{x}\partial\bar{y} = s_2O_1$, $\partial^2 f_3/\partial\bar{x}\partial\bar{z} = s_2O_1$, $\partial^2 f_3/\partial\bar{y}^2 = s_2O_1$, $\partial^2 f_3/\partial\bar{y}\partial\bar{z} = s_2O_1$.

The map (13) has a fixed point in $(\bar{x},\bar{y},\bar{z}) = O_1$ for $\bar{a} = O_1$ and $\bar{b} = -1/D_3y(0,0,1,0,0) + O_1$ with one eigenvalue less than 1 in absolute value and an eigenvalue equal to 1 with multiplicity two. We call these values of the parameters $\bar{a}_0$ and $\bar{b}_0$, respectively. Moreover, the centre manifold associated to the eigenvalue equal to 1 has the expression: $\bar{x} = (D_3x/D_3y)\bar{y} + O_1$, where the derivatives of x and y are evaluated at $(0,0,1,0,0)$. Note that $D_3y \neq 0$. The restriction of the map to the centre manifold, that we call again $F_{\bar{a},\bar{b}}$, is

$$\begin{pmatrix} \bar{y} \\ \bar{z} \end{pmatrix} \overset{F_{\bar{a},\bar{b}}}{\mapsto} \begin{pmatrix} D_3y\,\bar{z} + \tilde{f} \\ \bar{a} + \bar{b}\bar{y} + (1 - D_3y\,\bar{b})\bar{z} + \frac{1}{2}D_{33}z\,\bar{z}^2 + \tilde{g} \end{pmatrix},$$

where the derivatives are evaluated at $(0,0,1,0,0)$, $\tilde{f}$, $\tilde{g}$ are of first order in (s_1, s_2, s_3), and satisfy:

$$\frac{\partial\tilde{f}}{\partial\bar{y}} = D_2y\,s_2 + s_2O_1, \quad \frac{\partial^2\tilde{f}}{\partial\bar{y}^2} = s_2O_1, \quad \frac{\partial^2\tilde{f}}{\partial\bar{y}\partial\bar{z}} = s_2O_1, \quad \frac{\partial^2\tilde{f}}{\partial\bar{z}^2} = s_2O_1,$$

$$\frac{\partial^2\tilde{g}}{\partial\bar{y}^2} = s_2O_1, \quad \frac{\partial^2\tilde{g}}{\partial\bar{y}\partial\bar{z}} = D_{23}z\,s_2 + s_2O_1,$$

where the derivatives of y are evaluated at $(0,0,1,0,0)$. Moreover, the fixed point $\bar{p}_0 \in \mathbb{R}^2$ of $F_{\bar{a}_0,\bar{b}_0}$ with a double eigenvalue 1 satisfies $\bar{p}_0 = O_1$.

Proceeding as in the previous case (and with the same notation) we obtain $c_1 = -D_{33}z/D_3y + O_1$ and $c_2 = s_2[(D_{23}zD_3y - D_2yD_{33}z)/D_3y + O_1]$, where the derivatives of y and z are evaluated at $(0,0,1,0,0)$. Using the conditions 3(b) of proposition 2.20, we see that c_1 and c_2 are different from zero if n is large enough.

As before, in order to see that the BT bifurcation unfolds generically, we compute μ and ν and see that $|\frac{\partial(\mu,\nu)}{\partial(\bar{a},\bar{b})}| \neq 0$, but again this is an easy check. □

Then, by applying to the return maps, for n large enough, the results of Viana [16] and Palis and Viana [12], we obtain immediately the proof of theorem 1.

