ON SOLUTIONS OF THE SCHRÖDINGER EQUATION WITH RADIATION CONDITIONS AT INFINITY: THE LONG-RANGE CASE

Y.Gâtel and D.Yafaev
Université de Rennes

Abstract

We consider the homogeneous Schrödinger equation with a long-range potential and show that its solutions satisfying some a priori bound at infinity can asymptotically be expressed as a sum of incoming and outgoing distorted spherical waves. Coefficients of these waves are related by the scattering matrix. This generalizes a similar result obtained earlier in the short-range case.

1. INTRODUCTION

In this paper we obtain a complete description of solutions $u(x)$ of the homogeneous Schrödinger equation with a long-range potential V

$$-\Delta u(x) + V(x)u(x) = \lambda u(x), \quad x \in \mathbb{R}^d, \ \lambda > 0, \ V = \overline{V} \tag{1.1}$$

obeying the natural condition

$$\int_{|x|\leq\rho} |u(x)|^2 dx \leq C\rho, \ \text{ as } \rho \to \infty. \tag{1.2}$$

Our aim is to show that every function $u(x)$ satisfying (1.1) and (1.2) is asymptotically a sum of incoming and outgoing distorted spherical waves

$$w_\pm(x,\lambda) = |x|^{-(d-1)/2} e^{\pm i\varphi(x,\lambda)}, \tag{1.3}$$

where the phase φ is a suitably chosen approximate solution of the eikonal equation

$$|\nabla_x \varphi(x,\lambda)|^2 + V(x) = \lambda. \tag{1.4}$$

Clearly, for functions $w_\pm$, estimate (1.2) is fulfilled. Remainders $\epsilon(x)$ in our asymptotic formulas will satisfy the condition

$$\lim_{\rho\to\infty} \rho^{-1} \int_{|x|\leq\rho} |\epsilon(x)|^2 dx = 0, \tag{1.5}$$

which will be denoted as $\epsilon(x) = o_{av}(|x|^{-(d-1)/2})$. Of course, condition (1.5) is satisfied if $\epsilon(x) = o(|x|^{-(d-1)/2})$ in the usual sense.

The main result of our paper can be formulated as follows

A) For any function $a_- \in L^2(\mathbb{S}^{d-1})$, there exist a function $a_+ \in L^2(\mathbb{S}^{d-1})$ and a solution u of equation (1.1) with asymptotics

$$u(x) = a_+(\hat{x})w_+(x,\lambda) - a_-(\hat{x})w_-(x,\lambda) + o_{av}(|x|^{-(d-1)/2}), \qquad \hat{x} = x/|x| \tag{1.6}$$

$$\partial_r u(x) = i\sqrt{\lambda}\Big(a_+(\hat{x})w_+(x,\lambda) + a_-(\hat{x})w_-(x,\lambda)\Big) + o_{av}(|x|^{-(d-1)/2}), \tag{1.7}$$

as $|x| \to \infty$. Moreover one has $||a_+|| = ||a_-||$.

B) Every u satisfying (1.1) and (1.2) has the asymptotics (1.6), (1.7) with some functions $a_\pm \in L^2(\mathbb{S}^{d-1})$.

We emphasize that in the part A) the function a_+ and the solution u are determined uniquely by a_-. Of course, the roles of a_+ and a_- can be interchanged here. This implies that the operator $\Sigma(\lambda) : a_- \mapsto a_+$ is unitary in $L^2(\mathbb{S}^{d-1})$.

Usually the scattering matrix $\mathcal{S}(\lambda) : L^2(\mathbb{S}^{d-1}) \to L^2(\mathbb{S}^{d-1})$ for the Schrödinger operator $H = -\Delta + V$ is defined in terms of solutions of the time-dependent equation. Indeed, it can be shown that

$$(e^{-iHt}f)(x) = e^{i\Omega(x,t)}(2it)^{-d/2} f_\pm((2t)^{-1}x) + \epsilon_\pm(x,t), \tag{1.8}$$

where $\Omega(x,t) = -\Omega(x,-t)$ is a solution (perhaps approximate) of the time-dependent equation

$$\frac{\partial\Omega}{\partial t} + |\nabla\Omega|^2 + V = 0, \tag{1.9}$$

and $||\epsilon_\pm(\cdot,t)|| = o(1)$ as $t \to \pm\infty$. Let us write $f_\pm(x)$ as $f_\pm(\hat{x},|x|)$. The operator $\mathcal{S}(\lambda) : f_-(\cdot,\sqrt{\lambda}) \mapsto f_+(\cdot,\sqrt{\lambda})$ is known as the scattering matrix. It turns out that the operators $\mathcal{S}(\lambda)$ and $\Sigma(\lambda)$ essentially coincide. Namely, $\mathcal{S}(\lambda) = \Sigma(\lambda)\mathcal{T}$, where $\mathcal{T}$, $(\mathcal{T}a)(\omega) = a(-\omega)$, is the reflection operator on the sphere. Thus, in the long-range case, our approach provides a way to construct the scattering matrix in terms of solutions of the stationary equation (1.1).

We emphasize that neither relation (1.6) nor (1.8) define the scattering matrix $\mathcal{S}(\lambda)$ uniquely. Indeed, an arbitrary function $\Theta(\hat{x},\lambda)$ can be added to $\varphi(x,\lambda)$ in definition (1.3) or the function $-\operatorname{sgn} t\, \Theta(\hat{x}, |x|^2/(4t^2))$ can be added to $\Omega(x,t)$ in (1.8). This corresponds to the replacement of $\mathcal{S}(\lambda)$ by $e^{i\Theta_+(\lambda)}\mathcal{S}(\lambda)e^{i\Theta_-(\lambda)}$ where $\Theta_\pm(\lambda)$ is multiplication by $\Theta(\pm\hat{x},\lambda)$.

In the short-range case (when $\varphi(x,\lambda) = \sqrt{\lambda}|x| - (d-3)\pi/4$), the results formulated above were proven in [18], where the techniques of [2], [3] were extensively used.

Our proofs rely heavily on the well-elaborated machinery of long-range scattering theory (see [5], [15]). More precisely, we need the sharp form of the limiting absorption principle ([1], [9], [12]), and the asymptotics of the function $((H - \lambda \pm i0)^{-1}f)(x)$ as $|x| \to \infty$ ([6], [10]). A new point, however, is the one-to-one correspondence between functions $a_\pm \in L^2(\mathbb{S}^{d-1})$ and solutions of (1.1) satisfying condition (1.2).

