On the number of particles which a curved quantum waveguide can bind

Pavel Exner[a,b] and Simeon A. Vugalter[a]

a) Nuclear Physics Institute, Academy of Sciences, 25068 Řež near Prague,
b) Doppler Institute, Czech Technical University, Břehová 7, 11519 Prague,
exner@ujf.cas.cz, vugalter@ujf.cas.cz

Abstract

We discuss the discrete spectrum of N particles in a curved planar waveguide. If they are neutral fermions, the maximum number of particles which the waveguide can bind is given by a one–particle Birman–Schwinger bound in combination with the Pauli principle. On the other hand, if they are charged, *e.g.*, electrons in a bent quantum wire, the Coulomb repulsion plays a crucial role. We prove a sufficient condition under which the discrete spectrum of such a system is empty.

1 Introduction

A rapid progress of mesoscopic physics brought, in particular, interesting new problems concerning relations between geometry and spectral properties of quantum Hamiltonians. They involve models of quantum wires, dots, and similar systems. While in reality these are rather complicated systems composed of different semiconductor materials, experience tells us that their basic features can be explained using simple models in which electrons (regarded as free particles with an effective mass) are supposed to be confined to an appropriate spatial region, either by a potential or by a hard wall. A brief description of this approximation with a guide to further reading is given in Ref. [1]. In addition, such models apply not only to electrons in semiconductor microstructures; a different example is represented by atoms trapped in hollow optical fibers [2].

It is natural that most theoretical results up to date refer to the case of a single particle in the confinement. On the other hand, from the practical point of view it is rather an exception than a rule that an experimentalist is able to isolate a single electron or atom, and therefore many–body problems in this setting are of interest. For instance, two–dimensional quantum dots which can be regarded as artificial atoms have been studied recently, usually in presence of a magnetic field, either for a pair of electrons or in the semiclasical situation when a Thomas–Fermi–type approach is applicable — *cf.* [3-6] and references therein.

In these studies, however, geometry of the dot played a little role, because the confinement was realized by a harmonic potential or a circular hard wall. This is not the case for open systems modelling quantum wires where a deformation of a straight channel is needed to produce nontrivial spectral properties. In particular, a quantum waveguide exhibit bound states if it is bent [1, 7, 8], protruded [9-11] or allowing a leak to another duct [12-14], and the discrete spectrum depends substantially on the shape of the channel. With few exceptions such as Ref. [15], however, the known results refer to the one–particle case.

It is the aim of the present paper to initiate a rigorous investigation of many–particle effects in quantum waveguides. We are going to discuss here a system of N particles in a bent planar Dirichlet tube, *i.e.*, a hard–wall channel, and ask whether N–particle bound states exist for a given geometry. After collecting the necessary preliminaries in the next section, we shall derive first in Section 3 a simple bound for the neutral case which follows from the Birman–Schwinger estimate of the one–particle Hamiltonian in combination with the Pauli principle.

The main result of the paper is formulated and proved in Section 4. It concerns the physically interesting case of charged particles; the example we have in mind is, of course, electrons in a bent semiconductor quantum wire. The electrostatic repulsion makes spectral analysis of the corresponding Hamiltonian considerably more complicated. Using variational technique borrowed from atomic physics, we derive here a sufficient condition under which the discrete spectrum is empty. The condition is satisfied for N large enough and represents an implicit equation for the maximum number of charged particles which a waveguide of a given curvature and width can bind. Some other aspects of the result and open questions are discussed briefly in the concluding section.

2 Preliminaries

The waveguide in question will be modelled by a curved planar strip Σ in $I\!R^2$, of a constant width $d = 2a$. It can be obtained by transporting the perpendicular interval $[-a, a]$ along the curve Γ which is the axis of Σ. Up to Euclidean transformations, the strip is uniquely characterized by its halfwidth a and the (signed) curvature $s \mapsto \gamma(s)$ of Γ, where s denotes the arc length. We adopt the regularity assumptions of Refs. [1, 7]:

(i) Ω is not self–intersecting,

(ii) $a\|\gamma\|_\infty < 1$,

(iii) γ is piecewise C^2 with γ', γ'' bounded,

and restrict our attention to the case when the tube is curved in a bounded region only:

(iv) there is $b > 0$ such that $\gamma(s) = 0$ for $|s| > b$; without loss of generality we may assume that $2b > a$.

