Asymptotics of solutions in $nA + nB \to C$ reaction–diffusion systems

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Abstract

We analyze the long time behavior of initial value problems that model a process where particles of type A and B diffuse in some substratum and react according to $nA + nB \to C$. The case $n = 1$ has been studied before; it presents nontrivial behavior on the reactive scale only. In this paper we discuss in detail the cases $n > 3$, and prove that they show nontrivial behavior on the reactive and the diffusive length scale.

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Contents

1 Introduction and main results

We consider, for arbitrary but fixed $n \in \mathbf{N}$, $n \geq 1$, the reaction–diffusion problem

$$a_t = a_{xx} - \frac{1}{2}(4ab)^n \ , \tag{1}$$

$$b_t = b_{xx} - \frac{1}{2}(4ab)^n \ , \tag{2}$$

for $x \in \mathbf{R}$, $t \geq \tau \geq 0$, with initial conditions $a(x,\tau) = a_0(x)$, $b(x,\tau) = b_0(x)$, satisfying

$$\begin{aligned} &\lim_{x\to-\infty} a_0(x) = 1 \ , \\ &\lim_{x\to+\infty} b_0(x) = 1 \ , \\ &\lim_{x\to+\infty} a_0(x) = \lim_{x\to-\infty} b_0(x) = 0 \ . \end{aligned} \tag{3}$$

The choice of the initial time $t = \tau$, and a class of initial conditions a_0, b_0 will be described later on, but for the purpose of this introduction it is useful to have in mind the "natural" case: $\tau = 0$, $a_0(x) = 1$ for $x < 0$, $a_0(x) = 0$ for $x > 0$, and $b_0(x) = 1$ for $x > 0$, $b_0(x) = 0$ for $x < 0$.

This initial value problem models the time evolution of a chemical system of two (initially separated) substances A and B, that diffuse in some substratum and react according to $nA + nB \to C$, with a substance C that is supposed not to participate in the reaction anymore. The model is a mean–field description of such a situation where the functions a and b represent the densities of the substances A and B. For more details see [3].

Equations (1) and (2) are best studied in terms of the sum

$$v = a + b \ , \tag{4}$$

and the difference

$$u = a - b \ , \tag{5}$$

which satisfy the equations

$$u_t = u_{xx} \ , \tag{6}$$
$$v_t = v_{xx} - (v^2 - u^2)^n \ , \tag{7}$$

with initial conditions v_0 and u_0 (at time $t = \tau$) that satisfy

$$\lim_{x\to-\infty} u_0(x) = 1 \ ,$$
$$\lim_{x\to+\infty} u_0(x) = -1 \ , \tag{8}$$

and

$$\lim_{x\to\pm\infty} v_0(x) = 1 \ . \tag{9}$$

For initial conditions a_0, b_0 with

$$a_0(x) = b_0(-x) \ , \tag{10}$$

the functions v_0 and u_0 are even and odd, respectively, and the equations (6), (7) preserve this symmetry. Furthermore, for the special initial condition

$$u(x,\tau) = -\mu_1(x/\sqrt{\tau}) \ , \tag{11}$$

with μ_1 defined by the equation

$$\mu_1(y) = \mathrm{erf}(\frac{y}{2}) \equiv \frac{2}{\sqrt{\pi}} \int_0^{y/2} e^{-\sigma^2} d\sigma \ , \tag{12}$$

equation (6) has the explicit solution

$$u(x,t) = -\mu_1(x/\sqrt{t}) \ . \tag{13}$$

We note that the initial condition (11) for u (at time $t = \tau$) is simply the solution of equation (6) with the "natural" initial condition, $u(x,0) = 1$ for $x < 0$, $u(x,0) = -1$ for $x > 0$, evaluated at $t = \tau$. To keep this paper as simple as possible we now restrict the discussion to this case, i.e., we consider from now on equation (7) with initial conditions satisfying (9), and u given by (13). We note, however, that more general (asymmetric) initial conditions for u could be treated as well. This would lead to corrections to u of the order $\mathcal{O}(1/t)$, and such corrections do not change in any way the discussion of the equation for v that follows.

The reaction–diffusion problems considered here develop, in addition to the built–in diffusive length scale $\mathcal{O}(\sqrt{t})$, an additional shorter length scale, on which the reaction takes place. The function F,

$$F = \frac{1}{2}(4ab)^n \equiv \frac{1}{2}(v^2 - u^2)^n \ , \tag{14}$$

is called the reaction term or reaction front, and we are interested in describing the asymptotic behavior of the function F for large times. The knowledge of this behavior is useful, since it appears to be universal, in the sense that it is largely independent of the choice of the initial conditions and of the details of the model under consideration. As mentioned above, if v_0 is an even function, then v and as a consequence F are even functions of x. We will see that the critical point of F at $x = 0$ is a maximum, and that F decays (rapidly) for large x.

Before proceeding any further we note that the factor of $4^{n-1/2}$ in (1), (2) and (14) is just a normalization, and has been chosen for convenience to make the equation (7) for v look simple. In fact, any system of the form

$$a_t = D_a a_{xx} - k_a (ab)^n \ ,$$
$$b_t = D_b b_{xx} - k_b (ab)^n \ ,$$

with positive D_a, D_b, k_a, and k_b, and with initial conditions such that

$$\lim_{x\to-\infty} a(x,0) = a_\infty > 0 \ ,$$
$$\lim_{x\to\infty} b(x,0) = b_\infty > 0 \ ,$$
$$\lim_{x\to+\infty} a(x,0) = \lim_{x\to-\infty} b(x,0) = 0 \ ,$$

can be reduced, by scaling space and time and the amplitudes, to the problem

$$a_t = a_{xx} - \frac{1}{2}(4ab)^n \ ,$$
$$b_t = D b_{xx} - \frac{1}{2}(4ab)^n \ ,$$

with $D > 0$, and with initial conditions such that

$$\lim_{x\to-\infty} a(x,0) = 1 ,$$
$$\lim_{x\to\infty} b(x,0) = \beta > 0 ,$$
$$\lim_{x\to+\infty} a(x,0) = \lim_{x\to-\infty} b(x,0) = 0 .$$

In this paper we have limited the discussion to the case $\beta = 1$ and $D = 1$. The case $\beta \neq 1$ leads to a moving reaction front. A change of coordinates to a moving frame complicates the analysis, but the problem could still be treated with the methods presented here. Choosing $D = 1$ makes the equations mathematically simpler. As a consequence, as we have seen, the two equations for a and b can be reduced to just one equation for the sum $v = a + b$, since the equation for the difference $u = a - b$ can be solved explicitly. Even though we do not expect the asymptotic behavior of the solution to change in any relevant way if $D \neq 1$, the strategy of proof would have to be changed considerably, since the equations can not be decoupled anymore in that case.

Before we state our results, we briefly discuss the expected dependence of the results on the parameter n.

The case $n = 1$ has been studied in detail in [1], where it is proved that in this case the reaction term (14) satisfies, for all $z \in \mathbf{R}$,

$$\lim_{t\to\infty} t^{2\gamma} F(t^{\alpha} z, t) = \rho(|z|) ,$$

where $\alpha = 1/6$, and $\gamma = 1/3$, and where $\rho\colon \mathbf{R}_{+}\to \mathbf{R}_{+}$ is a smooth function that decays like $\exp(-\mathrm{const.}z^{3/2})$ for large values of z. It follows furthermore from the results in [1] that the function F is very small on the diffusive scale in the sense that for $n = 1$, $y \neq 0$, and all $p \geq 0$,

$$\lim_{t\to\infty} t^{p} F(\sqrt{t}y, t) = 0 . \tag{15}$$

The smallness of F on the diffusive scale is easily understood by realizing that, for $n = 1$ and for positive values of x on the diffusive scale, i.e., for $x/\sqrt{t} >> 1$, equation (1) essentially reduces to

$$a_t = a_{xx} - \lambda a , \tag{16}$$

with $\lambda > 0$. Therefore, the function a decays exponentially fast to zero on this scale, and similarly for b for negative values of x.

For $n > 1$, however, equation (1) reduces, for $x/\sqrt{t} >> 1$, essentially to

$$a_t = a_{xx} - \lambda a^{n} , \tag{17}$$

with $\lambda > 0$. The solution of (17) has an asymptotic behavior that is radically different from the solution of (16). In particular, for $n = 2$, the solution may even blow up in finite time if a is not a positive function. Note that, for n odd, the nonlinear term in (17) is always a "friction term", independent of the sign of a, and the case of n odd will therefore turn out to be easier to treat than the case of n even. It is well known [2] that for $n > 3$ and small bounded integrable initial conditions, the nonlinearity in (17) becomes irrelevant for large times in the sense that the solution converges to a multiple of $\exp(-x^2/4t)/\sqrt{t}$, which solves the linear equation $a_t = a_{xx}$. We would therefore expect that, for $n > 3$, the function F is of the order $\mathcal{O}(t^{-n/2})$ on the diffusive scale. This turns out to be wrong. As we will prove below, F is of the order $\mathcal{O}(t^{-n/(n-1)})$ for $n > 3$, because F converges on this scale pointwise to a function that is not integrable at the origin. This corresponds to a solution of (17) for which the nonlinear term is a marginal perturbation, i.e., a solution with an amplitude of the order $\mathcal{O}(t^{-1/(n-1)})$. We will see that one can take advantage of this fact, and a diffusive stability bound will be good enough to prove convergence of F to its limit, but as a consequence, our results will be limited to the case $n > 3$. The cases $n = 2$ and $n = 3$ are special and will not be discussed any further.

The following theorem is our main result.

Theorem 1 *For arbitrary but fixed $n \in \mathbf{N}$, $n \geq 4$, there exist $\tau > 0$, functions μ_1, μ_2, φ_1, φ_2, and a class of initial conditions (specified at $t = \tau$), such that (7) has a unique solution v that satisfies for all $t \geq \tau$ the bound*

$$|v(x,t) - v_{\infty}(x,t)| < \frac{\mathrm{const.}}{t^{4\gamma}} , \tag{18}$$

where

$$v_{\infty}(x,t) = \mu_1\left(\frac{|x|}{\sqrt{t}}\right) + t^{-\varepsilon}\mu_2\left(\frac{|x|}{\sqrt{t}}\right) + t^{-\gamma}\varphi_1\left(\frac{|x|}{t^{\alpha}}\right) + t^{-3\gamma}\varphi_2\left(\frac{|x|}{t^{\alpha}}\right) , \tag{19}$$

$\gamma = \frac{1}{2n+1}$, $\varepsilon = \frac{1}{n-1}$ *and* $\alpha = \frac{1}{2} - \gamma$.

Remark 2 *This theorem is a local result, in the sense that the class of initial conditions will be a set of functions in a (small) neighborhood of the function $v_{\infty,0}$, $v_{\infty,0}(x) = v_{\infty}(x,\tau)$. In particular, our methods do not allow us to show that the solution with the "natural" initial condition $v_0 \equiv 1$ at $t = 0$ belongs to this set at $t = \tau$. We do expect, however, that this is the case, as has been proved for $n = 1$ in [1].*

Remark 3 *We note that, if an initial condition v_0 is such that $v_0(x) - |u(x/\sqrt{\tau})| < 0$ for a certain x, then $a_0(x) < 0$, if $x > 0$, or $b_0(x) < 0$ if $x < 0$. A priori, we do not need to consider such initial conditions, since in our model a and b represent particle densities, and the solutions a and b are positive if the initial conditions a_0 and b_0 are positive. As we will see, for $n \geq 4$, it will not be necessary to impose that a_0 and b_0 be positive everywhere, and it will neither be necessary to impose that $v_0 = a_0 + b_0$ be an even function.*

As we will see, the functions φ_1 and φ_2 are small on the diffusive scale, i.e., for $x \approx \sqrt{t}y$, $y \neq 0$, and t large,

$$v_\infty(\sqrt{t}y, t) = \mu_1(|y|) + t^{-\varepsilon}\mu_2(|y|) + \mathcal{O}(t^{-2\varepsilon'}) \ , \tag{20}$$

where $\varepsilon' = \varepsilon$ if $n > 5$, and $2\gamma < \varepsilon' < \varepsilon$ if $n = 4$, 5. Using the definition (4), (5) for v and u, we therefore find that for $y > 0$ and t large,

$$a(\sqrt{t}y, t) = \frac{1}{2}t^{-\varepsilon}\mu_2(y) + \mathcal{O}(t^{-2\varepsilon'}) \ ,$$

and similarly for b, for $y < 0$. In contrast to the case $n = 1$, where only exponentially few particles reach the diffusive scale, the amount of particles decays only slowly for $n > 3$. Our results imply that, for large times, the density of the remaining particles is given by the function μ_2, i.e., it is independent of the initial conditions.

As a corollary to Theorem 1 we get a precise description of the reaction front F on the reactive and the diffusive scale. This description will be given in Section 4, once we have defined the functions μ_1, μ_2, φ_1 and φ_2 in Section 3. In Section 2 we explain our strategy for proving Theorem 1. This strategy is implemented in Section 5 and Section 6. The Appendix contains the proof of the existence of the functions μ_2, φ_1 and φ_2.

2 Strategy of the Proof

Consider functions v of the form

$$v(x,t) = v_\infty(x,t) + \psi(x,t) \ , \tag{21}$$

with v_∞ as in Theorem 1, and $\psi(x,\tau) = \psi_0(x)$, for some $\tau >> 1$, with $\psi_0 \in L_1 \cap L_\infty$. Substituting (21) into (7) leads to an equation for the function ψ of the form

$$\dot{\psi} = \psi'' - V\psi - I - T(\psi) \ , \tag{22}$$

for certain functions V and I, and for T some nonlinear map. We will show that if v_∞ is defined correctly, τ large enough and ψ_0 small enough, then V can be chosen positive and T will be small, so that the solution of equation (22) will be bounded for large times by the corresponding solution of the inhomogeneous heat equation $\dot{\psi} = \psi'' - I$. We will find that, with the right choice of v_∞,

$$\int dx \ \left|I(\sqrt{t}x, t)\right| \leq \text{const.}\ t^{-1-4\gamma} \ , \tag{23}$$

from which the bound (18) will follow. We note that $4\gamma < \frac{1}{2}$ for $n \geq 4 > \frac{7}{2}$, so that contributions of initial conditions will become irrelevant for large times, i.e., the solution ψ becomes what is called "slaved to the inhomogeneous term".