To conclude the introduction, let us mention the recent paper [14] that in spirit is not very far from the ours. They get in the general context of asymptotically Euclidean manifolds a result of type A) (but not of type B)). Moreover, they only deal with perturbations of the metrics which correspond to very short-range potentials and consider only functions $a_\pm \in C^\infty$.

2. PRELIMINARIES

1. Let $d \geq 2$. Any $x \in \mathbb{R}^d$ will be written as $x = |x|\hat{x} = r\omega$ in spherical coordinates. We denote $\partial_j = \partial/\partial x_j$, $j = 1, \ldots, d$. By C and c, we denote positive constants whose precise values are of no importance. We first recall the definition of useful spaces in our framework. Let $\mathfrak{S} = \mathfrak{S}(\mathbb{R}^d)$ be the Schwartz space of test functions. For every $G \subset \mathbb{R}^d$, let $L^2_\gamma(G)$ be the Hilbert space of measurable functions f on G such that

$$||f||^2_{\gamma,G} = \int_G (1+|x|^2)^\gamma |f(x)|^2 dx < \infty.$$

If $\gamma = 0$ ou $G = \mathbb{R}^d$, we write $L^2(G)$ or L^2_γ. We denote by $\mathbf{H}^2 = \mathbf{H}^2(\mathbb{R}^d)$ the Sobolev space and by $\mathbf{H}^2_{loc}$ the space of locally $\mathbf{H}^2$ functions. Let B be the space of functions f such that

$$||f||_B = \left(\int_{|x|\leq 1} |f(x)|^2 dx\right)^{1/2} + \sum_{n=0}^{\infty} \left(2^n \int_{2^n \leq |x| \leq 2^{n+1}} |f(x)|^2 dx\right)^{1/2} < \infty.$$

The space B^*, dual to B with respect to $L^2(\mathbb{R}^d)$, is a Banach space with one of its equivalent norms defined by

$$||f||_{B^*} = \sup_{\rho \geq 1} \left(\rho^{-1} \int_{|x|\leq \rho} |f(x)|^2 dx\right)^{1/2}. \tag{2.1}$$

We denote by $\langle \cdot, \cdot \rangle$ the corresponding duality symbol. Clearly,

$$\mathfrak{S} \subset L^2_\gamma \subset B \subset L^2_{1/2} \subset L^2(\mathbb{R}^d) \subset L^2_{-1/2} \subset B^* \subset L^2_{-\gamma}, \quad \forall \gamma > 1/2.$$

Let B^*_0 be the closure of $L^2(\mathbb{R}^d)$ in the norm of B^*. Then $f \in B^*_0$ if and only if f satisfies (1.5). The spaces B, B^*, B^*_0 were first introduced in the context of scattering theory in [2].

We need the following elementary property of the space B^*.

Lemma 2.1 *If $u \in \mathbf{H}^2_{loc}$, then*

$$||\nabla u||_{B^*} \leq C(||u||_{B^*} + ||\Delta u||_{B^*}). \tag{2.2}$$

Proof – For any real $\zeta \in C_0^\infty(\mathbb{R}^d)$, one has the identity

$$\int_{\mathbb{R}^d} |\nabla u|^2 \zeta dx = 2^{-1} \int_{\mathbb{R}^d} |u|^2 \Delta\zeta dx - \operatorname{Re} \int_{\mathbb{R}^d} \Delta u \overline{u} \zeta dx. \tag{2.3}$$

Indeed, integrating by parts we see that

$$-\int_{\mathbb{R}^d} \Delta u \overline{u} \zeta dx = \int_{\mathbb{R}^d} |\nabla u|^2 \zeta dx + \int_{\mathbb{R}^d} \nabla u \overline{u} \nabla \zeta dx, \tag{2.4}$$

and

$$\int_{\mathbb{R}^d} \nabla u \overline{u} \nabla \zeta dx = -\int_{\mathbb{R}^d} u \nabla \overline{u} \nabla \zeta dx - \int_{\mathbb{R}^d} |u|^2 \Delta \zeta dx.$$

The last equality is equivalent to

$$2\mathrm{Re} \int_{\mathbb{R}^d} \nabla u \overline{u} \nabla \zeta dx = -\int_{\mathbb{R}^d} |u|^2 \Delta \zeta dx. \tag{2.5}$$

Combining (2.4) and (2.5), we arrive at (2.3).

Suppose now that $\zeta(x) = 1$ for $|x| \leq 1$, $\zeta(x) = 0$ for $|x| \geq 2$, and set $\zeta_\rho(x) = \zeta(x/\rho)$. Then (2.3) yields

$$\int_{\mathbb{R}^d} |\nabla u|^2 \zeta(x/\rho) dx = 2^{-1} \rho^{-2} \int_{\mathbb{R}^d} |u|^2 (\Delta \zeta)(x/\rho) dx - \mathrm{Re} \int_{\mathbb{R}^d} \Delta u \overline{u} \zeta(x/\rho) dx,$$

and hence

$$\int_{|x| \leq \rho} |\nabla u|^2 dx \leq C \int_{|x| \leq 2\rho} (|\Delta u|^2 + |u|^2) dx, \tag{2.6}$$

with C independent of $\rho \geq 1$. $\square$

Corollary 2.2 *Let $u \in \mathbf{H}^2_{loc} \cap B^*$ be a solution of equation* (1.1) *with a bounded potential V. Then*

$$||\nabla u||_{B^*} \leq C ||u||_{B^*}. \tag{2.7}$$

We need also an abstract theorem of H. Hahn (see e.g. [19], chapter 7 §5) which we formulate in terms adapted to our case.