As usual we put $\hbar = 2m = 1$; then the one–particle Hamiltonian of such a waveguide is the Dirichlet Laplacian $-\Delta_D^\Sigma$ defined in the conventional way — *cf.* [16], Sec. XIII.15. Using the natural locally orthogonal curvilinear coordinates s, u in Σ one can map $-\Delta_D^\Sigma$ unitarily onto the operator

$$H_1 \;=\; -\partial_s \,(1+u\gamma)^{-2}\, \partial_s \;-\; \partial_u^2 \;+\; V(s,u) \tag{2.1}$$

on $L^2(I\!\!R\times(-a,a)\,)$ with the effective curvature–induced potential

$$V(s,u) \;:=\; -\,\frac{\gamma(s)^2}{4(1+u\gamma(s))^2} \;+\; \frac{u\gamma''(s)}{2(1+u\gamma(s))^3} \;-\; \frac{5}{4}\,\frac{u\gamma'(s)^2}{(1+u\gamma(s))^4} \tag{2.2}$$

which is e.s.a. on the core $D(H) \;=\; \{\,\psi \,:\, \psi\in C^\infty,\;\, \psi(s,\pm a)=0,\;\, H\psi\in L^2\,\}$ — *cf.* Refs. [1, 7] for more details.

If the waveguide contains N particles, the state Hilbert space is $L^2(\Sigma))^N$; the Pauli principle will be taken into account later. We assume that each particle has the charge e; using the same "straightening" transformation we are then able to rewrite the Hamiltonian as

$$\begin{aligned} H_N \;\equiv\; H_N(\gamma,a,e) \;&=\; \sum_{j=1}^{N} \Big\{ -\partial_{s_j}\,(1+u_j\gamma(s_j))^{-2}\,\partial_{s_j} \;-\; \partial_{u_j}^2 \;+\; V(s_j,u_j) \Big\} \\ &\quad+\; e^2 \sum_{1\le j<l\le N} |\vec r_j - \vec r_l|^{-1}\,, \end{aligned} \tag{2.3}$$

with the domain $\left(\mathcal{H}^2(I\!\!R)\otimes\mathcal{H}_0^2(-a,a)\right)^N$, where $\vec r_j \;=\; \vec r_j(s_j,u_j)$ are the Cartesian coordinates of the N–th particle.

As we have said our main aim in this paper is to estimate the maximum number of particles which a curved waveguide with given γ, a can bind, *i.e.*, to find conditions under which the discrete spectrum of H_N is empty. To this end, one has to determine first the bottom of the essential spectrum. In complete analogy with the usual HVZ theorem [16], we find

$$\sigma_{\rm ess}(H_N) \;=\; \left[\,\mu_{N-1} + \left(\frac{\pi}{2a}\right)^2,\, \infty\right)\,, \tag{2.4}$$

where $\mu_{N-1} := \inf\,\sigma(H_{N-1})$. Obviously,

$$\inf\,\sigma_{\rm ess}(H_N) \;\le\; \mu_{N-k} + k\left(\frac{\pi}{2a}\right)^2$$

holds for $k=1,\dots,N\!-\!1$, so

$$\inf\,\sigma_{\rm ess}(H_N) \;\le\; N\left(\frac{\pi}{2a}\right)^2. \tag{2.5}$$

In a straight tube the two expressions equal each other, while for $\gamma\ne 0$ we have a sharp inequality because $\mu_1 < \left(\frac{\pi}{2a}\right)^2$ holds in this case.

