THE COLE-HOPF AND MIURA TRANSFORMATIONS REVISITED

FRITZ GESZTESY AND HELGE HOLDEN

Dedicated with great pleasure to Ludwig Streit on the occasion of his 60th birthday

Abstract. An elementary yet remarkable similarity between the Cole-Hopf transformation relating the Burgers and heat equation and Miura's transformation connecting the KdV and mKdV equations is studied in detail.

1. Introduction

Our aim in this note is to display the close similarity between the well-known Cole–Hopf transformation relating the Burgers and the heat equation, and the celebrated Miura transform connecting the Korteweg–de Vries (KdV) and the modified KdV (mKdV) equation. In doing so we will introduce an additional twist in the Cole–Hopf transformation (cf. (1.28), (1.29)), which to the best of our knowledge, appears to be new. Moreover, we will reveal the history of this transformation and uncover several instances of its rediscovery (including those by Cole and Hopf).

We start with a brief introductory account on the KdV and mKdV equations. The KdV equation [42] was derived as an equation modeling the behavior of shallow water waves moving in one direction by Korteweg and his student de Vries in 1895[1]. The landmark discovery of the inverse scattering method by Gardner, Green, Kruskal, and Miura in 1967 [19] (cf. also [20]) brought the KdV equation to the forefront of mathematical physics, and started the phenomenal development involving multiple disciplines of science as well as several branches of mathematics.

The KdV equation (in a setting convenient for our purpose) reads

$$\mathrm{KdV}(V) = V_t - 6VV_x + V_{xxx} = 0, \tag{1.1}$$

while its modified counterpart, the mKdV equation, equals

$$\mathrm{mKdV}(\phi) = \phi_t - 6\phi^2\phi_x + \phi_{xxx} = 0. \tag{1.2}$$

Miura's fundamental discovery [46] was the realization that if ϕ satisfies the mKdV equation (1.2), then

$$V_\pm(x,t) = \phi(x,t)^2 \pm \phi_x(x,t), \quad (x,t) \in \mathbb{R}^2 \tag{1.3}$$

both satisfy the KdV equation. The transformation (1.3) has since been called the Miura transformation. Furthermore, explicit calculations by Miura showed the validity of the identity

$$\mathrm{KdV}(V_\pm) = (2\phi \pm \partial_x)\,\mathrm{mKdV}(\phi). \tag{1.4}$$

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[1] But the equation had been derived earlier by Boussinesq [7] in 1871, see Heyerhoff [36] and Pego [48].

The Miura transformation (1.3) was quite prominently used in the construction of an infinite series of conservation laws for the KdV equation, see [47], [16], Sect. 5.1. Miura's identity (1.4) then demonstrates how to transfer solutions of the mKdV equation to solutions of the KdV equation, but due to the nontrivial kernel of $(2\phi \pm \partial_x)$, it is not immediately clear how to reverse the procedure and to transfer solutions of the KdV equation to solutions of the mKdV equation. It was shown in [24] (see also [21], [22], [26], [31]) how to revert the process.

Following a similar treatment of the (modified) Kadomtsev-Petviashvili equation in [27], we use here a method that considerably simplies the proofs in [21], [24], [26], [31]. Introduce the first-order differential expression

$$\widetilde{P}(V) = 2V\partial_x - V_x. \tag{1.5}$$

Then one derives

$$\mathrm{mKdV}(\phi) = \partial_x \left(\frac{1}{\psi}(\psi_t - \widetilde{P}(V_\pm)\psi)\right), \tag{1.6}$$

where

$$\phi = \psi_x/\psi, \quad \psi > 0, \quad V_\pm = \phi^2 \pm \phi_x. \tag{1.7}$$

Next, let $V = V(x,t)$ be a solution of the KdV equation, $\mathrm{KdV}(V) = 0$, and $\psi > 0$ be a function satisfying

$$\psi_t = \widetilde{P}(V)\psi, \quad -\psi_{xx} + V\psi = 0. \tag{1.8}$$

Then one immediately deduces that ϕ solves the mKdV equation, $\mathrm{mKdV}(\phi) = 0$, and hence the Miura transformation has been "inverted".