Theorem 2.3 *Let $\mathcal{F} : B \to L^2(\mathbb{S}^{d-1})$ be a bounded operator. Suppose that the adjoint operator $\mathcal{F}^* : L^2(\mathbb{S}^{d-1}) \to B^*$ has a closed range. Then an element $u \in B^*$ belongs to the image $R(\mathcal{F}^*)$ of $\mathcal{F}^*$ if (and only if) $\langle u, f \rangle = 0$ for all f from the kernel $N(\mathcal{F})$ of the operator $\mathcal{F}$. Moreover, $R(\mathcal{F}) = L^2(\mathbb{S}^{d-1})$ if $N(\mathcal{F}^*) = \{0\}$.*

2. Let us now give precise conditions on the potential V. We always suppose that

$$V(x) = V_L(x) + V_S(x) \tag{2.8}$$

where $V_L(x)$ is a C^3-function such that

$$|\partial^\alpha V_L(x)| \leq C(1 + |x|)^{-\delta - |\alpha|}, \quad \delta \in (0, 1], \quad 0 \leq |\alpha| \leq 3, \tag{2.9}$$

while the short-range part $V_S(x)$ satisfies

$$|V_S(x)| \leq C(1 + |x|)^{-1-\delta}. \tag{2.10}$$

Note however that many intermediary results are valid under weaker assumptions.

We start with a reformulation ([18]) of a standard uniqueness result (see e.g. [16]) in terms convenient for us.

Proposition 2.4 *Suppose that u satisfies* (1.1) *and for some $a \in L^2(\mathbb{S}^{d-1})$ (and one of the signs "+" or "−")*

$$u(x) = a(\hat{x}) w_\pm(x,\lambda) + o_{av}(|x|^{-(d-1)/2}), \tag{2.11}$$

$$\partial_r u(x) = \pm i\sqrt{\lambda} a(\hat{x}) w_\pm(x,\lambda) + o_{av}(|x|^{-(d-1)/2}), \tag{2.12}$$

as $|x| \to \infty$. Then $u = 0$.

This assertion shows that a solution u of equation (1.1) is determined uniquely by a single asymptotic coefficient in (1.6) and (1.7).

3. The Schrödinger operator $H = -\Delta + V(x)$ is self-adjoint in $L^2(\mathbb{R}^d)$ on the domain $\mathbf{H}^2$. It follows from Proposition 2.4 that the operator H does not have positive eigenvalues. Construction of scattering theory relies on the following fundamental result on its resolvent $R(z) = (H-z)^{-1}$ known as the limiting absorption principle. Denote by $||R(z)||_{B,B^*}$ the norm of the operator $R(z) : B \to B^*$.

Proposition 2.5 *Under the assumptions* (2.8)-(2.10)

$$||R(z)||_{B,B^*} \le C, \qquad Im\, z \neq 0, \tag{2.13}$$

where the constant C does not depend on z as long as $0 < \lambda_0 \le Re\, z \le \lambda_1 < \infty$. Moreover, for any $f, g \in B$, the function $(R(z)f, g)$ (resp. the operator-function $R(z) : L^2_\gamma \to L^2_{-\gamma}$, $\gamma > 1/2$) is continuous (resp. continuous in norm) with respect to z in the complex plane cut along $[0,\infty)$ possibly with exception of the point $z = 0$.

The proof of this result, based on the Mourre estimate, can be found in [12] (see also [5] and [9]).

Corollary 2.6 *Under the assumptions of Proposition* 2.5

$$||\nabla R(z)||_{B,B^*} \le C, \quad Im\, z \neq 0. \tag{2.14}$$

Proof – Let $f \in B$ and $v = R(z)f \in B^*$. Since $\Delta v = (V-z)v - f$, we have that

$$||\Delta v||_{B^*} \le c(||v||_{B^*} + ||f||_{B^*}) \le C||f||_B,$$

by Proposition 2.5. To conclude, it remains to apply Lemma 2.1. □

3. GENERALIZED FOURIER TRANSFORMS

1. For our purpose we use an approximate solution of the eikonal equation (1.4) (see e.g. [10]).

Lemma 3.1 *There exists a real-valued function $\Phi(x,\lambda)$ satisfying for $|x|$ large enough*

$$2\sqrt{\lambda}\frac{\partial \Phi}{\partial |x|}(x,\lambda) = |\nabla_x \Phi(x,\lambda)|^2 + V_L(x), \tag{3.1}$$

and for all $x \in \mathbb{R}^d$, λ varying on any compact subinterval $\Lambda \subset \mathbb{R}_+$

$$|\partial_x^\alpha \partial_\lambda^\beta \Phi(x,\lambda)| \le C(1+|x|)^{1-|\alpha|-\delta}, \quad |\alpha| \le 3, \quad \beta \le 2. \tag{3.2}$$

Remark that the estimates (3.2) for $\beta > 0$ are used in Section 6. In this section we need only continuity of $\Phi(x,\lambda)$ with respect to λ. Relation (3.1) ensures that

$$\varphi(x,\lambda) = \sqrt{\lambda}|x| - \Phi(x,\lambda) - \pi(d-3)/4 \tag{3.3}$$

satisfies equation (1.4) with V replaced by V_L for sufficiently large $|x|$. In what follows $\Phi(x,\lambda)$ is any function satisfying (3.1) up to a term $O(|x|^{-2})$ as $|x| \to \infty$ and estimates (3.2). In fact, one can construct by iterations (see for example [15]) an explicit function $\Phi(x,\lambda)$ satisfying these conditions but, in general, this construction requires assumption (2.9) for $|\alpha|$ bigger than 3. Recall that the symbol $o_{av}(|x|^{-(d-1)/2})$ was defined by (1.5).

Lemma 3.2 *Let $u \in \mathbf{H}^2_{loc}$ be a solution of equation* (1.1) *satisfying, for some $a_\pm \in L^2(\mathbb{S}^{d-1})$, the conditions*

$$u(x) = a_+(\hat{x})w_+(x,\lambda) - a_-(\hat{x})w_-(x,\lambda) + o_{av}(|x|^{-(d-1)/2}), \tag{3.4}$$

$$(\partial_r u)(x) = i\sqrt{\lambda}\Big(a_+(\hat{x})w_+(x,\lambda) + a_-(\hat{x})w_-(x,\lambda)\Big) + o_{av}(|x|^{-(d-1)/2}). \tag{3.5}$$

as $|x| \to \infty$. Then $||a_+|| = ||a_-||$.