4.2 Case 2: $|\lambda_1| < 1$, $|\lambda_3| > |\lambda_2| > 1$.

In this section we suppose that there is a generalized homoclinic tangency and that we have performed the changes of variables and reparametrization of proposition 2.21. We proceed as in the other section, considering a change of variables and parameters for the map $f^n_{a,b}$ near the generalized homoclinic tangency. Before giving these changes we need more information about the points and the values of the parameters for which there are saddle-node and BT bifurcations:

Lemma 4.6 *There exists smooth maps $T_i = T_i(s_1, s_2, s_3, b)$, $i = 1,2,3$, $u = u(s_1, s_2, s_3, b)$ defined in a neighbourhood of $(0,0,0,0)$ and $v = v(s_1, s_2, s_3)$, defined in a neighbourhood of $(0,0,0)$ such that:*

1. *If we take $s_1 = \lambda_1^n$, $s_2 = \lambda_2^{-n}$ and $s_3 = \lambda_3^{-n}$ then the map $f_{u,b}$ has an n-periodic saddle-node bifurcation in (T_1, T_2, T_3), that is:*

$$\begin{array}{rcl} x(\lambda_1^n T_1, T_2, T_3, u, b) - T_1 & = & 0 \\ y(\lambda_1^n T_1, T_2, T_3, u, b) - \lambda_2^{-n} T_2 & = & 0 \\ z(\lambda_1^n T_1, T_2, T_3, u, b) - \lambda_3^{-n} T_3 & = & 0 \\ \lambda_2^{-n}\lambda_3^{-n}(1 - d_1 + d_2 - d_3) & = & 0, \end{array} \tag{14}$$

   and $T_i(0,0,0,b) = 1$, $i = 1,2,3$, $u(0,0,0,b) = 0$, for all n large enough and b small enough, where $d_i = d_i(Df_{u,b}(T_1, \lambda_2^{-n}T_2, \lambda_3^{-n}T_3))$.

2. *If we set $v = v(\lambda_1^n\lambda_2^n\lambda_3^n, \lambda_2^{-n}, \lambda_2^n\lambda_3^{-n})$, $\bar{T}_i = T_i(\lambda_1^n, \lambda_2^{-n}, \lambda_3^{-n}, v)$, $\bar{u} = u(\lambda_1^n, \lambda_2^{-n}, \lambda_3^{-n}, v)$ then $v(0,0,0) = 0$ and $f_{\bar{u},v}$ has an n-periodic BT bifurcation in $(\bar{T}_1, \bar{T}_2, \bar{T}_3)$. This means that*

$$\begin{array}{rcl} x(\lambda_1^n \bar{T}_1, \bar{T}_2, \bar{T}_3, \bar{u}, b) - T_1 & = & 0 \\ y(\lambda_1^n \bar{T}_1, \bar{T}_2, \bar{T}_3, \bar{u}, b) - \lambda_2^{-n} \bar{T}_2 & = & 0 \\ z(\lambda_1^n \bar{T}_1, \bar{T}_2, \bar{T}_3, \bar{u}, b) - \lambda_3^{-n} \bar{T}_3 & = & 0 \\ \lambda_2^{-n}\lambda_3^{-n}(1 - d_1 + d_2 - d_3) & = & 0, \\ \lambda_3^{-n}(2 - d_1 + d_3) & = & 0 \end{array} \tag{15}$$

To prove this lemma we only have to impose the conditions of saddle-node and BT bifurcation and apply the implicit function theorem, as before (see sections 3.1.2 and 3.3.2).

We will give two different changes of variables and reparametrizations for the two types of generalized tangency. Denote $(\bar{x}, \bar{y}, \bar{z})$ the new variables and $(\bar{a}, \bar{b})$ the new parameters.

Type I

We consider the change:

$$\begin{array}{rcl} t_1 &=& T_1(\lambda_1^n, \lambda_2^{-n}, \lambda_3^{-n}, b) + \lambda_2^{-n}\lambda_3^{-n}\bar{x} \\ t_2 &=& \lambda_2^{-n}T_2(\lambda_1^n, \lambda_2^{-n}, \lambda_3^{-n}, b) + \lambda_2^{-2n}\lambda_3^{-2n}\bar{y} + \frac{D_2 y}{D_2 z}\lambda_2^{-n}\lambda_3^{-2n}\bar{z} \\ t_3 &=& \lambda_3^{-n}T_3(\lambda_1^n, \lambda_2^{-n}, \lambda_3^{-n}, b) + \lambda_2^{-n}\lambda_3^{-2n}\bar{z} \\ a &=& u(\lambda_1^n, \lambda_2^{-n}, \lambda_3^{-n}, b) + \lambda_2^{-2n}\lambda_3^{-2n}\bar{a} \\ b &=& v(\lambda_1^n\lambda_2^n\lambda_3^n, \lambda_2^{-n}, \lambda_2^n\lambda_3^{-n}) + \lambda_3^{-n}\bar{b}, \end{array} \tag{16}$$