3 Asymptotic Expansion

In order to implement the strategy outlined in Section 2, we need a function v_∞ that approximates the solution v for large times sufficiently well, uniformly in x. Since we would like to control the time evolution of equation (22) on $L_1 \cap L_\infty$, this function v_∞ needs to satisfy $\lim_{x\to\pm\infty} v_\infty(x,t) = 1$ in order for v to satisfy the boundary conditions (9). Furthermore, the inhomogeneous term I in equation (22) contains second derivatives of v_∞, and the function I can therefore only be in $L_1 \cap L_\infty$ if v_∞ is at least twice differentiable. We now construct a function v_∞ satisfying these requirements through a two length–scale asymptotic expansion.

To simplify the notation later on we use the convention that, unless stated otherwise,

$$y \equiv \frac{x}{\sqrt{t}} \ , \tag{24}$$

and

$$z \equiv \frac{x}{t^\alpha} \ , \tag{25}$$

and we will refer to y as the diffusive length scale and to z as the reactive length scale.

The function v_∞ is given by the first and second order terms of a so called "matched asymptotic expansion". The "matched" refers to the fact that such an expansion contains functions that can not be associated uniquely with one of the length scales and can therefore be used to "match" the behavior at large distances of the shorter scale with the behavior at short distances of the larger scale. Let

$$\delta = \frac{n+2}{n-1} \ , \tag{26}$$

and let γ, α, and ε be as in Theorem 1. Then, the functions $y \equiv t^{-\gamma}z$, $t^{-\varepsilon}y^{-\delta} \equiv t^{-\gamma}z^{-\delta}$, $t^{-\varepsilon}y^{2-\delta} \equiv t^{-3\gamma}z^{2-\delta}$ and $t^{-2\varepsilon}y^{-2\delta+1} \equiv t^{-3\gamma}z^{-2\delta+1}$ are of this form and will naturally show up in the function v_∞. As a consequence, the representation (19) for v_∞ is not unique. If we choose (as we will) to compute the expansion for v_∞ in the order of decreasing amplitudes, i.e., if we first compute the term of order $\mathcal{O}(t^{-\gamma})$, then the term of order $\mathcal{O}(t^{-\varepsilon})$, and finally the term of order $\mathcal{O}(t^{-3\gamma})$, we get a representation of v_∞ of the form

$$v_\infty(x,t) = \mu_1(|y|) + t^{-\gamma}\eta(|z|) + t^{-\varepsilon}\left(\mu_2(|y|) - \lambda\,|y|^{-\delta}\right) + t^{-3\gamma}\varphi_2(|z|) \ , \tag{27}$$

where

$$\eta(z) = \varphi_1(z) + \lambda z^{-\delta} \ , \tag{28}$$

with λ a certain constant to be determined later.

We note that, by definition, v_∞ is a symmetric function, and it is therefore sufficient to consider positive values of x if we choose appropriate boundary conditions at $x = 0$ to ensure regularity.

Finally, since we will need to describe the asymptotic behavior of various functions near zero and infinity, we introduce the following notation. Let f be a continuous function from $\mathbf{R}_+$ to $\mathbf{R}$, k a positive integer and $p_1 < p_2 < \cdots < p_k$ real numbers. Then, we say that

$$f(x) = \sum_{i=1}^{k} f_i\ x^{p_i} + \ldots$$

near $x = 0$, if

$$\lim_{x\to 0^+} \frac{1}{x^{p_k}} \left| f(x) - \sum_{i=1}^{k} f_i\ x^{p_i} \right| = 0 \ ,$$

and we say that

$$f(x) = \sum_{i=1}^{k} f_i\ x^{-p_i} + \ldots$$

near $x = \infty$, if

$$\lim_{x\to\infty} x^{p_k} \left| f(x) - \sum_{i=1}^{k} f_i\ x^{-p_i} \right| = 0 \ .$$

3.1 Equation for μ_1

To lowest order the function v is asymptotic to $\mu_1(|y|)$, with μ_1 as defined in (12). We note that μ_1 has near $y = 0$ the expansion

$$\mu_1(y) = \kappa y + \kappa_3 y^3 + \ldots \ , \tag{29}$$

where $\kappa = \frac{1}{\sqrt{\pi}}$ and $\kappa_3 = -\frac{1}{12}\kappa$. Furthermore, $\lim_{y\to\infty}\mu_1(y) = 1$.

3.2 Equation for φ_1

We make the ansatz $v(x,t) = \mu_1(|y|) + t^{-\gamma}\eta(|z|)$ which we substitute into equation (7). We multiply the resulting equation with $t^{\gamma+2\alpha}$, and take then the limit $t \to \infty$, keeping z fixed. This leads to the differential equation

$$\eta'' = \left(2\kappa z\eta + \eta^2\right)^n \ , \tag{30}$$

where z is now considered a variable in $\mathbf{R}_+$. Since $\lim_{y\to\infty}\mu_1(y) = 1$, the correct boundary condition for η at infinity is

$$\lim_{z\to\infty} \eta(z) = 0 \ , \tag{31}$$

and at $z = 0$ we impose

$$\eta'(0) = -\kappa \ , \tag{32}$$

which makes the function $\mu_1(|y|) + t^{-\gamma}\eta(|z|)$ twice differentiable at $x = 0$, since μ_1 and η are twice differentiable at zero and $\partial_x\left(\mu_1(|y|) + t^{-\gamma}\eta(|z|)\right)(0) = 0$. A proof of the following proposition can be found in the appendix.

Proposition 4 *For $n \in \mathbf{N}$, $n \geq 2$, there exists a unique function $\eta \colon \mathbf{R}_+ \to \mathbf{R}$ that satisfies equation (30) with the boundary conditions (31) and (32). The function η is positive, and has near $z = 0$ the expansion*

$$\eta(z) = \eta_0 - \kappa z + \eta_2 z^2 - \eta_4 z^4 + \dots \ ,$$

with positive coefficients η_0, η_2 and η_4. For z large, η is of the form

$$\eta(z) = \frac{\lambda}{z^\delta} + \frac{\lambda_\infty}{z^{\delta'}} + \dots \ ,$$

for a certain constant λ_∞, with δ as in (26),

$$\lambda = \left(\frac{\delta(\delta+1)}{(2\kappa)^n} \right)^{1/(n-1)} \ , \tag{33}$$

and

$$\delta' = \begin{cases} \left(\sqrt{4n\delta(\delta+1)+1} - 1 \right)/2 & 2 \leq n \leq 5 \ , \\ 2\delta + 1 & n \geq 6 \ . \end{cases}$$

We note that $3 < \delta' \leq 2\delta + 1$. The constants η_0, η_2, η_4 and λ_∞ are given in the appendix. The function φ_1 is defined in (28) in terms of η.

3.3 Equation for μ_2

We make the ansatz $v(x,t) = \mu_1(|y|) + t^{-\gamma}\eta(|z|) + t^{-\varepsilon}(\mu_2(|y|) - \lambda\, |y|^{-\delta})$ which we substitute into equation (7). We multiply the resulting equation with $t^{1+\varepsilon}$, and take then the limit $t \to \infty$, keeping y fixed. Since $\eta(z) = \eta(t^\gamma y)$ and $\lim_{t\to\infty} t^{\varepsilon-\gamma}\eta(z) - \lambda y^{-\delta} = 0$, this leads to the differential equation for the function μ_2,

$$\mu_2'' + \frac{1}{2} y \mu_2' + \varepsilon \mu_2 = (2\mu_1\mu_2)^n \ , \tag{34}$$

where y is now considered as a variable in $\mathbf{R}_+$. At $y = 0$ we impose the boundary condition

$$\lim_{y \to 0} \mu_2(y) y^\delta = \lambda \ , \tag{35}$$

which removes the leading singularity of the function $\mu_2(|y|) - \lambda\, |y|^{-\delta}$ at $y = 0$. As we will see, the sub–leading singularity is proportional to $|y|^{2-\delta}$, which is not a twice differentiable function at $y = 0$ (except for $n = 4$ where $\delta = 2$). This singularity will be cancelled by imposing appropriate boundary conditions for the function φ_2.

The correct choice of boundary conditions for μ_2 at infinity is somewhat less obvious. In the appendix we show that the condition $\lim_{y\to\infty} \mu_2(y) = 0$ is not sufficient to single out a unique function μ_2. If μ_2 does converge to zero at infinity, then it is asymptotic to a solution of the equation

$$\mu'' + \frac{1}{2} y \mu' + \varepsilon \mu = 0 \ .$$

This linear equation is compatible with a (very slow) algebraic decay, $\mu_2(y) \approx y^{-2\varepsilon}$, or with a modified Gaussian decay, $\mu_2(y) \approx \exp(-y^2/4)/y^{1-2\varepsilon}$, with the algebraic decay being the generic case. It will be essential in later sections that μ_2 decays rapidly at infinity, and we therefore impose the boundary condition

$$\lim_{y \to \infty} \mu_2(y) y^{2\varepsilon} = 0 \ . \tag{36}$$

A proof of the following proposition can be found in the appendix.

Proposition 5 *For all $n \geq 4$, there exists a unique positive function $\mu_2 \colon \mathbf{R}_+ \to \mathbf{R}$ that satisfies equation (34) with the boundary conditions (35) and (36). For y small, the function μ_2 is of the form*

$$\mu_2(y) = \lambda y^{-\delta} + \lambda_0 y^{2-\delta} + \lambda_1 y^{4-\delta} + \dots \ , \tag{37}$$

with

$$\lambda_0 = \frac{1}{2}\frac{\lambda}{\kappa} \frac{-2n\kappa_3\delta(\delta+1) - \kappa(\delta - 2\varepsilon)}{(n-1)\,\delta(\delta+1) + 2(2\delta-1)} > 0 \ , \tag{38}$$

with $\lambda_1 \neq 0$ and with λ as in Proposition 4. For y large, the function μ_2 decays rapidly in the sense that

$$\mu_2(y) = \exp(-\frac{y^2}{4}) \left(\frac{C}{y^{1-2\varepsilon}} + \dots \right) \ , \tag{39}$$

for some constant $C > 0$.

3.4 Equation for φ_2

We make the ansatz $v(x,t) = \mu_1(|y|) + t^{-\gamma}\eta(|z|) + t^{-\varepsilon}(\mu_2(|y|) - \lambda |y|^{-\delta}) + t^{-3\gamma}\varphi_2(|z|)$ which we substitute into equation (7). We multiply the resulting equation with $t^{3\gamma+2\alpha}$, and take then the limit $t \to \infty$, keeping z fixed. This leads to the (linear) differential equation for φ_2,

$$\varphi_2'' + \gamma\eta + \alpha z\eta' + (2-\delta)(1-\delta)\lambda_0 z^{-\delta} = n\left(2\kappa z\eta + \eta^2\right)^{n-1}\left[(2\kappa z + 2\eta)(\varphi_2 + \lambda_0 z^{2-\delta}) + 2\kappa_3 z^3\eta\right] . \tag{40}$$

In order to compensate the sub–leading singular behavior of μ_2 near $x = 0$ we make the ansatz

$$\varphi_2(z) = -\lambda_0 z^{2-\delta} + h(z) , \tag{41}$$

which we substitute into equation (40). For the function h we get the equation

$$h'' + \gamma\eta + \alpha z\eta' = n\left(2\kappa z\eta + \eta^2\right)^{n-1}\left((2\kappa z + 2\eta)h + 2\kappa_3 z^3\eta\right) . \tag{42}$$

Since the function η is regular near $z = 0$, the solution h turns out to be regular near $z = 0$, too. Therefore, the function $z \mapsto h(|z|)$ is twice differentiable near $x = 0$ if we impose at $z = 0$ the boundary condition

$$h'(0) = 0 . \tag{43}$$

At infinity we need that $\lim_{z\to\infty}\varphi_2(z) = 0$. We therefore require that

$$\lim_{z\to\infty}(h(z) - \lambda_0 z^{2-\delta}) = 0 . \tag{44}$$

A proof of the following proposition can be found in the appendix.

Proposition 6 *For all $n \geq 4$, there exists a unique function $h\colon \mathbf{R}_+ \to \mathbf{R}$ that satisfies equation (42) with the boundary conditions (43) and (44). Near $z = 0$, the function h is of the form*

$$h(z) = h_0 + h_2 z^2 + \ldots ,$$

with certain coefficients h_0 and h_2, and for z large h is of the form

$$h(z) = \lambda_0 z^{2-\delta} + \frac{\lambda'}{z^{\delta'-2}} + \ldots ,$$

with λ_0 as defined in (38), for a certain constant λ', and with δ' as defined in Proposition 4.

4 The Reaction Front

Using the properties of the functions μ_1, μ_2, φ_1 and φ_2, we get from Theorem 1 the following behavior of the reaction front F.

Corollary 7 *Let v be as in Theorem 1, and F as defined in (14). Then, for all $z \in \mathbf{R}$,*

$$\lim_{t\to\infty} t^{2n\gamma}F(t^\alpha z,t) = \frac{1}{2}(2\kappa|z|\eta(|z|) + \eta(|z|)^2)^n = \frac{1}{2}\eta''(|z|) = \begin{cases} \eta_2 - 6\eta_4|z|^2 + \ldots & \textit{for} \quad |z| \approx 0 , \\ \frac{1}{2}(2\lambda\kappa)^n / |z|^{\delta+2} + \ldots & \textit{for} \quad |z| >> 1 , \end{cases}$$

and for all $y \neq 0$,

$$\lim_{t\to\infty} t^{n\varepsilon}F(\sqrt{t}y,t) = \frac{1}{2}(2\mu_1\ \mu_2)^n(|y|) = \begin{cases} \frac{1}{2}(2\lambda\kappa)^n / |y|^{\delta+2} + \ldots & \textit{for} \quad |y| \approx 0 , \\ \exp(-n|y|^2/4)(2^{n-1}C^n / |y|^{n(1-2\varepsilon)} + \ldots) & \textit{for} \quad |y| >> 1 . \end{cases}$$

Here, η_2, η_4 are as defined in Proposition 4 and C is as defined in (39).