Proof – It follows from (1.1) that

$$\operatorname{Im} \Delta u \overline{u} = 0. \tag{3.6}$$

Integration by parts now yields

$$\operatorname{Im} \int_{|x|=r} u_r \overline{u}\, dS_r = \operatorname{Im} \int_{|x|<r} \Delta u \overline{u}\, dx = 0. \tag{3.7}$$

Then, using (3.4), (3.5) and the identity

$$\operatorname{Im} i(a_-\overline{a}_+ w_- \overline{w_+} - a_+\overline{a}_- w_+\overline{w}_-) = 0,$$

we see that

$$\begin{aligned} \operatorname{Im} u_r\overline{u} &= \operatorname{Im} i\sqrt{\lambda}(a_+w_+ + a_-w_-)(\overline{a}_+\overline{w}_+ - \overline{a}_-\overline{w}_-) + \epsilon(x) \\ &= \sqrt{\lambda} r^{1-d}(|a_+|^2 - |a_-|^2) + \epsilon(x), \end{aligned} \tag{3.8}$$

where

$$\int_{|x|\le\rho} |\epsilon(x)| dx = o(\rho), \quad \rho \to \infty.$$

Inserting (3.8) into (3.7) and integrating over $r \in (0,\rho)$, we obtain that $||a_+||^2 - ||a_-||^2 = o(1)$ and hence $||a_+|| = ||a_-||$. $\square$

Let us denote $R_L(z) = (-\Delta + V_L - z)^{-1}$. Our paper relies essentially on the following result.

Theorem 3.3 *Let V_L satisfy* (2.9) *and let $w_\pm(x,\lambda)$ be given by* (1.3). *Then for every $f \in \mathfrak{S}$ there exists $a^L_\pm \in L^2(\mathbb{S}^{d-1})$ such that, as $|x| \to \infty$,*

$$(R_L(\lambda \pm i0)f)(x) = \pi^{1/2}\lambda^{-1/4}a^L_\pm(\pm\hat{x})w_\pm(x,\lambda) + o(|x|^{-(d-1)/2}), \tag{3.9}$$

$$(\partial_r R_L(\lambda \pm i0)f)(x) = \pm i\pi^{1/2}\lambda^{1/4}a^L_\pm(\pm\hat{x})w_\pm(x,\lambda) + o(|x|^{-(d-1)/2}). \tag{3.10}$$

This result was first proven in [6] for $\delta > 1/2$ and for suitably chosen sequences $|x_n| \to \infty$. These technical points have been independently overcome in [10] and [15].

Relation (3.9) allows us to define the mapping $\mathcal{F}^L_\pm(\lambda) : \mathfrak{S} \to L^2(\mathbb{S}^{d-1})$ by the equality

$$(\mathcal{F}^L_\pm(\lambda)f)(\hat{x}) = a^L_\pm(\hat{x}).$$

Set $v_\pm = R_L(\lambda \pm i0)f$ for $f \in \mathfrak{S}$ so that $(-\Delta + V_L - \lambda)v_\pm = f$. Integrating by parts and using relations (3.9), (3.10), we obtain that

$$\begin{aligned}\int_{|x|<r}(v_\pm\overline{f} - f\overline{v}_\pm)\,dx &= \int_{|x|<r}(\overline{v}_\pm\Delta v_\pm - v_\pm\Delta\overline{v}_\pm)\,dx = \int_{|x|=r}(\overline{v}_\pm\partial_r v_\pm - v_\pm\partial_r\overline{v}_\pm)\,dx \\ &= \pm 2\pi i\int_{\mathbb{S}^{d-1}}|a_\pm(\hat{x})|^2\,d\hat{x} + o(1),\end{aligned} \tag{3.11}$$

and hence, passing to the limit $r \to \infty$,

$$\langle R_L(\lambda + i0)f, f\rangle - \langle R_L(\lambda - i0)f, f\rangle = 2\pi i\int_{\mathbb{S}^{d-1}}|(\mathcal{F}^L_\pm(\lambda)f)(\hat{x})|^2 d\hat{x}. \tag{3.12}$$

Proposition 2.5 and relation (3.12) yield

Lemma 3.4 *The operator $\mathcal{F}^L_\pm(\lambda)$ extends by continuity to a bounded operator from B into $L^2(\mathbb{S}^{d-1})$.*

The asymptotics (3.9) and (3.10) can now be extended to an arbitrary $f \in B$. At the same time, the short-range part of the potential will be included.

Theorem 3.5 *Let V satisfy* (2.8)-(2.10) *and let $w_\pm(x,\lambda)$ be given by* (1.3). *Then, for every $f \in B$ there exists $a_\pm \in L^2(\mathbb{S}^{d-1})$ such that, as $|x| \to \infty$,*

$$\left(R(\lambda \pm i0)f\right)(x) = \pi^{1/2}\lambda^{-1/4}a_\pm(\pm\hat{x})w_\pm(x,\lambda) + o_{av}(|x|^{-(d-1)/2}), \tag{3.13}$$

$$\left(\partial_r R(\lambda \pm i0)f\right)(x) = \pm i\pi^{1/2}\lambda^{1/4}a_\pm(\pm\hat{x})w_\pm(x,\lambda) + o_{av}(|x|^{-(d-1)/2}). \tag{3.14}$$

Proof – Choose, for example, the sign " + ". Consider first (3.13) for $R_L(\lambda + i0)f$. Let $f_n \in \mathfrak{S}$ be such that $||f_n - f||_B \to 0$ as $n \to \infty$. Then

$$\begin{aligned}\rho^{-1}||R_L(\lambda + i0)f - c(\lambda)(\mathcal{F}^L_+(\lambda)f)w_+||^2_{L^2(|x|\le\rho)} &\le \rho^{-1}||R_L(\lambda + i0)(f - f_n)||^2_{L^2(|x|\le\rho)} \\ &+\rho^{-1}\alpha(\lambda)||(\mathcal{F}^L_+(\lambda)(f - f_n))w_+||^2_{L^2(|x|\le\rho)} \\ &+\rho^{-1}||R_L(\lambda + i0)f_n - \alpha(\lambda)(\mathcal{F}^L_+(\lambda)f_n)w_+||^2_{L^2(|x|\le\rho)},\end{aligned} \tag{3.15}$$

where $\alpha(\lambda)=\pi^{1/2}\lambda^{-1/4}$. Clearly, the first two terms on the right-hand side are bounded by

$$C\Big(||R_L(\lambda+i0)(f-f_n)||_{B^*}+||\mathcal{F}_+^L(\lambda)(f-f_n)||_{L^2(\mathbb{S}^{d-1})}\Big),$$

which is estimated by $||f-f_n||_B$ according to Proposition 2.5 and Lemma 3.4. The last term in (3.15) tends to zero for fixed n and $\rho\to\infty$ according to (3.9). This yields asymptotics (3.13) for $R_L(\lambda+i0)f$ and any $f\in B$.