where $D_2 y$ and $D_2 z$ are evaluated at $(\lambda_1^n T_1, T_2, T_3, u, b)$.

We want to express $f_{a,b}^n$ in terms of $\bar{x}, \bar{y}, \bar{z}, \bar{a}, \bar{b}$. Let $(\bar{x}, \bar{y}, \bar{z}, \bar{a}, \bar{b})$ denote a point. The (t_1, t_2, t_3, a, b) variables of this point are given by (16). After applying $f_{a,b}^n$ to this point, we get as (t_1, t_2, t_3) coordinates:

$$(t_1, t_2, t_3) = \mathbf{x}(\bar{t}_1, \bar{t}_2, \bar{t}_3, u + \lambda_2^{-2n}\lambda_3^{-2n}\bar{a}, v + \lambda_3^{-n}\bar{b}),$$

where

$$\begin{array}{rcl} \bar{t}_1 &=& \lambda_1^n T_1 + \lambda_1^n\lambda_2^{-n}\lambda_3^{-n}\bar{x}, \\ \bar{t}_2 &=& T_2 + \lambda_2^{-n}\lambda_3^{-2n}\bar{y} + \dfrac{D_2 y}{D_2 z}\lambda_3^{-2n}\bar{z}, \\ \bar{t}_3 &=& T_3 + \lambda_2^{-n}\lambda_3^{-n}\bar{z}, \end{array}$$

and $T_i = T_i(\lambda_1^n, \lambda_2^{-n}, \lambda_3^{-n}, b)$, $i = 1, 2, 3$, $u = u(\lambda_1^n, \lambda_2^{-n}, \lambda_3^{-n}, b)$, and $v = v(\lambda_1^n\lambda_2^n\lambda_3^n, \lambda_2^{-n}, \lambda_2^n\lambda_3^{-n})$.

Transforming this back to the $(\bar{x}, \bar{y}, \bar{z})$ coordinates, and denoting the values of these coordinates of the new point by $\bar{x}'$, $\bar{y}'$, $\bar{z}'$:

$$\begin{array}{rcl} \bar{x}' &=& (x - T_1(\lambda_1^n, \lambda_2^{-n}, \lambda_3^{-n}, b))\lambda_2^n\lambda_3^n, \\ \bar{y}' &=& \left(y - \lambda_2^{-n}T_2 - \dfrac{D_2 y}{D_2 z}(z - \lambda_3^{-n}T_3)\right)\lambda_2^{2n}\lambda_3^{2n}, \\ \bar{z}' &=& (z - \lambda_3^{-n}T_3)\lambda_2^n\lambda_3^{2n}, \end{array}$$

where x, y, z are evaluated in the same point as before.

Now, we use the following variables: $s_1 = \lambda_1^n\lambda_2^n\lambda_3^n$, $s_2 = \lambda_2^{-n}$ and $s_3 = \lambda_2^n\lambda_3^{-n}$. Performing the Taylor expansion of the map $\mathbf{x}(\cdot, b)$ about the point:

$$(\lambda_1^n T_1, T_2, T_3, u) = (s_1 s_2^2 s_3 T_1, T_2, T_3),$$

where $T_i = T_i(s_1 s_2^2, s_2, s_2 s_3, b)$, $u = u(s_1 s_2^2, s_2, s_2 s_3, b)$, we obtain, using the equations (14)