3 Neutral fermions

If the particles in question are neutral fermions, one can get a simple upper bound on the number of bound states using the one–particle Hamiltonian (2.1); it is sufficient to estimate the dimension of $\sigma_{\rm disc}(H_1)$ and to employ the Pauli principle. To this aim, one has to estimate H_1 from above by an operator with the transverse and longitudinal variables decoupled; its projections to transverse modes are then one–dimensional Schrödinger operators to which the modified Birman–Schwinger bound may be applied [17-19]. In Ref. [1] we used this argument in the situation where a is small so that only the lowest transverse mode and the leading term in (2.2) may be taken into account.

A modification to the more general case is straightforward. We introduce the function

$$\tilde W(s)\,:=\,\frac{\gamma(s)^2}{4\delta_-^2}\,+\,\frac{a|\gamma''(s)|}{2\delta_-^3}\,+\,\frac{5a^2\gamma'(s)^2}{4\delta_-^4}\,, \tag{3.1}$$

where

$$\delta_\pm\,:=\,1\pm a\|\gamma\|_\infty\,, \tag{3.2}$$

which majorizes the effective potential, $V(s,u)\le \tilde W(s)$. Furthermore, we set

$$\tilde W_j(s)\,:=\,\max\left\{0,\,\left(\frac{\pi}{2a}\right)^2(1-j^2)\right\} \tag{3.3}$$

for $j=2,3,\dots$; in view of the assumptions *(ii), (iii)* only finite number of them is different from zero.

Replacing V by $\tilde W$, and $(1+u\gamma)^{-2}$ by δ_+^{-2}, we get an estimating operator with separating variables, or in other words, a family of shifted one–dimensional Schrödinger operators; we are looking for the number of their eigenvalues below $\inf\sigma_{\rm ess}(H_1)=\left(\frac{\pi}{2a}\right)^2$. The mentioned modification of the Birman–Schwinger bound is based on splitting the rank–one operator corresponding to the singularity of the resolvent kernel $\frac{1}{2\kappa}\,e^{-\kappa|s-s'|}$ at $\kappa=0$ and applying a Hilbert-Schmidt estimate to the rest. In analogy with Refs. [17, 18, 19] we employ this trick for the lowest–mode component of the estimating operator, while for the higher modes we use the full resolvent at the values $\kappa_j\,:=\,\left(\frac{\pi}{2a}\right)\sqrt{j^2-1}$. In this way we arrive at the following conclusion:

Proposition 3.1 The number N of neutral particles of half–integer spin S which a curved quantum waveguide can bind satisfies the inequality

$$\begin{aligned} N\,\le\,(2S+1)\Bigg\{1\,+\,\delta_+^2\,&\frac{\int_{I\!\!R^2}\tilde W(s)|s-t|\,\tilde W(t)\,ds\,dt}{\int_{I\!\!R}\tilde W(s)\,ds} \\ &+\,\sum_{j=2}^\infty\frac{a\delta_+^2}{\pi\sqrt{j^2-1}}\int_{I\!\!R}\tilde W_j(s)\,ds\Bigg\}\,. \end{aligned} \tag{3.4}$$

Remarks 3.2 (a) As we have said, the number of nonzero term in the last sum is finite. More exactly, the index j runs up to the entire part of $\sqrt{1+\left(\frac{2a}{\pi}\right)^2\|\tilde W\|_\infty}\,$; hence if a is small enough this term is missing at all.
(b) The assumption *(iv)* is not needed here. It is sufficient, *e.g.*, that the functions γ, γ', and $|\gamma''|^{1/2}$ decay as $|s|^{-1-\varepsilon}$ as $|s|\to\infty$.

4 Main result: N charged particles

We have said in the introduction that the present study is motivated mainly by the need to describe electrons in curved quantum wires. Unfortunately, the above simple estimate have no straightforward consequences for the situation when the particles are charged. While the electrostatic repulsion adds a positive term to the Hamiltonian (2.3), it may move at the same time the bottom of the essential spectrum since the energies of the bound "clusters" are, of course, sensitive to the interaction change.

