SMOOTH DEPENDENCE OF THERMODYNAMIC LIMITS OF SRB-MEASURES

MIAOHUA JIANG AND RAFAEL DE LA LLAVE

ABSTRACT. The thermodynamic limits of SRB-measures for uniformly hyperbolic sets are smoothly dependent on the map in an appropriate functional space.

1. Introduction

In a recent paper [Rue97], Ruelle showed that, in an appropriate sense, the SRB measure μ_f of an Axiom A map f depends differentiably on the map f and computed explicit formulas for the derivative.

Since the motivation of that paper was to serve eventually as a justification of statistical mechanics out of equilibrium, it seems natural to extend these results to systems of many particles and study the dependence of the derivatives of the measures (out of which one can compute response functions, entropy production, etc.) as the number of particles, increases and specially, to study whether there is a thermodynamic limit as the number of particles tend to infinity. The existence of this thermodynamic limit of the derivatives of the measures implies that one can define the thermodynamic response functions.

In this paper we establish differentiability of thermodynamic limits of SRB-measures for lattice systems that have been often used as models in statistical mechanics ([Sim93]). We call attention to the fact that for these systems not only it makes sense to consider the limit of physical quantities as the number of particles tends to infinity, but also one can make sense of the system with an infinite number of particles as a dynamical system on a Banach manifold. Hence, we will show that the systems

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we consider (uniformly hyperbolic systems) have thermodynamic properties slightly away from equilibrium. Besides the uncontroversial fact that hyperbolic systems are a very natural first step to study thermodynamic behavior of deterministic systems [Kry79], we note that it has been argued by Ruelle and Gallavotti-Cohen that the thermodynamic properties developed under the mathematical hypothesis of hyperbolicity should hold for all the physically relevant systems. This hypothesis is called the "chaotic hypothesis" [Gal98].

We assume that to each point of $\mathbb{Z}^d$ we associate a dynamical system given by a phase space manifold M and a map f acting on M. So, the phase space of the whole system is $\mathcal{M} = \otimes_{\mathbb{Z}^d} M$ and we can define a map on $\mathcal{M}$ by $F = \otimes_{\mathbb{Z}^d} f$. These F are referred to as the *uncoupled systems*. We will be concerned with modifications of these uncoupled systems, which are obtained by small local couplings and that are also translation invariant (see later precise definitions).

Recently, it has been shown [JP98] that when f is Axiom A, one can define the SRB measures for such systems. The measure of the infinite system can be obtained by taking the limit (in an appropriate sense) of the SRB measures of finite dimensional systems. Moreover, the SRB measures of the infinite system also satisfy a variational principle that behaves naturally with respect to finite dimensional approximations. Even though some of the characteristics of SRB measures in finite dimensions, such as the approximation by periodic orbits [Bow75], do not even make sense, a certain number of them are still true. Notably, in [JP98] it was established that one can have a variational principle for the thermodynamic limit of SRB measures of finite subystems (hence, we will refer to these limits as the SRB measures of the infinite system) and, indeed, a significant part of thermodynamic formalism still holds. This characterization of SRB measures as equilibrium measures of a potential will play a very important role in our study. One question that, to the best of our knowledge remains open is whether the characterization of SRB measures of attractors as the weak limit of iterated Lebesgue measures carries through for the infinite dimensional system.

In this paper, we will show that indeed, the SRB measures of the infinite system depend smoothly on the map when the space of maps is given an appropriate topology that incorporates smoothness, locality of couplings, and translation invariance. Hence, the derivatives will also enjoy these physically natural properties. We will also show in a subsequent paper that the formulas for derivatives in [Rue97] can be adapted to the infinite dimensional system. Moreover, these derivatives are the limit of the derivative operators on finite number of sites as we take the number of sites considered to infinity.

The availability of thermodynamic formalism allows us to show that there are thermodynamic limits of many dynamical quantities such as the metric entropy or the dimension of basic sets. We will also show in this paper that these thermodynamic limits depend smoothly on the map.

We note that the smooth dependence of these dynamical quantities for finite dimensional systems have been considered in several places ([Mn90, KKPW90, Wei92]).

The proof of our results will be as follows: It turns out that much of the geometric theory of hyperbolic systems (structural stability, persistence of hyperbolicity and the parametric versions of them) goes through for diffeomorphisms of Banach manifolds.

Given a map Φ in a C^1 neighborhood of F, we can find a unique homeomorphism h close to the identity that satisfies $\Phi\circ h = h\circ F$. This h will depend smoothly on Φ (The proof of this result in finite dimensional case was done in [dlLMM86]). Moreover, the invariant exponential splitting of the tangent bundle, $T_x\mathcal{M} = \mathcal{E}^u_\Phi(\bar{x}) \oplus \mathcal{E}^s_\Phi(\bar{x})$ satisfies that $\mathcal{E}^u_\Phi(h(\bar{x}))$ and $\mathcal{E}^s_\Phi(h(\bar{x}))$ depend smoothly on Φ (This was proved in [Mn90] for finite dimensional systems). Modifying the proofs indicated above, we will show that, if Φ has the structure of local couplings, then h, $\mathcal{E}^u_\Phi(h(\bar{x}))$, $\mathcal{E}^s_\Phi(h(\bar{x}))$ and their derivatives with respect to Φ also have similar properties and they depend smoothly on the map when we give them a topology based on a norm that incorporates locality, etc.

By using the result above systematically, we can reduce the study of the SRB measure for Φ to the study of an equilibrium measure for F with a potential function that depends on Φ. As it is well-known, in finite dimensions, an appropriate potential is just the logarithm of the Jacobian of Φ restricted along the unstable manifold. This potential has the advantage that, besides having an SRB measure as its equilibrium, it admits a clear geometric interpretation. Nevertheless, it is not suitable for the considerations of the limits as the number of sites goes to infinity since it does not have a limit. In order to be able to take limits as the size of the system grows, one has to take advantage of the group structure of the underlying lattice $\mathbb{Z}^d$. The thermodynamic limit of SRB measures on finite dimensional spaces with respect to $\mathbb{Z}$-actions becomes an equilibrium measure for a meaningful potential with respect to a $\mathbb{Z}^{d+1}$-action induced by the dynamics on the infinite dimensional space and the translation maps of the lattice $\mathbb{Z}^d$. An appropriate potential φ_Φ suitable for considerations in infinite dimensions is constructed in [JP98]. We will show that, under the conditions of our theorem, the potential function φ_Φ depends smoothly on Φ.

The rest of the argument is a thermodynamic argument showing that the pressure depends smoothly on the potential. This result is well-known in finite dimensions since the underlying lattice gas model is one dimensional and thus, the pressure is analytic. In the infinite dimensional system case, the underlying lattice gas model is higher dimensional. The C^∞-smoothness of the pressure for the potential functions was proved in [BK97]. Hence, in our situation, $P(\varphi_\Phi)$ the pressure of φ_Φ depends smoothly on Φ.

The main result follows since the SRB measure is the derivative functional of the pressure and the metric entropy can be expressed in terms of the pressure and the SRB measure.

Since the precise definition of differentiability requires definitions of spaces of smooth diffeomorphisms with local coupling structures, translation invariance, we postpone the precise formulation of our results till these technical definitions are introduced.

2. Preliminaries

In this section, we will collect some results from smooth ergodic theory and from coupled map lattices. We will use them to establish the notation that we will use through out the paper.

2.1. **SRB-measures for finite dimensional maps.** Let M be a smooth compact Riemannian manifold and f be a C^r-diffeomorphism of M, $r > 1$. We assume that f possesses a locally maximal hyperbolic set Λ, i.e., f is uniformly hyperbolic on Λ and there exists an open neighborhood $U \supset \Lambda$ that does not contain any larger invariant hyperbolic set. Two important particular cases of this situation are when $\overline{f(U)} \subset U, \Lambda = \cap_{n\in\mathbb{N}} f^n(U)$, i.e., Λ is an attractor and, when $\Lambda = M$, i.e., the system is Anosov. Note that strictly speaking, an Anosov system is a particular case of an attractor.

When Λ is an attractor, a Sinai-Ruelle-Bowen measure μ for f can be described by the following limit:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \int g(f^k(x))dm = \int g d\mu,$$

where dm denotes the Lebesgue measure and the equality holds for any continuous function g on M [Bow75]. More generally, when Λ is a locally maximal hyperbolic set f is topologically transitive on Λ, the SRB measure μ is the unique invariant measure (called equilibrium state) at which the functional (called the pressure of ν)

$$h_\nu(f) + \int \varphi_f(x) d\nu$$

attains its maximal value, which is equal to the topological pressure $P(\varphi_f)$:

$$P(\varphi_f) = h_\mu(f) + \int \varphi_f(x) d\mu, \tag{1}$$

where $h_\mu(f)$ is the measure theoretical entropy, and $\varphi_f(x) = -\log J^u f(x)$ ($J^u f(x)$ is the Jacobian of the restriction of f along the unstable manifold at x). The fact that μ satisfies the equation (1) is called the variational principle.

To ensure uniqueness of SRB-measures for direct product spaces, we will further assume that f is topologically mixing on Λ. Note that a direct product of two topologically transitive systems is not necessarily topological transitive and a direct product of two topologically mixing ones is still topologically mixing.

2.2. **Coupled map lattices.** We will consider a compact manifold M of dimension n and we will assume that there is a C^r diffeomorphism f defined on M, $r \in \mathbb{N} \cup \{\infty\}$.

We will also assume that there is a locally maximal hyperbolic set $\Lambda \subset M$. That is, we can find $\gamma < 1$ so that for every $x \in \Lambda$,

$$
\begin{aligned}
(2) \qquad & T_x M = E^u_f(x) \oplus E^s_f(x) \\
& Df(x) : E^s_f(x) \to E^s_f(f(x)); \quad Df(x) : E^u_f(x) \to E^u_f(f(x))
\end{aligned}
$$

and that, for an appropriately chosen C^∞ Riemannian metric g (called adapted metric), we have for all $x \in \Lambda$

$$
(3) \qquad ||Df(x)|_{E^s_f(x)}|| \le \gamma; \quad ||Df(x)^{-1}|_{E^u_f(f(x))}|| \le \gamma
$$

We will denote by d the distance in M induced by the metric g.

The phase space of lattice maps will be $\mathcal{M} = \otimes_{i \in \mathbb{Z}^d} M$ endowed with some structures that we now describe.

We define a distance in $\mathcal{M}$ by

$$
(4) \qquad \rho(\bar{x}, \bar{y}) = \sup_{i \in \mathbb{Z}^d} d(x_i, y_i)
$$

where $\bar{x} = (x_i)$ and $\bar{y} = (y_i)$ are two points in $\mathcal{M}$ and d is the Riemannian distance on M.

As it is well-known, $\mathcal{M}$ can be given a structure of a Banach manifold with a Finsler metric as follows. We will consider

$$
T_{\bar{x}}\mathcal{M} = \oplus_{i \in \mathbb{Z}^d} T_{x_i} M = \{(v_i)_{i \in \mathbb{Z}_d} | v_i \in T_{x_i} M, ||\bar{v}|| \equiv \sup_i |v_i| < \infty\},
$$

(that is, we model $T\mathcal{M}$ on $\ell^\infty(\mathbb{R}^n)$, so that, in particular, it is not a separable space). This defines a Finsler metric on $T\mathcal{M}$ which is compatible with (4).

Note that we can still define the exponential map using the metric g on the manifold M. Given a vector $\bar{v} \in T_{\bar{x}}\mathcal{M}$, in the ith copy of M, we follow the geodesic flow for the metric g starting at x_i, in the direction of v_i for a unit of time. We note that using the exponential map, we can give $\mathcal{M}$ the structure of a Banach manifold modeled on

$\ell^\infty(\mathbb{R}^n)$. Since all components of the exponential map are uniformly C^∞ differentiable maps, the exponential map defined above is a C^∞ chart from a neighborhood of $T_{\bar{x}}\mathcal{M}$ to a neighborhood of the manifold $\mathcal{M}$. (Of course, since we define the topology on $\mathcal{M}$ through this charts, the fact that a neighborhood is covered is automatic. The only thing that needs to be checked is that the transitions from charts to charts are C^∞ but this follows easily from the fact that the geodesic flow is uniformly C^r for all r.)

Note that since the geodesic flow on each coordinate is independent, the exponential map maps a ball in $T_{\bar{x}}\mathcal{M}$ onto a product of neighborhoods in M of each point x_i.