The KdV equation (1.1) and the mKdV equation (1.2) are just the first (nonlinear) evolution equations in a countably infinite hierarchy of such equations (the (m)KdV hierarchy). The considerations (1.3)–(1.8) extend to the entire hierarchy of these equations, replacing the first-order differential expression $\widetilde{P}(V) = \widetilde{P}_1(V)$ by an appropriate first-order differential expression $\widetilde{P}_n(V)$ for $n \in \mathbb{N}$ (cf., e.g., [21], [22], [26], [31]). More precisely, denoting the nth KdV equation in the KdV hierarchy by

$$\mathrm{KdV}_n(V) = 0, \tag{1.9}$$

Lax [44] constructed differential expressions $P_{2n+1}(V)$ of order $2n+1$ with coefficients differential polynomials of V such that

$$\frac{d}{dt}L - [P_{2n+1}(V), L] = \mathrm{KdV}_n(V), \quad n \in \mathbb{N}. \tag{1.10}$$

Here L denotes the Schrödinger differential expression

$$L = -\partial_x^2 + V. \tag{1.11}$$

The KdV functional in (1.1) then corresponds to $n = 1$ and one obtains

$$P_3(V) = -4\partial_x^3 + 6V\partial_x + 3V_x \tag{1.12}$$

in this case. Restriction of $P_{2n+1}(V)$ to the (algebraic) nullspace of L then yields the first-order differential expression

$$\widetilde{P}_n(V) = P_{2n+1}(V)\big|_{\ker(L)}, \quad n \in \mathbb{N}. \tag{1.13}$$

Next we turn to the the Cole–Hopf transformation and its history. The classical Cole–Hopf transformation [13], [41], covered in most textbooks on partial differential equations, states that

$$V(x,t) = -2\frac{\psi_x(x,t)}{\psi(x,t)}, \quad (x,t) \in \mathbb{R} \times (0,\infty), \tag{1.14}$$

where $\psi > 0$ is a solution of the heat equation

$$\psi_t = \psi_{xx}, \tag{1.15}$$

satisfies the (viscous) Burgers equation

$$V_t + VV_x = V_{xx}. \tag{1.16}$$

However, already in 1906, Forsyth, in his multi-volume treatise on differential equations ([18], p. 100), discussed the equation (in his notation)

$$\frac{\partial}{\partial x}\left\{\frac{1}{\beta}\left(\gamma - \frac{\partial \alpha}{\partial x} - \alpha^2\right)\right\} - 2\frac{\partial \alpha}{\partial y} = 0, \tag{1.17}$$

where $\alpha = \alpha(x,y)$. Hence there exists a function θ such that

$$\alpha = \frac{\partial \theta}{\partial x}, \quad \gamma - \frac{\partial \alpha}{\partial x} - \alpha^2 = 2\beta\frac{\partial \theta}{\partial y}. \tag{1.18}$$

Assuming the function z satisfies

$$z_{xx} + 2\alpha z_x + 2\beta z_y + \gamma = 0, \tag{1.19}$$

an easy calculation shows that

$$\frac{\partial^2}{\partial x^2}(ze^\theta) + 2\beta\frac{\partial}{\partial y}(ze^\theta) = 0. \tag{1.20}$$

Introducing new variables $t = -y$ and $u(x,t) = -2\alpha(x,y)$ as well as fixing $\beta = 1/2$, $\gamma = 0$, and $z = 1$, one concludes that (1.17) indeed reduces to the viscous Burgers equation

$$u_t + uu_x = u_{xx}, \tag{1.21}$$

while (1.20) equals

$$(e^\theta)_t = (e^\theta)_{xx}, \tag{1.22}$$

with solutions related by

$$u = -2\theta_x. \tag{1.23}$$

However, Forsyth did not study the ramifications of this transformation, and no applications are discussed.

Shortly thereafter, in 1915, Bateman [1] introduced the model equation

$$u_t + uu_x = \nu u_{xx}. \tag{1.24}$$

He was interested in the vanishing viscosity limit, that is, when $\nu \to 0$. By studying solutions of the form $u = F(x+Ut)$, he concluded that "the question of the limiting form of the motion of a viscous fluid when the viscosity tends to zero requires very careful investigation".

