On Higher-order Melnikov Method for the Exponentially Small Splitting of Separatrices

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This note has appeared as an answer to a question posed by Dr. Gallavotti, who asked about the way of calculating higher order (with respect to the Melnikov method) corrections to the splitting of separatrices, which would be compatible with the methods presented in [Gel97a, DGJS97]. In fact, the method of these papers do not require these calculations, but they may be useful if the main term "vanishes". Moreover I feel that these calculations are useful for a better understanding of the method and are important for comparisons with the results provided by alternative approaches.

In the considered class of problems the splitting of separatrices is exponentially small, and the possibility of using the Melnikov method is not evident at all. In some sense the present paper is a step backwards with respect to the results of [Gel97b, Gel98], where the unique small parameter was related to the perturbation period. Compared with these papers the present note makes use of an additional small parameter, which is proportional to the perturbation amplitude. A method, similar to the present one, may be used to compute Taylor coefficients of the correction factor defined in [Gel97b].

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1 Parametrization of separatrices

For the sake of brevity we restrict the consideration by the Hamiltonian,

$$H = \frac{\omega I}{\varepsilon} + \frac{y^2}{2} + (1 + \mu m(\theta))(\cos x - 1) , \tag{1}$$

which was already considered in [DGJS97]. This example affords to illustrate all main features of the method. The generalization is straightforward. We assume that x and y are scalar variables, and $\theta \in \mathbf{T}^n$, $I \in D \subset \mathbf{R}^n$, ω is a fixed frequency vector. In [DGJS97] we restricted our consideration to the case 2 and $\omega = (1, \gamma)$, where γ is the golden mean number.

The system depends on two small parameters ε and μ.

The origin $x = y = 0$ is a partially hyperbolic invariant torus. Like [DGJS97] we study its whiskers in a parametric form. The parametrization is a function of the parameters s and θ, which satisfies the following system:

$$\begin{array}{rclrcl} D_t x^\sigma &=& y^\sigma , & D_t y^\sigma &=& (1 + \mu m(\theta)) \sin x^\sigma , \\ D_t \theta &=& \frac{\omega}{\varepsilon} , & D_t I^\sigma &=& -\mu \frac{\partial m}{\partial \theta} (\cos x^\sigma - 1) , \end{array} \tag{2}$$

where the operator $D_t = \lambda \frac{\partial}{\partial s} + \frac{\omega}{\varepsilon} \frac{\partial}{\partial \theta}$, the constant λ is the Lyapunov exponent of the torus, and $\sigma \in \{ -, + \}$. The symbol "+" refers to the stable whisker and "−" to the unstable one. The parametrization is defined by this system, the boundary conditions,

$$\lim_{s \to -\infty} x^-(s, \theta) = 0 , \qquad \lim_{s \to +\infty} x^+(s, \theta) = 2\pi ,$$

$$\lim_{s \to \sigma\infty} y^\sigma(s, \theta) = 0 , \qquad \lim_{s \to \sigma\infty} I^\sigma(s, \theta) = 0 ,$$

and the following analytic properties: the parametrizations are 2π-periodic in θ and 2πi-periodic in s. The last property assumes that the functions are analytic in the half-planes $\sigma \Re t > R$ for some positive constant R.

We describe the expansions for the parametrizations and λ in power series with respect to the parameter μ,

$$\begin{array}{rcl} x^\sigma(s, \theta) &=& x_0(s) + \mu x_1^\sigma(s, \theta) + \mu^2 x_2^\sigma(s, \theta) + \dots , \\ y^\sigma(s, \theta) &=& y_0(s) + \mu y_1^\sigma(s, \theta) + \mu^2 y_2^\sigma(s, \theta) + \dots , \\ I^\sigma(s, \theta) &=& I_0(s) + \mu I_1^\sigma(s, \theta) + \mu^2 I_2^\sigma(s, \theta) + \dots , \\ \lambda &=& \lambda_0 + \mu \lambda_1 + \mu^2 \lambda_2 + \dots . \end{array} \tag{3}$$