On the Banach manifold $\mathcal{M}$, we can define the *direct product* map or *uncoupled map* by $F = \otimes_{i\in\mathbb{Z}^d} f_i$. The map is C^r on the manifold $\mathcal{M}$.

Recall that, since all the component manifolds M are compact and the maps are identical copies, we have that all the maps f_i have derivatives of order up to r uniformly bounded and that the derivatives of order r have a modulus of continuity that is independent of i. The map F not only is a C^r map, but the r derivative is uniformly continuous.

More importantly, the map F possesses an infinite-dimensional hyperbolic set $\Delta_F = \otimes_{i\in\mathbb{Z}^d}\Lambda_i$, where f_i and Λ_i are copies of f and Λ, respectively. It is easy to check that setting $\mathcal{E}^s_F(\bar{x}) = \oplus_{i\in\mathbb{Z}^d} E^s_f(x_i)$, $\mathcal{E}^u_F(\bar{x}) = \oplus_{i\in\mathbb{Z}^d} E^u_f(x_i)$, then,

$$T_{\bar{x}}\mathcal{M} = \mathcal{E}^s_F(\bar{x}) \oplus \mathcal{E}^u_F(\bar{x})$$

is a hyperbolic splitting in Δ_F and, for every $\bar{x} \in \Delta_F$,

$$\begin{gathered} DF(\bar{x})\mathcal{E}^s_F(\bar{x}) = \mathcal{E}^s_F(F(\bar{x})); \quad DF(\bar{x})\mathcal{E}^u_F(\bar{x}) = \mathcal{E}^u_F(F(\bar{x})) \\ ||DF(\bar{x})|_{\mathcal{E}^s_F(\bar{x})}|| \le \gamma; \quad ||DF(\bar{x})^{-1}|_{\mathcal{E}^u_F(F(\bar{x}))}|| \le \gamma \end{gathered}$$

where the γ is the same number as in (3).

We call attention to the fact that, even if the definition of hyperbolic systems and splittings goes through without much problems, the definition of transitivity and the properties of approximation by periodic orbits are not quite straightforward to transport to the infinite dimensional context since, with the Banach manifold that we introduced, is not separable.

We recall the definitions of some objects discussed in [JP98].

Let S denote the spatial translations on $\mathcal{M}$ induced by the translations on the integer lattice $\mathbb{Z}^d$, i.e., for any $k \in \mathbb{Z}^d$ and $\bar{x} = (x_i) \in \mathcal{M}$, $S^k(\bar{x}) = (x_{i+k})$. Let the map G be a C^2-perturbation of the identity map on $\mathcal{M}$. G is said to be spatially translation invariant if $G \circ S = S \circ G$. It is said to have *short range* property if G, written in the form $G = (G_i)_{i\in\mathbb{Z}^d}$, where $G_i : \mathcal{M} \to M_i$, has the following property: there exist a *decay constant* θ, $0 < \theta < 1$ and a constant $C > 0$ such that for any fixed $k \in \mathbb{Z}^d$ and any points $\bar{x} = (x_j), \bar{y} = (y_j) \in \mathcal{M}$ with $x_j = y_j$ for all $j \in \mathbb{Z}^d, j \neq k$,

$$d(G_i(\bar{x}), G_i(\bar{y})) \leq C\theta^{|i-k|} d(x_k, y_k).$$

Define $\Phi = G \circ F$ (or equivalently, $\Phi = F \circ G$, since F is also a diffeomorphism). The map Φ is a perturbation of F. The infinite-dimensional dynamical system $(\mathcal{M}, (\Phi, S))$ is called a *coupled map lattice*. If $G = \mathrm{Id}$, the lattice is called *uncoupled* . When G is spatially translation invariant, Φ satisfies the same property and the pair (Φ, S) generates a $\mathbb{Z}^{d+1}$-action on $\mathcal{M}$.

Fix a point $\bar{x}^* \in \Delta_\Phi$, and a finite volume $V \subset \mathbb{Z}^d$, the map $\Phi_V : x_V \to \Phi_V(x_V)$ on $M_V = \otimes_{i\in V} M_i$ is defined coordinatewise by

$$\big(\Phi_V(x_V)\big)_i = \big(\Phi((x_V, x^*|_{\widehat{V}})\big)_i, \quad i \in V,$$

where the point $(x_V, x^*|_{\widehat{V}})$ denotes an element in $\mathcal{M}$ whose restrictions to V and its complement $\widehat{V}$ are x_V and $x^*|_{\widehat{V}}$, respectively.

The map Φ_V is a diffeomorphism of M_V when the perturbation G is sufficiently close to identity and it is C^1-close to the diffeomorphism $F_V = \otimes_{i\in V} f$. By the structural stability theorem Φ_V possesses a hyperbolic set Δ_{Φ_V} since F_V has a hyperbolic set $\Delta_{F_V} = \otimes_{i\in V}\Lambda$. There exists a conjugating homeomorphism $h_V : \Delta_{F_V} \to \Delta_{\Phi_V}$, $\Phi_V \circ h_V = h_V \circ F_V$.

The maps Φ_V and h_V provide finite-dimensional approximations for the infinite-dimensional maps Φ and h, respectively.

Let μ_V be the SRB-measure on the hyperbolic set Δ_{Φ_V} for the map Φ_V. Then, it is shown in [JP98] that the measure μ_V "weakly converges" to a measure μ_Φ on Δ_Φ. The measure μ_Φ is invariant and exponentially mixing under Φ and spatial translations S. It also satisfies the variational principle:

$$P_\tau(\varphi_\Phi) = h_{\mu_\Phi}(\tau) + \int \varphi_\Phi d\mu_\Phi, \tag{5}$$

where τ denotes the $\mathbb{Z}^{d+1}$ action on Δ_Φ induced by Φ and S, $P_\tau(\varphi_\Phi)$ is the topological pressure for the potential function φ_Φ, and $h_{\mu_\Phi}(\tau)$ is the measure theoretical entropy of μ_Φ with respect to τ. This "weak convergence" is in a rather special sense since the measures are defined on the different spaces and we need to consider the convergence on observables that admit natural restrictions to smaller systems. The limiting process is similar to the one in the thermodynamic limit of Gibbs ensembles on lattices when the underlying finite volume tends to infinity. We call μ_Φ the SRB-measure of the coupled map lattice (Φ, S). The main purpose of this paper is to show that the relation $\Phi \to \mu_\Phi$ is differentiable in a proper setup.

3. Structural stability and smooth dependence on the perturbation

3.1. **Introduction.**

In this section, we will study structural stability and properties of the hyperbolic splitting for lattice systems.

As it turns out, one can generalize strategies that worked for finite dimensional systems but we also have to pay attention to the spatial structure and prove not only regularity properties, but also properties of decay. Part of this strategy have already been considered in [BS88] [PS91] [Jia95].

For our purposes, it seems that the most convenient proof of structural stability is that of [Mos69], as modified in [dlLMM86]. This proof reduces structural stability to an application of the implicit function theorem and then, all the work goes into establishing that the operator we consider is differentiable. In our case, we will obtain the spatial properties by introducing spaces that incorporate these local properties.

After we study the dependence of structural stability on the perturbation, we will study the properties of the invariant hyperbolic splitting and the dependence on the perturbation. We will do that by formulating the invariance of the splitting as a fixed point problem in an appropriate space of sections.

3.2. **Geometry of lattice bundles and spaces of sections.**

Definition 3.1. *We will consider the following spaces of vector fields (i.e. sections of the tangent bundle*

$$C^0(\mathcal{M}, T\mathcal{M}) = \{\bar{v}(\bar{x}) : \ \bar{x} \to v(\bar{x}) \text{ is continuous}, ||\bar{v}||_{C^0} \equiv \sup_{i\in\mathbb{Z}^d} \sup_{\bar{x}\in\mathcal{M}} |v_i(\bar{x})| < \infty\} \tag{6}$$

For $0 < \alpha < 1$, given a trivialization of TM by a finite number of coordinate charts, $\{P_1, \dots, P_l\}$ we denote

$$C^\alpha(\mathcal{M}, T\mathcal{M}) = \{\bar{v}(\bar{x}) : \ ||\bar{v}||_{C^\alpha} \equiv \max(||\bar{v}||_{C^0}, \sup_{i\in\mathbb{Z}^d} \max_k \sup_{\bar{x}\neq\bar{y}\in\mathcal{M}, x_i, y_i \in P_k} \frac{|v_i(\bar{x}) - v_i(\bar{y})|}{d(\bar{x},\bar{y})^\alpha} < \infty\} \tag{7}$$

Following the usual convention, we will also allow $\alpha = \mathrm{Lip}$ in the definition above. In that case, we set $\alpha = 1$ in (7). It is convenient and standard to think of Lip *as a special symbol that* $\mathrm{Lip} < 1$, $\mathrm{Lip} > \alpha$, $\forall \alpha \in \mathbb{R}^+ < 1$ *and* Lip *enters in arithmetic expressions as* 1. *For convenience, we assume that the maximum distance on M is* 1. *Thus, we have $\|v\|_{C^{\alpha_1}} \le \|v\|_{C^{\alpha_2}}$ when $0 \le \alpha_1 \le \alpha_2$.*

Remark 3.2. We note that the definition of the Hölder norm depends on the choice of the trivialization. Nevertheless, any two of these norms are equivalent. Note that we choose the trivialization in M, which is finite dimensional and compact, and not in $\mathcal{M}$.

Definition 3.3. *We will denote by C^0_S, C^α_S the subspaces of the spaces $C^0(\mathcal{M}, T\mathcal{M})$, $C^\alpha(\mathcal{M}, T\mathcal{M})$, respectively, that also satisfy*

$$v(\bar{x}) \in T_{\bar{x}}\mathcal{M}.$$

Before we describe the space of diffeomorphisms that capture the idea of localized (or *short-ranged* [PS91, Jia95]) interactions, we will need to introduce a technical proposition that will simplify some of the arguments in the definitions as well as the proof of the structural stability.

Definition 3.4. *We say that a positive valued function* $\Gamma : \mathbb{Z}^d \to \mathbb{R}^+$ *is a decay function when:*

1. $\sum_{i\in\mathbb{Z}^d} \Gamma(i) < \infty$,
2. $\sum_{j\in\mathbb{Z}^d} \Gamma(i-j)\Gamma(j-k) \leq \Gamma(i-k)$

The importance of decay functions is that infinite matrices $A = (a_{ij})_{i,j\in\mathbb{Z}^d}$ endowed with the norm

$$\|A\| = \sup_{i,j} |a_{ij}|\Gamma^{-1}(i-j)$$

form a Banach algebra. This, in turn, will make it possible to define spaces of maps that behaves well under composition. Roughly speaking, our spaces of diffeomorphisms will contain maps where the influence of the i^{th} coordinate of the argument on the j^{th} coordinate of the map is bounded by a decay function.

Concrete examples of decay functions are the following.

Proposition 3.5. *Given* $\alpha > d$, $\theta \geq 0$, *for some* $a > 0$ *(depending on* α, θ, *the function defined by*

$$\Gamma(i) = \begin{cases} a|i|^{-\alpha}\exp(-\theta|i|) & i \neq 0, \\ a & i = 0, \end{cases}$$

is a decay function.

Proof. When $i = k$ the desired result amounts to $\sum_j \Gamma^2(i-j) \leq a$. Since the left hand side is a convergent sum multiplied by a^2, this can always be achieved taking a sufficiently small.

When $i \neq k$

(8)

$$\begin{aligned} &e^{\theta|i-k|}\sum_j \Gamma(i-j)\Gamma(j-k) \\ &= a^2 \sum_{\substack{i\neq j \\ j\neq k}} |i-j|^{-\alpha}|j-k|^{-\alpha}e^{-\theta\left(|i-j|+|j-k|-|i-k|\right)} + 2a^2\Gamma(i-k)e^{\theta|i-k|} \end{aligned}$$

It suffices to show that the right hand side of (8) is smaller than $a|i-k|^{-\alpha}$. Note that the second term in (8) is $2a^2|i-k|^{-\alpha}$.