Only in 1940 did Burgers ([9], p. 8) introduce[2] what has later been called the (viscous) Burgers equation, as a simple model of turbulence, and did some preliminary investigation on properties of the solution.

Taking advantage of the later rediscovered Forsyth transformation by Cole and Hopf, Burgers continued the investigations of what he called the nonlinear diffusion equation, focusing mainly on statistical aspects of the equation. The results of these investigations were collected in his book [11].

In 1948, Florin [17], in the context of applications to watersaturated flow, rediscovered Forsyth's transformation, which would become well-known under the name Cole-Hopf transformation only some 44 years later.

Although the Cole–Hopf transformation had already been published in 1906, it was only with the seminal papers by Hopf [41][3] in 1950[4] and by Cole [13] in 1951 that the full impact of the simple transformation was seen. In particular the careful study by Hopf concerning the vanishing viscosity limit represented a landmark in the emerging theory of conservation laws. Although the Cole–Hopf transformation is restricted to the Burgers equation, the insight and the motivation from this analysis has been of fundamental importance in the theory of conservation laws. Furthermore, Cole states the generalization of the Cole–Hopf transformation to a particular multi-dimensional system. More precisely, if $\psi = \psi(x,t)$, $(x,t) \in \mathbb{R}^n \times (0,\infty)$, satisfies the n-dimensional heat equation

$$\psi_t = \nu \Delta \psi, \quad \nu > 0, \tag{1.25}$$

and one defines

$$V = -2\nu \nabla \ln(\psi), \tag{1.26}$$

then V satisfies

$$V_t + (V \cdot \nabla) V = \nu \Delta V, \tag{1.27}$$

and the vector-valued function $V = V(x,t) \in \mathbb{R}^n$ has as many components (i.e., n) as the dimension of the underlying space. Observe, in particular, that V is irrotational (i.e., $V = \nabla W$ for some W, or equivalently, $\operatorname{curl} V = 0$,). The multi-dimensional extension was rediscovered by Kuznetsov and Rozhdestvenskii [43] in 1961.

In this note we show the following relations,

$$V_t + VV_x - \nu V_{xx} = 2\nu \left(-\frac{1}{\psi}\partial_x + \frac{\psi_x}{\psi^2} \right) (\psi_t - \nu \psi_{xx}) \tag{1.28}$$

$$= -2\nu \partial_x \left(\frac{1}{\psi} (\psi_t - \nu\psi_{xx}) \right), \tag{1.29}$$

whenever $V = -2\nu\psi_x/\psi$ for a positive function ψ. This clearly displays the nature of the Cole–Hopf transformation and closely resembles Miura's identity (1.4) and the relation (1.6). Even though identities (1.28) and (1.29) are elementary observations, much to our surprise, they appear to have escaped notice in the extensive

[2]Frequently Burgers equations is quoted from his 1948 paper [10], but he had already introduced it in 1940.

[3]With a misprint in the title, writing $u_t + uu_x = \mu_{xx}$ rather than $u_t + uu_x = \mu u_{xx}$.

[4]Hopf [41] states in a footnote (p. 202) that he had the "Cole–Hopf transformation" already in 1946, but "it was not until 1949 that I became sufficiently acquainted with the recent development of fluid dynamics to be convinced that a theory based on (1.24) could serve as an instructive introduction into some of the mathematical problems involved".

literature on the Cole-Hopf transformation thus far. While both the KdV and mKdV equations are nonlinear partial differential equations, the case of the Burgers and heat equations just considered is a bit different since it relates a nonlinear and a linear partial differential equation (see also [6], Sect. 6.4).

One can also extend the Cole–Hopf transformation to the case of a potential term F in the heat equation, see, for instance, [37]. Here the relation (1.29) reads as follows,

$$V_t + VV_x - \nu V_{xx} + 2\nu F_x = -2\nu\partial_x\left(\frac{1}{\psi}(\psi_t - \nu\psi_{xx} - F\psi)\right), \tag{1.30}$$

whenever $V = -2\nu\partial_x \ln(\psi)$ for a positive function ψ. The case of Burgers' equation externally driven by a random potential term recently generated particular interest, see, for instance, [3], [4], [35], [37], [38], [39] and the references therein. We also mention a very interesting application of the Cole-Hopf transformation to the pair of the telegraph and a nonlinear Boltzmann equation in [40], generalizing the pair of the heat and Burgers equation considered in this note.