We dedicate special attention to the terms of the first and second order, which we write explicitly. We use

$$x_0(s) = 4 \arctan \mathrm{e}^s , \qquad y_0(s) = \frac{2}{\cosh s} , \qquad I_0(s) = 0 , \qquad \lambda_0 = 1 , \tag{4}$$

as the zero order approximation. At this order the stable and unstable separatrices coincide. We substitute the series (3) into the equations (2) and collect the terms of the same order in μ. We denote

$$D_t^0 = \frac{\partial}{\partial s} + \frac{\omega}{\varepsilon}\frac{\partial}{\partial\theta} \ .$$

We obtain the following system for the first order terms:

$$\begin{aligned} D_t^0 x_1^\sigma + \lambda_1 \frac{\partial x_0}{\partial s} &= y_1^\sigma \ , & (5)\\ D_t^0 y_1^\sigma + \lambda_1 \frac{\partial y_0}{\partial s} &= \cos x_0 x_1^\sigma + m(\theta)\sin x_0 \ , & (6)\\ D_t^0 I_1 &= -\frac{\partial m}{\partial\theta}(\theta)(\cos x_0 - 1) \ . & (7)\end{aligned}$$

The analysis of the first order approximation leads to the almost standard Melnikov theory. We are mainly interested in the second order, which is defined by

$$\begin{aligned} D_t^0 x_2^\sigma + \lambda_1 \frac{\partial x_1}{\partial s} + \lambda_2 \frac{\partial x_0}{\partial s} &= y_1^\sigma \ , & (8)\\ D_t^0 y_2^\sigma + \lambda_1 \frac{\partial y_1}{\partial s} + \lambda_2 \frac{\partial y_0}{\partial s} &= \cos x_0 x_1^\sigma + m(\theta)\cos x_0 x_1 - \tfrac{1}{2}\sin x_0 x_1^2 \ , & (9)\\ D_t^0 I_2 + \lambda_1 \frac{\partial I_1}{\partial s} &= \frac{\partial m}{\partial\theta}(\theta)\sin x_0 x_1 \ . & (10)\end{aligned}$$

The procedure may be continued. At the n^{th} order in μ we obtain a linear nonhomogeneous system for the n^{th} order approximation for the separatrices. All these equations have to be supplied with the boundary conditions. We provide the explicit solutions later after explaining the way of solving these equations.

Homological equations

The basic homological equation has the form

$$D_t^0 f = g \ ,$$

where $g = g(s,\theta)$ is assumed to be periodic in θ. This equation appears at each order of the I component. It is easy to solve this equation, provided the right-hand side decreases at infinity ($s \to +\infty$ or $-\infty$). The desired solution is given by

$$f(s,\theta) = \int_{\sigma\infty}^{0} g(s+t, \theta + \tfrac{\omega t}{\varepsilon})\,\mathrm{d}t \ .$$

This integral preserves the periodicity and gives a function, which decreases at infinity too.

The equations for the x and y components have the following form:

$$D_t^0 x = y + g_1(s,\theta) \ , \tag{11}$$

$$D_t^0 y = \cos x_0 x + g_2(s,\theta) \ . \tag{12}$$

We know two basic solutions of the corresponding homogeneous system:

$$\vec{u}_1(s) = \begin{pmatrix} \xi_1 \\ \eta_1 \end{pmatrix} = \begin{pmatrix} \frac{2}{\cosh s} \\ -\frac{2\sinh s}{\cosh^2 s} \end{pmatrix} , \tag{13}$$

$$\vec{u}_2(s) = \begin{pmatrix} \xi_2 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \sinh s + \frac{s}{\cosh s} \\ \cosh s + \frac{1}{\cosh s} - \frac{s \sinh s}{\cosh^2 s} \end{pmatrix} . \tag{14}$$

The first solution, $\vec{u}_1$, is the derivative of $(x_0, y_0)(s)$. The Wronskian of these tow solutions is constant,

$$\begin{vmatrix} \xi_1 & \xi_2 \\ \eta_1 & \eta_2 \end{vmatrix} = 4 \ .$$

We solve the nonhomogeneous system using the method of variation of parameters. Indeed, if C_1 and C_2 satisfy

$$D_t^0 C_1 = \frac{1}{4} \begin{vmatrix} g_1 & \xi_2 \\ g_2 & \eta_2 \end{vmatrix} , \qquad D_t^0 C_2 = \frac{1}{4} \begin{vmatrix} \xi_1 & g_1 \\ \eta_1 & g_2 \end{vmatrix} , \tag{15}$$

then the functions,

$$x = C_1(s,\theta)\xi_1(s) + C_2(s,\theta)\xi_2(s) \ , \tag{16}$$

$$y = C_1(s,\theta)\eta_1(s) + C_2(s,\theta)\eta_2(s) \ , \tag{17}$$

solve the nonhomogeneous system (11), (12).