Since $e^{-\theta(|i-j|+|j-k|-|i-k|)} \le 1$, we can bound the first term of (8) by:

$$a^2 \sum_{\substack{i\neq j\\ j\neq k}} |i-j|^{-\alpha}|j-k|^{-\alpha}$$

We consider the sets

$$\mathcal{B} = \{j \in \mathbb{Z}^d - \{i,k\} : \ |i-j| \le |j-k|\}$$

and

$$\mathcal{B}^c = \{j \in \mathbb{Z}^d - \{i,k\} : \ |i-j| > |j-k|\}.$$

Since $\max(|i-j|,|j-k|) \ge |i-k|/2$ for $j \in \mathcal{B}$, we have $|j-k|^{-\alpha} \le 2^\alpha |i-k|^{-\alpha}$. Hence,

$$|i-j|^{-\alpha}|j-k|^{-\alpha} \le 2^\alpha |i-j|^{-\alpha}|i-k|^{-\alpha}.$$

Similarly, for $j \in \mathcal{B}^c$ we have

$$|i-j|^{-\alpha}|j-k|^{-\alpha} \le 2^\alpha |j-k|^{-\alpha}|i-k|^{-\alpha}.$$

Hence, we have

$$\begin{aligned}
&a^2 \sum_{\substack{i\neq j\\ j\neq k}} |i-j|^{-\alpha}|j-k|^{-\alpha} e^{-\theta[|i-j|+|j-k|-|i-k|}\\
&\le a^2 \sum_{j\in\mathcal{B}} |i-j|^{-\alpha}|j-k|^{-\alpha} + a^2 \sum_{j\in\mathcal{B}^c} |i-j|^{-\alpha}|j-k|^{-\alpha}\\
&\le a^2 \sum_{j\in\mathcal{B}} 2^\alpha |i-k|^{-\alpha}|i-j|^{-\alpha} + a^2 \sum_{j\in\mathcal{B}^c} 2^\alpha |i-k|^{-\alpha}|j-k|^{-\alpha}
\end{aligned}$$

Bounding the sums over $\mathcal{B}$ and $\mathcal{B}^c$ by sums over $\mathbb{Z}^d$ we obtain that the right hand side of (8) can be bounded by:

$$a^2(2\cdot 2^\alpha K_{d,\alpha} + 2)|i-k|^{-\alpha} \tag{9}$$

where $K_{d,\alpha} = \sum_{j\in\mathbb{Z}^d-\{0\}} |j|^{-\alpha}$. Since the constant in (9) contains a factor a^2, by taking a sufficiently small, we can achieve that the bound has the desired form. $\square$

Remark 3.6. Note that $\Gamma(i) = \exp(-\theta|i|), 0 < \theta < 1$ is not a decay function. Nevertheless, it has been customary in many papers to use spaces in which the dependence on distant coordinates is bounded by exponential decay functions. Of course, since

$$\exp(-(\theta+\epsilon)|i|) \le |i|^{-\alpha}\exp(-\theta|i|) \le \exp(-\theta|i|),$$

the results obtained using the decay function $|i|^{-\alpha}\exp(-\theta|i|)$ imply results for exponential decay functions. In the later part of this paper, we will use exponential decay since it is more convenient for some calculations and to be able to use theorems from the literature as stated.

Remark 3.7. Much of the results of this paper, in particular, all the results in Sections 3, 4 can be generalized to the situation when we have a graph H in place of the lattice $\mathbb{Z}^d$. The only requirement is that we have a decay function $\Gamma(i,j)$ satisfying the properties of Definition 3.4.

In some graphs e.g. the Bethe Lattice where one can define a natural distance d satisfying the *ultrametric property* $d(j,k) \le \max(d(i,j), d(i,k))$, it is easy to introduce such Γ. It suffices to take $\Gamma(i,j) = a\varphi(d(i,j))$ where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a monotone decreasing function that tends to zero sufficiently fast and a is a sufficiently small constant.

If we define the sets $\mathcal{B}$ and $\mathcal{B}^c$ as above we have by the ultrametric property of the distance and the monotonicity of φ, because all the terms are positive.

$$\begin{aligned}
&\sum_j \Gamma(i,j)\Gamma(j,k) \le \sum_{j\in\mathcal{B}} \Gamma(i,j)\Gamma(j,k) + \sum_{j\in\mathcal{B}^c} \Gamma(i,j)\Gamma(j,k) \\
&\le \sum_{j\in\mathcal{B}} \Gamma(i,j)\Gamma(i,k) + \sum_{j\in\mathcal{B}^c} \Gamma(i,k)\Gamma(j,k) \le \Gamma(i,k)[\sum_{j\in\mathcal{B}} \Gamma(i,j) + \sum_{j\in\mathcal{B}^c} \Gamma(j,k)].
\end{aligned}$$

Note that if φ decreases fast enough, the sums in the brackets converge and, by choosing a sufficiently small we can make sure that they are smaller than 1.

Definition 3.8. *Let $h : \mathcal{M} \to \mathbb{R}^n$ be a function. For the convenience of later use, $\mathbb{R}^n$ can be considered as a finite dimensional Banach space. Define*

$$\gamma_{\alpha,i}(h) = \sup_{(x_j)_{j\neq i}} \sup_{(z_j=y_j=x_j)_{j\neq i}, y_i\neq z_i} \frac{\|h(\bar{y}) - h(\bar{z})\|}{d^\alpha(y_i, z_i)}.$$

Clearly, $\gamma_{\alpha,i}(h)$ is a semi-norm. We use $\alpha = \mathrm{Lip}$ to denote the case when $\alpha = 1$.

Later, we will use this semi-norm to define spaces of C^r diffeomorphisms of $\mathcal{M}$ with local interactions. When h is a function from $\mathcal{M}$ to $L(\mathbb{R}^n)$, the space of all linear operators on $\mathbb{R}^n$, the norm $\|\cdot\|$ on $L(\mathbb{R}^n)$ is the one induced by the norm in $\mathbb{R}^n$. The definition can be adapted to the coordinate functions of sections of product bundles by taking a finite number of coordinate charts of M. If we fix a coordinate chart in $M_j, j \in \mathbb{Z}^d$, given a section v of $T\mathcal{M}$, we can identify v_j, the jth variable of $v = (v_j)$, with a function from $\mathcal{M}$ to $\mathbb{R}^n$ and take the maximum of the norm of the function corresponding to each chart as the norm of v_j. From now on, we assume that a fixed finite coordinate chart of M is chosen.

Definition 3.9. *Given a decay function Γ, we introduce the following Banach spaces for $0 \le \alpha \le \mathrm{Lip}$,*

$$C^{\alpha}_{\Gamma}(\mathcal{M}, T\mathcal{M}) = \{v : \mathcal{M} \to T\mathcal{M}, v \in C^0_S \ \|v\|_{C^{\alpha}_{\Gamma}} \equiv \max\{\|v\|_{C^0}, \sup_{i,j \in \mathbb{Z}^d} \gamma_{\alpha,j}(v_i)\Gamma^{-1}(i-j)\}\}.$$

Remark 3.10. Note that, even though the norm $\|\cdot\|_{C^{\alpha}_{\Gamma}}$ depends on the choice of the finite coordinate chart of M, all such norms are equivalent and thus, define the same Banach space. Again, we emphasize that we choose a finite chart on M and not charts on $\mathcal{M}$.

Now we define the Banach space of differentiable sections.

Definition 3.11. *For each $r \in \mathbb{N}$, we denote*

$$\begin{aligned} &C^r_{\Gamma}(\mathcal{M}, T\mathcal{M}) = \big\{v \in C^0_S : \mathcal{M} \to T\mathcal{M}, \\ &\|v\|_{C^r_{\Gamma}} \equiv \max_{0 \le k \le r} \sup_{i_1,\cdots,i_k \in \mathbb{Z}^d} \sup_{i \in \mathbb{Z}^d} \Big\| \frac{\partial^k}{\partial x_{i_1} \cdots \partial x_{i_k}} v_i(\bar{x}) \Big\|_{C^0} \max\{\Gamma^{-1}(i-i_1), \cdots \Gamma^{-1}(i-i_k)\} \\ &< \infty\big\}. \end{aligned}$$

Note that the derivative $\frac{\partial}{\partial x_{i_1}} v_i(\bar{x})$ (will be denoted by $\partial_{i_1} v_i$, for short) is a linear operator on the tangent space TM. Its norm can be defined using the norm induced by the Finsler metric The norm for the multilinear operator $\frac{\partial^k}{\partial x_{i_1} \cdots \partial x_{i_k}} v_i(\bar{x})$ is similarly defined.

Using the Riemannian geometry exponential map, we can define spaces of maps close to the identity completely analogous to those of vector fields. For every map $G : \mathcal{M} \to \mathcal{M}$ close to the identity map, we identify it with the map $\widetilde{G} : \mathcal{M} \to T\mathcal{M}$, $\widetilde{G}(\bar{x}) \equiv \mathcal{A}_{\bar{x}}^{-1} G(\bar{x})$, where $\mathcal{A}_{\bar{x}} = \otimes_{i \in \mathbb{Z}^d} \mathcal{A}_{x_i}$ denotes the exponential map from $T_{\bar{x}}\mathcal{M}$ to $\mathcal{M}$ induced by the Riemannian metric g. It is easy to see that the map $G \to \widetilde{G}$ is one-to-one and the identity map corresponds to the zero section. Under the norms just defined, we obtain open sets in these corresponding Banach spaces that are open neighborhoods of the identity map.

Similar definitions of Banach spaces work for maps from $\mathcal{M}$ to other bundles over $\mathcal{M}$ that are direct sums of bundles on M on which we have a metric. In particular, the definition can be extended to Grassmannian of the tangent bundle, which will be used when we prove the smooth dependence of stable and unstable subbundles.

We state several simple properties about the norm just defined. The proofs are omitted since they are straightforward verifications.

Lemma 3.12. 1. *If G is a C^1_Γ map close to identity, then G is C^α_Γ for any $0 \le \alpha \le \mathrm{Lip}$.*

2. *Let h be a C^α_Γ map on $\mathcal{M}$ and G is C^1_Γ. Then, the composition $G \circ h$ is C^α_Γ and $\|G \circ h\|_{C^\alpha_\Gamma} \le \|G\|_{C^1_\Gamma} \|h\|_{C^\alpha_\Gamma}$.*
3. *G is $C^r_\Gamma, r > 1$ if and only if DG is C^{r-1}_Γ. When h is a C^α_Γ vector field on $\mathcal{M}$ and G is C^2_Γ, the vector field $DG \cdot h$ is C^α_Γ and $\|DG \cdot h\|_{C^\alpha_\Gamma} \le \|DG\|_{C^1_\Gamma} \|h\|_{C^\alpha_\Gamma}$.*

Next, we start to study the regularity of the composition of maps. The following result establishes continuity. Presumably, it is not optimal and one can use almost one derivative less. Nevertheless, it does not seem worth the effort to investigate this question at the present time.