Equation (1.30) extends to the multi-dimensional case corresponding to (1.27) and one obtains

$$V_t + \alpha(V\cdot\nabla)V - \nu\Delta V + \frac{2\nu}{\alpha}\nabla F = -\frac{2\nu}{\alpha}\nabla\left(\frac{1}{\psi}(\psi_t - \nu\Delta\psi - F\psi)\right), \tag{1.31}$$

whenever $\alpha \in \mathbb{R}\backslash\{0\}$ and $V = -(2\nu/\alpha)\nabla\ln(\psi)$ for a positive function ψ.

Obviously there is a close similarity between the heat and the Burgers equation expressed by (1.28), and Miura's identity (1.4) relating the mKdV and the KdV equation. The principal idea underlying these considerations being that one (hierarchy of) evolutions equation(s) can be represented as a linear differential expression acting on another (hierarchy of) evolution equation(s). As long as the null space of this linear differential expression can be analyzed in detail, it becomes possible to transfer solutions, in fact, entire classes of solutions (e.g., rational, soliton, algebro-geometric solutions, etc.) between these evolution equations. In concrete applications, however, it turns out to be simpler to rewrite a relationship between two evolution equations, such as (1.4) and (1.28), in a form analogous to (1.6) and (1.29), rather than analyzing the nullspaces of $(2\phi \pm \partial_x)$ and $(-\partial_x + (\psi_x/\psi))$ in detail. These strategies relating (hierarchies of) evolution equations and their modified analogs is not at all restricted to the Burgers and heat equations and the KdV and mKdV hierarchies, respectively, but applies to a large number of evolution equations including the Boussinesq [29], and more generally, the Gelfand–Dickey hierarchy and its modified counterpart, the Drinfeld–Sokolov hierarchy [34], the Toda and Kac–van Moerbeke hierarchies [8], [33], the Kadomtsev–Petviashvili and modified Kadomtsev–Petviashvili hierarchies [27], [32], etc.

For simplicity we restrict ourselves to classical solutions throughout this note. The case of distributional solutions of Burgers equation is considered, for instance, in [15].

Throughout this note we abbreviate by $C^{p,q}(\Omega\times\Lambda)$, $\Omega\subset\mathbb{R}^n$, $\Lambda\subset\mathbb{R}$ open, $n\in\mathbb{N}$, $p,q\in\mathbb{N}_0$, the linear space of continuous functions $f(x,t)$ with continuous partial derivatives with respect to $x=(x_1,\dots,x_n)$ up to order p and q partial derivatives with respect to t. $C^{p,q}(\Omega\times\Lambda;\mathbb{R}^n)$ is then defined analogously for $f(x,t)\in\mathbb{R}^n$.

2. The Miura transformation

We turn to the precise formulation of the relations between the KdV and the mKdV equation and omit details of a purely calculational nature.

Lemma 2.1. *Let $\psi = \psi(x,t) > 0$ be a positive function such that $\psi \in C^{4,0}(\mathbb{R}\times\mathbb{R})$, $\psi_t \in C^{1,0}(\mathbb{R}\times\mathbb{R})$. Define $\phi = \psi_x/\psi$. Then $\phi \in C^{3,1}(\mathbb{R}\times\mathbb{R})$ and*

$$\mathrm{mKdV}(\phi) = \partial_x \left(\frac{1}{\psi}\big(\psi_t - \widetilde{P}(V_\pm)\psi\big) \right), \tag{2.1}$$

where

$$\widetilde{P}(V) = 2V\partial_x - V_x \tag{2.2}$$

and

$$V_\pm = \phi^2 \pm \phi_x. \tag{2.3}$$

Proof. A straightforward calculation. □

The application to the KdV equation then reads as follows.