We need therefore another approach which would allow to take the repulsion term in (2.3) into account. An inspiration can be found in analysis of atomic N–body Hamiltonians. To formulate the result we need some notation. Given a positive β we denote by $\{\lambda_m\}_{m=1}^\infty$ the ordered sequence of eigenvalues of Dirichlet Laplacian at the rectangle

$$R_\beta := \left[-\frac{3}{2}\beta\delta_+,\, \frac{3}{2}\beta\delta_+\right] \times [-a,a]\,, \tag{4.1}$$

and set

$$T_\beta(N) := \begin{cases} 2\sum_{m=1}^n \lambda_m & \dots \quad N=2n \\ \\ 2\sum_{m=1}^n \lambda_m + \lambda_{n+1} & \dots \quad N=2n+1 \end{cases} \tag{4.2}$$

We have in mind here electrons and assume that the spin is $\frac{1}{2}$, otherwise $T_\beta(N)$ has to be replaced by the sum of the first N eigenvalues of $2S+1$ identical copies of the Laplacian. Now we are able state our main result:

Theorem 4.1 Assume *(i)–(iv)*. $\sigma_{\rm disc}\left(H_N(\gamma,a,e)\right)=\emptyset$ for $N\ge 2$ if the condition

$$T_\beta(N) + \frac{e^2}{2\beta\sqrt{7}}N(N-1) \ge \|\tilde W\|_\infty N + \left(\frac{\pi}{2a}\right)^2 N + \frac{e^2}{18\beta\sqrt{2}} \tag{4.3}$$

is valid for some $\beta\ge\max\{2b, 596\,e^{-2}\}$.

Proof: We use a variational argument which relies on a suitable decomposition of the configuration space. Consider a pair of smooth functions $v,\, g$ from $\mathbb{R}_+$ to $[0,1]$ such that

$$v(t) = \begin{cases} 0 & \dots \quad t\le 1 \\ \\ 1 & \dots \quad t\ge \frac{3}{2} \end{cases} \tag{4.4}$$

and

$$v(t)^2 + g(t)^2 = 1\,. \tag{4.5}$$

Elements of the configuration space are (s,u) with $s = \{s_1, \dots, s_N\}$ and $u = \{u_1, \dots, u_N\}$. We denote $\|s\|_\infty := \max\{s_1, \dots, s_N\}$ and employ the functions

$$s \mapsto v(\|s\|_\infty \beta^{-1}),\; g(\|s\|_\infty \beta^{-1})\,,$$

where $\beta > 2b > a$ is a parameter to be specified later. By abuse of notation, we use the symbols $v,\, g$ again both for these functions and the corresponding operators of multiplication. It is straightforward to evaluate $([H_N, v]\psi, v\psi)$ and the analogous expression with v replaced by g for a vector $\psi \in D(H_N)$; in both cases it is only the longitudinal kinetic part in (2.3) which contributes. This yields the identity

$$\begin{aligned} (H_N\psi,\psi) &= (H_N v\psi, v\psi) + (H_N g\psi, g\psi) \\ &+ \sum_{j=1}^{N} \left\{ \left\| (1+u_j\gamma_j)^{-1} v_j \psi \right\|^2 + \left\| (1+u_j\gamma_j)^{-1} g_j \psi \right\|^2 \right\}, \end{aligned}$$

where we have used the shorthands $v_j := \frac{\partial v}{\partial s_j}$, $g_j := \frac{\partial g}{\partial s_j}$, and $\gamma_j := \gamma(s_j)$. Notice further that the factors $(1+u_j\gamma_j)^{-1}$ may be neglected, because $v_j\, g_j$ are nonzero only if $s_j \ge \beta > 2b$ in which case $\gamma_j = 0$. Furthermore, with the exception of the hyperplanes where two or more coordinates coincide (which is a zero measure set) the norm $\|s\|_\infty$ coincides with just one of the coordinates $s_1, \dots, s_n$, and therefore

$$\sum_{j=1}^{N} \left\{ \|v_j\psi\|^2 + \|g_j\psi\|^2 \right\} \le \|\psi\|^2 \max_{1\le j\le N} \left\{ \|v_j\|_\infty^2 + \|g_j\|_\infty^2 \right\} \le \beta^{-2} C_0 \|\psi\|^2\,, \tag{4.6}$$