First order terms

Now we can calculate the first order terms. We have from (7)

$$I_1^\sigma(s,\theta) = -\int_{\sigma\infty}^0 (\cos x_0(s+t) - 1)\frac{\partial m}{\partial \theta}(\theta + \tfrac{\omega t}{\varepsilon})\,\mathrm{d}t \ .$$

The first order splitting is

$$\Delta I_1 = I_1^- - I_1^+ = -\int_{-\infty}^{\infty} (\cos x_0(s+t) - 1)\frac{\partial m}{\partial \theta}(\theta + \tfrac{\omega t}{\varepsilon})\mathrm{d}t \ .$$

The last integral can be easily calculated using residues. This integral coincides with the usual Melnikov function. It has some special analytical properties to be discussed later.

To obtain the x and y components we use the method of variation of parameters, which was described in the previous subsection. We put the following terms onto the right-hand side of the equations (5), (6):

$$\vec{g} = \begin{pmatrix} 0 \\ m(\theta)\sin x_0 \end{pmatrix} - \lambda_1 \begin{pmatrix} \frac{\partial x_0}{\partial s} \\ \frac{\partial y_0}{\partial s} \end{pmatrix} .$$

After the substitution of this $\vec{g}$ the equation (15) takes the form,

$$\begin{aligned} D_t^0 C_1 &= -\tfrac{1}{4} m(\theta) \sin x_0(s) \xi_2(s) - \lambda_1 \ , && (18) \\ D_t^0 C_2 &= \tfrac{1}{4} m(\theta) \sin x_0(s) \xi_1(s) \ , && (19) \end{aligned}$$

or, equivalently,

$$\begin{aligned} D_t^0 C_1 &= \frac{m(\theta)}{2} - \lambda_1 + \frac{m(\theta)}{2\cosh^2 s}\left(\frac{s \sinh s}{\cosh s} - 1\right) \ , && (20) \\ D_t^0 C_2 &= -\frac{m(\theta)\sinh s}{\cosh^3 s} \ . && (21) \end{aligned}$$

The absence of secular terms (the periodicity properties) proposes $\lambda_1 = <\frac{m}{2}> = 0$. We have to take into account that the right-hand side of (20) is not absolutely summable at infinity. The equation is linear, and we may represent the solution in the form

$$\begin{aligned} C_1^\sigma &= \frac{\tilde{m}(\theta)}{2} - \frac{1}{4}\int_{\sigma\infty}^{0} m(\theta + \tfrac{\omega t}{\varepsilon}) \left(\sin x_0(s+t)\xi_2(s+t) - 2\right) \mathrm{d}t \ , && (22) \\ C_2^\sigma &= \frac{1}{4}\int_{\sigma\infty}^{0} m(\theta + \tfrac{\omega t}{\varepsilon}) \sin x_0(s+t)\xi_1(s+t)\, \mathrm{d}t \ , && (23) \end{aligned}$$

where $\tilde{m}$ is a periodic solution of the following equation:

$$\frac{\omega}{\varepsilon}\frac{\partial \tilde{m}}{\partial \theta}(\theta) = m(\theta) \ .$$

We subtracted 2 under the first integral to ensure its convergence. An alternative approach can be based on the improper integrals technics of [Gal94]. Then the desired solutions, x_1^σ and y_1^σ, may be obtained by (16) and (17),

$$\begin{aligned} x_1^\sigma &= C_1^\sigma(s,\theta)\xi_1(s) + C_2^\sigma(s,\theta)\xi_2(s) \ , && (24) \\ y_1^\sigma &= C_1^\sigma(s,\theta)\eta_1(s) + C_2^\sigma(s,\theta)\eta_2(s) \ . && (25) \end{aligned}$$