Theorem 2.2. *Let $V = V(x,t)$ be a solution of the KdV equation, $\mathrm{KdV}(V) = 0$, with $V \in C^{3,1}(\mathbb{R}\times\mathbb{R})$, and let $\psi > 0$ be a positive function satisfying $\psi \in C^{2,0}(\mathbb{R}\times\mathbb{R})$, $\psi_t \in C^{1,0}(\mathbb{R}\times\mathbb{R})$ and*

$$\psi_t = \widetilde{P}(V)\psi, \quad -\psi_{xx} + V\psi = 0, \tag{2.4}$$

with $\widetilde{P}(V)$ given by (2.2). Define $\phi = \psi_x/\psi$ and $\widehat{V} = \phi^2 - \phi_x$. Then $V = \phi^2 + \phi_x$ and ϕ satisfies $\phi \in C^{4,1}(\mathbb{R}\times\mathbb{R})$ and the mKdV equation,

$$\mathrm{mKdV}(\phi) = 0. \tag{2.5}$$

Moreover, $\widehat{V}$ satisfies $\widehat{V} \in C^{3,1}(\mathbb{R}\times\mathbb{R})$ and the KdV equation,

$$\mathrm{KdV}(\widehat{V}) = 0. \tag{2.6}$$

Proof. A computation based on Lemma 2.1. □

Originally, Theorem 2.2 was proved in [24] (see also [21], [22], [25], [26], [31]) using supersymmetric methods. The above arguments, following [27] in the context of the (modified) Kadomtsev-Petviashvili equation, result in considerably shorter calculations. The "if part" in Theorem 2.2 also follows from prolongation methods developed in [49]. A different approach to Theorem 2.2, assuming rapidly decreasing solutions of the KdV equation, can be found in Sect. 38 of [2].

Remark 2.3. The chain of transformations

$$V \to \phi \to -\phi \to \widehat{V} \tag{2.7}$$

reveals a Bäcklund transformation between the KdV and mKdV equations ($V \to \phi$) and two auto-Bäcklund transformations for the KdV ($V \to \widehat{V}$) and mKdV equations ($\phi \to -\phi$), respectively.

Remark 2.4. For simplicity we assumed $\psi(x,t) > 0$ for all $(x,t) \in \mathbb{R}^2$ in Theorem 2.2. However, as proven by Lax [45], one can show that $\psi(x,t_0) > 0$ for some $t_0 \in \mathbb{R}$ and all $x \in \mathbb{R}$ actually implies $\psi(x,t) > 0$ for all $(x,t) \in \mathbb{R}^2$ (see also [24]). Moreover, in case $V(x,t_0)$ is real-valued, we note that $L(t_0)\psi(x,t_0) = 0$ has a positive solution $\psi(x,t_0) > 0$ if and only if the Schrödinger differential expression

$L(t_0) = -\partial_x^2 + V(x,t_0)$ is nonoscillatory at $\pm\infty$ (cf. [28]). While the system of equations (2.4) always has a solution $\psi(x,t)$, (cf. Lemma 3 in [24]), it is the additional requirement $\psi(x,t) > 0$ for all $(x,t) \in \mathbb{R}^2$ which renders ϕ (and hence $\widehat{V}$) nonsingular. Without the condition $\psi > 0$, Theorem 2.2 describes (auto)Bäcklund transformations for the KdV and mKdV equations with characteristic singularities (cf. [25]).

3. The Cole–Hopf transformation

Finally we return to relations (1.28), (1.29), and (1.30). Since they are all proved by explicit calculations we may omit these details and focus on a precise formulation of the results instead.

Lemma 3.1. *Let $\psi = \psi(x,t) > 0$ be a positive function with $\psi \in C^{3,0}(\mathbb{R}\times(0,\infty))$, $\psi_t \in C^{1,0}(\mathbb{R}\times(0,\infty))$. Define $V = -2\nu\psi_x/\psi$ with $\nu > 0$. Then $V \in C^{2,1}(\mathbb{R}\times(0,\infty))$ and*

$$V_t + VV_x - \nu V_{xx} = 2\nu\left(-\frac{1}{\psi}\partial_x + \frac{\psi_x}{\psi^2}\right)(\psi_t - \nu\psi_{xx}) \tag{3.1}$$

$$= -2\nu\partial_x\left(\frac{1}{\psi}(\psi_t - \nu\psi_{xx})\right). \tag{3.2}$$

The extension to the case with a potential term F in the heat equation reads as follows.