where $C_0 := \|v'\|_\infty^2 + \|g'\|_\infty^2$. We arrive at the estimate

$$(H_N\psi,\psi) \ge L_1[v\psi] + L_1[g\psi] \tag{4.7}$$

with

$$L_1[\phi] := (H_N\phi,\phi) - \frac{C_0}{\beta^2}\, \|\phi\|_{\mathcal{N}_\beta}^2\,, \tag{4.8}$$

where the last index symbolizes the norm of the vector ϕ restricted to the subset $\mathcal{N}_\beta := \left\{ s :\; \beta \le \|s\|_\infty \le \frac{3\beta}{2} \right\}$ of the configuration space.

Next one has to estimate separately the contributions from the inner and outer parts. Let us begin with the exterior. We introduce the following functions:

$$\begin{aligned} f_1(s) &= v\left(2s_1\|s\|_\infty^{-1}\right), \\ f_j(s) &= v\left(2s_j\|s\|_\infty^{-1}\right) \prod_{n=1}^{j-1} g\left(2s_n\|s\|_\infty^{-1}\right), \qquad j = 2, \dots, N-1 \\ f_N(s) &= \prod_{n=1}^{N-1} g\left(2s_n\|s\|_\infty^{-1}\right). \end{aligned}$$

It is clear from the construction that

$$\sum_{j=1}^{N} f_j(s)^2 = 1\,. \tag{4.9}$$

Moreover, the functions

$$s_j \mapsto v(2s_j\|s\|_\infty),\ g(2s_j\|s\|_\infty)$$

have a non–zero derivative only if $|s_j| \geq \frac{1}{2}\|s\|_\infty^{-1}$. Hence on the support of $s \mapsto v(\|s\|_\infty\beta^{-1})$ the derivative is non–zero if $|s_j| \geq \frac{1}{2}\beta > b$. In other words, the function $s \mapsto f_j(s)^2 v(\|s\|_\infty\beta^{-1})$ has zero derivative in all the parts of the configuration space where at least one of the electrons dwells in the curved part of the waveguide. Commuting the (longitudinal kinetic part of) H_N with f_j, we get in the same way as above the identity

$$L_1[v\psi] = \sum_{j=1}^{N} \Big\{ L_1[f_j v\psi] - \|(\nabla_s f_j)v\psi\|^2 \Big\}\,, \tag{4.10}$$

where $\nabla_s := (\partial_{s_1}, \dots, \partial_{s_1})$. Next we need a pointwise upper bound on $\sum_{j=1}^{N}(\nabla_s f_j)^2$: denoting $\sigma_j := 2s_j\|s\|_\infty$, we can write

$$\begin{aligned}
\sum_{j=1}^{N} |(\nabla_s f_j)(s)|^2 \;=\; & \frac{4}{\|s\|_\infty^2}\Big\{ v'(\sigma_1)^2 \\
& + g'(\sigma_1)^2 v(\sigma_2)^2 + g(\sigma_1)^2 v'(\sigma_2)^2 + \cdots \\
& + g'(\sigma_1)^2 g(\sigma_2)^2 \dots g(\sigma_N)^2 + \cdots + g(\sigma_1)^2 \dots g(\sigma_{N-1})^2 g'(\sigma_N)^2 \Big\}\,,
\end{aligned}$$

which gives after a partial resummation

$$\begin{aligned}
&= \frac{4}{\|s\|_\infty^2}\Big\{ v'(\sigma_1)^2 + g'(\sigma_1)^2 + g(\sigma_1)^2 g'(\sigma_2)^2 + \cdots \\
&\qquad + g(\sigma_1)^2 \dots g(\sigma_{N-1})^2 g'(\sigma_N)^2 \Big\} \\
&\leq \frac{4}{\|s\|_\infty^2}\left\{ v'(\sigma_1)^2 + \sum_{j=1}^{N} g'(\sigma_j)^2 \right\} \leq \frac{4NC_0}{\|s\|_\infty^2}\,;
\end{aligned}$$