Taking into account the resolvent identity and Proposition 2.5, we see that

$$R(\lambda+i0)f=R_L(\lambda+i0)g,\ \text{where}\ g=f-V_SR(\lambda+i0)f\in B.$$

Therefore (3.13) is an immediate consequence of the same relation for $R_L(\lambda+i0)g$.

The proof of (3.14) can be obtained quite similarly on the basis of (3.10) and the boundedness of the operator $\partial_r R_L(\lambda+i0):B\to B^*$. $\square$

We define now the mapping $\mathcal{F}_\pm(\lambda):B\to L^2(\mathbb{S}^{d-1})$ by the equality

$$(\mathcal{F}_\pm(\lambda)f)(\hat{x})=a_\pm(\hat{x}). \tag{3.16}$$

it follows from (3.13) and (3.14) that equality (3.11) holds for some sequence $r_n\to\infty$ which implies the equality

$$\langle R(\lambda+i0)f,f\rangle-\langle R(\lambda-i0)f,f\rangle=2\pi i\int_{\mathbb{S}^{d-1}}|(\mathcal{F}_\pm(\lambda)f)(\hat{x})|^2d\hat{x}. \tag{3.17}$$

This gives us

Proposition 3.6 *The operator $\mathcal{F}_\pm(\lambda)$ is bounded from B into $L^2(\mathbb{S}^{d-1})$, and its adjoint $\mathcal{F}_\pm^*(\lambda)$ is a bounded operator from $L^2(\mathbb{S}^{d-1})$ into B^*.*

2. Our aim is now to derive a convenient representation for the operator $\mathcal{F}_\pm^*(\lambda)$ on functions $a\in C^\infty(\mathbb{S}^{d-1})$. Let $\eta\in C^\infty(\mathbb{R}^d)$ be such that $\eta(x)=0$ in a neighbourhood of $x=0$ and $\eta(x)=1$ for large $|x|$. Obviously, the functions

$$u_\pm(x,\lambda)=a(\pm\hat{x})w_\pm(x,\lambda)\eta(x), \tag{3.18}$$

belong to the space B^*. It follows from (3.2) that

$$\partial_r u_\pm(x,\lambda)=\pm i\lambda^{1/2}u_\pm(x,\lambda)+o(|x|^{-(d-1)/2-\delta}). \tag{3.19}$$

Set also

$$g_\pm(\lambda)=(-\Delta+V-\lambda)u_\pm(\lambda)=:G_\pm(\lambda)a. \tag{3.20}$$

Straightforward calculations (which can be found for example in [10]) show that

$$g_\pm(\lambda)\in L^2_\gamma(\mathbb{R}^d)\quad\text{for some }\gamma>1/2. \tag{3.21}$$

Of course, the definition of $G_\pm(\lambda)$ depends on η, but as we will see below, this dependence is inessential.

Lemma 3.7 *Let $a \in C^\infty(\mathbb{S}^{d-1})$, $w_\pm(x,\lambda)$ be defined by (1.3) and $u_\pm$, $g_\pm$ be given by (3.18), (3.20). Then (as elements of B^*)*

$$\pm 2i\pi^{1/2}\lambda^{1/4}\mathcal{F}^*_\pm(\lambda)a = u_\pm(\lambda) - R(\lambda \mp i0)g_\pm(\lambda). \tag{3.22}$$

Proof – We consider only the " + " case, proofs being similar. It suffices to check that for any $f \in B$

$$\langle u_+, f\rangle - \langle g_+, R(\lambda + i0)f\rangle = 2i\pi^{1/2}\lambda^{1/4}(a, \mathcal{F}_+(\lambda)f)_{L^2(\mathbb{S}^{d-1})}. \tag{3.23}$$

Let us apply the Green formula

$$\int_{|x|<r} (\overline{v}\Delta u_+ - u_+\Delta\overline{v})\, dx = \int_{|x|=r} (\overline{v}\partial_r u_+ - u_+\partial_r\overline{v})\, dS_r \tag{3.24}$$

to $v = R(\lambda + i0)f$. It follows from (3.13) and (3.14) that

$$\int_{|x|=r_n} (|v - bw_+|^2 + |\partial_r v - i\lambda^{1/2}bw_+|^2) dS_{r_n} = o(1),$$

where $b = \pi^{1/2}\lambda^{-1/4}\mathcal{F}_+(\lambda)f$, for some sequence $r_n \to \infty$. Therefore, by the Schwarz inequality,

$$\int_{|x|=r_n} ((\overline{v} - \overline{bw_+})\partial_r u_+ - u_+(\partial_r\overline{v} + i\lambda^{1/2}\overline{bw_+}))dS_{r_n} = o(1). \tag{3.25}$$

Comparing the right-hand side of (3.24) with (3.25), we thus obtain

$$\lim_{n\to\infty} \int_{|x|=r_n} (\overline{v}\partial_r u_+ - u_+\partial_r\overline{v})\, dS_{r_n} = 2i\pi^{1/2}\lambda^{1/4}(a, \mathcal{F}_+(\lambda)f)_{L^2(\mathbb{S}^{d-1})}.$$

So to conclude, it remains to note that the left-hand side of (3.24) equals

$$\int_{|x|<r_n} (u_+(-\Delta + V - \lambda)\overline{v} - \overline{v}(-\Delta + V - \lambda)u_+)dx = \int_{|x|<r_n} (u_+\overline{f} - g_+\overline{R(\lambda + i0)f})dx,$$

and hence tends to the left-hand side of (3.23) as $r_n \to \infty$. □

Corollary 3.8 *For any $a \in C^\infty(\mathbb{S}^{d-1})$, the function $\mathcal{F}^*_\pm(\lambda)a \in \mathbf{H}^2_{loc}$ and*

$$(-\Delta + V - \lambda)\mathcal{F}^*_\pm(\lambda)a = 0. \tag{3.26}$$

Taking also into account Corollary 2.2 we deduce the following result

Proposition 3.9 *The operators $\Delta\mathcal{F}^*_\pm(\lambda)$ and $\partial_j\mathcal{F}^*_\pm(\lambda)$, $j = 1, \ldots, d$, defined on the set $C^\infty(\mathbb{S}^{d-1})$ extend by continuity to bounded operators from $L^2(\mathbb{S}^{d-1})$ into B^*. For any $a \in L^2(\mathbb{S}^{d-1})$, the function $\mathcal{F}^*_\pm(\lambda)a \in \mathbf{H}^2_{loc} \cap B^*$ and the equation (3.26) is satisfied.*