$$\begin{array}{rcl} \bar{x}' &=& D_3 x\, \bar{z} + O_1 \\ \bar{y}' &=& D_2 y\left(s_2^{-1}\bar{y} + \frac{D_2 y}{D_2 z}s_2^{-2}\bar{z}\right) + D_3 y s_2^{-2}s_3^{-1}\bar{z} + \bar{a} + \frac{1}{2}D_{33}y\,\bar{z}^2 - \frac{D_2 y}{D_2 z}\bar{z}' s_2^{-1} + O_1, \\ \bar{z}' &=& D_2 z\,\bar{y} + D_2 y s_2^{-1}\bar{z} + D_3 z s_2^{-1}s_3^{-1}\bar{z} + \frac{1}{2}D_{33}z s_2\bar{z}^2 + s_2 O_1, \end{array}$$

where the derivatives of x, y and z are evaluated at $(s_1 s_2^2 s_3 T_1, T_2, T_3, u, b)$, and O_1 means first order in (s_1, s_2, s_3).

We note that if we substitute the value of $\bar{z}'$ in the second equation of the last system, we obtain

$$\bar{y}' = \left(D_3 y - \frac{D_2 y}{D_2 z} D_3 z\right) s_2^{-2} s_3^{-1} \bar{z} + \bar{a} + \frac{1}{2}\left(D_{33} y - \frac{D_2 y}{D_2 z} D_{33} z\right) \bar{z}^2.$$

We define

$$\bar{T}_i = \bar{T}_i(s_1, s_2, s_3) = T_i(s_1 s_2^2 s_3, s_2, s_2 s_3, v(s_1, s_2, s_3)), \quad i = 1, 2, 3,$$

$$\bar{u} = \bar{u}(s_1, s_2, s_3) = u(s_1 s_2^2 s_3, s_2, s_2 s_3, v(s_1, s_2, s_3)).$$

So, if $b = v + \bar{b} s_2 s_3$, and we consider T_i and u as a function of s_1, s_2 and s_3 we claim that

$$T_i - \bar{T}_i = \alpha_i \bar{b} s_2^2 s_3 + s_2^2 s_3\, O_1, \quad i = 1, 2, 3,$$

$$u - \bar{u} = \alpha_4 \bar{b} s_2^2 s_3 + s_2^2 s_3\, O_1,$$

where α_i, $i = 1, \ldots, 4$ are real numbers. Indeed, by the previous lemma we can write $T_i = 1 + s_2 h_i(b, s_1, s_2, s_3)$, $u = s_2 h_4(b, s_1, s_2, s_3)$, where h_i are smooth maps, and this implies the claim.

Moreover, taking into account that, by proposition 2.21, $D_3 z(0, 1, 1, a, b) = b$ and $\bar{T}_i = 1 + s_2 h_i$, we have:

$$D_2 y(s_1 s_2^2 s_3 T_1, T_2, T_3, u, b) - D_2 y(s_1 s_2^2 s_3 \bar{T}_1, \bar{T}_2, \bar{T}_3, \bar{u}, v) = s_2 s_3 O_0, \tag{17}$$

$$D_3 z(s_1 s_2^2 s_3 T_1, T_2, T_3, u, b) - D_3 z(s_1 s_2^2 s_3 \bar{T}_1, \bar{T}_2, \bar{T}_3, \bar{u}, v) = \bar{b} s_2 s_3 + s_2 s_3\, O_1 \tag{18}$$

and if we define $G := D_3 y D_2 z - D_2 y D_3 z$, then

$$G(s_1 s_2^2 s_3 T_1, T_2, T_3, u, b) - G(s_1 s_2^2 s_3 \bar{T}_1, \bar{T}_2, \bar{T}_3, \bar{u}, v) = D_5 G s_2 s_3 \bar{b} + s_2^2 s_3\, O_0,$$

where $D_5 G = D_5 G(s_1 s_2^2 s_3 \bar{T}_1, \bar{T}_2, \bar{T}_3, \bar{u}, v)$.