Lemma 3.13. *The mapping $\mathcal{C}$ defined by*

$$\mathcal{C}(G, h) = G \circ h$$

is Lipschitz when it is considered as

$$(1) \quad \mathcal{C} : C^1_\Gamma \times C^0 \to C^0;$$

$$(2) \quad \mathcal{C} : C^2_\Gamma \times C^\alpha_\Gamma \to C^\alpha_\Gamma;$$

Proof. Lipschitz in G is obvious since the map $\mathcal{C}$ is linear in G and clearly bounded. To establish the first conclusion, we have the following estimation.

$$\begin{aligned} &\|G \circ h - G \circ \widetilde{h}\| = \sup_{i,\bar{x}} \|G_i \circ h(\bar{x}) - G_i \circ \widetilde{h}(\bar{x})\| \\ &\le \sup_{i,\bar{x}} \big(\sum_{j \in \mathbb{Z}^d} \|\partial_j G_i\|\big) \sup_{j,\bar{x}} \|h_j(\bar{x}) - \widetilde{h}_j(\bar{x})\|. \end{aligned}$$

Note that $\sum_{j\in\mathbb{Z}^d} \|\partial_j G_i\| \le \sum_{j\in\mathbb{Z}^d} \Gamma(i-j)\|G\|_{C^1_\Gamma}$. Then, we have :

$$\|G \circ h - G \circ \widetilde{h}\|_{C^0} \le \sum_{j\in\mathbb{Z}^d} \Gamma(i-j)\|G\|_{C^1_\Gamma}\|h - \widetilde{h}\|_{C^0}.$$

To prove the second conclusion, we start by estimating $\|\partial_j G_i(h(\bar{x})) - \partial_j G_i(h(\bar{y}))\|$, where $\bar{x}, \bar{y}$ differ only at lattice site $l \in \mathbb{Z}^d$.

$$\begin{aligned} \|\partial_j G_i(h(\bar{x})) - \partial_j G_i(h(\bar{y}))\| &\le \sum_{k\in\mathbb{Z}^d} \|\partial_{kj} G_i\| \|h_k(\bar{x}) - h_k(\bar{y})\| \\ &\le \sum_{k\in\mathbb{Z}^d} \Gamma(i-k)\|G\|_{C^2_\Gamma}\Gamma(k-l)\|h\|_{C^\alpha_\Gamma} d^\alpha(\bar{x}_l, \bar{y}_l) \\ &\le \|G\|_{C^2_\Gamma}\|h\|_{C^\alpha_\Gamma}\Gamma(i-l) d^\alpha(\bar{x}_l, \bar{y}_l) \end{aligned}$$

Hence, we conclude that

$$(\partial_j G) \circ h \in C^\alpha_\Gamma,$$

and

$$\|(\partial_j G) \circ h\|_{C^\alpha_\Gamma} \le \|G\|_{C^2_\Gamma}\|h\|_{C^\alpha_\Gamma}.$$

Now, we use the finite increment formula to estimate $\|G_i(h+\hbar) - G_i(h)\|$.

$$\begin{aligned}
\|G_i(h+\hbar) - G_i(h)\| &= \| \int_0^1 DG_i \circ (h+s\hbar)\hbar \, ds\|_{\alpha,j} \leq \sup_s \|DG_i \circ (h+s\hbar)\hbar\| \\
&\leq \sum_{k\in\mathbb{Z}^d} \|\partial_k G_i \circ (h+s\hbar)\hbar_k\| \leq \sum_{k\in\mathbb{Z}^d} \|G\|_{C^2_\Gamma} \|h\|_{C^\alpha_\Gamma} \Gamma(i-k) \|\hbar_k\|_{\alpha,j} \\
&\leq \sum_{k\in\mathbb{Z}^d} \|G\|_{C^2_\Gamma} \|h\|_{C^\alpha_\Gamma} \Gamma(i-k)\Gamma(k-j) \|\hbar\|_{C^\alpha_\Gamma} \\
&\leq \|G\|_{C^2_\Gamma} \|h\|_{C^\alpha_\Gamma} \Gamma(i-j) \|\hbar\|_{C^\alpha_\Gamma}.
\end{aligned}$$

This proves the second conclusion. □

Once we have that the composition is continuous in those spaces, it is easy to prove higher order differentiability. Again, we note that the proof just given could go through under somewhat more general hypotheses of regularity. To obtain Lipschitz continuity in the first case, we only need $\sum_i \Gamma(i) < \infty$ and the second property of a decay function Γ is not needed. Moreover, we do not really need Lipschitz continuity to prove the differentiability in the lemma below but only continuity.

Lemma 3.14. *The mapping $\mathcal{C}$ defined by*

$$\mathcal{C}(G,h) = G \circ h$$

is C^1 when it is considered as

$$(1) \quad \mathcal{C} : C^2_\Gamma \times C^0 \to C^0;$$

$$(2) \quad \mathcal{C} : C^3_\Gamma \times C^\alpha_\Gamma \to C^\alpha_\Gamma.$$

Moreover, we have the following derivative formulas.

$$D_1\mathcal{C}(G\circ h)\mathcal{G} = \mathcal{G} \circ h;$$

$$D_2\mathcal{C}(G\circ h)\hbar = (DG\circ h)\hbar.$$

Proof. The first conclusion is obvious since the map is linear in the first variable.

To prove the second we note that by the finite increment formula, we have

$$G(h+\hbar) - G(h) = \int_0^1 DG(h+s\hbar)\hbar ds.$$

Applying Lemma (3.13) to DG, we have that DG is Lipschitz in the norm $\|\cdot\|_{C^\alpha_\Gamma}$ when $G \in C^3_\Gamma$. Thus, we have the following estimation.

$$\begin{aligned}\|G(h+\hbar) - G(h) - (DG(h))\hbar\|_{C^\alpha_\Gamma} &= \|\int_0^1 (DG(h+s\hbar) - DG(h))\hbar ds\|_{C^\alpha_\Gamma} \\ &\leq \|DG(h+s\hbar) - DG(h)\|_{C^\alpha_\Gamma}\|\hbar\|_{C^\alpha_\Gamma} \\ &\leq L_1(\|\hbar\|_{C^\alpha_\Gamma})^2,\end{aligned}$$

where L_1 denotes the Lipschitz constant for DG. This proves the C^1-differentiability. □

Corollary 3.15. *The mapping* $\mathcal{C}$ *defined by*

$$\mathcal{C}(G,h) = G \circ h$$

is C^r *when it is considered as*

$$(1) \quad \mathcal{C} : C^{r+1}_\Gamma \times C^0 \to C^0;$$

$$(2) \quad \mathcal{C} : C^{r+2}_\Gamma \times C^\alpha_\Gamma \to C^\alpha_\Gamma.$$

Moreover, the derivatives are what one would obtain by formal computation.

Proof. The proof can be obtained easily by induction based on Lemmas (3.13) and (3.14). □

Now we state and prove the result on the smooth dependence of the conjugating map h_G.

Theorem 3.16. *For any* $C^r, r \geq 1$ *hyperbolic map* f *of* M *with a hyperbolic set* Λ *and a decay function* Γ, *there exists* $\epsilon > 0$ *(We will make* ϵ *somewhat explicit in terms of properties of* F *at the end of the proof) such that if* G *is in the* C^{r+2}_Γ*-neighborhood of the identity map* Id *from* $\mathcal{M} \to \mathcal{M}$, *then, the coupled map* $\Phi = G \circ F$ *is topologically conjugate to* F *on the hyperbolic set* $\Delta = \otimes_{i\in\mathbb{Z}^d}\Lambda$, *where* $\mathcal{M} = \otimes_{i\in\mathbb{Z}^d} M$ *and* $F = \otimes_{i\in\mathbb{Z}^d} f$*: i.e., there exists a unique* h_G *in the* C^0*-neighborhood of* Id *satisfying*

$$\Phi \circ h_G = h_G \circ F.$$

Moreover,

1. *h_G is C^α_Γ, where $0 < \alpha < 1$ is close to the Hölder exponent of stable and unstable invariant subspaces for the unperturbed map.*
2. *The map $G \to h_G$ is C^r from C^{r+2}_Γ to C^α_Γ.*

Remark 3.17. We are not claiming, at the moment, that h_G is a homeomorphism. In finite dimensions this requires some arguments based either on index theory or on the fact that $\mathrm{Id} - \Phi_*$ is also invertible on C^0 sections. For details of the proof that h_G is a homeomorphism, see [Jia95]. We do not need this property in the considerations of smooth dependence of invariant measures.

Proof. The results of the theorem are conclusions from the Implicit Function Theorem. Define a nonlinear function $\mathcal{L} : C^{r+2}_\Gamma \times C^\alpha_\Gamma \to C^\alpha_\Gamma$ by:

$$\mathcal{L}(G,h)(\bar{x}) = h(\bar{x}) - \mathcal{A}_{\bar{x}}^{-1}\Phi(\mathcal{A}_{F^{-1}(\bar{x})}h(F^{-1}(\bar{x})))$$

Note that the first argument is a diffeomorphism and the second a section.

For typographical reasons, identifying sections and the homeomorphisms that they generate via the exponential mapping $\mathcal{A}_{\bar{x}}$, we will write the map as:

$$\mathcal{L}(G,h) = h - G \circ F \circ h \circ F^{-1}.$$

Lemma (3.14) implies that $\mathcal{L}$ is differentiable in both arguments G and h.

Note that $\mathcal{L}(\mathrm{Id}, 0) = 0$.

Moreover, we have the following proposition on the invertibility of the derivative operator.

Proposition 3.18. *$D_2\mathcal{L}(\mathrm{Id}, 0) = \mathrm{Id} - F_*$ is invertible as an operator on the space of sections C^α_Γ when $\alpha \le \alpha^*$, where $0 < \alpha^* \le 1$ is the Hölder exponent of the stable and unstable invariant subbundles for Df.*

Proof. Note that the equation for ξ given η

$$(\mathrm{Id} - F_*)\xi = \eta$$

can be written component-wise on each of the copies of the manifold as:

$$\begin{aligned}\xi_i(\dots, x_{i-1}, x_i, x_{i+1}, \dots) - Df|_{f^{-1}(x_i)}\xi_i(\dots, x_{i-1}, f^{-1}(x_i), x_{i+1}, \dots)\\ = \eta_i(\dots, x_{i-1}, x_i, x_{i+1}, \dots)\end{aligned}$$

In this equation, we can consider the variables x_j, $j \neq i$ as parameters. The theory of finite dimensional hyperbolic systems (see e.g [KKPW90]) tells us that if we fix x_j for $j \neq i$ the equation is solvable and we have for $0 \leq \alpha \leq \alpha^*$, $\alpha^* > 0$ depending only on f, we have

$$||\xi_i(\dots, x_{i-1}, \cdot, x_{i+1}, \dots)||_{C^\alpha(M,TM)} \leq K||\eta_i(\dots, x_{i-1}, \cdot, x_{i+1}, \dots)||_{C^\alpha(M,TM)} \tag{10}$$

Now, recalling that by the definition of C^α_Γ, we have

$$\begin{aligned}||\eta_i(\dots, x_{i-1}, \cdot, x_{i+1}, \dots, x_j, \dots) - \eta_i(\dots, x_{i-1}, \cdot, x_{i+1}, \dots, \bar{x}_j, \dots)||_{C^0(M,TM)}\\ \leq ||\eta||_{C^\alpha_\Gamma}\Gamma(i-j)d(x_j, \bar{x}_j)^\alpha\end{aligned}$$

and by (10) we have:

$$\begin{aligned}||\xi_i(\dots, x_{i-1}, \cdot, x_{i+1}, \dots, x_j, \dots) - \xi_i(\dots, x_{i-1}, \cdot, x_{i+1}, \dots, \bar{x}_j, \dots)||_{C^0(M,TM)}\\ \leq K||\eta||_{C^\alpha_\Gamma}\Gamma(i-j)d(x_j, \bar{x}_j)^\alpha\end{aligned} \tag{11}$$

By the definition of C^α_Γ, (10) and (11) imply that

$$||\xi||_{C^\alpha_\Gamma} \leq K||\eta||_{C^\alpha_\Gamma}$$

This finishes the proof of Proposition 3.18. □

Thus, the Implicit Function Theorem implies that there exists $\epsilon > 0$, for any G in the C^{r+2}_Γ-neighborhood of Id, there exists a unique map h_G is the C^α_Γ neighborhood of Id such that $\mathcal{L}(G, h_G) = 0$, i.e,

$$G \circ F \circ h_G = \Phi \circ h_G = h_G \circ F.$$

By Corollary 3.15, when G is C^{r+2}, the map $G \to h_G$ is C^r.

□

4. Smooth dependence of invariant hyperbolic splittings

To state and prove our results precisely we will need to endow the space of sections of the Grassmannian bundle $\mathcal{G}$ with a Banach manifold structure that also captures the ideas of locality and translation invariance. Similar treatments can be found in [PS91].

We will follow the standard practice in hyperbolic theory of representing linear subspaces close to a given one as the graph of a linear map from this space to its complement (see e.g., [HP70]).

A section of the Grassmannian bundle close to the stable subspace $\mathcal{E}^s(\bar{x})$ can be identified with a section of the space of linear bundle maps from $\mathcal{E}^s(\bar{x})$ to $\mathcal{E}^u(\bar{x})$, the unstable subspace. That is, given a family of linear maps

$$H^{s,u}(\bar{x}) : \mathcal{E}^s(\bar{x}) \to \mathcal{E}^u(\bar{x}),$$

we associate to it the family of spaces $\mathrm{Gr}(H^{s,u}(\bar{x})) \subset T_{\bar{x}}\mathcal{M}$

$$\mathrm{Gr}(H^{s,u}(\bar{x})) = \{(v, H^{s,u}(\bar{x})v) \mid v \in \mathcal{E}^s(\bar{x})\}$$

It is easy to see and indeed standard that we can identify sections of the Grasmannian in $\mathcal{E}^s \otimes \mathcal{E}^u$ close to $\mathcal{E}^s$ and sections of linear bundle maps from $\mathcal{E}^s$ to $\mathcal{E}^u$. Therefore, if we give spaces of sections of linear bundle maps appropriate Banach norms, which capture the notions of locality, invariance, regularity, we can define a Banach manifold structure in sections of the Grasmannian bundle which captures the same ideas. Analogous definitions and considerations work, of course, for perturbations of the standard bundle.