Lemma 3.2. *Let $F \in C^{1,0}(\mathbb{R}\times(0,\infty))$ and assume $\psi = \psi(x,t) > 0$ to be a positive function such that $\psi \in C^{3,0}(\mathbb{R}\times(0,\infty))$, $\psi_t \in C^{1,0}(\mathbb{R}\times(0,\infty))$. Define $V = -2\nu\psi_x/\psi$ with $\nu > 0$. Then $V \in C^{2,1}(\mathbb{R}\times(0,\infty))$ and*

$$V_t + VV_x - \nu V_{xx} + 2\nu F_x = -2\nu\partial_x\left(\frac{1}{\psi}(\psi_t - \nu\psi_{xx} - F\psi)\right). \tag{3.3}$$

We can exploit these relations as follows.

Theorem 3.3. *Let $F \in C^{1,0}(\mathbb{R}\times(0,\infty))$ and $\nu > 0$.*
(i) Suppose V satisfies $V \in C^{2,1}(\mathbb{R}\times(0,\infty))$, and

$$V_t + VV_x - \nu V_{xx} + 2\nu F_x = 0 \tag{3.4}$$

for some $\nu > 0$. Define

$$\psi(x,t) = \exp\left(-\frac{1}{2\nu}\int^x dy\, V(y,t)\right). \tag{3.5}$$

Then ψ satisfies $0 < \psi \in C^{3,1}(\mathbb{R}\times(0,\infty))$ and

$$\frac{1}{\psi}(\psi_t - \nu\psi_{xx} - F\psi) = C(t) \tag{3.6}$$

for some x-independent $C \in C(\mathbb{R})$.
(ii) Let $\psi > 0$ be a positive function satisfying $\psi \in C^{3,0}(\mathbb{R}\times(0,\infty))$, $\psi_t \in C^{1,0}(\mathbb{R}\times(0,\infty))$ and suppose

$$\psi_t = \nu\psi_{xx} + F\psi \tag{3.7}$$

for some $\nu > 0$. Define

$$V = -2\nu\frac{\psi_x}{\psi}. \tag{3.8}$$

Then $V \in C^{2,1}(\mathbb{R}\times(0,\infty))$ satisfies (3.4).

Remark 3.4. One can "scale away" $C(t)$ in Theorem 3.3 (i) by introducing a new function $\tilde{\psi}$. In fact, the function $\tilde{\psi}(x,t) = \psi(x,t)\exp(-\int_0^t ds\, C(s))$ satisfies

$$\tilde{\psi} = \nu\tilde{\psi}_{xx} + F\tilde{\psi}. \tag{3.9}$$

Remark 3.5. Using the standard representation of solutions of the heat equation initial value problem,

$$\psi_t = \nu\psi_{xx}, \quad \psi(x,0) = \psi_0(x), \tag{3.10}$$

assuming

$$\psi_0 \in C(\mathbb{R}), \quad \psi_0(x) \le C_1 \exp(C_2|x|^{1+\gamma}) \text{ for } |x| \ge R \tag{3.11}$$

for some $R > 0$, $C_j \ge 0$, $j = 1,2$, and $0 \le \gamma < 1$, given by (cf. [12], Ch. 3; [14], Ch. V)

$$\psi(x,t) = \frac{1}{2(\pi\nu t)^{1/2}} \int_{\mathbb{R}} dy\, \exp(-(x-y)^2/4\nu t)\psi_0(y) > 0, \tag{3.12}$$

the corresponding initial value problem for the Burgers equation

$$V_t + VV_x - \nu V_{xx} = 0, \quad V(x,0) = V_0(x), \tag{3.13}$$

reads

$$V(x,t) = \frac{\int_{\mathbb{R}} dy\, (x-y)t^{-1} \exp\big(-(2\nu)^{-1}\int_0^y d\eta V_0(\eta) - (x-y)^2(4\nu t)^{-1}\big)}{\int_{\mathbb{R}} dy\, \exp\big(-(2\nu)^{-1}\int_0^y d\eta V_0(\eta) - (x-y)^2(4\nu t)^{-1}\big)}, \tag{3.14}$$

assuming $V_0 \in C(\mathbb{R})$ and

$$V_0(x) \ge 0 \text{ or } \left| \int_0^x dy\, V_0(y) \right| \underset{|x|\to\infty}{=} O(|x|^{1+\gamma}) \text{ for some } \gamma < 1. \tag{3.15}$$