We know, by proposition 2.21, that $G(0, 1, 1, a, b) = 0$ for all a and b small enough. This means that $D_5 G = s_2(A_1 + O_1)$, where A_1 does not depend on (s_1, s_2, s_3). If we substitute d_1 in the fourth equation of the system (15) by its value obtained isolating it in the fifth equation, we see that

$$s_2^2 s_3 (1 - d_2 + 2 d_3) = 0.$$

Substituting d_i, $i = 1, 2, 3$, by their values, we get

$$G(s_1 s_2^2 s_3 \bar{T}_1, \bar{T}_2, \bar{T}_3, \bar{u}, v) = s_2^2 s_3 (A + O_1).$$

Hence, finally, we have

$$G(s_1 s_2^2 s_3 T_1, T_2, T_3, u, b) = s_2^2 s_3 (A + O_1),$$

where $A = A(\bar{b})$ does not depend on (s_1, s_2, s_3).

As $D_2 z(0, , 1, 1, 0, 0) \neq 0$ then this means that

$$\bar{y}' = A \bar{z} + \bar{a} + \frac{1}{2}(D_{33} y - \frac{D_2 y}{D_2 z} D_{33} z) \bar{z}^2 + O_1,$$

where $A = A(\bar{b})$.

On the other hand, computing explicitly the last condition of equation (15) we obtain:

$$D_3 z(s_1 s_2^2 s_3 \bar{T}_1, \bar{T}_2, \bar{T}_3, \bar{u}, v) = (2 + O_1) s_2 s_3 - s_3 D_2 y(s_1 s_2^2 s_3 \bar{T}_1, \bar{T}_2, \bar{T}_3, \bar{u}, v).$$

Combining the latter equation with (17) and (18) we get:

$$\bar{z}' = D_2 z(0,1,1,0,0)\, \bar{y} + (\bar{b} + 2)\bar{z} + O_1.$$

Summarizing, we have that the limit return map is

$$\begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} \mapsto \begin{pmatrix} D_3 x\, \bar{z} \\ A\bar{z} + \bar{a} + \frac{1}{2}(D_{33} y - \frac{D_2 y}{D_2 z} D_{33} z)\bar{z}^2 \\ D_2 z\, \bar{y} + (\bar{b} + 2)\bar{z} \end{pmatrix},$$

where $D_2 z \neq 0$, $B := D_3 x \neq 0$, $\beta := D_{33} y - \frac{D_2 y}{D_2 z} D_{33} z \neq 0$ and they are evaluated at $(0,1,1,0,0)$.

Using the change $\bar{y} = AB^{-1}\bar{x} + \bar{a} + \beta B^{-2}\bar{x}^2 + \tilde{y}$, and performing a change of scale and a reparametrization, we obtain:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} z \\ 0 \\ a + bz + x^2 \end{pmatrix}. \tag{19}$$

Type II

We use the same notation as in the previous case. In this case, we consider the following change of variables and reparametrization for the map $f^n_{a,b}(t_1, t_2, t_3) = \mathbf{x}(\lambda_1^n t_1, \lambda_2^n t_2, \lambda_3^n t_3, a, b)$:

$$\begin{aligned} t_1 &= T_1 + \lambda_2^{-n}\lambda_3^{-n}\bar{x} \\ t_2 &= \lambda_2^{-n} T_2 + \lambda_2^{-2n}\lambda_3^{-n}\bar{y} \\ t_3 &= \lambda_3^{-n} T_3 + \lambda_2^{-2n}\lambda_3^{-2n}\bar{z} - \frac{D_2 y}{D_3 y}\lambda_2^{-n}\lambda_3^{-2n}\bar{y} \\ a &= u + \lambda_2^{-2n}\lambda_3^{-2n}\bar{a} \\ b &= v + \lambda_3^{-n}\bar{b} \end{aligned}$$