So, our next task will be to introduce appropriate spaces of sections of linear bundle maps that capture the desired properties.

Recalling that $T_{\bar{x}}\mathcal{M} = \oplus_{j\in\mathbb{Z}^d} T_{x_j}M$, we decompose the linear map $H^s(\bar{x})$ into blocks $H^s_{ij}(\bar{x}) : T_{x_j}M \to T_{x_i}M$ defined by

$$H^s_{ij}(\bar{x}) = \Pi_{T_{x_i}M}\big(H^s\big|_{T_{x_j}M}\big),$$

where $\Pi_{T_{x_i}M}$ is the projection associated to the direct sum decomposition $T_{\bar{x}}\mathcal{M} = \oplus_{j\in\mathbb{Z}^d} T_{x_j}M$.

Given a decay function Γ we define

$$\|H^s(\bar{x})\|_\Gamma = \sup_{\bar{x},i,j} \|H^s_{ij}(\bar{x})\|\Gamma^{-1}(i-j).$$

This is clearly a Banach norm in a space of linear maps $\mathcal{E}^s(\bar{x}) \to \mathcal{E}^u(\bar{x})$. Note that the fact that this norm is finite captures the idea that when j and i are very far apart, the jth coordinate of v has very little influence on the ith coordinate of $H^s(\bar{x})v$.

We denote by $\mathcal{L}_\Gamma$ the space of linear maps from with finite $\|\ \ \|_\Gamma$. Similarly, we define $\mathcal{L}_\Gamma^{s,u}$ as the maps form $\mathcal{E}^s$ to $\mathcal{E}^u$ with finite norm $||\ \ ||_\Gamma$. Similar notation for maps from one subbundle to another. We use the notation $\mathcal{L}_\Gamma^{\alpha,\beta}$ to refer to the possibilities.

It will be very important to note that, because of the properties of decay functions the norm $||\ \ ||_\Gamma$ makes the space $\mathcal{L}_\Gamma$ into a Banach algebra. Moreover, if $A \in \mathcal{L}_\Gamma^{\alpha,\beta}$, $B \in \mathcal{L}_\Gamma^{\beta,\gamma}$, we have:

$$||BA||_\Gamma \le ||B||_\Gamma ||A||_\Gamma$$

If we introduce a trivialization of $\mathcal{E}^s(\bar{x})$, $\mathcal{E}^u(\bar{x})$ induced by a trivialization of $T_{x_i}M$ we can define:

$$C^0(\mathcal{L}_\Gamma), \quad C^\alpha(\mathcal{L}_\Gamma);$$

as the Banach spaces of continuous and Hölder sections, respectively. In particular, the C^α_Γ norm is defined by

(12)
$$\|H^s(\bar{x})\|_{C^\alpha_\Gamma} = \max\{\sup_{\bar{x},i,j} \|H^s_{ij}\|\Gamma^{-1}(i-j), \sup_{k,i,j} \gamma_{\alpha,j}(H^s_{ik}(\bar{x}))\Gamma^{-1}(i-j)\}$$

We also introduce similarly spaces of Hölder sections for all the other bundles of linear maps considered above.

We emphasize that we only use trivializations on finite dimensional objects and make them induce trivializations in the infinite dimensional ones. We also emphasize that the trivializations are only used to introduce Hölder norms. The objects that we are considering – sections of linear bundle maps – have an intrinsic meaning. All the functional equations we will derive are expressed in terms of geometric objects and their natural operations. Of course, contraction, etc. will be expressed in norms that use trivializations.

Again, it is important to note that the spaces with the norms (12) are Banach algebras under the pointwise multiplication of operators. Using that the Hölder norm induces a Banach algebra, this reduces to properties of the decay functions.

Translation invariance can be readily incorporated into consideration. We note that the translation S^k introduced in section (2) induces a transformation (still denoted by S^k) on linear maps such that

$$(S^k H^s)_{ij}(\bar{x}) = H^s_{i+k,j+k}(S^k \bar{x})$$

Clearly, S^k does not change the $C^0_\Gamma, C^\alpha_\Gamma$ norms of a section.

The space of sections which are invariant under the translations S^k is a closed subspace of $C^0_\Gamma, C^\alpha_\Gamma$. We will denote them by $C^0_{\Gamma,S}, C^\alpha_{\Gamma,S}$ respectively.

Following the hint of [Mn90], we will find it convenient to study the hyperbolic splittings for Φ evaluated at the point of the conjugacy given by structural stability. That is we will study $\mathcal{E}^s(h(\bar{x})) \oplus \mathcal{E}^u(h(\bar{x}))$. Since h is close to Id, we can use same trivialization chart containing both points $\bar{x}$ and $h(\bar{x})$. Thus, the tangent space $T_{h(\bar{x})}\mathcal{M}$ is identified with $T_{\bar{x}}\mathcal{M}, \bar{x} \in \Delta$. Therefore, we can apply the derivative operator $D\Phi$ to sections over Δ. The action on sections denoted by $\mathcal{T}$ is defined by

$$\mathcal{T} v(\bar{x}) = D\Phi\big|_{h \circ F^{-1}(\bar{x})} v(F^{-1}(\bar{x})).$$

This operator has the advantage that $v(\bar{x})$ always depends on $v(F^{-1}(\bar{x}))$ independent of what Φ is. This is an important advantage with respect to the naively more natural push forward. See for example [Mn90], [Rue97] for a more geometric justification.

The main result of this section is the following theorem.

Theorem 4.1. *For any $C^r, r \geq 4$ hyperbolic map f of M with a hyperbolic set Λ and a decay function Γ, there exists $\epsilon > 0$ such that if G is in an ϵ C^r_Γ-neighborhood of the identity map* Id *from $\mathcal{M} \to \mathcal{M}$, then, denoting as usual, $F = \otimes_{i \in \mathbb{Z}^d} f$, $\Phi = G \circ F$, we have*

1. *there exits a hyperbolic splitting of $T\mathcal{M}$ over the hyperbolic set $\Delta_\Phi = h_G(\Delta_F)$:*

$$T\mathcal{M} = \mathcal{E}^u_\Phi(h_G(\bar{x})) \oplus \mathcal{E}^s_\Phi(h_G(\bar{x}))$$

invariant under the derivative operator $D\Phi = DG \circ F \cdot DF$*;*

2. $\mathcal{E}^u_\Phi(h_G(\bar{x}))$ *are* C^α_Γ *sections of the Grassmannian close to* $\mathcal{E}^u_F(\bar{x})$ *in* C^α_Γ *norm. Similar result holds for* $\mathcal{E}^s_\Phi(h_G(\bar{x}))$*;*
3. $D\Phi$ *is expanding on* $\mathcal{E}^u_\Phi(h_\Phi(\bar{x}))$ *and contracting on* $\mathcal{E}^s_\Phi(h_\Phi(\bar{x}))$ *in both* C^0 *and* C^α_Γ *norms;*
4. *the map from* $G \in C^r_\Gamma \to \mathcal{E}^{u,s}_\Phi(h_G(\bar{x}))$ *is* C^{r-3} *when the sections of the Grasmannian are given the* C^α_Γ *norm.*

Moreover, if G *commutes with translations, so do the invariant subspaces.*

Proof. We will just give the proof for the unstable bundle. The result for the stable bundle follows by applying the result on unstable bundles to F^{-1}. (Of course, a direct proof for the stable follows along the same lines as the proof for the unstable.)

We consider the operator $D\Phi \circ h$ in the components induced by the coordinates $\mathcal{E}^s_F(\bar{x}) \oplus \mathcal{E}^u_F(\bar{x})$, $\mathcal{E}^s_F(F(\bar{x})) \oplus \mathcal{E}^u_F(F(\bar{x}))$ in the domain and the range, respectively. It can be represented by a matrix

$$D\Phi \circ h(x) = \begin{pmatrix} A^{ss}_G(\bar{x}) & A^{su}_G(\bar{x}) \\ A^{us}_G(\bar{x}) & A^{uu}_G(\bar{x}) \end{pmatrix}$$

We note that $A^{su}_F = 0$, $A^{us}_F = 0$ because the splitting is invariant under F and A^{ss}_F, A^{uu}_F are block diagonal matrices.

Notice that since A_F is a direct product of finite dimensional systems, we have that

$$\begin{aligned} \|A^{uu}_F\|_{C^0} &= \|Df^{uu}\|_{C^0(M)} \\ \|A^{ss}_F\|_{C^0} &= \|Df^{ss}\|_{C^0(M)} \\ \|A^{uu}_F\|_{C^\alpha_\Gamma} &= \|Df^{uu}\|_{C^\alpha(M)} \\ \|A^{ss}_F\|_{C^\alpha_\Gamma} &= \|Df^{ss}\|_{C^\alpha(M)} \end{aligned} \tag{13}$$

Therefore, when G is in a sufficiently small C^r_Γ-neighborhood $r \geq 4$ of F, we can assume

$$\begin{aligned} &\|A^{su}_G\|_{C^0_\Gamma} \leq \epsilon \quad \|A^{su}_G\|_{C^0_\Gamma} \leq \epsilon \\ &\|A^{ss}_G\|_{C^0_\Gamma} \leq \mu < 1 \quad \|(A^{uu}_G)^{-1}\|_{C^0_\Gamma} \leq \mu < 1 \end{aligned} \tag{14}$$

with ϵ arbitrarily small and μ as close to λ as desired.

We will consider sections of Grassmannians close to $\mathcal{E}_F^u$ as the graph of a section of linear maps $\mathcal{E}_F^u(\bar{x}) \to \mathcal{E}_F^s(\bar{x})$. Given $U_{\bar{x}} \in \mathcal{L}_\Gamma^u$, the space of linear maps from $\mathcal{E}_F^u(\bar{x}) \to \mathcal{E}_F^s(\bar{x}), \bar{x} \in \Delta$, we consider

$$\operatorname{Gr} U_{\bar{x}} = \{(U_{\bar{x}} v, v) \mid v \in \mathcal{E}_F^u(\bar{x})\}$$

Since

$$D\Phi|_{h(\bar{x})}(U_{\bar{x}}v, v) = ([A_G^{ss}(\bar{x})U_{\bar{x}} + A_G^{su}(\bar{x})]v, [A_G^{uu}(\bar{x}) + A_G^{us}(\bar{x})U_{\bar{x}}]v)$$

we see that the graph of $U_{\bar{x}}$ is invariant if and only if

$$U_{F(\bar{x})}[A_G^{uu}(\bar{x}) + A_G^{us}(\bar{x})U_{\bar{x}}] = [A_G^{ss}(\bar{x})U_{\bar{x}} + A_G^{su}(\bar{x})]$$

Equivalently

$$\begin{aligned} U_{\bar{x}} \;=\; & \left[A_G^{ss}(F^{-1}(\bar{x}))U_{F^{-1}(\bar{x})} + A_G^{su}(F^{-1}(\bar{x}))\right] \\ & \cdot \left[A_G^{uu}(F^{-1}(\bar{x})) + A_G^{us}(F^{-1}(\bar{x}))U_{F^{-1}(\bar{x})}\right]^{-1} \end{aligned} \tag{15}$$

We now define a map $\mathcal{F}$: $(G, U) \in C^r \times C^\alpha(\mathcal{L}_\Gamma^u) \to \mathcal{F}(G, U) \in C^\alpha(\mathcal{L}_\Gamma^u)$:

(16)

$$\mathcal{F}(G, U) = U_{\bar{x}} -$$

$$\left[A_G^{ss}(F^{-1}(\bar{x}))U_{F^{-1}(\bar{x})} + A_G^{su}(F^{-1}(\bar{x}))\right] \left[A_G^{uu}(F^{-1}(\bar{x})) + A_G^{us}(F^{-1}(\bar{x}))U_{F^{-1}(\bar{x})}\right]^{-1}$$

Note that this map can be defined when U is in a neighborhood of 0 and G is in a neighborhood of the identity. $A_G^u u \circ F^{-1}$ is boundedly invertible and A_G^{us} has very small norm. So that, indeed the second factor in (16) can be inverted.