Finally, we note that, according again to Proposition 2.5, all objects introduced here are continuous (in a weak sense) with respect to λ. For example,

Lemma 3.10 *The function $\lambda \mapsto (\mathcal{F}_\pm(\lambda)f, a)$ is continuous in λ for all $f \in B$ and $a \in L^2(\mathbb{S}^{d-1})$.*

4. THE MAIN RESULT

1. Now we are able to obtain the spatial asymptotics of functions $(\mathcal{F}^*_\pm(\lambda)a)(x)$. For $a\in C^\infty(\mathbb{S}^{d-1})$, let us set

$$\Sigma_+(\lambda)a=\pi^{1/2}\lambda^{-1/4}\mathcal{F}_+(\lambda)G_-(\lambda)\mathcal{T}a,\quad \Sigma_-(\lambda)a=\pi^{1/2}\lambda^{-1/4}\mathcal{T}\mathcal{F}_-(\lambda)G_+(\lambda)a, \tag{4.1}$$

where $\mathcal{T}$, $(\mathcal{T}a)(\omega)=a(-\omega)$, is the reflection operator on the sphere and $G_\pm$ is defined by (3.18), (3.20). Recall that, according to (3.21), $G_\pm(\lambda):C^\infty(\mathbb{S}^{d-1})\to L^2_\gamma(\mathbb{R}^d)$ for some $\gamma>1/2$ while $\mathcal{F}_\pm(\lambda):L^2_\gamma(\mathbb{R}^d)\to L^2(\mathbb{S}^{d-1})$ for any $\gamma>1/2$. Hence $\Sigma_\pm(\lambda):C^\infty(\mathbb{S}^{d-1})\to L^2(\mathbb{S}^{d-1})$ is well defined.

Combining Lemma 3.7 and relations (3.13), (3.14) we arrive at

Proposition 4.1 *Let $w_\pm(x,\lambda)$ be given by* (1.3). *Then for any $a\in C^\infty(\mathbb{S}^{d-1})$, we have*

$$(\mathcal{F}^*_+(\lambda)a)(x)=\beta(\lambda)\Big(a(\hat{x})w_+(x,\lambda)-(\Sigma_-(\lambda)a)(\hat{x})w_-(x,\lambda)\Big)+o_{av}(|x|^{-(d-1)/2}), \tag{4.2}$$

$$(\mathcal{F}^*_-(\lambda)\mathcal{T}a)(x)=\beta(\lambda)\Big((\Sigma_+(\lambda)a)(\hat{x})w_+(x,\lambda)-a(\hat{x})w_-(x,\lambda)\Big)+o_{av}(|x|^{-(d-1)/2}), \tag{4.3}$$

and

$$(\partial_r\mathcal{F}^*_+(\lambda)a)(x)=i\lambda^{1/2}\beta(\lambda)\Big(a(\hat{x})w_+(x,\lambda)+(\Sigma_-(\lambda)a)(\hat{x})w_-(x,\lambda)\Big)+o_{av}(|x|^{-(d-1)/2}), \tag{4.4}$$

$$(\partial_r\mathcal{F}^*_-(\lambda)\mathcal{T}a)(x)=i\lambda^{1/2}\beta(\lambda)\Big((\Sigma_+(\lambda)a)(\hat{x})w_+(x,\lambda)+a(\hat{x})w_-(x,\lambda)\Big)+o_{av}(|x|^{-(d-1)/2}), \tag{4.5}$$

as $|x|\to\infty$, where $\beta(\lambda)=-i2^{-1}\pi^{-1/2}\lambda^{-1/4}$.

Lemma 4.2 *The operator $\Sigma_\pm(\lambda)$ extends by continuity to an isometric operator in $L^2(\mathbb{S}^{d-1})$.*

Proof – Isometricity of $\Sigma_+(\lambda)$ (resp. $\Sigma_-(\lambda)$) on the set $C^\infty(\mathbb{S}^{d-1})$ is a direct consequence of Lemma 3.2 applied to relations (4.3), (4.5) (resp. (4.2), (4.4)). $\square$

The boundedness of the operators $\mathcal{F}^*_\pm(\lambda)$, $\partial_r\mathcal{F}^*_\pm(\lambda)$ and $\Sigma_\pm(\lambda)$ allows us, with help of Theorem 3.5, to extend Proposition 4.1 to an arbitrary $a\in L^2(\mathbb{S}^{d-1})$.

Proposition 4.3 *The asymptotics* (4.2)- (4.5) *hold true for all $a\in L^2(\mathbb{S}^{d-1})$.*

In fact, we have concluded the proof of the part A) of the main result formulated in Section 1.

2. Our proof of the part B) relies on the following properties of the operators $\mathcal{F}_\pm(\lambda)$ and $\mathcal{F}^*_\pm(\lambda)$.

Proposition 4.4 *If $a\in L^2(\mathbb{S}^{d-1})$, then the limit*

$$\lim_{\rho\to\infty}\rho^{-1}\int_{|x|\le\rho}|(\mathcal{F}^*_\pm(\lambda)a)(x)|^2\,dx=(2\pi)^{-1}\lambda^{-1/2}||a||^2_{L^2(\mathbb{S}^{d-1})} \tag{4.6}$$

exists.

Proof – According to (4.2) or (4.3), the left-hand side of (4.6) equals

$$4\pi\lambda^{1/2}\lim_{\rho\to\infty}\rho^{-1}\int_{|x|\leq\rho}|(\mathcal{F}_\pm^*(\lambda)a)(x)|^2\,dx = ||a||^2_{L^2(\mathbb{S}^{d-1})} + ||\Sigma_\mp(\lambda)a||^2_{L^2(\mathbb{S}^{d-1})}$$
$$-2\rho^{-1}\mathrm{Re}\left(\int_{\mathbb{S}^{d-1}}(\Sigma_\mp(\lambda)a)(\omega)\overline{a}(\omega)d\omega\int_0^\rho \exp(\mp 2i\varphi(r\omega,\lambda))\,dr\right)+o(1).$$

Integrating with the help of (3.2) by parts, we see that

$$\lim_{\rho\to 0}\rho^{-1}\int_0^\rho \exp(\mp 2i\varphi(r\omega,\lambda))\,dr = 0.$$

Therefore (4.6) follows from isometricity of the operator $\Sigma_\mp(\lambda)$. $\square$

Combining Propositions 3.6 and 4.4, we obtain

Proposition 4.5 *One has the two-sided estimate*

$$(2\pi)^{-1/2}\lambda^{-1/4}||a||_{L^2(\mathbb{S}^{d-1})} \leq ||\mathcal{F}_\pm^*(\lambda)a||_{B^*} \leq C||a||_{L^2(\mathbb{S}^{d-1})}. \tag{4.7}$$

Hence the kernel $N(\mathcal{F}_\pm^*(\lambda))$ of the operator $\mathcal{F}_\pm^*(\lambda):\ L^2(\mathbb{S}^{d-1})\to B^*$ is trivial and its image $R(\mathcal{F}_\pm^*(\lambda))$ is closed.