5 The Equation for ψ

In order to simplify the notation we define the function $\overline{u}$,

$$\overline{u}(x,t) = \mu_1(|y|) ,$$

the function μ_3,

$$\mu_3(y) = \mu_2(y) - \lambda y^{-\delta} ,$$

the function ϕ,

$$\phi(x,t) = t^{-\gamma}\eta(|z|) + t^{-\varepsilon}\mu_3(|y|) + t^{-3\gamma}\varphi_2(|z|) \ , \tag{45}$$

and the function ϕ_1,

$$\phi_1(x,t) = \phi(x,t) - \kappa\frac{|x|}{\sqrt{t}} \ . \tag{46}$$

The function v_∞ in Theorem 1 and in (27) can then be written as $v_\infty = \overline{u} + \phi$.

Let now $v = v_\infty + \psi$. Then,

$$\begin{aligned}\left(v^2 - u^2\right)^n &= \left((\overline{u} + \phi + \psi)^2 - u^2\right)^n = \left((2\overline{u}\phi + \phi^2) + (2(\overline{u} + \phi) + \psi)\,\psi\right)^n \\ &= \sum_{k=0}^{n} \binom{n}{k} \left(2\overline{u}\phi + \phi^2\right)^{n-k} (2(\overline{u} + \phi) + \psi)^k\, \psi^k \ .\end{aligned}$$

Therefore, substituting the ansatz $v = v_\infty + \psi$ into (7) leads to the following equation for the function ψ,

$$\dot{\psi} = \psi'' - \widehat{V}\psi - I - \widehat{T}(\psi) \ , \tag{47}$$

with the function $\widehat{V}$,

$$\widehat{V} = 2n \ (2\overline{u}\phi + \phi^2)^{n-1} \ (\overline{u} + \phi) \ , \tag{48}$$

the function I,

$$I = -\dot{\phi} + \phi_1'' - \left(2\overline{u}\phi + \phi^2\right)^n \ , \tag{49}$$

and the map $\widehat{T}$,

$$\widehat{T}(\psi) = n\left(2\overline{u}\phi + \phi^2\right)^{n-1}\psi^2 + \sum_{k=2}^{n} \binom{n}{k} \left(2\overline{u}\phi + \phi^2\right)^{n-k} (2(\overline{u} + \phi) + \psi)^k\, \psi^k \ . \tag{50}$$

5.1 The function V

The function $\widetilde{\mu}$, $\widetilde{\mu}(y) = \mu_1(y)/y$ is strictly decreasing on $\mathbf{R}_+$, and therefore $\mu_1(y)/y \geq \mu_1(t^\gamma y)/(t^\gamma y)$ for $t \geq 1$. Furthermore, the functions η and μ_1 are strictly positive and μ_1 is strictly increasing. These properties imply that, for $t \geq \tau \geq 1$, $\overline{u}(x,t) + t^{-\gamma}\eta(|z|) = \mu_1(|y|) + t^{-\gamma}\eta(|z|) \geq t^{-\gamma}(\mu_1(|z|) + \eta(|z|)) \geq t^{-\gamma}c_0 > 0$, where $c_0 = \inf_{z>0}(\mu_1(z) + \eta(z))$. Next, since the functions μ_3 and φ_2 are bounded and since $3\gamma \geq \varepsilon$, for $n \geq 4$, we have that $\left|t^{-\varepsilon}\mu_3(|y|) + t^{-3\gamma}\varphi_2(|z|)\right| <$ const. $t^{-\varepsilon}$, and as a consequence $(\overline{u} + \phi)$ and $(2\overline{u} + \phi)$ are positive functions of x for all fixed $t \geq \tau_0$, if τ_0 large enough.

Proposition 8 *For n odd, $n \geq 5$, there exists $\tau_0 \geq 1$, such that for all $t \geq \tau_0$ the function $\widehat{V}$ is positive.*

Proof. The function $(2\overline{u}\phi + \phi^2)^{n-1}$ is positive, for n odd. ■

As a consequence, for n odd, equation (47) is of the form indicated in Section 2, with $V = \widehat{V}$ and $T = \widehat{T}$. The rest of this section treats the case of n even, which, as indicated in the introduction, is slightly more delicate. It can be skipped in a first reading or if the reader is only interested in the case of n odd.

So let n be even. The idea is to split $\widehat{V}$ into its positive part $V = \widehat{V}_+$ and its negative part $V_1 = \widehat{V}_-$, and to show that $\widehat{V}_-$ is small enough so that it can be treated together with the nonlinear term. Consider the function ϕ defined in (45). The problem is that ϕ becomes negative for large values of x, and that therefore $\widehat{V}$ becomes negative for large values of x. To understand why ϕ becomes negative, we note that the leading order term $t^{-\gamma}\lambda z^{-\delta}$ in the large z asymptotics of $t^{-\gamma}\eta(z)$ is compensated by the leading order term $-t^{-\varepsilon}\lambda y^{-\delta}$ in the large y asymptotics of $t^{-\varepsilon}\mu_3(y)$. The leading order of ϕ at x large is therefore given by the second order term in the large z asymptotics of η and the leading term in the large z asymptotics of φ_2. The first of these terms is proportional to $t^{-\gamma}z^{-\delta'}$, and the second one is proportional to $t^{-3\gamma}z^{2-\delta'} \equiv t^{-\gamma}y^2 z^{-\delta'}$. The corresponding proportionality constants λ_∞ and λ' can be computed for $n > 5$ and turn out to be negative. For $n = 4, 5$ these constants can not be obtained from asymptotic expansions, but numerical results show that they are in fact also negative in these cases. We do not need a proof of this numerical fact, because the following proposition is also correct for positive $\widehat{V}$.

Proposition 9 *For n even, $n \geq 4$, there exists $\tau_1 \geq 1$, such that the function V_1, satisfies for all $t \geq \tau_1$ the bound*

$$\sup_{x\in\mathbf{R}} |V_1(x,t)| \leq \text{const.}\ t^{-\gamma(n-1)(\delta'+1)} \ . \tag{51}$$

Proof. The idea is to write ϕ as the sum of a function ϕ_0 that is positive and a function ϕ_∞ that absorbs the asymptotic behavior at infinity. Since $\mu_3(y) \approx \lambda_0 y^{2-\delta}$ for y small, with $\lambda_0 > 0$, there exists $y_0 > 0$ such that $\mu_3(|y|) \geq 0$, for all $|y| \leq y_0$. Let $c > 0$, to be chosen below, and let θ be the Heaviside step function, i.e., $\theta(x) = 1$ for $x > 0$, and $\theta(x) = 0$ for $x < 0$. Then, we define the function ϕ_∞ by the equation

$$\phi_\infty(x,t) = -ct^{-\gamma}\theta(|y| - y_0)\,|y|^2\,|z|^{-\delta'} \quad ,$$

and we set $\phi_0 = \phi - \phi_\infty$. In order to prove that ϕ_0 is positive, for c large enough, we write $\phi_0 = \phi_0^{(1)} + \phi_0^{(2)}$, where

$$\phi_0^{(1)}(x,t) = t^{-\gamma}(\eta(|z|) - \lambda|z|^{-\delta}\theta(|y| - y_0)) + t^{-3\gamma}\varphi_2(|z|) + c\;\theta(|y| - y_0)t^{-3\gamma}|z|^{2-\delta'} \; ,$$

and

$$\phi_0^{(2)}(x,t) = t^{-\varepsilon}\left(\mu_3(|y|) + \lambda|y|^{-\delta}\theta(|y| - y_0)\right) \; .$$

$\phi_0^{(2)}$ is positive for $|y| > y_0$, since in this case $\phi_0^{(2)}(x,t) = \mu_2(|y|) > 0$, and $\phi_0^{(2)}$ is positive for $|y| < y_0$ by definition of y_0. Next we consider $\phi_0^{(1)}$. For $|z| < t^\gamma y_0$ we have that $\phi_0^{(1)}(x,t) = t^{-\gamma}\eta(|z|) + t^{-3\gamma}\varphi_2(|z|)$. But $t^{-\gamma}\eta(z) + t^{-3\gamma}\varphi_2(z) > 0$ for all $z \geq 0$, and all $t \geq \tau$, if τ is sufficiently large, since $\eta > 0$, since φ_2 is bounded, and since $|\varphi_2(z)| < \eta(z)$ for z large enough. Finally, using the asymptotic properties of η and φ_2 we see that $\phi_0^{(1)} > 0$ for $|z| > t^\gamma y_0$ if c is chosen large enough.

We now estimate the function V_1. From the definition of ϕ_∞ we get that

$$|\phi_\infty(x,t)| \leq \text{const.}\; t^{-\gamma(\delta'+1)} \; ,$$

and therefore, since ϕ_0 is positive, we have the lower bound

$$\phi(x,t)^{n-1} \geq \overline{c}\; t^{-\gamma(n-1)(\delta'+1)} \; ,$$

for some constant $\overline{c} < 0$, from which (51) follows. ■

5.2 The function I

Theorem 10 *Let I be as defined in (49), and let $n \geq 4$. Then, there exists a constant $c_I > 0$, such that for all $t \geq 1$,*

$$\int_{-\infty}^{\infty} dx\; |I(\sqrt{t}x, t)| \leq c_I\; t^{-1-4\gamma} \; . \tag{52}$$

The function I is even, and it is therefore sufficient to bound it for $x \geq 0$. The strategy of the proof will be to rewrite the function I as a sum of functions of the form $t^{-\sigma}G(y)F(z)$, with $\sigma > 0$, and with G and F functions with appropriate asymptotic behavior at zero and infinity. Each of the terms in the sum can then be estimated with the help of Lemma 13 below. In order to keep the notation as simple as possible, we suppress in what follows the arguments of functions whenever there is no risk of confusion.

Proposition 11 *For $x \geq 0$, the function $-I$ is of the form*

$$-I = \sum_{p=2}^{n}\sum_{q=0}^{p} A_{p,q} + \sum_{i=2}^{8} A_i \; , \tag{53}$$

where

$$\begin{aligned}
A_2 &= \sum_{p=1}^{n-1}\binom{n-1}{p} t^{-2\gamma(n-1-p)-p\varepsilon} T_1^{n-1-p} T_2^p \left(t^{-4\gamma}T_3\right) \; , \\
A_3 &= n(t^{-2\gamma}T_1 + t^{-\varepsilon}T_2)^{n-1}\left(t^{-\gamma(1+\delta')}T_4 + t^{-2\varepsilon}T_5 + t^{-6\gamma}T_6\right) \; , \\
A_4 &= \sum_{p=2}^{n}\binom{n}{p}(t^{-2\gamma}T_1 + t^{-\varepsilon}T_2)^{n-p}(t^{-4\gamma}T_3 + t^{-\gamma(1+\delta')}T_4 + t^{-2\varepsilon}T_5 + t^{-6\gamma}T_6)^p \; , \\
A_5 &= -t^{-1-3\gamma}\left(3\gamma\varphi_2 + \alpha z\varphi_2'\right) \; , \\
A_6 &= -t^{-2n\gamma-2\gamma} n T_1^{n-1} 2z^{2-\delta}\eta\left(\lambda_0 - \mu_3 y^{\delta-2}\right) \; , \\
A_7 &= t^{-2n\gamma+2\gamma-\varepsilon} n T_1^{n-1}(2\mu_1\mu_3 - 2\kappa\lambda_0 y^{3-\delta}) - t^{-2n\gamma+2\gamma-\varepsilon} n(2\kappa\lambda z^{1-\delta})^{n-1}(2\mu_1\mu_3 - 2\kappa\lambda_0 y^{3-\delta}) \; , \\
A_8 &= t^{-2n\gamma+\gamma} n T_1^{n-1} 2\eta((\mu_1 - \kappa y) - \kappa_3 y^3) - t^{-2n\gamma+\gamma} n(2\kappa\lambda z^{1-\delta})^{n-1} 2\lambda z^{-\delta}(\mu_1 - \kappa y - \kappa_3 y^3) \; , \\
A_{p,q} &= \binom{n}{p}\binom{p}{q}\left(R_2^{p,q} - R_1^{p,q}\right) \; ,
\end{aligned}$$

where

$$R_1^{p,q} = t^{-2n\gamma+2p\gamma-p\varepsilon}(2\kappa\lambda z^{1-\delta})^{n-p}(2(\mu_1-\kappa y)\lambda y^{-\delta})^{p-q}(2\mu_1\mu_3)^q \ ,$$
$$R_2^{p,q} = t^{-2n\gamma+2p\gamma-p\varepsilon}T_1^{n-p}(2(\mu_1-\kappa y)y^{-\delta}z^{\delta}\eta)^{p-q}(2\mu_1\mu_3)^q \ ,$$

and where

$$T_1(z) = 2\kappa z\eta(z)+\eta(z)^2 \ , \tag{54}$$
$$T_2(y,z) = 2(\mu_1(y)-\kappa y)y^{-\delta}z^{\delta}\eta(z)+2\mu_1(y)\mu_3(y) \ , \tag{55}$$
$$T_3(y,z) = (2\kappa z+2\eta(z))\,\varphi_2(z)+2\mu_3(y)y^{\delta-2}z^{2-\delta}\eta(z) \ , \tag{56}$$
$$T_4(y,z) = 2(\mu_1(y)-\kappa y)y^{2-\delta'}z^{\delta'-2}\varphi_2(z) \ , \tag{57}$$
$$T_5(y) = \mu_3(y)^2 \ , \tag{58}$$
$$T_6(y,z) = 2\mu_3(y)y^{\delta-2}z^{2-\delta}\varphi_2(z)+\varphi_2(z)^2 \ . \tag{59}$$