For the splitting we have:

$$\begin{aligned} \Delta x_1 &= \xi_1(s)\Delta C_1(s,\theta) + \xi_2(s)\Delta C_2(s,\theta) \ , && (26) \\ \Delta y_1 &= \eta_1(s)\Delta C_1(s,\theta) + \eta_2(s)\Delta C_2(s,\theta) \ , && (27) \end{aligned}$$

where

$$\Delta C_1 = -\frac{1}{4}\int_{-\infty}^{\infty} m(\theta + \tfrac{\omega t}{\varepsilon})\,(\sin x_0(s+t)\xi_2(s+t) - 2)\,\mathrm{d}t\;, \tag{28}$$

$$\Delta C_2 = \frac{1}{4}\int_{-\infty}^{\infty} m(\theta + \tfrac{\omega t}{\varepsilon})\sin x_0(s+t)\xi_1(s+t)\,\mathrm{d}t\;. \tag{29}$$

Second order terms

We consider the second order approximation for the I components only. ¿From the equation (10) we obtain

$$I_2^\sigma(s,\theta) = \int_{\sigma\infty}^{0} \frac{\partial m}{\partial\theta}(\theta + \tfrac{\omega t}{\varepsilon})\sin x_0(s+t)x_1^\sigma(s+t,\theta+\tfrac{\omega t}{\varepsilon})\,\mathrm{d}t\;, \tag{30}$$

where we used $\lambda_1 = 0$. Thus ΔI_2 is described by a combination of two double integrals,

$$\begin{aligned}\Delta I_2(s,\theta) &= \int_{-\infty}^{0} \frac{\partial m}{\partial\theta}(\theta + \tfrac{\omega t}{\varepsilon})\sin x_0(s+t)x_1^-(s+t,\theta+\tfrac{\omega t}{\varepsilon})\,\mathrm{d}t \\ &\quad - \int_{\infty}^{0} \frac{\partial m}{\partial\theta}(\theta + \tfrac{\omega t}{\varepsilon})\sin x_0(s+t)x_1^+(s+t,\theta+\tfrac{\omega t}{\varepsilon})\,\mathrm{d}t\;,\end{aligned} \tag{31}$$

In a quite similar situation [GGM98] it was shown, that the properties of ΔI_2 and ΔI_1 are different. In our context this difference comes from the fact, that the functions under these two integrals are not analytical continuations of each other. This is a very delicate point, because this difference is exponentially small in ε.

The expressions for ΔI_2 may be rewritten in the following form:

$$\begin{aligned}\Delta I_2(s,\theta) &= \int_{-\infty}^{\infty} \frac{\partial m}{\partial\theta}(\theta + \tfrac{\omega t}{\varepsilon})\sin x_0(s+t)x_1^-(s+t,\theta+\tfrac{\omega t}{\varepsilon})\,\mathrm{d}t \\ &\quad -4\begin{vmatrix} \frac{\partial C_1^+}{\partial\theta} & \Delta C_1 \\ \frac{\partial C_2^+}{\partial\theta} & \Delta C_2 \end{vmatrix}\;.\end{aligned} \tag{32}$$

To see this one should replace $x_1^+ = x_1^- - \Delta x_1$ and use the explicit formula for Δx_1. In the formula above the integral is not absolutely convergent, a special care should be taken to understand this integral in the proper way.

2 Observables

The choice of convenient observables may notably simplify the study of the splitting. In [DGJS97] it was shown that the Hamiltonian (1) may be transformed to a especially simple form,

$$H' = H \circ \mathcal{C} = \frac{\omega J}{\varepsilon} + E \ ,$$

by an analytic canonical coordinate change,

$$\mathcal{C} : (s, \varphi, E, J) \mapsto (x, \theta, y, I) \ ,$$

in a neighborhood of the stable whisker. The change is identical on the angles, $\theta = \varphi$, but we use different notations for these angles, because the canonically conjugate momenta are not the same, they are I and J, respectively. The normalization condition for the change is

$$\mathcal{C}(s, \theta, 0, 0) = P^+(s, \theta) \ ,$$

where

$$P^+(s, \theta) = (x^+(s, \theta), \theta, y^+(s, \theta), I^+(s, \theta))$$

represents the stable separatrix. In other words, this means that the stable whisker is given by $E = 0$, $J = 0$, and the s and φ coordinates coincide with the corresponding parameters on the stable separatrix. Note, that the coordinate s, restricted onto the unstable whisker, do not coincide with the parameter s (such an error was done in [DS92], but had no consequences for the main result of that paper).