Let us return to equation (1.1).

Lemma 4.6 *Let $u\in\mathbf{H}^2_{loc}\cap B^*$ be a solution of (1.1). For any $f\in B$ such that $\mathcal{F}_\pm(\lambda)f=0$, we have $\langle u,f\rangle=0$.*

Proof – By Corollary 2.2, $\partial_r u\in B^*$ and hence

$$\int_{|x|\leq\rho}(|u(x)|^2+|\partial_r u(x)|^2)\,dx\leq C\rho,\quad \rho\geq 1. \tag{4.8}$$

If $f\in B$ and $\mathcal{F}_\pm(\lambda)f=0$ then, according to Theorem 3.5, the function $v_\pm=R(\lambda\pm i0)f$ satisfies the condition

$$\int_{|x|\leq\rho}(|v_\pm(x)|^2+|\partial_r v_\pm(x)|^2)\,dx=o(\rho), \tag{4.9}$$

as $\rho\to\infty$. By the Schwarz inequality, it follows from (4.8) and (4.9) that

$$\int_{|x|\leq\rho}(\partial_r u\overline{v}_\pm - u\partial_r\overline{v}_\pm)dx=o(\rho),$$

as $\rho\to\infty$. Consequently, there exists a sequence $r_n\to\infty$ such that

$$\lim_{r_n\to\infty}\int_{|x|=r_n}(\partial_r u\overline{v}_\pm - u\partial_r\overline{v}_\pm)dS_{r_n}=0. \tag{4.10}$$

Now integrating by parts and taking into account equation (1.1) for u, we find that

$$\langle u,f\rangle=\lim_{r_n\to\infty}\int_{|x|\leq r_n}u(-\Delta+V-\lambda)\overline{v}_\pm\,dx=\lim_{r_n\to\infty}\int_{|x|=r_n}(\overline{v}_\pm\partial_r u-u\partial_r\overline{v}_\pm)dS_{r_n}.$$

Therefore $\langle u, f \rangle = 0$ by virtue of (4.10). $\Box$

It follows from Proposition 4.5 that Theorem 2.3 can be applied to the operator $\mathcal{F}_{\pm}(\lambda)$. Therefore, by Lemma 4.6, every $u \in \mathbf{H}^2_{loc} \cap B^*$ belongs to the range of $\mathcal{F}^*_{\pm}(\lambda)$. Taking also into account Proposition 3.9, we obtain a description of solutions of equation (1.1) in terms of the operator $\mathcal{F}^*_{\pm}(\lambda)$.

Theorem 4.7 *For $u \in \mathbf{H}^2_{loc} \cap B^*$ to be a solution of equation* (1.1)*, it is necessary and sufficient that $u \in R(\mathcal{F}^*_{\pm}(\lambda))$.*

According to Proposition 4.3, Theorem 4.7 leads immediately to the part B) of our main result. Let us reformulate both parts adding informations contained in Propositions 2.4, 4.4 and 4.5. Note that functions $a_{\pm}$ and u in this formulation are related by the equalities

$$u(x) = 2i\pi^{1/2}\lambda^{1/4}(\mathcal{F}^*_{+}(\lambda)a_{+})(x), \qquad u(x) = 2i\pi^{1/2}\lambda^{1/4}(\mathcal{F}^*_{-}(\lambda)\mathcal{T}a_{-})(x). \tag{4.11}$$

Theorem 4.8 *Let assumptions* (2.8)-(2.10) *hold and let $w_{\pm}(x,\lambda)$ be given by* (1.3)*. For any $a_{-} \in L^2(\mathbb{S}^{d-1})$, there exist a function $a_{+} \in L^2(\mathbb{S}^{d-1})$ and a solution $u \in \mathbf{H}^2_{loc} \cap B^*$ of* (1.1) *with asymptotics* (1.6), (1.7) *as $|x| \to \infty$. The function a_{+} and the solution u are determined uniquely by a_{-}. Moreover, $||a_{-}|| = ||a_{+}||$ and the roles of a_{+} and a_{-} in this formulation can be interchanged.*

Conversely, any $u \in \mathbf{H}^2_{loc} \cap B^*$ satisfying (1.1) *has the asymptotics* (1.6), (1.7) *as $|x| \to \infty$ for some functions $a_{\pm} \in L^2(\mathbb{S}^{d-1})$. Furthermore*

$$||u||_{B^*} \leq C||a_{\pm}||_{L^2(\mathbb{S}^{d-1})},$$

and the following limit exists

$$\lim_{\rho\to\infty} \rho^{-1} \int_{|x|\leq\rho} |u(x)|^2 dx = 2||a_{\pm}||^2_{L^2(\mathbb{S}^{d-1})}. \tag{4.12}$$

It follows from (4.11) and Proposition 4.1 that the functions $a_{\pm}$ in relations (1.6), (1.7) are related by equalities $a_{+} = \Sigma_{+}(\lambda)a_{-}$ and $a_{-} = \Sigma_{-}(\lambda)a_{+}$. Therefore

$$\Sigma_{+}(\lambda)\Sigma_{-}(\lambda) = \Sigma_{-}(\lambda)\Sigma_{+}(\lambda) = I. \tag{4.13}$$

In view of the isometricity of $\Sigma_{\pm}(\lambda)$, this implies

Proposition 4.9 *The operators $\Sigma_{\pm}(\lambda)$ are unitary on $L^2(\mathbb{S}^{d-1})$. Moreover*

$$\Sigma_{+}(\lambda) = \Sigma^*_{-}(\lambda), \qquad \Sigma_{-}(\lambda) = \Sigma^*_{+}(\lambda).$$

As an easy consequence of Theorem 4.8, we obtain

Proposition 4.10 *In order that the equation* (1.1) *have a solution with asymptotics*

$$u(x) = a_{+}(\hat{x})w_{+}(x,\lambda) - a_{-}(\hat{x})w_{-}(x,\lambda) + o_{av}(|x|^{-(d-1)/2}), \quad a_{\pm} \in L^2(\mathbb{S}^{d-1}),$$

as $|x| \to \infty$, it is necessary and sufficient that $a_{+} = \Sigma_{+}(\lambda)a_{-}$ or $a_{-} = \Sigma_{-}(\lambda)a_{+}$.