Proof. In terms of the functions (54)–(59) we get that, for $x>0$,

$$2\overline{u}\phi+\phi^2 = t^{-2\gamma}T_1+t^{-\varepsilon}T_2+t^{-4\gamma}T_3+t^{-\gamma(1+\delta')}T_4+t^{-2\varepsilon}T_5+t^{-6\gamma}T_6 \ ,$$

and therefore $(2\overline{u}\phi+\phi^2)^n = B_1+\sum_{i=2}^4 A_i$, where

$$B_1 = (t^{-2\gamma}T_1+t^{-\varepsilon}T_2)^n+n(t^{-2\gamma}T_1)^{n-1}(t^{-4\gamma}T_3) \ ,$$

and where A_2, A_3 and A_4 are as defined above. Since $-I=\dot{\phi}-\phi_1''+\left(2\overline{u}\phi+\phi^2\right)^n$, it remains to be shown that

$$B_1-\phi_1''+\dot{\phi} = \sum_{p=2}^{n}\sum_{q=0}^{p}A_{p,q}+\sum_{i=5}^{8}A_i \ .$$

Using the differential equations for μ_2, η and φ_2, we find that

$$B_1-\phi_1''+\dot{\phi} = R_1+R_2+S_3+A_5 \ ,$$

where A_5 as defined above, where

$$R_1 = -t^{-n\varepsilon}\left(-(2\kappa\lambda y^{1-\delta})^n+(2\mu_1(\lambda y^{-\delta}+\mu_3))^n\right) \ ,$$
$$R_2 = (t^{-2\gamma}T_1+t^{-\varepsilon}T_2)^n-(t^{-2\gamma}T_1)^n \ ,$$

and where

$$S_3 = -t^{-2n\gamma-2\gamma}\left[-n(2\kappa\lambda)^n(\frac{\kappa_3}{\kappa}+\frac{\lambda_0}{\lambda})z^{-\delta}+n(T_1)^{n-1}\left((\varphi_2+\lambda_0z^{2-\delta})(2\kappa z+2\eta)+2\kappa_3z^3\eta\right)\right]$$
$$+n(t^{-2\gamma}T_1)^{n-1}(t^{-4\gamma}T_3) \ .$$

The functions R_1 and R_2 can be further decomposed as follows

$$R_1 = S_1-\sum_{p=2}^{n}\sum_{q=0}^{p}\binom{n}{p}\binom{p}{q}R_1^{p,q} \ ,$$
$$R_2 = S_2+\sum_{p=2}^{n}\sum_{q=0}^{p}\binom{n}{p}\binom{p}{q}R_2^{p,q} \ ,$$

where $R_1^{p,q}$ and $R_2^{p,q}$ are as defined above, and where

$$S_1 = -nt^{-2\gamma(n-1)-\varepsilon}(2\kappa\lambda z^{1-\delta})^{n-1}(2(\mu_1-\kappa y)\lambda y^{-\delta}+2\mu_1\mu_3) \ ,$$
$$S_2 = nt^{-2\gamma(n-1)-\varepsilon}T_1^{n-1}T_2 \ .$$

It remains to be shown that

$$\sum_{i=1}^{3}S_i = \sum_{i=6}^{8}A_i \ ,$$

but this follows using the definitions. ∎

5.2.1 Proof of Theorem 10

In order to characterize the behavior of a function near zero and infinity we introduce the following family of vector spaces.

Definition 12 *Let p and q be two real numbers with $p+q \geq 0$. Then, we define $\mathcal{V}(p,q)$ to be the vector space of continuous functions F from $\mathbf{R}_+$ to $\mathbf{R}$, for which the norm*

$$\|F\|_{p,q} = \sup_{x \geq 0} |F(x)| \left(x^{-p} + x^{q}\right)$$

is finite.

Note that, if a function is in $\mathcal{V}(p,q)$, then it is also in $\mathcal{V}(p',q')$ for any pair of numbers $p' \leq p, q' \leq q$ for which $p'+q' \geq 0$. Furthermore, if F_1 is in $\mathcal{V}(p_1,q_1)$, and F_2 is in $\mathcal{V}(p_1,q_2)$, then the product F_1F_2 is in $\mathcal{V}(p_1+p_2,q_1+q_2)$.

The following Lemma provides the tool that we use to estimate the terms on the right hand side of (53).

Lemma 13 *Let $F \in \mathcal{V}(F_0,F_1)$ and $G \in \mathcal{V}(G_0,G_1)$, and assume that*

$$1 - G_1 < F_1 \ , \tag{60}$$
$$1 + G_0 > -F_0 \ , \tag{61}$$

and that

$$F_1 \neq 1 + G_0 \ . \tag{62}$$

Then, there is a constant $C > 0$, such that for all $t \geq 1$,

$$\int_0^\infty |G(x)F(t^\gamma x)| \ dx \leq Ct^{-\xi} \ , \tag{63}$$

where

$$\xi = \gamma \cdot \min\{F_1, 1+G_0\} \ . \tag{64}$$

Proof. From (62) it follows that either $F_1 < 1 + G_0$ or $F_1 > 1 + G_0$. In the first case we get using (60) that $1 - G_1 < F_1 < 1 + G_0$, and therefore

$$\begin{aligned}\int_0^\infty |G(x)F(t^\gamma x)| \ dx &\leq \left(\sup_{x\geq 0} x^{F_1} |F(t^\gamma x)|\right) \int_0^\infty \frac{1}{x^{F_1}} |G(x)| \ dx \\ &\leq \text{const. } t^{-\gamma F_1} \ ,\end{aligned}$$

and in the second case we get using (61) that $-F_0 < 1 + G_0 < F_1$, and therefore

$$\begin{aligned}\int_0^\infty |G(x)F(t^\gamma x)| \ dx &\leq \left(\sup_{x\geq 0} \frac{1}{x^{G_0}} |G(x)|\right) \int_0^\infty x^{G_0} |F(t^\gamma x)| \ dx \\ &\leq \text{const. } t^{-\gamma(1+G_0)} \ .\end{aligned}$$

■

We now show that the right hand side in (53) can be bounded by a sum of terms of the form $t^{-\sigma}G(y)F(z)$. For each of these terms we then show that the corresponding functions G and F satisfy the hypothesis of Lemma 13. This then implies that $\int_0^\infty t^{-\sigma}G(x)F(t^\gamma x) \ dx \leq$ const. $t^{-(\sigma+\xi)}$, for a certain ξ depending on F and G. It is therefore sufficient to prove that $\sigma + \xi \geq 1 + 4\gamma$ for all these terms in order to prove the inequality (52).

Proposition 14 *For y, $z > 0$ we have the bounds*

$$\begin{aligned}
T_2(y,z) &\leq \widehat{T_2}(y) \equiv 2\left|\mu_1 - \kappa y\right| y^{-\delta} \left(\sup_{z>0} \left|z^\delta \eta(z)\right|\right) + 2\left|\mu_1\mu_3\right| \ , \\
T_3(y,z) &\leq \widehat{T_3}(z) \equiv \left|(2\kappa z + 2\eta)\,\varphi_2\right| + 2\left(\sup_{y>0}\left|\mu_3(y)y^{\delta-2}\right|\right) z^{2-\delta}\eta \ , \\
T_4(y,z) &\leq T_{4,1}(y)\cdot T_{4,2}(z) \equiv \left(2\left|\mu_1(y) - \kappa y\right| y^{2-\delta'}\right) \cdot \left(z^{\delta'-2}\left|\varphi_2(z)\right|\right) \ , \\
T_6(y,z) &\leq \widehat{T_6}(z) \equiv 2\left(\sup_{y>0}\left|\mu_3(y)y^{\delta-2}\right|\right) z^{2-\delta}\left|\varphi_2\right| + \left|\varphi_2\right|^2 \ ,
\end{aligned} \tag{65}$$

and $T_1 \in \mathcal{V}(0,\delta-1)$, $\widehat{T_2} \in \mathcal{V}(3-\delta,\delta-1)$, $\widehat{T_3} \in \mathcal{V}(2-\delta,\delta'-3)$, $T_{4,1} \in \mathcal{V}(5-\delta',\delta'-3)$, $T_{4,2} \in \mathcal{V}(\delta'-\delta,0)$, $T_5 \in \mathcal{V}(4-2\delta,2\delta)$, $\widehat{T_6} \in \mathcal{V}(4-2\delta,\delta'+\delta-4)$.

Proof. The inequalities (65) follow by using the triangle inequality and the asymptotic properties of the functions μ_1, μ_2, η and φ_1. ■

Bound on the function A_2. We have the bound

$$|A_2| \le \text{const.} \sum_{p=1}^{n-1} t^{-\sigma} \widehat{T_2}^p \left(T_1^{n-1-p} \widehat{T_3} \right) ,$$

where $\sigma = 1 + \gamma + 3p\varepsilon\gamma$. The function $G = \widehat{T_2}^p$ is in $\mathcal{V}((3-\delta)p, 3\varepsilon p)$, and the function $F = T_1^{n-1-p}\widehat{T_3}$ is in $\mathcal{V}(2-\delta,$ $3\varepsilon(n-1-p) - 3 + \delta')$. Since $\delta' > 1$, the inequalities (60) and (61) are satisfied and, since $\delta' < 5$ for $n \ge 3$,

$$\xi/\gamma = \begin{cases} \delta' - 3\varepsilon p & \text{if } p \ge 2 \\ 3 - 3\varepsilon p & \text{if } p = 1 \end{cases} \ge 3 - 3\varepsilon p .$$

Therefore, $\sigma + \xi \ge 1 + \gamma + 3p\varepsilon\gamma + (3 - 3p\varepsilon)\gamma = 1 + 4\gamma$ as required.

Bound on the function A_3. We have that $A_3 = t^{-\gamma(1+\delta')} B_{3,4} + t^{-2\varepsilon} B_{3,5} + t^{-6\gamma} B_{3,6}$, where $B_{3,i} = n(t^{-2\gamma}T_1 + t^{-\varepsilon}T_2)^{n-1}T_i$, $i = 4, \ldots, 6$. Since $|T_2/T_1| \le$ const. $t^{3\gamma\varepsilon}$, and $\varepsilon - 3\gamma\varepsilon = 2\gamma$, we have the bound

$$t^{-\gamma(1+\delta')}|B_{3,4}| \le \text{const. } t^{-\sigma} \left(T_1^{n-1} T_{4,2} \right) \ T_{4,1} ,$$

with $\sigma = 2\gamma(n-1) + \gamma(1+\delta')$. The function $G = T_{4,1}$ is in $\mathcal{V}(5-\delta', \delta'-3)$ and the function $F = T_1^{n-1}T_{4,2}$ is in $\mathcal{V}(\delta'-\delta, 3)$. Since $\delta' > 3$ the inequalities (60) and (61) are satisfied and $\xi/\gamma = 6 - \delta'$. Therefore $\sigma + \xi = 1 - 3\gamma + \gamma(1+\delta') + (6-\delta')\gamma = 1 + 4\gamma$ as required.

Similarly, we have that

$$t^{-2\varepsilon}|B_{3,5}| \le \text{const. } t^{-\sigma} T_1^{n-1} \ T_5 ,$$

with $\sigma = 2\gamma(n-1) + 2\varepsilon$. The function $G = T_5$ is in $\mathcal{V}(4-2\delta, 2\delta)$ and the function $F = T_1^{n-1}$ is in $\mathcal{V}(0,3)$. The inequalities (60) and (61) are satisfied and $\xi/\gamma = 5 - 2\delta = 3 - 6\varepsilon$. Therefore, $\sigma + \xi = 1 - 3\gamma + 2\varepsilon + (3 - 6\varepsilon)\gamma = 1 + 4\gamma$ as required.

Finally,

$$t^{-6\gamma}|B_{3,6}| \le \text{const. } t^{-\sigma} T_1^{n-1} \widehat{T_6} ,$$

where $\sigma = 2(n-1)\gamma + 6\gamma$. The function $G \equiv 1$ is in $\mathcal{V}(0,0)$, and the function $F = T_1^{n-1}\widehat{T_6}$ is in $\mathcal{V}(4-2\delta, 3(n-1)\varepsilon + \delta' + \delta - 4)$. The inequalities (60) and (61) are satisfied and $\xi/\gamma = 1$. Therefore, $\sigma + \xi = 1 - 3\gamma + 6\gamma + \gamma = 1 + 4\gamma$ as required.

Bound on the function A_4. Since the functions T_3/T_1 and T_6/T_1 are bounded, $T_4/T_1 \le$ const. $t^{3\varepsilon\gamma}$ and $T_5/T_1 \le$ const. $t^{3\varepsilon\gamma}$ we have that

$$|A_4| \le \text{const.} \sum_{p=2}^{n} t^{-2\gamma(n+p)} T_1^n \le \text{const. } t^{-\sigma} T_1^n ,$$

where $\sigma = 2n\gamma + 4\gamma$. The function $G \equiv 1$ is in $\mathcal{V}(0,0)$, and the function $F = T_1^n$ is in $\mathcal{V}(0, 3n\varepsilon)$. The inequalities (60) and (61) are satisfied and $\xi/\gamma = 1$. Therefore, $\sigma + \xi = (1-\gamma) + 4\gamma + \gamma = 1 + 4\gamma$ as required.

Bound on the function A_5. We have the bound

$$|A_5| \le t^{-\sigma} \left| 3\gamma\varphi_2 + \alpha z \varphi_2' \right| , \tag{66}$$

where $\sigma = 2n\gamma + 4\gamma$. The function $G \equiv 1$ is in $\mathcal{V}(0,0)$, and the function $F = |3\gamma\varphi_2 + \alpha z\varphi_2'|$ is in $\mathcal{V}(2-\delta, \delta'-2)$. The inequalities (60) and (61) are satisfied and $\xi/\gamma = 1$. Therefore, $\sigma + \xi = (1-\gamma) + 4\gamma + \gamma = 1 + 4\gamma$ as required.