Without going into further details we mention the existence of a hierarchy of Burgers equations which are related to the linear partial differential equations

$$\psi_t = \partial_x^{n+1}\psi, \quad n \in \mathbb{N} \tag{3.16}$$

by the Cole-Hopf transformation $V = -2\psi_x/\psi$ (see [5]).

The multi-dimensional extension of Lemma 3.2 reads as follows.

Lemma 3.6. *Let $F \in C^{1,0}(\mathbb{R}^n \times (0,\infty))$ and assume $\psi = \psi(x,t) > 0$ to be a positive function such that $\psi \in C^{3,0}(\mathbb{R}^n \times (0,\infty))$, $\psi_t \in C^{1,0}(\mathbb{R}^n \times (0,\infty))$. Define $V = -(2\nu/\alpha)\nabla \ln(\psi)$ with $\alpha \in \mathbb{R}\backslash\{0\}$, $\nu > 0$. Then $V \in C^{2,1}(\mathbb{R}^n \times (0,\infty); \mathbb{R}^n)$ and*

$$V_t + \alpha(V\cdot\nabla)V - \nu\Delta V + \frac{2\nu}{\alpha}\nabla F = -\frac{2\nu}{\alpha}\nabla\left(\frac{1}{\psi}\big(\psi_t - \nu\Delta\psi - F\psi\big)\right). \tag{3.17}$$

Our final result shows how to transfer solutions between the multi-dimensional Burgers equation and the heat equation.

Theorem 3.7. *Let $F \in C^{1,0}(\mathbb{R}^n \times (0,\infty))$, $\alpha \in \mathbb{R}\backslash\{0\}$, and $\nu > 0$.*
(i) Assume that $V \in C^{2,1}(\mathbb{R}^n \times (0,\infty); \mathbb{R}^n)$ satisfies

$$V = \nabla\Phi \tag{3.18}$$

for some potential $\Phi \in C^{3,1}(\mathbb{R}^n \times (0,\infty))$ and

$$V_t + \alpha(V\cdot\nabla)V - \nu\Delta V + \frac{2\nu}{\alpha}\nabla F = 0. \tag{3.19}$$

Define

$$\psi = \exp\left(-\frac{\alpha}{2\nu}\Phi\right). \tag{3.20}$$

Then $\psi \in C^{3,1}(\mathbb{R}^n \times (0,\infty))$ *and*

$$\frac{1}{\psi}\big(\psi_t - \nu\Delta\psi - F\psi\big) = C(t), \tag{3.21}$$

for some x*-independent* $C \in C((0,\infty))$.
(ii) Let $\psi > 0$ *be a positive function satisfying* $\psi \in C^{3,0}(\mathbb{R}^n \times (0,\infty))$, $\psi_t \in C^{1,0}(\mathbb{R}^n \times (0,\infty))$ *and suppose*

$$\psi_t = \nu\Delta\psi + F\psi. \tag{3.22}$$

Define

$$V = -\frac{2\nu}{\alpha}\nabla\ln(\psi). \tag{3.23}$$

Then $V \in C^{2,1}(\mathbb{R}^n \times (0,\infty); \mathbb{R}^n)$ *satisfies* (3.19).

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Department of Mathematics, University of Missouri, Columbia, MO 65211, USA
E-mail address: fritz@math.missouri.edu
URL: http://www.math.missouri.edu/people/fgesztesy.html

Department of Mathematical Sciences, Norwegian University of Science and Technology, N–7034 Trondheim, Norway
E-mail address: holden@math.ntnu.no
URL: http://www.math.ntnu.no/~holden/