Note that $\mathcal{F}(\mathrm{Id}, 0) = 0$. Therefore, by the Implicit Function Theorem, Theorem 4.1 is proved once we show that the operator $\mathcal{F}$ defined by (16) is differentiable in G and U and $D_2\mathcal{F}$ is invertible at $(\mathrm{Id}, 0)$. We note that, by Corollary 3.15 the mappings

$$G \to A_G^{ss} \circ F^{-1}, A_G^{su} \circ F^{-1}, A_G^{uu} \circ F^{-1}, A_G^{us} \circ F^{-1} \tag{17}$$

are C^{r-3} when the G is considered in the manifold with the C_Γ^r topology and the target spaces with the C_Γ^α topology.

Note that the map $\mathcal{F}$ defined in (16) is formed out of the maps (17) by applying Banach algebra operations, which are, of course, analytic.

Hence, the map $\mathcal{F}$ is C^{r-3} when we consider it as a map of spaces of sections to spaces of sections endowed with the indicated topologies. Since we are assuming $r \geq 4$, this means that the map is C^1.

To apply the implicit function theorem in the indicated spaces, we only need to check that the derivative with respect to the second argument is invertible at the origin.

Given the form of the derivative of composition, and the well known formulas for the derivatives of the Banach algebra operations, we have:

$$D_2\mathcal{F}(\mathrm{Id}, 0)U_{\bar{x}} = \mathrm{Id} - A_F^{ss}(F^{-1}(\bar{x}))U_{F^{-1}(\bar{x})}(A_F^{uu}(F^{-1}(\bar{x})))^{-1} \tag{18}$$

To finish the proof of Theorem 4.1, we just need to show that when $0 < \alpha < 1$ is appropriately chosen, the operator in (18) is invertible. We will do it by showing that the norm of $\mathcal{S} : U_{\bar{x}} \to SU$

$$(\mathcal{S}U)_{\bar{x}} = A_F^{ss}(F^{-1}(\bar{x}))U_{F^{-1}(\bar{x})}(A_F^{uu}(F^{-1}(\bar{x})))^{-1}$$

is strictly smaller than 1.

We note that the operator $U \to U_{F^{-1}}$ acting on C_Γ^α has norm $||DF^{-1}||_{C^0}^\alpha$. This norm is equal to $||Df^{-1}||_{C^0(M)}^\alpha$.

Hence, recalling that C_Γ^α is a Banach algebra and the expressions for norms (13) we have that:

$$||\mathcal{S}|| \leq ||DF^{-1}||_{C^0(\mathcal{M})}\, ||(A_F^{uu})^{-1}||_{C_\Gamma^\alpha}||(A_F^{ss})||_{C_\Gamma^\alpha} \tag{19}$$

$$= ||Df^{-1}||_{C^0(M)}^\alpha||(Df^{uu})^{-1}||_{C^0(M)}||(Df^{ss})||_{C^0(M)} \tag{20}$$

Clearly, for $\alpha > 0$ adequately small, the RHS of (19) is smaller than 1. Indeed, the condition that α makes the RHS of (19) smaller than 1 is the same condition for the regularity of the splitting applying the invariant section theorem of [HP70].

□

To prepare for the proof of smooth dependence of the equilibrium state, we need further to express the action of $D\Phi$ on the invariant unstable subbundle $\mathcal{E}_\Phi^u(h_G(\bar{x}))$ in terms of an infinite matrix. Because of the invariance, there exists an infinite matrix

$B = (\mathbf{b}_{ij})_{i,j\in\mathbb{Z}^d}$ such that

$$\begin{pmatrix} A_G^{ss}\bar{x}) & A_G^{su}(\bar{x}) \\ A_G^{us}(\bar{x}) & A_G^{uu}(\bar{x}) \end{pmatrix} \begin{pmatrix} U_{\bar{x}} \\ \mathrm{Id} \end{pmatrix} = \begin{pmatrix} U_{F(\bar{x})} \\ \mathrm{Id} \end{pmatrix} B.$$

Clearly, the matrix B is the matrix representation of $D\Phi$ along the invariant unstable subbundle $\mathcal{E}^u_\Phi(h_G(\bar{x}))$ under the chosen bases. So, we have

$$B = A_G^{uu}(\bar{x}) + A_G^{us}(\bar{x})U_{\bar{x}}.$$

Note that $D\Phi = DG \cdot DF$. If we write DG blockwise according to the hyperbolic splitting, we have

$$A_G^{uu}(\bar{x}) = G^{uu}(F \circ h_G(\bar{x}))(D^u f(h_G(\bar{x}))),$$

where G^{uu} denotes the block matrix corresponding to the action of DG on the unstable subbundle $\mathcal{E}^u_F(h_G)$ and $(D^u f(h_G(\bar{x})))$ is a diagonal block matrix with $D^u f((h_G)_i(\bar{x}))$ on the main diagonal. Note that $\|G^{uu} - \mathrm{Id}\,\|_{C^r_\Gamma}$ is small and $\|h_G - \mathrm{Id}\,\|_{C^\alpha_\gamma}$ is small. We can rewrite the matrix in the form

$$B = (D^u f(x_i))(\mathrm{Id} + A_G). \tag{21}$$

By the theorem just proved, we have that the map $G \to A_G$ is C^{r-3} with respect to the C^r_Γ norm for G and C^α_Γ norm for A_G. Further more, $\|A_G\|_{C^\alpha_\Gamma} \le \delta(\epsilon)$, where $\epsilon = \|G\|_{C^r_\Gamma}$ and $\lim_{\epsilon\to 0}\delta(\epsilon) = 0$.

5. Smooth dependence of potential functions on infinite matrices

We first describe the Banach space containing all the infinite matrices that are matrix representations of $D^u\Phi$, the restriction of the derivative $D\Phi$ on the unstable spaces. We denote this space by $\mathcal{B}$.

An infinite matrix function $A(\bar{x}), \bar{x} \in \Delta_F$ is an element of $\mathcal{B}$ if it satisfies the following criteria:

(1) $A(\bar{x}) = (a_{ij}(\bar{x}))_{i,j\in\mathbb{Z}^d}$ is an infinite matrix. Each entry $a_{ij}(\bar{x})$ is a matrix function of finite size $p \times p$, where p is the dimension of the unstable space of Df.

(2) For matrix functions $a_{ij}(\bar{x})$, we define its norm $\|\cdot\|$ using the following formula:

$$\|a_{ij}(\bar{x})\| = \max_{1\le k,l\le p}\sup_{\bar{x}\in\Delta_F} |(a_{ij}(\bar{x}))_{kl}|.$$

We recall the C^α_Γ-norm for $A(\bar{x})$ is defined by (refHalpha) in the previous section.

$$\|A(\bar{x})\| = \max\{\sup_{ij\in\mathbb{Z}^d} \|a_{ij}(\bar{x})\|\Gamma^{-1}(i-j), \sup_{k,i,j} \gamma_{\alpha,j}(a_{ik}(\bar{x}))\Gamma^{-1}(i-j)\}.$$

An infinite matrix $A(\bar{x})$ belongs to $\mathcal{B}$ iff $\|A(\bar{x})\| < \infty$. It is easy to check that $\mathcal{B}$ is a Banach space. In order to have a spatial translation invariant SRB measure, it is necessary that Φ is spatial translation invariant. Note that we do not need to assume the spatial translation invariance until the very last section when we introduce equilibrium states for potential functions.

Next, we describe the Banach space of potential functions defined on the hyperbolic set Δ_F. Let $\psi(\bar{x})$ be a real function on Δ_F. We define its norm

$$\|\psi(\bar{x})\| = \max\{\sup_{\bar{x}\in\Delta_F} |\psi(\bar{x})|, \gamma_{\alpha,j}(\psi)\Gamma^{-1}(j).\}$$

We denote $\mathcal{H}$ all such functions with finite norm. It is also a Banach space.

In order to define the map from the infinite matrix space $\mathcal{B}$ to the potential function space $\mathcal{H}$, we need to define a special linear functional on $\mathcal{B}$, tr^0: $\mathcal{B} \to \mathcal{H}$:

$$\overset{0}{\mathrm{tr}}(A(\bar{x})) = \mathrm{trace}(a_{00}(\bar{x})).$$

The map $\mathcal{L} : \mathcal{B} \to \mathcal{H}$ is defined by

$$\psi_A(\bar{x}) =: \mathcal{L}(A(\bar{x})) =: \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \overset{0}{\mathrm{tr}}(A^k(\bar{x})).$$

In [JP98], it is proven that when $\Phi = G \cdot F$ is C^1-close to F and G is at least C^2 and has *short-range* property, or equivalently, C^α_Γ close to Id for some Γ, there exists a measure μ_ϕ on the hyperbolic set Δ_Φ invariant under both Φ and the translation S. The measure is obtained as a thermodynamic limit of SRB-measures for Φ_V, the finite dimensional approximations of Φ. The pull-back measure $(h_G)^*\mu_\Phi$ is an equilibrium state for the $\mathbb{Z}^{d+1}$-action generated by (F, S) on Δ for a potential function φ_Φ and

$$\varphi_\Phi(h_G(\bar{x})) = -\log J^u f(x_0) + \psi_{A_G}(\bar{x}),$$

where A_G is defined by (21) in the last section.

Since $-\log J^u f(x_0)$ is independent of $\Phi = G \cdot F$, the next step we need to show is that $\psi_A(\bar{x})$ depends on A smoothly. In fact, we have the following theorem.

Theorem 5.1. *The map $\mathcal{L}$ is well-defined in a small neighborhood of the origin of the Banach space $\mathcal{B}$ and it is analytic.*

Proof. Let t be a real number. We have

$$\mathcal{L}(tA(\bar{x})) = \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \overset{0}{\operatorname{tr}}((tA(\bar{x}))^k) = \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \overset{0}{\operatorname{tr}}(A^k(\bar{x}))t^k.$$

It is clear that $\operatorname{tr}^0(A_1 A_2 \cdots A_k)$ is a multilinear operator from $\mathcal{B}$ to $\mathcal{H}$. We need to show that it is bounded. We start with $k = 1$.

$$|\overset{0}{\operatorname{tr}}(A(\bar{x}))| = |\operatorname{trace}(a_{00}(\bar{x}))| \leq p\|A(\bar{x})\|.$$

Assume that $\bar{x}$ and $\bar{y}$ differ only at $j \in \mathbb{Z}^d$.

$$|\overset{0}{\operatorname{tr}}(A(\bar{x})) - \overset{0}{\operatorname{tr}}(A(\bar{y}))| = |\operatorname{trace}(a_{00}(\bar{x}) - a_{00}(\bar{y}))| \leq p\|A(\bar{x})\| d^{\alpha}(x_j, y_j)\Gamma(j).$$

These two inequalities imply that $\|\operatorname{tr}^0\| \leq p$.

When $k = 2$,

$$|\overset{0}{\operatorname{tr}}(AB) = |\operatorname{trace}(\sum_{i\in\mathbb{Z}^d} a_{0i} b_{i0})| \leq p\|A\|\|B\|.$$

For $\bar{x}$ and $\bar{y}$ with $x_i = y_i, i \neq j, i \in \mathbb{Z}^d$,

$$|\overset{0}{\operatorname{tr}}(A(\bar{x})B(\bar{x})) - \overset{0}{\operatorname{tr}}(A(\bar{y})B(\bar{y}))| = |\operatorname{trace} \sum_{i\in\mathbb{Z}^d} a_{0i}(\bar{x})b_{i0}(\bar{x}) - a_{0i}(\bar{y})b_{i0}(\bar{y})|$$

$$\leq |\operatorname{trace} \sum_{i\in\mathbb{Z}^d} a_{0i}(\bar{x})(b_{i0}(\bar{x}) - b_{i0}(\bar{y}))| + |\operatorname{trace} \sum_{i\in\mathbb{Z}^d} (a_{0i}(\bar{x}) - a_{0i}(\bar{y}))b_{i0}(\bar{y})|$$

$$\leq p\big(\sum_{i\in\mathbb{Z}^d} \Gamma(i)\Gamma(i-j) + \sum_{i\in\mathbb{Z}^d} \Gamma(i)\Gamma(j)\big)\|A\|\|B\| d^{\alpha}(x_j, y_j)$$

$$\leq 2ps\|A\|\|B\| d^{\alpha}(x_j, y_j)\Gamma(j),$$

where $s = \sum_{j\in\mathbb{Z}^d} \Gamma(j)$.