Bound on the function A_6. We have the bound

$$|A_6| \le \text{const. } t^{-\sigma} \left(T_1^{n-1} z^{2-\delta} \eta \right) \ |\lambda_0 - \mu_3 y^{\delta-2}| ,$$

where $\sigma = 1 + \gamma$. The function $G = |\lambda_0 - \mu_3 y^{\delta-2}|$ is in $\mathcal{V}(2,2)$ and the function $F = T_1^{n-1} z^{2-\delta}\eta$ is in $\mathcal{V}(2-\delta, 1+2\delta)$. The inequalities (60) and (61) are satisfied and $\xi/\gamma = 3$. Therefore, $\sigma + \xi = 1 + \gamma + 3\gamma = 1 + 4\gamma$ as required.

Bound on the function A_7. We have the bound

$$|A_7| \le \text{const. } t^{-\sigma} |T_1^{n-1} - (2\kappa\lambda z^{1-\delta})^{n-1}| \ |2\mu_1\mu_3 - 2\kappa\lambda_0 y^{3-\delta}| ,$$

where $\sigma = 2n\gamma - 2\gamma + \varepsilon$. The function $G = |2\mu_1\mu_3 - 2\kappa\lambda_0 y^{3-\delta}|$ is in $\mathcal{V}(5-\delta, \delta-3)$ and the function $F = |T_1^{n-1} - (2\kappa\lambda z^{1-\delta})^{n-1}|$ is in $\mathcal{V}(-3, 3+\delta'-\delta)$. The inequalities (60) and (61) are satisfied and $\xi/\gamma = 6-\delta$. Therefore, $\sigma + \xi = (1-\gamma) - 2\gamma + \varepsilon + (6-\delta)\gamma = 1 + 4\gamma$ as required.

Bound on the function A_8. We have the bound

$$|A_8| \leq \text{const. } t^{-\sigma} |T_1^{n-1}\eta - (2\kappa\lambda z^{1-\delta})^{n-1}\lambda z^{-\delta}| \; |\mu_1 - \kappa y - \kappa_3 y^3| \; ,$$

where $\sigma = 2n\gamma - \gamma$. The function $G = |\mu_1 - \kappa y - \kappa_3 y^3|$ is in $\mathcal{V}(5, -3)$ and the function $F = |T_1^{n-1}\eta - (2\kappa\lambda z^{1-\delta})^{n-1}\lambda z^{-\delta}|$ is in $\mathcal{V}(-3-\delta, 3+\delta')$. The inequalities (60) and (61) are satisfied and $\xi/\gamma = 6$. Therefore, $\sigma + \xi = (1-\gamma) - \gamma + 6\gamma = 1 + 4\gamma$ as required.

Bound on the functions $A_{p,q}$. We have the bound

$$|A_{p,q}| \leq \text{const. } t^{-\sigma} \; |(z^\delta \eta)^{p-q} \; T_1^{n-p} - \lambda^{p-q}(2\kappa\lambda z^{1-\delta})^{n-p}| \; \left(\left|2(\mu_1 - \kappa y) y^{-\delta}\right|^{p-q} |2\mu_1\mu_3|^q \right) \; ,$$

where $\sigma = 2n\gamma - 2p\gamma + p\varepsilon$. The function $G = \left|2(\mu_1 - \kappa y)y^{-\delta}\right|^{p-q} |2\mu_1\mu_3|^q$ is in $\mathcal{V}(p(3-\delta), 3\varepsilon p + q)$ and the function $F = |(z^\delta\eta)^{p-q} \; T_1^{n-p} - \lambda^{p-q}(2\kappa z^{1-\delta})^{n-p}|$ is in $\mathcal{V}(-3\varepsilon(n-p), 2+\delta' - 3p\varepsilon)$. The inequalities (60) and (61) are satisfied, and

$$\xi/\gamma = \begin{cases} 5 - 3p\varepsilon & \text{if } p = 2 \; , \\ 2 + \delta' - 3p\varepsilon & \text{if } p \geq 3 \; . \end{cases}$$

Therefore, $\sigma + \xi = 1 + 4\gamma$, for $p = 2$ and $\sigma + \xi = 1 + \gamma(1 + \delta') > 1 + 4\gamma$, for $p \geq 3$, as required.

This completes the proof of Theorem 10. ■

5.3 The Map T

Equation (47) is of the form (22) if we define the map T by the equation

$$T(\psi) = \begin{cases} \widehat{T}(\psi) & \text{for } n \text{ odd} \; , \\ \widehat{T}(\psi) + V_1\psi & \text{for } n \text{ even} \; , \end{cases} \tag{67}$$

with $\widehat{T}$ as defined in (50) and V_1 as defined in Section 5.1. Using the definitions, we see that T can be written as,

$$T(\psi) = \sum_{p=1}^{n} \sum_{q=0}^{p} V_{p,q} \; \psi^{p+q} \; , \tag{68}$$

with

$$V_{p,q} = \begin{cases} 0 & \text{for } (p,q) = (1,0) \text{ and } n \text{ odd,} \\ V_1 & \text{for } (p,q) = (1,0) \text{ and } n \text{ even,} \\ \binom{n}{p}\binom{p}{q}(2\overline{u}\phi + \phi^2)^{n-p}(2\overline{u} + 2\phi)^{p-q} & \text{for } p + q \geq 2 \; . \end{cases} \tag{69}$$

Proposition 15 *Let $V_{p,q}$ as in (69). Then, for all $t \geq 1$,*

$$\sup_{x \in \mathbf{R}} |V_{p,q}(x,t)| \leq \text{const. } t^{-e(p,q)} \; , \tag{70}$$

where

$$e(p,q) = \begin{cases} \gamma(n-1)(\delta'+1) & \text{for } (p,q) = (1,0) \; , \\ 2\gamma(n-2) + 2\gamma & \text{for } (p,q) = (2,0) \; , \\ 2\gamma(n-p) & \text{for } (p,q) \neq (2,0) \text{ and } p + q \geq 2 \; . \end{cases} \tag{71}$$

Proof. The case $(p,q) = (1,0)$ follows from (51). Let now $(p,q) \neq (1,0)$. Since $\varepsilon - \gamma - \nu\gamma \geq 0$, for all ν, $0 \leq \nu \leq \delta$, we find that

$$\begin{aligned} \sup_{x \in \mathbf{R}} |z|^\nu \; |\phi(x,t)| &\leq t^{-\gamma}(\sup_{z \in \mathbf{R}_+} |z^\nu \; \eta(z)| + t^{-(\varepsilon - \gamma - \nu\gamma)} \sup_{y \in \mathbf{R}_+} |y^\nu \; \mu_3(y)| + t^{-2\gamma} \sup_{z \in \mathbf{R}_+} |z^\nu \; \varphi_2(z)|) \\ &\leq \text{const. } t^{-\gamma} \; . \end{aligned}$$

Furthermore, since $\mu_1(y) = \mathcal{O}(y)$ near $y = 0$,

$$\left|2\overline{u}\phi + \phi^2\right| \leq t^{-\gamma} \; 2\frac{\mu_1(|y|)}{|y|} \; |z\phi| + |\phi|^2 \leq \text{const. } t^{-2\gamma} \; .$$

Since the function $|\overline{u}+\phi|$ is bounded, it follows that $|V_{p,q}(x,t)| \leq \text{const.}\ t^{-e(p,q)}$, with $e(p,q) = 2\gamma(n-p)$. For $(p,q) = (2,0)$ we improve this bound using additional properties of the function $\overline{u}+\phi$. Namely, since $2/(n-2) \leq \delta - 1$, we have that

$$\begin{aligned}
|V_{2,0}(x,t)| &\leq \text{const.} \sup_{x\in\mathbf{R}} \left|(2\overline{u}\phi + \phi^2)^{n-2}(\overline{u}+\phi)^2\right| \\
&\leq \text{const.} \sup_{x\in\mathbf{R}} \left|(2t^{-\gamma}\frac{\overline{u}}{y}z\phi + \phi^2)^{n-2}(t^{-\gamma}z\frac{\overline{u}}{y}+\phi)^2\right| \\
&\leq \text{const.} \sup_{x\in\mathbf{R}} \left|2t^{-\gamma}\frac{\overline{u}}{y}z^{1+2/(n-2)}\phi + \left(z^{2/(n-2)}\phi\right)\ \phi\right|^{n-2} \quad \left|t^{-\gamma}\frac{\overline{u}}{y}\right|^2 \\
&\qquad + \text{const.} \sup_{x\in\mathbf{R}} \left|2t^{-\gamma}\frac{\overline{u}}{y}z\phi + \phi^2\right|^{n-2} \quad \left(2t^{-\gamma}\left|\frac{\overline{u}}{y}\right| |z\phi| + |\phi|^2\right) \\
&\leq \text{const.}\ t^{-2(n-2)\gamma - 2\gamma}\ .
\end{aligned}$$

■

6 Proof of the main result

For functions f in $\mathcal{J} = L_1(\mathbf{R}) \cap L_\infty(\mathbf{R})$ we use the norms $\|f\|_1 = \int |f(x)|\, dx$, $\|f\|_\infty = \sup_{x\in\mathbf{R}} |f(x)|$ and $\|f\| = \|f\|_1 + \|f\|_\infty$, and we denote by $\mathcal{B}$ the Banach space of functions φ in $L_\infty([1,\infty)) \times \mathcal{J}$ for which the norm $\|\ \|_{\mathcal{B}}$,

$$\|\varphi\|_{\mathcal{B}} = \sup_{t\geq 1} t^{4\gamma} \left\|\varphi(\sqrt{t}\ .\ ,t)\right\|\ ,$$

is finite. Let τ_0 as in Proposition 8 and τ_1 as in Proposition 9, and consider, for fixed $\tau > \max\{\tau_0, \tau_1\}$, functions ψ of the form

$$\psi(x,t) = \tau^{-4\gamma}\varphi(x/\sqrt{\tau}, t/\tau)\ ,$$

with $\varphi \in \mathcal{B}$. Let K be the fundamental solution of the differential operator $\partial_t - \partial_x^2 - \tau V(\sqrt{\tau}x, \tau t)$, and let, for given $\nu \in \mathcal{J}$, the map $\mathcal{R}$ be defined by the equation

$$\mathcal{R}(\varphi)(x,t) = \varphi_{0,1}(x,t) + \varphi_{0,2}(x,t) + \mathcal{N}(\varphi)(x,t)\ ,$$

where

$$\begin{aligned}
\varphi_{0,1}(x,t) &= \int_{\mathbf{R}} K(x,t;y,1)\ \nu(y)\ dy\ , \\
\varphi_{0,2}(x,t) &= \tau^{4\gamma}\tau \int_1^t ds \int_{\mathbf{R}} dy\ K(x,t;y,s)\ I(\sqrt{\tau}y, \tau s)\ ,
\end{aligned}$$

and where

$$\mathcal{N}(\varphi)(x,t) = \sum_{p=1}^{n}\sum_{q=0}^{p} \mathcal{N}_{p,q}(\varphi)(x,t)\ ,$$

with

$$\mathcal{N}_{p,q}(\varphi)(x,t) = \tau^{4\gamma}\tau \int_1^t ds \int_{\mathbf{R}} dy\ K(x,t;y,s)\ V_{p,q}(\sqrt{\tau}y, \tau s)\ \tau^{-4\gamma(p+q)}\varphi(y,s)^{p+q}\ .$$

The integral equation $\varphi = \mathcal{R}(\varphi)$ is equivalent to the differential equation (22) with initial condition $\psi_0(x) = \psi(x,\tau) = \tau^{-4\gamma}\nu(x/\sqrt{\tau})$. We note that, since the function V is positive, the kernel K is bounded pointwise by the fundamental solution K_0 of the heat equation,

$$K_0(x,t;y,s) = \frac{1}{\sqrt{4\pi}}\frac{1}{\sqrt{t-s}} \exp\left(-\frac{1}{4}\frac{(x-y)^2}{(t-s)}\right)\ . \tag{72}$$

The following Proposition makes Theorem 1 precise.