recall that $C_0 := \|v'\|_\infty^2 + \|g'\|_\infty^2$. Consequently,

$$\begin{aligned}
L_1[v\psi] \;&\geq\; \sum_{j=1}^{N} L_1[f_j v\psi] \;-\; 4NC_0 \left\| v\psi\|s\|_\infty^{-1} \right\|^2 \\
&= \sum_{j=1}^{N} \left\{ L_1[f_j v\psi] - 4NC_0 \left\| f_j v\psi\|s\|_\infty^{-1} \right\|^2 \right\} \\
&= \sum_{j=1}^{N} L_2[f_j v\psi]\,,
\end{aligned} \tag{4.11}$$

where

$$L_2[\phi] \;:=\; L_1[\phi] \;-\; 4NC_0 \left\| \phi \|s\|_\infty^{-1} \right\|^2 . \tag{4.12}$$

Hence we have to find a lower bound to $L_2(\psi_j)$ with $\psi_j := f_j v\psi)$. Since $s_j \geq \frac{1}{2}\|s\|_\infty \geq \frac{1}{2}\beta > b$ holds on the support of ψ_j, we have $V(s_j, u_j) = 0$ there. This allows us to write

$$(H_N\psi_j, \psi_j) \;=\; (H_{N-1}\psi_j, \psi_j) + \left\|\partial_{s_j}\psi_j\right\|^2 + \left\|\partial_{u_j}\psi_j\right\|^2 + e^2 \sum_{j\neq l=1}^{N} \left(|\vec{r}_j - \vec{r}_l|^{-1}\psi_j, \psi_j \right) ,$$

where H_{N-1} refers to the system with the j–th electron excluded, and therefore

$$(H_N\psi_j, \psi_j) \;\geq\; \left(\mu_{N-1} + \left(\frac{\pi}{2a}\right)^2 \right) \|\psi_j\|^2 + e^2 \sum_{j\neq l=1}^{N} \left(|\vec{r}_j - \vec{r}_l|^{-1}\psi_j, \psi_j \right) .$$

Since $|\vec{r}_j - \vec{r}_l| \leq \sqrt{(s_j - s_l)^2 + 4a^2} \;\leq\; 2\sqrt{\|s\|_\infty^2 + a^2}$, we have

$$(H_N\psi_j, \psi_j) \;\geq\; \left(\mu_{N-1} + \left(\frac{\pi}{2a}\right)^2 \right) \|\psi_j\|^2 + \frac{e^2(N-1)}{2} \left((\|s\|^2 + a^2)^{-1/2}\psi_j, \psi_j \right) .$$

The sought lower bound then follows from (4.12) and (4.8):

$$\begin{aligned} L_2[\psi_j] \;&\geq\; \left(\mu_{N-1} + \left(\frac{\pi}{2a}\right)^2 \right) \|\psi_j\|^2 - 4NC_0 \left\| \psi_j \|s\|_\infty^{-1} \right\|^2 \\ &\quad -\; C_0\beta^{-2}\|\psi_j\|_{\mathcal{N}_\beta}^2 + \frac{e^2(N-1)}{2} \left((\|s\|^2 + a^2)^{-1/2}\psi_j, \psi_j \right) \;; \end{aligned}$$

recall that $\mathcal{N}_\beta := \left\{ s:\; \beta \leq \|s\|_\infty \leq \frac{3\beta}{2} \right\}$. The second and the third term at the *rhs* can be combined using

$$4NC_0 \left\| \psi_j \|s\|_\infty^{-1} \right\|^2 + C_0\beta^{-2}\|\psi_j\|_{\mathcal{N}_\beta}^2 \;\leq\; (4N+1)C_0 \left\| \psi_j \|s\|_\infty^{-1} \right\|^2 .$$