Thus, we have $\|\operatorname{tr}^0(AB)\| \leq 2ps\|A\|\|B\|$. By the Banach algebra property, we have

$$\|\overset{0}{\operatorname{tr}}(A_1 A_2 \cdots A_k)\| \leq 2ps\|A_1\|\|A_2\| \cdots \|A_k\|.$$

Therefore, the operator $\mathcal{L}$ is analytic in the open neighborhood $\{A : \|A\| < 1\} \subset \mathcal{B}$. $\square$

6. Smooth dependence of equilibrium states on potential functions

In this last section we prove the smooth dependence of equilibrium states on their potential functions. The main tool used here is the transition from equilibrium states of coupled map lattices to invariant Gibbs states on higher dimensional lattice spin systems for corresponding potentials. The main results of this section were proved over the last several years in several related articles [BK95, JM96, BK97]. However, we shall give sufficient details to make this article self-contained and refer to [BK95, JM96, BK97] for some technical steps.

We first state the main result of this section. We will use a slightly different Banach space of potential functions: the one with exponential decay properties. As we explained earlier, the results are equivalent. We first define a new family of metrics ρ_q on $\mathcal{M}$: for every $0 < q < 1$, and $\bar{x}, \bar{y} \in \mathcal{M}$

$$\rho_q(\bar{x}, \bar{y}) = \sup_{i \in \mathbb{Z}^d} q^{|i|} d(x_i, y_i).$$

Then, the potential function space is defined to be the space of all Hölder continuous functions with respect to the metric ρ_q:

$$\mathcal{H} = \{\varphi(\bar{x}) : \Delta \to \mathbb{R}, \sup_{\bar{x} \neq \bar{y}} \frac{|\varphi(\bar{x}) - \varphi(\bar{y})|}{\rho_q^\alpha(\bar{x}, \bar{y})} < \infty\}. \tag{22}$$

The norm on this Banach space is

$$\|\varphi\| = \sup_{\bar{x} \neq \bar{y}} \frac{|\varphi(\bar{x}) - \varphi(\bar{y})|}{\rho_q^\alpha(\bar{x}, \bar{y})}.$$

By choosing an appropriate $0 < q < 1$, the inclusion map from the Banach space defined via a decay function γ to $\mathcal{H}$ is bounded, and thus analytic.

Let $\varphi_0(\bar{x}) \in \mathcal{H}$ be a potential function whose value depends only on the coordinate x_0, which means it is actually a Hölder continuous function on $\Lambda \subset M$. We denote the ϵ-neighborhood of $\varphi_0(\bar{x})$ in $\mathcal{H}$ by $O_\epsilon(\varphi_0)$. Let τ denote the $\mathbb{Z}^{d+1}$-action on Δ induced by the map F and translations S. A measure μ invariant under τ is called an equilibrium state for $\varphi \in O_\epsilon(\varphi_0)$, if it satisfies the *variational principle* equation:

$$P_\tau(\varphi) = h_\tau(\mu) + \int \varphi d\mu,$$

where $P_\tau(\varphi)$ and $h_\tau(\mu)$ are the topological pressure for φ and the measure theoretical entropy of μ with respect to τ, respectively. The next theorem is a summary of results from [Jia95, JP98, BK97].

Theorem 6.1. *For every φ_0 and constants $0 < \alpha, q < 1$ in the definition of the space $\mathcal{H}$, there exists an ϵ such that the equilibrium state μ_φ for every function $\varphi \in O_\epsilon(\varphi_0)$ is unique. This unique equilibrium state denoted by μ_φ is mixing with respect to both the map F and spatial translations S.*

By the results of the previous sections, we conclude that the SRB-measure for Φ is unique and mixing with respect to Φ and S when Φ satisfies the conditions in Theorem (4.1). To complete the proof of smooth dependence of the SRB-measure for Φ, we only need to show the the equilibrium state depends smoothly on the potential function in $O_\epsilon(\varphi_0) \subset \mathcal{H}$.

Let $\mathcal{H}^*$ denote the dual space of $\mathcal{H}$. Clearly, for any $\varphi \in O_\epsilon(\varphi_0) \subset \mathcal{H}$, $\mu_\varphi \in \mathcal{H}^*$. Now we state the main theorem of this section.

Theorem 6.2. *For any φ_0 and constants $0 < \alpha, q < 1$ in the definition of the space $\mathcal{H}$, there exists an $\epsilon > 0$ such that the map from $O_\epsilon(\varphi_0) \subset \mathcal{H}$ to $\mathcal{H}^*$: $\varphi \to \mu_\varphi$ is C^∞ Frechet differentiable.*

The strategy to prove the theorem is to use the symbolic representation of the uncoupled map lattice. To prove the smooth dependence of equilibrium states is then equivalent to proving the smooth dependence of invariant Gibbs states on potentials, which is proven by showing that the topological pressure is C^∞ Frechet differentiable in the corresponding potential space.

6.1. **Symbolic Representation of the uncoupled map lattice.** Since we assume that f possesses a locally maximal hyperbolic set Λ, there exists a Markov partition that induces a semi-conjugating map π from a subshift of finite type Σ_A onto Λ [Bow75]. The subshift is determined by an aperiodic matrix A since f is assumed to be topologically mixing. Let σ_t denote the left shift map on the subshift. We have

$f \circ \pi = \pi \circ \sigma_t$. Using this map π, we can obtain naturally a semi-conjugating map $\bar{\pi} = \otimes_{i \in \mathbb{Z}^d} \pi$ from $\otimes_{i \in \mathbb{Z}^d} \Sigma_A$ (denoted by $\Sigma_A^{\mathbb{Z}^d}$) onto the infinite dimensional hyperbolic set $\Delta = \otimes_{i \in \mathbb{Z}^d} \Lambda$ for the uncoupled map F. Let σ_s denote the maps on $\Sigma_A^{\mathbb{Z}^d}$ induced by translations on $\mathbb{Z}^d$. We have

$$F \circ \bar{\pi} = \bar{\pi} \circ \otimes_{i \in \mathbb{Z}^d} \sigma_t, \quad S \circ \bar{\pi} = \bar{\pi} \circ \sigma_s.$$

The corresponding metric ρ_q on $\Sigma_A^{\mathbb{Z}^d}$ is defined by

$$\rho_q(\bar{\xi}, \bar{\eta}) = \sup_{i \in \mathbb{Z}^d, j \in \mathbb{Z}} q^{|i|+|j|} d(\xi_i(j), \eta_i(j)),$$

where $d(\cdot,\cdot)$ denotes the discrete distance on the space of finite symbols.

Let $\widetilde{\mathcal{H}}$ denote the Banach space of all Hölder continuous functions on $\Sigma_A^{\mathbb{Z}^d}$. The norm is defined similarly:

$$\|\varphi\| = \max\{\sup_{\bar{\xi} \in \Sigma_A^{\mathbb{Z}^d}} |\varphi(\bar{\xi}), \sup_{\bar{\xi} \neq \bar{\eta}} \frac{|\varphi(\bar{\xi}) - \varphi(\bar{\eta})|}{\rho_q^{\alpha}(\bar{\xi}, \bar{\eta})}\}.$$

Proposition 6.3. [Jia95]

1. *the map φ to $\varphi \circ \bar{\pi}$ is a bounded linear operator.*
2. *the $\mathbb{Z}^{d+1}$-action topological pressures for both functions φ and $\varphi \circ \bar{\pi}$ are equal;*
3. *For any $\varphi \in O_\epsilon(\varphi_0) \subset \mathcal{H}$, μ_φ is its unique equilibrium state if and only if $\mu^*_{\varphi \circ \bar{\pi}} = \bar{\pi}^*(\mu_\varphi)$, the pull-back measure under $\bar{\pi}$, is the unique equilibrium state for $\varphi \circ \bar{\pi}$.*

By this proposition, to prove the main theorem, we need only to show that the topological pressure $P_\tau(\varphi \circ \bar{\pi})$ on $\widetilde{\mathcal{H}}$ is C^∞ Frechet differentiable since the unique equilibrium state μ_φ is the Frechet derivative of $P_\tau(\cdot)$ at point φ.

6.2. **localization of the potential functions.** For the convenience of utilizing directly the results in [BK95, BK97], we introduce the Banach space of potentials that are localization of potential functions. For simplicity, we shall also drop the map $\bar{\pi}$ in our notation. We assume $\varphi_0(\bar{\xi}), \bar{\xi} = (\xi_i)_{i \in \mathbb{Z}^d}, \xi_i \in \Sigma_A$ is a potential function on $\Sigma_A^{\mathbb{Z}^d}$ whose value depends only on the coordinate ξ_0. $\varphi, \psi \in \widetilde{\mathcal{H}}$.

Localization of φ_0 and φ We consider $\Sigma_A^{\mathbb{Z}^d}$ as a subset of the full shift of dimension $d+1$. The potential U^0 obtained from localization of φ_0 is a translation invariant longitudinal potential on the intervals of Σ_A. Let $I_n = [-n, n]$. $\widehat{I_n} = \mathbb{Z} \setminus I_n$. $(\xi_I, \eta^*_{\widehat{I}})$ denotes the element in Σ_A whose values in I agree with those of ξ and whose values in $\widehat{I}$ agree with those of η^*

When $n = 0, I_0 = \{0\}$, for a configuration ξ_{I_0}, choose any η^*_0 such that $(\xi_{I_0}, (\eta^*_0)_{\widehat{I_0}}) \in \Sigma_A$ and define

$$U^0(\xi_{I_0}) = \varphi_0(\xi_{I_0}, (\eta^*_0)_{\widehat{I_0}}).$$

Assume that $U^0(\xi_{I_{n-1}})$ is defined for all configurations over I_{n-1}. For a configuration ξ_{I_n}, choose any η^*_n (depending on the configuration ξ_{I_n}) such that $(\xi_{I_n}, (\eta^*_n)_{\widehat{I_n}}) \in \Sigma_A$ and define

$$\begin{aligned} U^0(\xi_{I_n}) &= \varphi_0(\xi_{I_n}, (\eta^*_n)_{\widehat{I_n}}) - \varphi_0(\xi_{I_{n-1}}, (\eta^*_{n-1})_{\widehat{I_{n-1}}}) \\ &= \varphi_0(\xi_{I_n}, (\eta^*_n)_{\widehat{I_n}}) - \sum_{k=0}^{n-1} U^0(\xi_{I_k}). \end{aligned}$$

In this way, we have

$$\varphi_0(\xi) = \sum_{n=0}^{\infty} U^0(\xi_{I_n}).$$

For all other types of configurations over finite volumes, the potential is defined to be zero. Since the function φ_0 is Hölder continuous, the corresponding longitudinal potential decays exponentially to zero as the length of the interval increases.

The procedure to localize φ is similar. The potential U is now defined for all configurations over $d+1$-dimensional cubes. Because of the translation invariance, it is completely determined by its values for configurations over cubes centered at the origin $Q_n = \otimes_{i \in \mathbb{Z}^d, j \in \mathbb{Z}}[-n, n]$.

$$\varphi(\bar{\xi}) = \sum_{n=0}^{\infty} U(\bar{\xi}_{Q_n}).$$

Let Ω_{Q_n} denote the space of all configurations over the finite volume Q_n. We define a real function U_n on Ω_{Q_n}: $U_n(\bar{\xi}) \equiv U_n(\bar{\xi}_{Q_n}) \equiv U(\bar{\xi}_{Q_n})$. Formally, we can write $U = \sum_{k=0}^{\infty} U_n$.

Banach space of potentials Let

$$\|U_n\| = \sup_{\bar{\xi}_{Q_n} \in \Omega_{Q_n}} |U_n(\bar{\xi}_{Q_n})|.$$

For $0 < \theta < 1$, define a norm for U:

$$\|U\| = \sup_{0 \leq n < \infty} \theta^{-n} \|U_n\|.$$

It is easy to see that all such translation invariant potentials with a finite norm form a Banach space. This Banach space is denoted by $\mathcal{P}$. One can also easily verify that the map from the potential functions to the corresponding potentials is a bounded linear map when $\theta = q^{\alpha}$.

Note that the longitudinal potential is fixed. Thus, to prove our main theorem, it suffices to show that the topological pressure for potentials $U^0 + U$, $P(U^0 + U)$ is Frechet differentiable with respect to U in a small neighborhood of the origin of the Banach space $\mathcal{P}$.