where $D_2 y$ and $D_3 y$ are evaluated at $(\lambda_1^n T_1, T_2, T_3, u, b)$, $T_i = T_i(\lambda_1^n, \lambda_2^{-n}, \lambda_3^{-n}, b)$, for $i = 1,2,3$, $u = u(\lambda_1^n, \lambda_2^{-n}, \lambda_3^{-n}, b)$, $v = v(\lambda_1\lambda_2\lambda_3, \lambda_2^{-n}, \lambda_2^n\lambda_3^{-n})$. Using similar estimates to the previous case, we can see that the limit return map is:

$$\begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} \mapsto \begin{pmatrix} (D_2 x - \frac{D_2 y}{D_3 y})\bar{y} \\ D_3 y \bar{z} \\ \bar{a} + \bar{z} + A\bar{y}^2 \end{pmatrix},$$

where $A = \frac{1}{2(D_3 y)^2}(D_{33} z (D_2 y)^2 - 2D_{23} z D_2 y D_3 y + D_{33} z (D_3 y)^2) \neq 0$. It is also easy to see that this map is conjugate to the map (19).

Then, we have proven the following proposition:

Proposition 4.7 *Let $\{f_{a,b}\}_{a,b}$ be a two-parameter family of three dimensional diffeomorphisms having a dissipative fixed point p_0 for $(a,b)=(0,0)$, with eigenvalues λ_1, λ_2, λ_3 such that: $|\lambda_1|<1$, $|\lambda_3|>|\lambda_2|>1$. Suppose that the fixed point has a generalized homoclinic tangency n p_1 which unfolds generically. Then, after a change of variables and a reparametrization depending on n, the map $f^n_{a,b}$ for (a,b) near $(0,0)$ and near p_1 has a limit return map when $n\to\infty$ of the form:*

$$(\bar{x},\bar{y},\bar{z})\mapsto(0,\bar{a}+\bar{b}\bar{y}+\bar{z}^2,\bar{y}),$$

where $\bar{a}$ and $\bar{b}$ are the new parameters after the reparametrization.

Finally, we are in the conditions we needed to prove theorem 2. We perform the proof of this theorem item by item:

1. We know that the Bogdanov-Takens bifurcations exist. In order to see that are generic and unfolds generically, we only have to see that this is true for the limit return map, and this is an easy computation.
2. This is consequence of the properties of the BT bifurcation.
3. Again a consequence of the existence of BT bifurcations.
4. Working with the limit return map, it is possible to obtain values of the parameters for which there is a quadratic homoclinic tangency of a sectionally dissipative fixed point for the return map, if n is large enough. This can be made using the results concerning the existence of homoclinic tangencies near a Bogdanov-Takens bifurcation (see again Broer et al. [3]). Then we can apply the results of Viana [16].
5. Proceed as before but now applying Palis and Viana[12].

5 Quadratic maps in dimensions 2 and 3

The objective of this section is to characterize the quadratic maps $f:\mathbb{R}^i\longrightarrow\mathbb{R}^i$, with $i=2,3$ such that $f\circ f$ is quadratic, and, if $i=3$ f has also constant jacobian. We will see that any map that does not have any invariant linear foliation is linearly conjugate to the map $(x,y)\longrightarrow(a+bx-y^2,x)$, in the first case, and to the map $(x,y,z)\longrightarrow(a+bx+z-y^2,x,cy)$, in the second case. In the two cases we also suppose that the jacobian of the map is not identically equal to zero. This proves theorem 3.

Case i=2

We shall suppose that $\det Df$ is not identically equal to zero. We can write the map as $f(x,y)=v_0+xv_1+yv_2+x^2v_{11}+y^2v_{22}+2xyv_{12}$, where $v_i,v_{ij}\in\mathbb{R}^2$. As $f\circ f$ is a quadratic map, then the map $P(x,y)=\det Df(f(x,y))\det Df(x,y)$ has to be at most quadratic. We can write

$$P(x,y)=(Q(x,y)+P_1(x,y))\det Df(f(x,y)),$$

where P_1 is at most linear and

$$Q(x,y)=4(\det(v_{11},v_{12})x^2+\det(v_{12},v_{22})y^2+\det(v_{11},v_{22})xy).$$