Proposition 16 *Let $\beta \geq \max\left\{1,\ 3\ c_I \int_0^1 \left(1 + \frac{1}{\sqrt{1-s}}\right) \frac{ds}{s^{1/2+4\gamma}}\right\}$, with c_I as defined in (52), and let τ be sufficiently large. Then, for all $\nu \in \mathcal{J}$ with $\|\nu\| < \beta/6$, the equation $\varphi = \mathcal{R}(\varphi)$ has a unique solution φ^* in the ball $\mathcal{U}(\beta) = \{\varphi \in \mathcal{B} |\ \|\varphi\|_{\mathcal{B}} < \beta\}$.*

Proof. Since $4\gamma < 1/2$, the solution of the integral equation will be dominated by $\varphi_{0,2}$, and, as we will see, β has been chosen such that $\|\varphi_{0,2}\|_{\mathcal{B}} \leq \beta/3$. The idea is therefore to show that, if τ is large enough to make the nonlinear part of the map $\mathcal{R}$ small, and if $\|\nu\| < \beta/6$, then the map $\mathcal{R}$ contracts the ball $\mathcal{U}(\beta)$ into itself, which by the contraction mapping principle implies the theorem. We first show that $\mathcal{R}$ maps the ball $\mathcal{U}(\beta)$ into itself. For the contribution coming from the initial condition we have

$$\left\|\varphi_{0,1}(\sqrt{t}\ .\ ,t)\right\| \leq \frac{2}{\sqrt{t}}\,\|\nu\| \ ,$$

and therefore

$$\|\varphi_{0,1}\|_{\mathcal{B}} \leq 2\,\|\nu\| < \beta/3 \ .$$

We next estimate the norm $\|\varphi_{0,2}\|_{\mathcal{B}}$. Let $c(t,s) = \frac{1}{\sqrt{t}} + \frac{1}{\sqrt{t-s}}$. Then,

$$\begin{aligned}
\left\|\varphi_{0,2}(\sqrt{t}\ .\ ,t)\right\| &\leq \tau^{4\gamma}\tau \int_1^t ds\ c(t,s) \int_{\mathbf{R}} dy\ \ \left|I(\sqrt{\tau}y,\tau s)\right| \\
&= \tau^{4\gamma}\tau \int_1^t \sqrt{s}\ c(t,s)\ ds\ \int_{\mathbf{R}} dx\ \ \left|I(\sqrt{\tau s}x,\tau s)\right| \\
&\leq c_I\ \tau^{4\gamma}\tau \int_1^t \sqrt{s}\ c(t,s)\ ds\ \ (\tau s)^{-(1+4\gamma)} \\
&\leq c_I\ t^{-4\gamma} \int_0^1 c(1,s)\ \frac{ds}{s^{1/2+4\gamma}} \\
&\leq \frac{\beta}{3} t^{-4\gamma} \ ,
\end{aligned}$$

and therefore

$$\|\varphi_{0,2}\|_{\mathcal{B}} < \beta/3 \ .$$

It remains to be shown that the nonlinearity is also bounded by $\beta/3$, for τ large enough. For $\varphi \in \mathcal{U}(\beta)$ we have,

$$\left\|\mathcal{N}(\varphi)(\sqrt{t}\ .\ ,t)\right\| \leq \text{const.}\ \tau^{4\gamma}\tau \int_1^t c(t,s)\ \sqrt{s}\ ds \sum_{p=1}^{n}\sum_{q=0}^{p} (\tau s)^{-e(p,q)}\ \ s^{-4\gamma(p+q)}\tau^{-4\gamma(p+q)}\ \|\varphi\|_{\mathcal{B}}^{p+q} \ . \tag{73}$$

For $(p,q) = (1,0)$ we get, since $\delta_1 \equiv \gamma(n-1)(\delta'+1) - 1 > 0$,

$$\begin{aligned}
\left\|\mathcal{N}_{1,0}(\varphi)(\sqrt{t}\ .\ ,t)\right\| &\leq \text{const.}\ \tau^{-\delta_1}\beta \int_1^t c(t,s)\ \sqrt{s}\ ds\ s^{-1-\delta_1-4\gamma} \\
&\leq \text{const.}\ \tau^{-\delta_1}\beta\ t^{-4\gamma} \int_0^1 \frac{c(1,s)}{s^{1/2+4\gamma}}\ ds \ ,
\end{aligned}$$

and for $(p,q) = (2,0)$ we get

$$\begin{aligned}
\left\|\mathcal{N}_{2,0}(\varphi)(\sqrt{t}\ .\ ,t)\right\| &\leq \text{const.}\ \tau^{4\gamma+1-8\gamma-(2\gamma(n-2)+2\gamma)}\beta^2 \int_1^t c(t,s)\ \sqrt{s}\ ds\ s^{-8\gamma-2\gamma-2\gamma(n-2)} \\
&\leq \text{const.}\ \tau^{-\gamma}\beta^2\ t^{-4\gamma} \int_0^1 \frac{c(1,s)}{s^{1/2+4\gamma}}\ ds \ ,
\end{aligned}$$

and for the other cases we have

$$\begin{aligned}
\left\|\mathcal{N}_{p,q}(\varphi)(\sqrt{t}\ .\ ,t)\right\| &\leq \text{const.}\ \tau^{4\gamma+1-2\gamma(n-p)-4\gamma p-4\gamma q}\beta^{2n} \int_1^t c(t,s)\ \sqrt{s}\ ds\ s^{-2\gamma(n-p)-4\gamma p-4\gamma q} \\
&\leq \text{const.}\ \tau^{-\gamma}\beta^{2n}\ t^{-4\gamma} \int_0^1 \frac{c(1,s)}{s^{1/2+4\gamma}}\ ds \ ,
\end{aligned}$$

and therefore $\|\mathcal{N}(\varphi)\|_{\mathcal{B}} \leq \beta/3$ if τ is large enough. Using the triangle inequality we get that $\|\mathcal{R}(\varphi)\|_{\mathcal{B}} \leq \beta$, which proves that $\mathcal{R}\left(\mathcal{U}(\beta)\right) \subset \mathcal{U}(\beta)$ as claimed. We now show that $\mathcal{R}$ is Lipschitz. Let φ_1 and φ_2 be in $\mathcal{U}(\beta)$. We have

$$\begin{aligned}
\left\|\mathcal{N}(\varphi_1)(\sqrt{t}\ .\ ,t) - \mathcal{N}(\varphi_2)(\sqrt{t}\ .\ ,t)\right\| &\leq \text{const.}\ \tau^{4\gamma}\tau \int_1^t ds\ c(t,s)\ \sqrt{s}\ \cdot \\
&\sum_{p=1}^{n}\sum_{q=0}^{p} (\tau s)^{-e(p,q)}\ \ s^{-4\gamma(p+q)}\tau^{-4\gamma(p+q)}\ \ \beta^{p+q-1}\ \ \|\varphi_1 - \varphi_2\|_{\mathcal{B}} \ ,
\end{aligned}$$

and therefore we get, using the same estimates as for (73), that

$$\|\mathcal{R}(\varphi_1) - \mathcal{R}(\varphi_2)\|_{\mathcal{B}} = \|\mathcal{N}(\varphi_1) - \mathcal{N}(\varphi_2)\|_{\mathcal{B}} \leq \frac{1}{2}\,\|\varphi_1 - \varphi_2\|_{\mathcal{B}} \ ,$$

provided τ is large enough. This completes the proof of Theorem 16. ■

7 Appendix

7.1 Proof of Proposition 4

We first prove the existence of a unique positive solution of equation (30) satisfying the boundary conditions (31) and (32). Then, we derive the results on the asymptotic behavior near zero and infinity.

7.1.1 Existence of the function η

Proposition 17 *Let, for $\rho > 0$, η_ρ be the solution of the initial value problem on $\mathbf{R}_+$,*

$$\begin{aligned} \eta'' &= (2\kappa z\eta + \eta^2)^n \ , \\ \eta'(0) &= -\kappa \ , \\ \eta(0) &= \rho > 0 \ . \end{aligned} \tag{74}$$

Then, there exists a unique $\bar{\rho}$ such that the function $\eta_{\bar{\rho}}$ is positive and satisfies $\lim_{x\to\infty} \eta_{\bar{\rho}}(x) = 0$.

Proof. We first prove that $\bar{\rho}$ is unique. Given a function η from $\mathbf{R}_+$ to $\mathbf{R}$ we define the function $\mathcal{F}(\eta)$, $\mathcal{F}(\eta)(z) = (\kappa z\eta + \eta^2)^n$. Assume that there are two values $\rho_1 > \rho_2 > 0$, such that the functions $\eta_1 \equiv \eta_{\rho_1}$ and $\eta_2 \equiv \eta_{\rho_2}$ are positive and satisfy $\lim_{x\to\infty} \eta_1(x) = \lim_{x\to\infty} \eta_2(x) = 0$. We first show that the function $\eta_{12} = \eta_1 - \eta_2$ is positive for all $x \geq 0$. Namely, if we assume the contrary, then because $\eta_{12}(0) > 0$, there must be a first $x_0 > 0$ such that $\eta_{12}(x_0) = 0$. Furthermore, if $\eta_{12}(x) > 0$ then $\eta_{12}''(x) = \mathcal{F}(\eta_1)(x) - \mathcal{F}(\eta_2)(x) > 0$, and therefore $\eta_{12}(x_0) = \rho_1 - \rho_2 + \int_0^{x_0} dx \int_0^x dy\ \eta_{12}''(y) > 0$, a contradiction. Therefore η_{12}, and as a consequence η_{12}'', are positive for all x, from which it follows that $\lim_{x\to\infty} \eta_{12}(x) > 0$, in contradiction with $\lim_{x\to\infty} \eta_{12}(x) = \lim_{x\to\infty} \eta_1(x) - \lim_{x\to\infty} \eta_2(x) = 0$.

To prove the existence of a $\bar{\rho}$ for which $\eta_{\bar{\rho}}$ is positive and for which $\lim_{x\to\infty} \eta_{\bar{\rho}}(x) = 0$, we use the so called shooting method. Note that, for any $\rho > 0$, the initial value problem (74) has a unique solution η_ρ, and since $\eta_\rho'(0) = -\kappa$, the function η_ρ is strictly decreasing on $[0, x_\rho)$ for x_ρ small enough. We will show that for small enough $\rho > 0$, the graph of η_ρ intersects the real axis and η_ρ becomes negative, whereas for ρ large enough, η_ρ has a minimum and then diverges to plus infinity. The (unique) point between those two sets is $\bar{\rho}$. Define the two subsets I_1 and I_2 of $\mathbf{R}_+$,

$$\begin{aligned} I_1 &= \{\rho \in \mathbf{R}_+ | \ \exists\ x_1, \eta_\rho(x_1) = 0 \text{ and } \eta_\rho(x) > 0 \text{ for } x \in [0, x_1)\} \ , \\ I_2 &= \{\rho \in \mathbf{R}_+ | \ \exists\ x_2, \eta_\rho'(x_2) = 0 \text{ and } \eta_\rho'(x) < 0, \eta_\rho(x) > 0 \text{ for } x \in [0, x_2)\} \ . \end{aligned}$$

We note that if $\eta_\rho'(x_0) = 0$ and $\eta_\rho(x_0) > 0$, for some x_0, then $\eta_\rho' > 0$ on any interval (x_0, x) on which η_ρ is defined, and a function η_ρ with $\rho \in I_2$ can therefore not converge to zero at infinity. Furthermore, since the function $\eta \equiv 0$ is a solution of the differential equation (74), it follows, since solutions are unique, that $\eta_\rho(x_0) > 0$ if $\eta_\rho'(x_0) = 0$, and therefore the intersection of I_1 with I_2 is empty. The sets I_1 and I_2 are open, by continuity of the solution η_ρ as a function of the initial data ρ. We now show that I_1 is non empty and bounded, which shows that $\bar{\rho} \equiv \sup I_1 < \infty$. This $\bar{\rho}$ is neither in I_1 nor in I_2, and therefore the function $\eta_{\bar{\rho}}$ is at the same time strictly positive and strictly decreasing, and therefore $\lim_{x\to\infty} \eta_{\bar{\rho}}(x) = 0$. To prove that I_1 is non empty, we fix any ρ_1 positive and choose $x_0 > 0$ small enough such that on $[0, x_0]$ the solution $\eta_1 \equiv \eta_{\rho_1}$ exists and is strictly decreasing. Then, $\rho_1 - \eta_1(x_0) > 0$. Choose now $0 < \rho_2 < \rho_1 - \eta_1(x_0)$ and let $\eta_2 \equiv \eta_{\rho_2}$ be the corresponding solution. As before, we have that the function $\eta_{12} = \eta_1 - \eta_2$, and its second derivative η_{12}'', are positive on the interval $[0, x_0)$, and therefore, since $\eta_2(x_0) = \rho_2 + \int_0^{x_0} dx \int_0^x dy\ \eta_2''(y) = \rho_2 + \eta_1(x_0) - \rho_1 - \int_0^{x_0} dx \int_0^x dy\ \eta_{12}''(y)$, we find that $\eta_2(x_0) < \rho_2 - \rho_1 + \eta_1(x_0)$. Using the definition of ρ_2 we therefore find that $\eta_2(x_0) < 0$. Therefore $\rho_2 \in I_1$. We now prove that I_1 is bounded. For $\rho > 0$, let x_ρ be the largest value (possibly infinite) such that on $[0, x_\rho)$ the solution η_ρ exists and is strictly positive. Then, $\eta_\rho'' = \mathcal{F}(\eta_\rho)$ is positive on $(0, x_\rho)$ and, therefore $\eta_\rho(x) > \rho - \kappa x$ for $x \in (0, x_\rho)$. As a consequence, if the function η_ρ exists on $[0, \rho/\kappa]$, then $x_\rho \geq \rho/\kappa$. Using again that $\eta_\rho(x) > \rho - \kappa x$ we then find that $\eta_\rho(x) > \rho/2$ for $x \in [0, \rho/2\kappa]$, and therefore $\mathcal{F}(\eta_\rho) > (\rho/2)^{2n}$ on $[0, \rho/2\kappa]$, which implies that $\eta_\rho'(\rho/2) > -\kappa + (\rho/2)^{2n+1}$, which is positive if $\rho > 2\kappa^{1/2n+1}$. Therefore $\eta'(x)$ must be equal to zero for some $x < \rho/\kappa$. Any such ρ therefore belongs to I_2. If the function η_ρ ceases to exist before $x = \rho/\kappa$ it must have been diverging to plus infinity for some $x < \rho/\kappa$ which again implies that $\eta_\rho'(x)$ must have been equal to zero for some $x < \rho/\kappa$, and the corresponding ρ is in I_2. ■

7.1.2 Asymptotic behavior of the function η

The function η is regular at zero, and the coefficients of its Taylor series at zero can be computed recursively. We have $\eta(0) = \eta_0 > 0$ and $\eta'(0) = -\kappa$, and therefore we get using the differential equation that $\eta_2 = \eta''(0)/2 = \eta_0^{2n}/2$, $\eta'''(0) = 0$ and $\eta_4 = -\eta^{iv}(0)/4! = \frac{n}{12}\eta_0^{2n-2}(\kappa^2 - \eta_0^{2n+1})$. The asymptotic behavior of η at infinity is obtained as follows. Assuming that η behaves like λ/z^δ at infinity we get from the differential equation that δ and λ are as defined in (26) and (33), respectively. That this is indeed the correct leading behavior of η at infinity can now be proved by using standard techniques based on repeated applications of l'Hôpital's rule. See for example [4]. Since the proof is simple, but lengthy and quite uninteresting, we do not give the details here.