6.3. **Differentiability of Topological Pressure.** We recall the definition of the topological pressure function for potential $U^0 + U$ with respect to the $\mathbb{Z}^{d+1}$-action.

$$P(U^0 + U) = \lim_{\Lambda \to \mathbb{Z}^{d+1}} \frac{1}{|\Lambda|} \ln Z_{\Lambda}(U^0 + U),$$

where

$$Z_{\Lambda}(U^0 + U) = \sum_{\bar{\xi} \in \Omega_{\Lambda}} \exp \sum_{I \subset \Lambda, Q \subset \Lambda} U^0(\bar{\xi}_I) + U(\bar{\xi}_Q).$$

Theorem 6.4. *For any fixed exponential decay longitudinal potential U^0 and $0 < \theta < 1$, there exists $\epsilon > 0$ such that the pressure function $P(U^0 + U)$ is C^{∞} Frechet differentiable in the $\epsilon > 0$-neighborhood of the origin of the Banach space $\mathcal{P}$.*

Proof. The proof of the theorem is based on the following two lemmas.

Lemma 6.5. *Let $P(x)$ be a real function in a bounded convex open set $\mathcal{U}$ of a Banach space.*

1. *If $P(x)$ is Gateaux differentiable and its Gateaux derivative $D_x P$ as a bounded linear operator is continuous in x, then $P(x)$ is Frechet differentiable.*
2. *If the Gateaux derivative $D_x P$is bounded for all $x \in \mathcal{U}$, then $P(x)$ is Lipschitz continuous in $\mathcal{U}$.*

Lemma 6.6. *Let $f_n(t), n = 1, 2, \cdots, f_\infty(t)$ be real functions on interval $[-\delta, \delta]$. If $\lim_{n\to\infty} f_n(t) = f_\infty(t)$ for each $t \in [-\delta, \delta]$ and $\sup_{n,t} |\frac{d^k f_n(t)}{dt^k}| < \infty$ for each k, then $f_\infty(t)$ is C^∞ and $\lim_{n\to\infty} \frac{d^k f_n(t)}{dt^k} = \frac{d^k f_\infty(t)}{dt^k}$.*

We shall omit the proofs of these lemmas since they are standard. Lemma 6.6 is taken from [Sim93]. As a direct corollary from the first lemma, we have that if the (n+2)th order of the Gateaux derivative of $P(x)$ is bounded in $\mathcal{U}$, the $P(x)$ is Frechet differentiable up to order $n, n \geq 1$.

Let

$$P_\Lambda(U^0 + U) = \frac{1}{|\Lambda|} \ln \sum_{\bar{\xi}_\Lambda \in \Omega_\Lambda} \exp \sum_{I,Q \subset \Lambda} U^0(\bar{\xi}_I) + U(\bar{\xi}_Q).$$

We have that $\lim_{\Lambda \to \mathbb{Z}^{d+1}} P_\Lambda(U^0 + U) = P(U^0 + U)$ for all $U \in \mathcal{P}$. According to the lemmas, in order to show that $P(U^0 + U)$ is Frechet differentiable in a neighborhood of the origin of $\mathcal{P}$, it suffices to prove that

$$\sup_{\Lambda,t} |\frac{d^k P_\Lambda(U^0 + U + tV)}{dt^k}| < \infty \tag{23}$$

for each $k = 1, 2, \cdots$ and uniformly for $U \in O_\epsilon(\mathcal{P})$ and $\|V\| = 1, V \in \mathcal{P}$. To prove this boundedness, we use several results that are nicely presented in $II.12$ of [Sim93].

Computation of $|\frac{d^k}{dt^k} P_\Lambda(U^0 + U + tV)|$:

Note that for any C^k function $h(t)$,

$$\frac{d^k}{dt^k} h(t) = \frac{\partial^k}{\partial t_1 \cdots \partial t_k} h(t_1 + \cdots + t_k)|_{t_i = t/k}.$$

So,

$$\frac{d^k P_\Lambda(U^0 + U + tV)}{dt^k} = \frac{\partial^k}{\partial t_1 \cdots \partial t_k} \frac{1}{\Lambda} \ln E_\Lambda(\exp \sum_{i=1}^k t_i H_i)|_{t_i = t/k},$$

where E_Λ denotes the integral with respect to the Gibbs distribution for the potential U^0+U over the finite space Ω_Λ and H_i are functions (Hamiltonians) on Ω_Λ: $H_i(\bar{\xi}_\Lambda) = H(\bar{\xi}_\Lambda) = \sum_{Q\subset\Lambda} V(\bar{\xi}_Q)$. Using the notation of Ursell functions (see II.12 of [Sim93]), we have

$$\frac{d^k}{dt^k}P_\Lambda(U^0+U+tV) = \frac{1}{\Lambda}u_{k,t/k,\Lambda}(H_1, H_2, \cdots, H_k).$$

Let $c(Q)$ denotes the center of the cube Q. We can rewrite the function $H(\bar{\xi}_\Lambda)$ into a sum over lattice points in Λ.

$$H(\bar{\xi}_\Lambda) = \sum_{x\in\Lambda}\sum_{c(Q)=x, Q\subset\Lambda} V(\bar{\xi}_Q).$$

We define a family of functions indexed by $x \in \Lambda$

$$g_x = \sum_{c(Q)=x, Q\subset\Lambda} V(\bar{\xi}_Q).$$

By the multi-linearity of the Ursell functions, we have

$$\frac{d^k P_\Lambda(U^0+U+tV)}{dt^k} = \frac{1}{\Lambda}\sum_{x_1,x_2,\cdots,x_k\in\Lambda} u_{k,t/k,\Lambda}(g_{x_1}, g_{x_2}, \cdots, g_{x_k}).$$

According to Theorem II.12.10 and Corollary II.12.8, to show $\frac{d^k}{dt^k}P_\Lambda(U^0+U+tV)$ is bounded for each k, we need only to prove that the following condition holds: there exists some constant $m, C_0, > 0$, where m and C_0 may depend on k and U^0, but are independent of U, V, and Λ, such that the truncated correlation function satisfy the condition

$$|< g_{x_1}\cdots g_{x_j}; g_{x_{j+1}}\cdots g_{x_k} >_\Lambda| \le C_0 e^{-ml}, \tag{24}$$

as long as there are two coordinate hyperplanes a distance of l apart separating $x_1, \cdots, x_j$ from $x_{j+1}, \cdots, x_k$. We recall that the definition of the truncated correlation function $< g_1; g_2 >$ for two functions g_1, g_2 is $< g_1 g_2 > - < g_1 >< g_2 >$. The integral $< \cdot >_\Lambda$ is again with respect to the Gibbs distribution for the potential U^0+U over the finite space Ω_Λ.

Estimation of the truncated correlation functions:

To estimate the truncated correlation functions, we need the following theorem from [BK97] (Theorem 1).

Theorem 6.7. *For each U^0 and $0 < \theta < 1$, there exist $\epsilon > 0, m > 0, C > 0$ such that if $\|U\| < \epsilon, U \in \mathcal{P}$, the truncated correlation functions satisfy, for all functions $h_1 : \Omega_{X_1} \to \mathbb{R}$, $h_2 : \Omega_{X_2} \to \mathbb{R}$, $X_1, X_2 \subset \Lambda \subset \mathbb{Z}^{d+1}$,*

$$\| < h_1 h_2 >_\Lambda - < h_1 >_\Lambda < h_2 >_\Lambda \| \leq C \min(|X_1|, |X_2|) \|h_1\| \|h_2\| e^{-md(X_1, X_2)},$$

where $d(X_1, X_2)$ is the distance between the sets X_1 and X_2 and $< \cdot >_\Lambda$ is the integral with respect to the Gibbs distribution from the potential $U^0 + U$.

We now estimate $| < g_{x_1} \cdots g_{x_j}; g_{x_{j+1}} \cdots g_{x_k} >_\Lambda |$. We rewrite

$$g_x = \sum_{c(Q)=x} V(\bar{\xi}_Q) = \sum_{n=0}^{\infty} V_n(\bar{\xi}_{Q_n(x)}),$$

where $Q_n(x)$ are cubes centered at x with $2n+1$ lattice points on each side and $V_n, n = 0, 1, 2, \cdots$ are functions from $\Omega_{Q_n(x)} \to \mathbb{R}$ from the definition of the potential V. Note also that we have removed the restriction $Q \subset \Lambda$ in the expression of g_x for the convenience of estimation. Since we assumed $V \in \mathcal{P}$ and $\|V\| = \sup_{n=0}^{\infty} \theta^{-n} \|V_n\| = 1$, we have $\|V_n\| \leq \theta^n$. It is clear that $< g_{x_1} \cdots g_{x_j}; g_{x_{j+1}} \cdots g_{x_k} >_\Lambda$ is multi-linear in (g_{x_i}). Therefore, we have

$$\begin{aligned} & | < g_{x_1} \cdots g_{x_j}; g_{x_{j+1}} \cdots g_{x_k} >_\Lambda | \\ = & \sum_{n_1, \cdots, n_k = 0}^{\infty} < V_{n_1}(\bar{\xi}_{Q_n(x_1)}) \cdots V_{n_j}(\bar{\xi}_{Q_n(x_j)}); V_{n_{j+1}}(\bar{\xi}_{Q_n(x_{j+1})}) \cdots V_{n_k}(\bar{\xi}_{Q_n(x_k)}) >_\Lambda . \end{aligned}$$

We consider the terms of this sum in two cases:

Case one: when $n_1, \cdots, n_k \leq \frac{l}{4}$, where l is the separation constant between $\{x_1, \cdots, x_j\}$ and $\{x_{j+1}, \cdots, x_k\}$.

According to Theorem 6.7, we have

$$\begin{aligned} & | < V_{n_1}(\bar{\xi}_{Q_n(x_1)}) \cdots V_{n_j}(\bar{\xi}_{Q_n(x_j)}); V_{n_{j+1}}(\bar{\xi}_{Q_n(x_{j+1})}) \cdots V_{n_k}(\bar{\xi}_{Q_n(x_k)}) >_\Lambda | \\ & \leq \theta^{n_1 + n_2 + \cdots + n_k} C k (l/2 + 1)^{d+1} e^{-ml/2} \\ & \leq \theta^{n_1 + n_2 + \cdots + n_k} C' e^{-ml/4}, \end{aligned}$$

where

$$C' = \sup_{0 \le l < \infty} Ck(l/2+1)^{d+1} e^{-ml/4}.$$

Let $\sum_1$ denote the sum of all such terms in case one. Then, we have

$$|\sum_1| \le C' e^{-ml/4} \sum_{n_1,\cdots,n_k=0}^{\infty} \theta^{n_1+n_2+\cdots+n_k} = \frac{C'}{(1-\theta)^k} e^{-ml/4}.$$

Case two: when at least one $n_i > l/4$ in the term determined by the sequence $(n_1, n_2, \cdots n_k)$.

Note that $| < h_1; h_2 >_\Lambda | \le 2\|h_1\|\|h_2\|$. So we have

$$| < V_{n_1}(\bar{\xi}_{Q_n(x_1)}) \cdots V_{n_j}(\bar{\xi}_{Q_n(x_j)});\ V_{n_{j+1}}(\bar{\xi}_{Q_n(x_{j+1})}) \cdots V_{n_k}(\bar{\xi}_{Q_n(x_k)}) >_\Lambda |$$

$$\le 2\|V_{n_1}\|\|V_{n_2}\| \cdots \|V_{n_k}\|$$

If we let $\sum_2$ denote the sum of all such terms in case two, we have

$$\sum_2 \le 2k\theta^{l/4} \sum_{n_1,\cdots,n_{k-1}=0}^{\infty} \theta^{n_1+n_2+\cdots+n_{k-1}} = \frac{2k}{(1-\theta)^{k-1}} e^{(\ln\theta)l/4}.$$

Combining these two cases, we have the desired estimation (23).

It seems to us that the arguments from Theorem 6.7 to the main Theorem are standard. However, we can not find exact references.

□

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Department of Mathematics and Computer Science, Wake Forest University, Winston-Salem, NC 27109

Department of Mathematics, University of Texas, Austin, TX 78712-1802

E-mail address, M. Jiang: `jiang@mthcsc.wfu.edu`

E-mail address, R. de la Llave: `llave@math.utexas.edu`