Then we have two possibilities: Either

$$\det(v_{11}, v_{12}) = \det(v_{12}, v_{22}) = \det(v_{11}, v_{22}) = 0$$

or there is a nonzero constant C such that $\det Df(f(x,y)) = C$ for all $(x,y) \in \mathbb{R}^2$. As a consequence of the inverse function theorem, $\det Df(x,y) = C$, for (x,y) belonging to some open set. This means that the jacobian is constant in the whole $\mathbb{R}$. But then we arrive to the same condition on the vectors v_{ij} as before.

It is easy to see that by means of a linear change of coordinates, we can suppose that the map f satisfies:

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_0 + a_1x + a_2y + a_{11}x^2 + 2a_{12}xy + a_{22}y^2 \\ b_0 + b_1x + b_2y \end{pmatrix}.$$

If we impose again that $f \circ f$ is quadratic, we get $a_{11} = a_{12} = 0$. As $b_1 \neq 0$ we can perform the change $x = \bar{x} + \alpha\bar{y} + \beta$, $y = \bar{y} + \gamma$, with suitable constants α, β and γ, arriving to the map that we wanted.

Case i=3

In this case, it is possible to see (Bosch and Tatjer [2], Fornaess and Wu [4]) that any map with constant jacobian and without linear invariant foliations, is conjugate to one of the following maps:

$$\begin{aligned} x' &= a_0 + a_1x + a_2y + a_3z + a_{11}x^2 + a_{22}y^2 + 2a_{12}xy \\ y' &= b_0 + b_1x + b_2y + b_{11}x^2 \\ z' &= c_0 + c_1x \end{aligned} \tag{20}$$

$$\begin{aligned} x' &= a_0 + a_1x + a_2y + a_3z + a_{11}x^2 + a_{22}y^2 + 2a_{12}xy \\ y' &= b_0 + b_1x + b_2y + b_{22}y^2 \\ z' &= c_0 + c_2y \end{aligned} \tag{21}$$

$$\begin{aligned} x' &= a_0 + a_1x + a_2y + a_3z + a_{11}x^2 + a_{33}z^2 + 2a_{13}xz \\ y' &= b_0 + b_1x + b_3z + b_{11}x^2 \\ z' &= c_0 + c_1x \end{aligned} \tag{22}$$

$$\begin{aligned} x' &= a_0 + a_1x + a_2y + a_3z + a_{22}y^2 + a_{33}z^2 + 2a_{23}yz \\ y' &= b_0 + b_1x + b_3z + b_{22}y^2 \\ z' &= c_0 + c_2y \end{aligned} \tag{23}$$

If we impose that $f \circ f$ is quadratic, it is easy to see that the last case has a linear invariant foliation, and the other cases, after a suitable linear change, can be always be written as:

$$\begin{aligned} x' &= a_0 + a_1x + a_2y + a_3z + a_{11}x^2 + a_{22}y^2 + 2a_{12}xy \\ y' &= b_0 + b_1x + b_2y \\ z' &= c_0 + c_1x + c_2y \end{aligned} \tag{24}$$

Indeed, in equation (20) we have that $b_{11} = 0$ or $a_{11} = a_{12} = a_{22} = 0$. In the first case we have the result we want. In the second case, we perform de change

$$\bar{x} = y - a_3/a_2z, \quad \bar{y} = x, \quad \bar{z} = z.$$

Note that as there is no invariant linear foliation we have that $a_2 \neq 0$ in this case.

In equation (21) we have that $b_{22} = 0$.

In equation (22) we have $b_{11} = 0$ or $a_{11} = a_{33} = a_{13} = 0$. In the first case we consider the change $(\bar{x}, \bar{y}, \bar{z}) = (x, z, y)$. The second case is the same as the second case corresponding to equation (20).