Once the leading behavior of η at infinity has been established we make the ansatz $\eta(z) = \lambda z^{-\delta} + s(z)$. To leading order we get for the function s the linear equation

$$s'' - \frac{n}{\lambda}(2\kappa\lambda)^n z^{-2} s = n(2\kappa\lambda)^{n-1}\lambda^2 z^{-3-2\delta} \ . \tag{75}$$

There is a certain constant λ_p, such that the function s_p, $s_p(z) = \lambda_p z^{-1-2\delta}$ is a particular solution of equation (75). The solutions of the homogeneous equation associated with (75) are of the form $s_h^{\pm}(z) = z^{p_\pm}$, and using the definition (33) for λ we find that

$$p_\pm = \frac{1}{2}\left(1 \pm \sqrt{1 + 4n\delta(\delta+1)}\right) . \tag{76}$$

For $n \geq 6$ we have that $|p_-| > 2\delta + 1$, and the asymptotic behavior of s is therefore for $n \geq 6$ of the form $\lambda_\infty / z^{2\delta+1}$, with $\lambda_\infty = \lambda_p$, and of the form $\lambda_\infty / z^{|p_-|}$ with some unknown coefficient λ_∞ for $n \leq 5$. It is tedious, but not difficult, to prove that this is indeed the correct second order behavior of η at infinity. We omit the details.

7.2 Proof of Proposition 5

In order to study equation (34) with boundary conditions (35) and (36), we make the ansatz $\mu_2(x) = m(x)/x^\delta$. For the function m we get the differential equation

$$m'' + (\frac{x}{2} - \frac{2\delta}{x})m' + \left(\delta(\delta+1)\frac{1}{x^2} - \left(\frac{\delta}{2} - \varepsilon\right)\right) m = \frac{1}{x^2}(2\frac{\mu_1(x)}{x}m)^n , \tag{77}$$

and the boundary conditions for m are

$$\lim_{x\to 0} m(x) = \lambda , \tag{78}$$

$$\lim_{x\to\infty} m(x)x^{2\varepsilon-\delta} = 0 . \tag{79}$$

7.2.1 Asymptotic behavior of the function μ_2

As indicated in Section 3.3, a solution of equation (34) that is defined on $\mathbf{R}_+$ behaves at infinity either like $x^{-2\varepsilon}$ or like $\exp(-x^2/4)/x^{1-2\varepsilon}$. The proof is similar to the one in [4]. We omit the details. Given the asymptotic behavior of μ_2 at infinity, we find for the function m at infinity either a behavior proportional to $x^{\delta-2\varepsilon}$, or a behavior proportional to $x^{5\varepsilon}\exp(-x^2/4)$. Since $\delta - 2\varepsilon > 0$, we find that

$$\lim_{x\to\infty} m(x) = 0 , \tag{80}$$

if and only if the boundary condition (79) is satisfied, and we will impose (80) from now on. We now discuss the asymptotic behavior of the function m near zero. From equation (77) we see that $m''(0)$ exists if and only if $\delta(\delta+1)m(0) = (\kappa m(0))^n$, i.e., if $m(0) = \lambda$, and if $m'(0) = 0$. We then find, that $m''(0)/2 = \lambda_0$, with λ_0 as defined in (38). By taking derivatives of equation (77) we find that $m'''(0) = 0$, and that $m^{iv}(0)/4! = \lambda_1$, for some constant $\lambda_1 \neq 0$. By taking further derivatives, one can recursively compute the Taylor coefficients of a solution m_0 of equation (77) that is regular (in fact, analytic) in a neighborhood of zero. The solution m_0 does however not satisfy the boundary condition (80). The solution of (77) that does satisfy (80) is of the form

$$m(x) = m_0(x) + x^p m_1(x) , \tag{81}$$

where $p = p_+ + \delta$, with p_+ as defined in (76). Here, $m_1(x) = m_1(0) + \ldots$, with $m_1'(0) = 0$, and with $m_1(0)$ to be determined. The asymptotic form (81) can be obtained by substituting the ansatz (81) for m into equation (77). Since $p > 7$ we find from (81) that near zero $m_0(x) + x^p m_1(x) = \lambda + \lambda_0 x^2 + \lambda_1 x^4 + \ldots$. We omit the details of the proof that the asymptotic behavior is as indicated.

7.2.2 Existence of the function μ_2

We now prove the existence of a function m that satisfies equation (77) with the boundary conditions (78) and (80). Since the second derivative of the solution m at zero is positive, and since m converges to zero at infinity, there must be a first $\xi \in \mathbf{R}_+$, such that $m'(\xi) = 0$. The basic idea is now to use this position ξ, and the value ρ of m at ξ, as parameters in shooting arguments towards zero and infinity. The first shooting argument will allow us to define a curve c_0 of initial conditions (ξ, ρ) in $\mathbf{R}_+^2$, for which the boundary condition at zero is satisfied, and the second shooting argument will allow us to find on this curve an initial condition for which the boundary condition at infinity is satisfied as well.

So, let (ξ, ρ) be an initial condition. Locally, i.e., near ξ, there exists a solution $m_{\xi,\rho}$ of equation (77). By definition, $m_{\xi,\rho}(\xi) = \rho$, $m'_{\xi,\rho}(\xi) = 0$, and therefore we get for the second derivative of $m_{\xi,\rho}$ at ξ,

$$m''_{\xi,\rho}(\xi) = \omega_1(\xi)\rho^n + \omega_2(\xi)\rho ,$$

where

$$\omega_1(\xi) = \frac{\left(2\frac{\mu_1(\xi)}{\xi}\right)^n}{\xi^2} , \tag{82}$$

and

$$\omega_2(\xi) = \frac{n}{2}\varepsilon - \frac{\delta(\delta+1)}{\xi^2} \ . \tag{83}$$

For initial conditions such that $\rho = c_2(\xi)$, where

$$c_2(\xi) = \left(\frac{\frac{n}{2}\varepsilon}{\left(2\frac{\mu_1(\xi)}{\xi}\right)^n} \left(\xi_0^2 - \xi^2\right) \right)^{\varepsilon} \tag{84}$$

and $\xi_0 = \sqrt{\delta(\delta+1)/\left(\frac{n\varepsilon}{2}\right)}$, we therefore have that $m''_{\xi,\rho}(\xi) = 0$. See Fig. 1 for the graph of the function c_2. The function c_2 has a maximum at the point ξ_m that satisfies the equation

$$\omega_1'(\xi_m) c_2(\xi_m)^{n-1} + \omega_2'(\xi_m) = 0 \ , \tag{85}$$

and the line c_2 divides the set of initial conditions into two subsets, a subset A where $m''_{\xi,\rho}(\xi) < 0$, and a subset B where $m''_{\xi,\rho}(\xi) > 0$. For initial conditions on c_2 we can compute $m'''_{\xi,c_2(\xi)}(\xi)$,

$$m'''_{\xi,c_2(\xi)}(\xi) = \omega_1'(\xi) c_2(\xi)^n + \omega_2'(\xi) c_2(\xi) \ .$$

Comparing with (85) we find that $m'''_{\xi_m,c_2(\xi_m)}(\xi_m) = 0$, and we have that $m'''_{\xi,c_2(\xi)}(\xi) < 0$ for $0 < \xi < \xi_m$. We now construct the line c_0 for $0 < \xi < \xi_m$.

Proposition 18 *Fix ξ, $0 < \xi < \xi_m$. Then, there exists a unique number $c_0(\xi)$, $c_2(\xi) > c_0(\xi) > \lambda$, such that $m_{\xi,c_0(\xi)}$ is positive and satisfies $\lim_{x\to 0} m_{\xi,c_0(\xi)}(x) = \lambda$. Furthermore, the function c_0 is continuous.*

Proof. The proof is similar to the one in Section 7.1.1. Define the two subsets I_1 and I_2 of the interval $I = (\lambda, c_2(\xi))$,

$$\begin{aligned} I_1 &= \{\rho \in I |\ \exists\, 0 < \xi_1 < \xi,\ m_{\xi,\rho}(\xi_1) = \lambda \text{ and } \lambda < m_{\xi,\rho}(x) < c_2(x) \text{ for } x \in (\xi_1, \xi)\} \ , \\ I_2 &= \{\rho \in I |\ \exists\, 0 < \xi_2 < \xi,\ m_{\xi,\rho}(\xi_2) = c_2(\xi_2) \text{ and } \lambda < m_{\xi,\rho}(x) < c_2(x) \text{ for } x \in (\xi_2, \xi)\} \ . \end{aligned}$$

The intersection of I_1 with I_2 is by definition empty, and the sets I_1 and I_2 are open, by continuity of the solution $m_{\xi,\rho}$ as a function of the initial data ρ. We now show that all ρ sufficiently close to λ are in I_1, and that all ρ sufficiently close to $c_2(\xi)$ are in I_2. This implies that $c_0(\xi) = \sup I_1 < c_2(\xi)$, and $c_0(\xi)$ is neither in I_1 nor in I_2, and therefore the function $m_{\xi,c_0(\xi)}$ satisfies $\lambda < m_0(x) < m_2(x)$ for all $0 < x < \xi$, and therefore $\lim_{x\to 0} m_{\xi,c_0(\xi)}(x) = \lambda$, since $\lim_{x\to 0} c_2(x) = \lambda$. So let (ξ, ρ) be an initial condition. Then, $m_{\xi,\rho}$ satisfies the integral equation

$$m_{\xi,\rho}(x) = \rho + \int_\xi^x \frac{dy}{p(y)} \int_\xi^y p(z)\ \Omega(m_{\xi,\rho}(z), z)\ dz \ , \tag{86}$$

where

$$p(z) = \frac{\exp(z^2/4)}{z^{2\delta}} \ ,$$

and where

$$\Omega(s, z) = \omega_1(z) s^n + \omega_2(z) s \ .$$

$\Omega(s,z)$ is strictly negative for $0 < z < \xi_m$ and $s \approx \lambda$, and therefore we find, like in the proof in Section 4 that any solution with an initial condition ρ sufficiently close to λ will cross the line $m \equiv \lambda$. Similarly, for an initial condition (ξ, ρ) close to $(\xi, c_2(\xi))$ we can use that $\Omega(s,z) \approx 0$, and that $\partial_z \Omega(c_2(z), z)$ is strictly negative to show that the corresponding solution will cross the line c_2. This completes the proof of the existence of $c_0(\xi)$. To prove uniqueness it is sufficient to use that $\partial_s \Omega(s,z) > 0$ for (s,z) in the set C (see Fig. 1), and to integrate the difference of two solutions from their respective initial condition to zero, which leads to a contradiction, since both solutions have to be equal to λ at zero. Finally, that c_0 is a continuous function follows from the continuity of $m_{\xi,\rho}$ as a function of ρ and ξ using the uniqueness of $c_0(\xi)$. ■

We now prove with a second shooting argument that solutions with initial conditions $(\xi, c_0(\xi))$, with $\xi \approx 0$, become negative somewhere in the interval $(\xi, 2)$, and that solutions with initial conditions $(\xi, c_0(\xi))$, with $\xi \approx \xi_m$, stay positive and diverge to plus infinity.

Proposition 19 *There exists a unique initial condition $(\xi^*, c_0(\xi^*))$ such that the corresponding solution $m_{\xi^*,c_0(\xi^*)}$ is positive and satisfies $\lim_{x\to\infty} m_{\xi^*,c_0(\xi^*)}(x) = 0$.*

Proof. Define the two subsets I_1 and I_2 of the interval $I=(0,\xi_m)$,

$$I_1 = \{\xi \in I |\ \exists\ \xi_1 > \xi,\ m_{\xi,c_0(\xi)}(\xi_1)=0 \text{ and } m_{\xi,c_0(\xi)}(x)>0,\ m'_{\xi,c_0(\xi)}(x)<0 \text{ for } x\in(\xi,\xi_1)\}\ ,$$
$$I_2 = \{\xi \in I |\ \exists\ \xi_2 > \xi,\ m'_{\xi,c_0(\xi)}(\xi_2)=0 \text{ and } m_{\xi,c_0(\xi)}(x)>0,\ m'_{\xi,c_0(\xi)}(x)<0 \text{ for } x\in(\xi,\xi_2)\}\ .$$