The coordinates E and J are local integrals of motion. It is convenient to use them as observables to measure the splitting. Let Δ be a splitting vector, we can define

$$\Delta E = \Omega'(\Delta, \tfrac{\partial}{\partial s}) \ , \qquad \Delta J = \Omega'(\Delta, \tfrac{\partial}{\partial \varphi}) \ ,$$

where $\Omega' = \mathrm{d}E \wedge \mathrm{d}s + \mathrm{d}J \wedge \mathrm{d}\varphi$ is the symplectic form. It is easy to see that ΔE and ΔJ are projections of the splitting vector Δ onto the corresponding coordinate axes. These quantities are not independent: the relation,

$$\frac{\omega \Delta J}{\varepsilon} + \Delta E = 0 \ ,$$

is due to the conservation of the energy H' (the stable and unstable whiskers belong to one energy level).

We may come back to the original variables. The normalizing condition and the canonicity of the change imply

$$\Delta E = \Omega(\Delta P, \tfrac{\partial P^+}{\partial s}) + O_2 \ , \qquad \Delta J = \Omega(\Delta P, \tfrac{\partial P^+}{\partial \theta}) + O_2 \ , \tag{33}$$

where ΔP is the splitting vector in the original coordinates, $\Omega = \mathrm{d}y \wedge \mathrm{d}x + \mathrm{d}I \wedge \mathrm{d}\theta$, and O_2 denotes the quadratic in ΔP terms. The last term appears because the splitting vector ΔP transforms as a tangent vector in the linear approximation only. But since this vector is exponentially small, the quadratic terms are not relevant for our study.

It is interesting to write explicitly the first terms of the expansion for the J components. Expanding the functions we obtain from (33):

$$\Delta J = \mu \Delta I_1 + \mu^2 \left(\Delta I_2 + \Delta y_1 \tfrac{\partial x_1^+}{\partial \theta} - \Delta x_1 \tfrac{\partial y_1^+}{\partial \theta} \right) + \dots , \tag{34}$$

where we used (3) and (4). Of course, the error term depends on ε too. The methods of [DGJS97] allow to prove that this residue is exponentially small.

The expression for the quadratic in μ term may be considerably simplified. We note, that

$$\begin{vmatrix} \frac{\partial x_1^+}{\partial \theta} & \Delta x_1 \\ \frac{\partial y_1^+}{\partial \theta} & \Delta y_1 \end{vmatrix} = \begin{vmatrix} \frac{\partial C_1^+}{\partial \theta} \xi_1 + \frac{\partial C_2^+}{\partial \theta} \xi_2 & \Delta C_1 \xi_1 + \Delta C_2 \xi_2 \\ \frac{\partial C_1^+}{\partial \theta} \eta_1 + \frac{\partial C_2^+}{\partial \theta} \eta_2 & \Delta C_1 \xi_1 + \Delta C_2 \eta_2 \end{vmatrix} = 4 \begin{vmatrix} \frac{\partial C_1^+}{\partial \theta} & \Delta C_1 \\ \frac{\partial C_2^+}{\partial \theta} & \Delta C_2 \end{vmatrix} .$$

Finally, we have

$$\begin{aligned} \Delta J_1 &= \Delta I_1 , \\ \Delta J_2 &= \Delta I_2 + 4 \begin{vmatrix} \frac{\partial C_1^+}{\partial \theta} & \Delta C_1 \\ \frac{\partial C_2^+}{\partial \theta} & \Delta C_2 \end{vmatrix} . \end{aligned}$$

Comparing with (32) we obtain

$$\Delta J_2 = \int_{-\infty}^{\infty} \frac{\partial m}{\partial \theta} (\theta + \tfrac{\omega t}{\varepsilon}) \sin x_0(s+t) x_1^-(s+t, \theta + \tfrac{\omega t}{\varepsilon}) \, \mathrm{d}t .$$

Note that the last integral is not absolutely convergent.

3 The Poincaré section $x = \pi$

Sometimes it is interesting to study the splitting on the natural Poincaré section $x = \pi$. For example, this may be necessary to compare the results obtained by different methods. Of course, this is not necessary for establishing the transversality of the separatrices.