In equation (23) we have $b_{22} = 0$. Suppose that there is not an invariant linear foliation. Then, in particular, $b_1 \neq 0$. We can perform the change $(\bar{x}, \bar{y}, \bar{z}) = (x - (a_1/b_1)y, y, z)$. Then we obtain a map like (23), but with $a_1 = 0$. But then we have again an invariant linear foliation, which contradicts the hypothesis.

Coming back to equation (24), it is immediate that $a_{11} = a_{12} = 0$. If we perform a change

$$\begin{aligned} x &= \bar{x} + \alpha_1 \bar{y} + \alpha_0 \\ y &= \bar{y} + \beta_0 \\ z &= \bar{z} + \gamma_1 \bar{y} + \gamma_0 \end{aligned}$$

and after a change of scale, we obtain the desired map.

Remark 5.1 *As a consequence of this theorem, we have that a quadratic map f without an invariant linear foliation, with constant jacobian and satisfying that $f \circ f$ is quadratic has a quadratic inverse. This means that these maps belong to the set of quadratic maps with constant jacobian and with quadratic inverse. Compare with Lomelí and Meiss [7].*

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References

[1] A. Banyaga, R. de la Llave, C. E. Wayne: Cohomology equations near hyperbolic points and geometric versions of Sternberg linearization theorem. *J. Geom. Anal.* **6** (1996), no. 4, 613–649.

[2] M. Bosch, J. C. Tatjer: Difeomorfismos cuadráticos de $\mathbb{R}^n$ con jacobiano constante. Actas VIII CEDYA, Universidad de Santander (1985). To appear.

[3] H. Broer, R. Roussarie, C. Simó: Invariant circles in the Bogdanov-Takens bifurcation for diffeomorphisms. *Ergodic Theory and Dynam. Systems* (1996), **16**, 1147-1172.

[4] J. E. Fornaess, H. Wu: Classification of degree 2 polynomial automorphisms of $\mathbb{C}^3$. *Publicacions Matemàtiques,* **42**, 1998, pp. 195-210.

[5] S. V. Gonchenko, D. Tuarev, L. P. Shil'nikov: Dynamical Phenomena in multidimensional systems with a structurally unstable homoclinic Poincaré curve. *Russian Acad. Sci. Dokl. Math.*, **47**(3) (1993), 410-415.

[6] S. V. Gonchenko, L. P. Shil'nikov, D. Tuarev: Quasiattractors and homoclinic tangencies. *Computers Math. Applic.* **34**(2-4) (1997), 195-227.

[7] H. Lomelí, J. D. Meiss: Quadratic Volume Preserving Maps. *Nonlinearity* **11** (1998), 557-574.

[8] L. Mora, M. Viana: Abundance of strange attractors. *Acta Math.* **171**, 1-71 (1993).

[9] S. Newhouse: The abundance of wild hyperbolic sets and nonsmooth stable sets for diffeomorphisms. *Publ. Math. I. H. E. S.* **50** (1979), 101-151.

[10] J. Palis, W. de Melo: *Geometric Theory of Dynamical Systems. An Introduction.* Springer-Verlag, New-York (1982).

[11] J. Palis, F. Takens: *Hyperbolicity & sensitive chaotic dynamics at homoclinic bifurcations.* Cambridge University Press, Cambridge (1993).

[12] J. Palis, M. Viana: High dimension diffeomorphisms displaying infinitely many sinks. *Ann. Math.* **140** (1994), 91-136.

[13] N. Romero: Persistence of homoclinic tangencies in higher dimensions. *Ergodic Theory and Dynam. Systems* **15** (1995), 735-757.

[14] J. C. Tatjer: Codimension two bifurcations near a cubic homoclinic tangency. *ECIT 91* (Singapore: World Scientific) pp. 308-318.

[15] J. C. Tatjer, C. Simó: Basins of attraction near homoclinic tangencies. *Ergodic Theory Dynam. Systems* **14** (1994), no. 2, 351-390

[16] M. Viana: Strange attractors in higher dimensions. *Bol. Soc. Bras. Mat.* **24**, 13-62 (1993).