By definition, the intersection of I_1 with I_2 is empty, and the sets I_1 and I_2 are open, by continuity of the solution $m_{\xi,c_0(\xi)}$ as a function of the initial data ξ. We now show that all ξ sufficiently close to 0 are in I_1, and that all ξ sufficiently close to ξ_m are in I_2. This implies that $\xi^*=\sup I_1<\xi_m$, is neither in I_1 nor in I_2, and therefore the function $m_{\xi^*,c_0(\xi*)}$ is positive and decreasing for $x>\xi^*$ which implies that $\lim_{x\to\infty} m_{\xi^*,c_0(\xi^*)}(x)=0$. So let $(\xi,c_0(\xi))$ be an initial condition with $0<\xi<x_0$, with $x_0\ll 1$. The proof that such an initial condition is in I_1 is rather lengthy and we therefore do not give the details here, but on a heuristic level it is easy to understand why such a solution is in I_1. Namely, near zero the asymptotics of the solution $m_{\xi,c_0(\xi)}$ is $m_{\xi,c_0(\xi)}(x)=\lambda+\lambda_0x^2+\cdots+(m_1)_{\xi,c_0(\xi)}(0)x^p+\ldots$, where $p=p_++\delta$, and where $(m_1)_{\xi,c_0(\xi)}(0)$ is such that $m'_{\xi,c_0(\xi)}(\xi)=0$. Neglecting higher order terms we find that $m'_{\xi,c_0(\xi)}(\xi)\approx 2\lambda_0\xi+(m_1)_{\xi,c_0(\xi)}(0)p\xi^{p-1}$, and we conclude that $(m_1)_{\xi,c_0(\xi)}(0)\approx -(2\lambda_0/p)/\xi^{p-2}$. Therefore, $m'_{\xi,c_0(\xi)}(\xi^{(p-5/2)/(p-1)})\approx -2\lambda_0/\xi^{1/2}\ll 0$ and $m_{\xi,c_0(\xi)}(\xi^{(p-5/2)/(p-1)})\approx\lambda$, if ξ is small enough. Therefore, since $m''_{\xi,c_0(\xi)}(x)<0$ for all $0<x\ll 1$, we find that $m_{\xi,c_0(\xi)}(x)<c_0(\xi)-2\lambda_0(x-\xi)/\xi^{1/2}$, and therefore $m_{\xi,c_0(\xi)}(x)=0$ for some $x\le c_0(\xi)\xi^{1/2}/(2\lambda_0)+\xi$, as claimed. Next, let $(\xi,c_0(\xi))$ be an initial condition with $\xi_m-x_1<\xi<\xi_m$, with $0<x_1<1$ to be chosen below. We now show that such an initial condition is in I_2. For all $\xi_0\ge\xi'\ge\xi$ we have the lower bounds $m_{\xi,c_0(\xi)}(\xi')\ge m_{\xi,c_0(\xi)}(\xi)k_1$ and $p(\xi')m'_{\xi,c_0(\xi)}(\xi')\ge m_{\xi,c_0(\xi)}(\xi)k_2$, where

$$k_1 = 1+\delta(\delta+1)\frac{\exp(\xi_0^2/4)}{\exp(\xi^2/4)}\int_{\xi_m-x_1}^{\xi_0} y^{2\delta}\ dy \int_{\xi_m-x_1}^{y} z^{-2\delta}\ (\frac{1}{\xi_0^2}-\frac{1}{z^2})\ dz\ ,$$

$k_1>0$, and

$$k_2 = \delta(\delta+1)\exp(-\xi_0^2/4)\int_{\xi_m-x_1}^{\xi_0} z^{-2\delta}\ (\frac{1}{\xi_0^2}-\frac{1}{z^2})\ dz\ ,$$

and for $x\ge\xi_0$ we therefore have the lower bound

$$m_{\xi,c_0(\xi)}(x)\ge m_{\xi,c_0(\xi)}(\xi)\left(k_1+k_2\int_{\xi_0}^{\infty}\frac{dy}{p(y)}\right)\ ,$$

and it follows, using again the integral equation (86), that $m_{\xi,c_0(\xi)}$ diverges at (or before) infinity, that therefore $m'_{\xi,c_0(\xi)}(x)=0$ for some $x>\xi$, which implies that $\xi\in I_2$, provided

$$k_1+k_2\int_{\xi_0}^{\infty}\frac{dy}{p(y)}>0\ . \tag{87}$$

For x_1 small enough and for n large enough (87) can be verified without too much difficulty. With the help of a computer one can show that (87) is satisfied for the remaining $n\ge 5$. For $n=4$ (87) is not satisfied, since the above bounds on $m_{\xi,c_0(\xi)}(\xi_0)$ and $m'_{\xi,c_0(\xi)}(\xi_0)$ are too weak. Sufficiently good bounds can be obtained by dividing the interval (ξ,ξ_0) in two pieces and by integrating lower bounds on each of the subintervals. We omit the details. Finally, uniqueness of ξ^* can be proved by integrating the difference of two solutions from ξ_0 to infinity, which, using the positivity of $\partial_s\Omega(s,z)$, leads to a contradiction with the fact that both of the solutions converge to zero at infinity. ■

7.3 Proof of Proposition 6

We first proof the existence of a unique solution of equation (42) with the boundary conditions (43) and (44). Then, we derive the results on the asymptotic behavior near zero and infinity.

7.3.1 Existence of the function φ_2

The equation (42) for h is linear. We therefore first construct two linearly independent solutions h_1 and h_2 for the corresponding homogeneous equation, which we then use to construct, using standard methods, a solution of (42) that satisfies the boundary conditions (43) and (44). The homogeneous equation is

$$h''-q\ h=0\ , \tag{88}$$

where

$$q(z)=n\left(2\kappa z\eta(z)+\eta(z)^2\right)^{n-1}(2\kappa z+2\eta(z))\ . \tag{89}$$

Since the equation (88) is linear, the integral equation for h_1,

$$h_1(x)=1+\int_0^x dy\int_0^y q(z)\ h_1(z)\ dz\ , \tag{90}$$

has a positive solution that exists for all x in $\mathbf{R}_+$. By definition, we have near $x=0$ the behavior $h_1(x) = 1 + \mathcal{O}(x^2)$. At infinity, the solution h_1 is asymptotic to a solution of the equation

$$h''(x) - \frac{n}{\lambda}(2\kappa\lambda)^n \frac{1}{x^2} h(x) = 0 \ .$$

This equation is the same as the homogeneous part of equation (75), and the leading order behavior of h_1 at infinity is therefore either proportional to x^{p_+} or to x^{p_-}, with $p_\pm$ as defined in (76). Since h_1 is positive, we find using (90), that $h_1(x) > 1$ for all x in $\mathbf{R}_+$, and therefore h_1 is near infinity of the form $h_1(x) = d_1 x^{p_+} + \dots$, for some constant $d_1 > 0$. A second solution of the homogeneous equation (88) is

$$h_2(x) = h_1(x) \int_0^x \frac{1}{h_1(y)^2} \, dy \ .$$

Near $x = 0$ we have that $h_2(x) = x + \dots$, and near infinity we find that

$$h_2(x) = h_1(x) \left(d - d_2 x^{1-2p_+} + \dots\right) \ , \tag{91}$$

where $d = \int_0^\infty 1/h_1(y)^2 \, dy$, and $d_2 = (1/d_1)^2 / (2p_+ - 1)$. We note that $h_1 h_2' - h_1' h_2 \equiv 1$. Therefore, the function h_p,

$$h_p(x) = c_1(x) \ h_1(x) + c_2(x) \ h_2(x) \ ,$$

where

$$c_1(x) = -\int_0^x h_2(y) \ f(y) dy \ ,$$
$$c_2(x) = \int_0^x h_1(y) \ f(y) dy \ ,$$

and where

$$f(x) = -\gamma\eta(x) - \alpha x \eta'(x) + n \left(2\kappa x \eta(x) + \eta(x)^2\right)^{n-1} 2\kappa_3 x^3 \eta(x) \ ,$$

satisfies equation (42). Near zero, the function f is of the form $f(x) = -\gamma\eta_0 + \dots$, and therefore, using the behavior of h_1 and h_2 near zero, we find that c_1 is near zero of order $\mathcal{O}(x^2)$, and c_2 is near zero of order $\mathcal{O}(x)$. The function h_p is therefore of order $\mathcal{O}(x^2)$ near zero. At infinity, the function f is of the form $f(x) = f_\infty x^{-\delta} + \dots$, where $f_\infty = -\gamma\lambda + \alpha\lambda\delta + n(2\kappa\lambda)^{n-1} 2\kappa_3 \lambda$, and therefore the function c_2 is near infinity of the form $c_2(x) = d_1 f_\infty \ x^{p_+ + 1 - \delta}/(p_+ + 1 - \delta) + \dots$, and c_1 is near infinity of the form $c_1(x) = -d \ c_2(x) + h_\infty + d_1 d_2 f_\infty \ x^{2 - p_+ - \delta}/(2 - p_+ - \delta) + \dots$, for some constant h_∞. Using these asymptotic behavior for c_1, h_1, c_2, and h_2, we find for the function h_p near infinity the behavior,

$$\begin{aligned}
h_p(x) &= (-d \ c_2(x) + h_\infty + \frac{d_1 d_2 f_\infty}{(2 - p_+ - \delta)} x^{2 - p_+ - \delta} + \dots) h_1(x) + c_2(x) h_1(x) \left(d - d_2 x^{1 - 2p_+} + \dots\right) \\
&= h_\infty h_1(x) - \frac{d_1 f_\infty}{p_+ + 1 - \delta} d_1 d_2 x^{p_+ + 1 - \delta} x^{p_+} x^{1 - 2p_+} + \frac{d_1 d_2 f_\infty}{2 - p_+ - \delta} d_1 x^{2 - p_+ - \delta} x^{p_+} + \dots \\
&= h_\infty h_1(x) + \frac{f_\infty}{2p_+ - 1} \left(\frac{-1}{p_+ + 1 - \delta} + \frac{1}{2 - p_+ - \delta} \right) x^{2 - \delta} + \dots \\
&= h_\infty h_1(x) + \lambda_0 x^{2 - \delta} + \dots \ .
\end{aligned} \tag{92}$$

In the last equality we have used the definition (38) for λ_0. The function h,

$$h(z) = h_p(z) - h_\infty \ h_1(z) \ ,$$

solves the equation (42), satisfies the boundary condition (43), and since, as we show in the next section, the higher order terms in (92) converge to zero at infinity, it also satisfies the boundary condition (44).

7.3.2 Asymptotic behavior of the function φ_2

By construction, the leading behavior of h at infinity is $h(z) = \lambda_0 z^{2-\delta} + \dots$. We therefore make the ansatz $h(z) = \lambda_0 z^{2-\delta} + k(z)$, and to leading order, we get for the function k the linear equation

$$k'' - n\delta(\delta + 1) z^{-2} k = c_k z^{-\delta'} \ , \tag{93}$$

for a certain constant c_k. The general solution of equation (93) is

$$k(z) = \frac{\lambda'}{z^{\delta' - 2}} + \text{const.} \ z^{p_-} + \text{const.} \ z^{p_+} \ ,$$

with a certain constant λ' and with p_+, p_- as defined in (76). Since $\lim_{z\to\infty} k(z)/z^{2-\delta} = 0$ but $p_+ > 2-\delta$, the coefficient of the term proportional to z^{p_+} must be zero. Therefore, since$|p_-| > \delta' - 2$ for all n, the asymptotic behavior of k is always given by $\lambda'/z^{\delta'-2}$. We omit the details of the proof that this is indeed the correct second order behavior of k at infinity.

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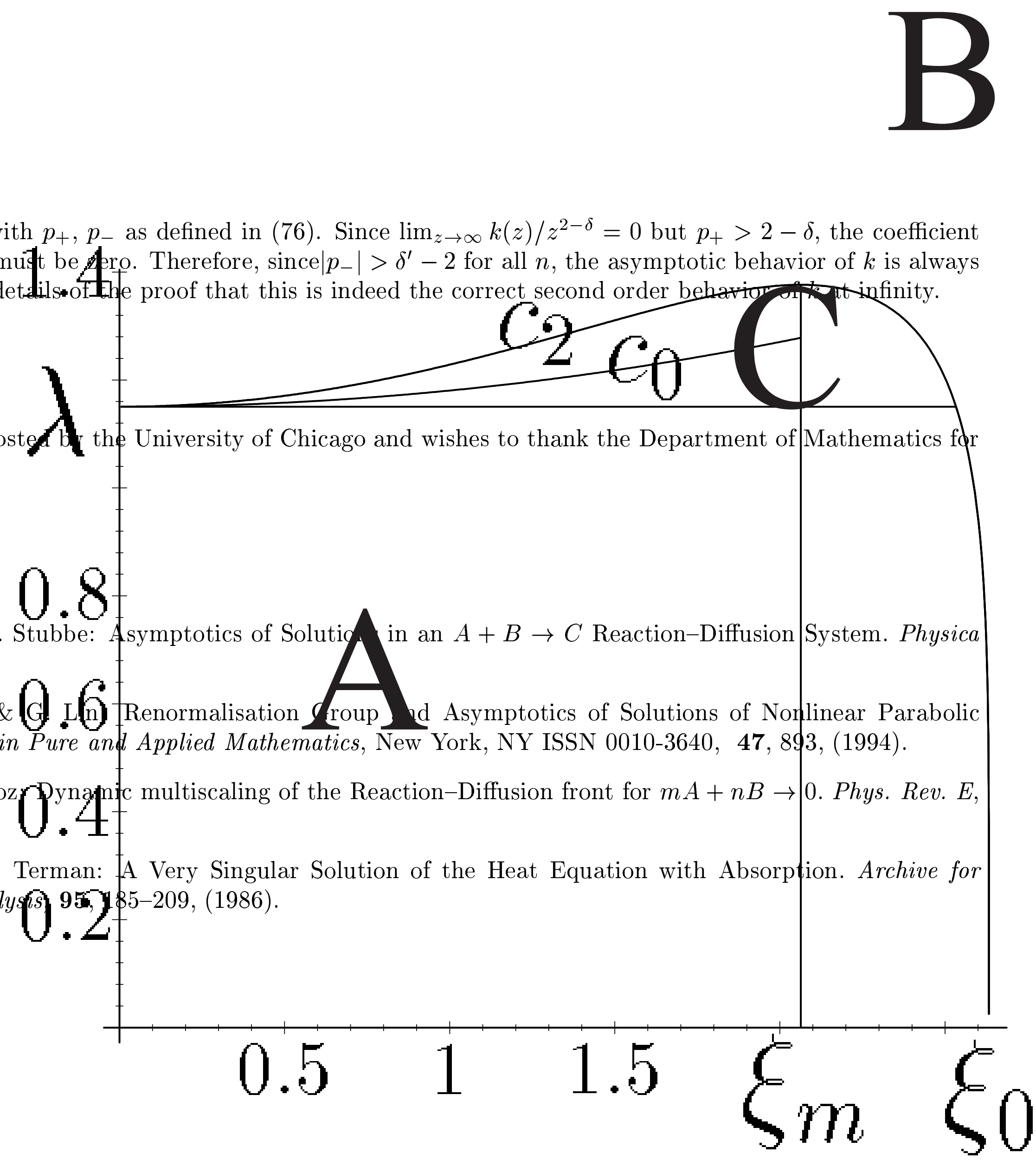

Fig. 1. Initial conditions for $m_{\xi,\rho}$ for $n = 6$.