Furthermore, $\|s\|_\infty \geq \beta > 2b > a$ yields $(\|s\|^2 + a^2)^{1/2} \leq \sqrt{2}\,\|s\|_\infty$ and

$$\begin{aligned} L_2[\psi_j] \;&\geq\; \left(\mu_{N-1} + \left(\frac{\pi}{2a}\right)^2 \right) \|\psi_j\|^2 \\ &\quad+\; \left(\frac{e^2(N-1)}{2\sqrt{2}} - \frac{C_0(4N+1)}{\beta} \right) \left\| \psi_j \|s\|_\infty^{-1} \right\|^2 . \end{aligned} \tag{4.13}$$

We are interested in the situation when the second term at the *rhs* is positive. This is achieved if

$$\frac{e^2(N-1)}{2\sqrt{2}} \;>\; \frac{C_0(4N+1)}{\beta}$$

which is ensured if we choose β in such a way that

$$\beta \;>\; \frac{18\sqrt{2}C_0}{e^2} \;; \tag{4.14}$$

recall that $N \geq 2$. Owing to the identity (4.11) we then have

$$L_1[v\psi] \;\geq\; \left(\mu_{N-1} + \left(\frac{\pi}{2a} \right)^2 \right) \|v\psi\|^2 \,, \tag{4.15}$$

which means in view of (2.4) that the external part of ψ does not contribute to the discrete spectrum.

Let us turn now to the inner part. The corresponding quadratic form in the decomposition (4.7) can be estimated with the help of (2.3) and (4.8) by

$$\begin{aligned} L_1[g\psi] \;&\geq\; \delta_+^{-2}\|\nabla_s g\psi\|^2 + \|\nabla_u g\psi\|^2 + \sum_{j=1}^{N} (V(s_j,u_j)g\psi, g\psi) \\ &+\; e^2 \sum_{1\leq<k\leq N} \left(\|\vec r_j - \vec r_k\|^{-1} g\psi, g\psi \right) \;-\; \frac{C_0}{\beta^2}\, \|g\psi\|^2 \,; \end{aligned} \tag{4.16}$$

recall that $\delta_+ := 1+a\|\gamma\|_\infty$. Using the function $\tilde W$ defined by (3.1) we find $|V(s_j,u_j)| \leq \tilde W(s_j)$, so

$$\max\{\, V(s,u) \,:\, (s,u) \in I\!\!R \times [-a,a] \,\} \;\leq\; \|\tilde W\|_\infty \,.$$

Consequently, the curvature–induced potential term can be estimated by

$$\sum_{j=1}^{N} (V(s_j,u_j)g\psi, g\psi) \;\leq\; \|\tilde W\|_\infty\, N \|g\psi\|^2 \,.$$

Furthermore, on the support of g we have

$$|\vec r_j - \vec r_k| \leq 2\sqrt{\|s\|_\infty^2 + a^2} \;\leq\; \sqrt{3\beta^2 + 4a^2} \,,$$

because $\|s\|_\infty \leq \frac{3}{2}\beta$ holds there. At the same time, $\beta > 2b > a$, so we arrive at the estimate

$$|\vec r_j - \vec r_k| \leq \sqrt{7}\beta \,,$$

which yields

$$\sum_{1\leq<k\leq N} \left(\|\vec r_j - \vec r_k\|^{-1} g\psi, g\psi \right) \;\geq\; \frac{N(N-1)}{2\beta\sqrt{7}}\, \|g\psi\|^2 \,.$$

Now we can combine the above estimates with the inequality $\frac{C_0}{\beta} < \frac{e^2}{18\sqrt{2}}$ which follows from (4.14) to get the bound

$$\begin{aligned} L_1[g\psi] \;&\geq\; \delta_+^{-2}\|\nabla_s g\psi\|^2 + \|\nabla_u g\psi\|^2 \\ &+\; \left[-N\|\tilde W\|_\infty + \frac{e^2 N(N-1)}{2\beta\sqrt{7}} \;-\; \frac{e^2}{18\beta\sqrt{2}} \right] \|g\psi\|^2 \,. \end{aligned} \tag{4.17}$$