Here we describe how to derive the formulas for the separatrix splitting on the Poincaré section $x = \pi$.

Our parametrizations may be normalized by the condition $x^\sigma(0,0) = \pi$. On the other hand, the condition, $x^\sigma(0,\theta) = \pi$ for all θ, is too restrictive and it is not compatible with some of the key analytical properties.

First, we define delay functions, $B^\sigma(\theta)$, by the equation,

$$x^\sigma(B^\sigma(\theta), \theta) = \pi .$$

Expanding the function in power series,

$$B^\sigma(\theta) = \mu B_1^\sigma(\theta) + \mu^2 B_2^\sigma(\theta) + \dots \ ,$$

substituting to the previous equation and collecting the terms of the first order in μ, we obtain

$$\frac{\partial x_0}{\partial s}(0) B_1^\sigma(\theta) + x_1^\sigma(0,\theta) = 0 \ .$$

Since $\frac{\partial x_0}{\partial s}(0) = 2$, this immediately gives us the following expression:

$$B_1^\sigma(\theta) = -\tfrac{1}{2} x_1^\sigma(0,\theta) = -C_1^\sigma(0,\theta) \ ,$$

where we used (26) and the equalities, $\xi_1(0) = 2$ and $\xi_2(0) = 0$. It is easy to see that B_1^σ is not exponentially small in ε. On the other hand, the delay function is related to the time, which is required for a point, which starts on the section $x = \pi$, to reach the torus $s = 0$ on the whisker. Since the stable and unstable whiskers are exponentially close, the difference $B^- - B^+$ has to be exponentially small too:

$$\Delta B_1 = C_1^+ - C_1^- = -\frac{1}{4} \int_{-\infty}^{\infty} m(\theta + \tfrac{\omega t}{\varepsilon}) \left(\sin x_0(t) \xi_2(t) - 2\right) \, \mathrm{d}t \ .$$

Let us compute the splitting on the section $x = \pi$:

$$\Delta I|_{x=\pi} = I^-(B^-(\theta), \theta) - I^+(B^+(\theta), \theta) \ .$$

We can expand in powers of μ:

$$\begin{aligned} \Delta I|_{x=\pi} &= \mu \Delta I_1(0,\theta) \\ &\quad + \mu^2 \left(\Delta I_2(0,\theta) + \frac{\partial I_1^-}{\partial s}(0,\theta) B_1^-(\theta) - \frac{\partial I_1^+}{\partial s}(0,\theta) B_1^+(\theta) \right) + \dots \ . \end{aligned}$$

All terms from the right-hand side have been computed.

Discussion

The described calculations are formal. Taking into account the analyticity of the whiskers in μ it is not too difficult to see that the errors of the approximation are of the first missing order of μ. Of course, the approximation errors also depend on ε. Studying this dependence is the central point, since the main terms of the splitting are exponentially small compared with ε.

We described the second order Melnikov method, which is compatible with the methods used in [Gel97a, Gel97b, DGJS97]. That is this method can be used to show that for $\mu = \mu_0 \varepsilon^p$ the estimates for the errors are exponentially small and to establish the conditions, which guaranty that the Melnikov function dominates the error. In fact, the exponentially small upper bounds for the error can be directly obtained for the special observables E and J only. That is the dots in (34) refer to an exponentially small residue. These functions and their approximations are quasi-periodic functions with fast-decreasing Fourier modes.

I say that $F(s,\theta) = \sum_k F_k \mathrm{e}^{\mathrm{i}k\theta + \mathrm{i}sk\omega\varepsilon^{-1}}$ is a *quasi-periodic function with fast-decreasing Fourier modes*, if $|F_k| \le A \exp(-r|k| - \rho|k\omega|\varepsilon^{-1})$ for some positive vector r and positive constants ρ and A (comp. with Lemma 3 from Sect. 7 of [DGJS97]). This property can be derived from the study of the form of the analyticity domain of the function F. The Fourier modes of the functions $\Delta I_1 = \Delta J_1$ and ΔJ_2 are fast-decreasing. But most of the functions described in the present note do not possess this property. In particular, the function ΔI_2 does not have this property.

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