Now we can put the above results together. In view of the inequality (4.15) and of (2.5), the last bound tells us that H_N has no discrete spectrum for $N \geq 2$ provided

$$\delta_+^{-2}\|\nabla_s g\psi\|^2 \;+\; \|\nabla_u g\psi\|^2 + \Bigg[\frac{e^2 N(N-1)}{2\beta\sqrt{7}} \;-\; \frac{e^2}{18\beta\sqrt{2}}$$
$$-\; N\|\tilde{W}\|_\infty - N\left(\frac{\pi}{2a}\right)^2\Bigg]\|g\psi\|^2 \;\geq\; 0 \tag{4.18}$$

for some β which satisfies the condition

$$\beta \;\geq\; \max\left\{2b,\, \frac{18\sqrt{2}C_0}{e^2}\right\}. \tag{4.19}$$

The first two terms in (4.18) are nothing else than the quadratic form of the $2N$–dimensional Laplacian on R_β^N — *cf.* (4.1). By Pauli principle each eigenvalue may appear only twice, thus one has to take the orthogonal sum of two copies of the Laplacian on R_β and to summ the first N eigenvalues of such an operator. This is exactly the quantity which we have called $T_\beta(N)$.

To finish the proof, it remains to estimate C_0 which appears in the conditions (4.14) and (4.19). We will not attempt an optimal bound and put simply

$$v(\xi) \;:=\; \sin\left(4\pi\xi^2(1-2\xi^2)\right)$$

for $t-1 =: \xi \in \left(0, \frac{1}{2}\right)$, then

$$v'(\xi)^2 + g'(\xi)^2 \;=\; (8\pi)^2\xi^2(1-4\xi^2)^2$$

has the maximum value $2\sqrt{2}(8\pi)^2/3 \approx 595.5$. ∎

5 Conclusions

Since the present study is rather a foray into an unchartered territory, the result is naturally far from optimal. Let us add a few remarks. First of all, it is clear that the overall size of the curved region affects substantially the number of particles which the waveguide can bind. We know that *any* curved tube has a one–particle bound state [1, 8], hence a tube with N slight bends which very far from each other (so far that the repulsion is much smaller that the gap between the bound state energy and the continuum) can certainly bind N particles for N arbitrarily large.

The method we use is borrowed from atomic physics where it yields bounds on ionization of an atom. Of course, there are differences. The binding is due to the curved hard wall of the waveguide rather than by the electrostatic attraction to the nucleus, and the spectrum of our one–particle operator (2.1) is finite. Consequently, there is a maximum number of particles which a given curved tube can bind as long as the particles are fermions. Bosons can occupy naturally a single state, and the idea of a *Bose condensate* of neutral spin–zero atoms in a curved hollow optical fiber is rather appealing.

On the other hand, a non–zero particle charge changes the picture, and even the number of bosons bind by a curved tube is limited: notice that the condition (4.3) is satisfied for large enough N without respect to the Pauli–principle term $T_\beta(N)$. Of course, the fermionic nature reduces the maximum number N further, since $T_\beta(N)$ growth for large N is between $o(N^3)$ in the limit $a \to 0$ and $o(N^2)$ for $2b \sim a$. At the same time, the maximum number also depends on the value of the charge. Since $\frac{1}{\sqrt{7}} - \frac{1}{18\sqrt{2}} > 0$ and the remaining terms in (4.3) are independent od e we see that $\sigma_{\rm disc}(H_N) = \emptyset$ for any $N \geq 2$ provided e is large enough. Thus our result confirms the natural expectation that for a given curved tube and sufficiently charged particles just one–particle bound states can survive.

We have not addressed in this paper the question about the minimum number of particles which a curved quantum waveguide can bind. The gap between the trivial result which follows from the one–particle theory [1, 7, 8] and the condition (4.3) leaves a lot of space for improvements. Moreover, it is a natural question whether strongly curved tubes which can bind many particles allow for some semiclassical description analogous to the case of the quantum dots [6]. This is a task for a future work.

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