Corollary 4.11 *If u satisfies equation* (1.1) *and*

$$u(x) = a(\hat{x}) w_\pm(x,\lambda) + o_{av}(|x|^{-(d-1)/2}), \quad a \in L^2(\mathbb{S}^{d-1}), \tag{4.14}$$

as $|x| \to \infty$ (for one of the signs " $+$ " or " $-$ "), then $u = 0$.

Since the relation (4.14) is not *a priori* supposed differentiable, Corollary 4.11 improves Proposition 2.4.

Set $\mathcal{S}(\lambda) = \Sigma_+(\lambda)\mathcal{T}$. By Proposition 4.9, this operator is unitary on $L^2(\mathbb{S}^{d-1})$ and $\mathcal{S}^*(\lambda) = \mathcal{T}\Sigma_-(\lambda)$. The following result shows that the operator $\mathcal{S}(\lambda)$ known as the scattering matrix "relates" $\mathcal{F}_+(\lambda)$ and $\mathcal{F}_-(\lambda)$.

Proposition 4.12 *For any $f \in B$,*

$$\mathcal{F}_+(\lambda) f = \mathcal{S}(\lambda)\mathcal{F}_-(\lambda) f. \tag{4.15}$$

Proof – Let $a \in L^2(\mathbb{S}^{d-1})$ be arbitrary. By Proposition 3.9, the function

$$u(x) = (\mathcal{F}_+^*(\lambda) a)(x) - (\mathcal{F}_-^*(\lambda)\mathcal{T}\Sigma_-(\lambda) a)(x)$$

satisfies equation (1.1). Moreover, by Proposition 4.3 and (4.13), $u(x) = o_{av}(|x|^{-(d-1)/2})$ as $|x| \to \infty$. Therefore Corollary 4.11 implies that $u = 0$ or, equivalently, that $\mathcal{F}_+^*(\lambda) = \mathcal{F}_-^*(\lambda)\mathcal{S}^*(\lambda)$. □

On the set $C^\infty(\mathbb{S}^{d-1})$, the scattering matrix can be expressed in terms of the operators $G_\pm(\lambda)$ defined by (3.18)-(3.20). Indeed, it follows from (4.1) that

$$\mathcal{S}(\lambda) a = \pi^{1/2}\lambda^{-1/4}\mathcal{F}_+(\lambda) G_-(\lambda) a, \quad \mathcal{S}^*(\lambda) a = \pi^{1/2}\lambda^{-1/4}\mathcal{F}_-(\lambda) G_+(\lambda) a, \tag{4.16}$$

which coincides with the representation for the scattering matrix obtained in [16]. The first of these equalities shows that, for any $a, b \in C^\infty(\mathbb{S}^{d-1})$,

$$\begin{aligned} (\mathcal{S}(\lambda) a, b) &= \pi^{1/2}\lambda^{-1/4}\langle G_-(\lambda) a, \mathcal{F}_+^*(\lambda) b\rangle \\ &= -i2^{-1}\lambda^{-1/2}\langle G_-(\lambda) a, u_+(\lambda) - R(\lambda - i0) G_+(\lambda) b\rangle. \end{aligned}$$

Since $G_-(\lambda) a$ and $G_+(\lambda) b$ are continuous functions of λ in $L^2_\gamma(\mathbb{R}^d)$ for some $\gamma > 1/2$ and $u_+(\lambda)$ is continuous in $L^2_{-\gamma'}$ for some $\gamma' > 1/2$, the continuity of $(\mathcal{S}(\lambda) a, b)$ with respect to λ follows from Proposition 2.5. Taking into account that the operators $\mathcal{S}(\lambda)$ and $\mathcal{S}^*(\lambda)$ are isometric, we obtain

Proposition 4.13 *The operators $\mathcal{S}(\lambda)$ and $\mathcal{S}^*(\lambda)$ are strongly continuous functions of λ.*

3. Here we formulate a corollary of our considerations for the non-homogeneous Schrödinger equation.

Proposition 4.14 *For any $a \in L^2(\mathbb{S}^{d-1})$, there exist functions $f_\pm \in B$ and $u_\pm \in \mathbf{H}^2_{loc}$ with asymptotics* (2.11)-(2.12) *satisfying the equation*

$$-\Delta u_\pm + V u_\pm = \lambda u_\pm + f_\pm. \tag{4.17}$$

Proof – According to the last statement of Theorem 2.3, it follows from Proposition 4.5 that $R(\mathcal{F}_\pm(\lambda)) = L^2(\mathbb{S}^{d-1})$. Therefore there exist functions $f_\pm \in B$ such that $\mathcal{F}_+(\lambda) f_= a$ or $\mathcal{F}_-(\lambda) f_- = \mathcal{T} a_-$. Clearly, functions $u_\pm = R(\lambda \pm i0) f_\pm$ satisfy (4.17) and have asymptotics (2.11)-(2.12) by virtue of Theorem 3.5. □

5. EXPANSION THEOREM

1. Let $\tilde{\mathcal{H}} = L^2(\mathbb{R}_+, L^2(\mathbb{S}^{d-1}))$ be the Hilbert space of $L^2(\mathbb{S}^{d-1})$-valued square integrable functions on $\mathbb{R}_+$ with the Lebesgue measure. Recall that the operator $\mathcal{F}_\pm(\lambda) : B \to L^2(\mathbb{S}^{d-1})$ was defined by equalities (3.13), (3.14) and (3.16). Now we introduce the mapping $F_\pm : \ B \to \tilde{\mathcal{H}}$ by the formula

$$(F_\pm f)(\lambda) = \mathcal{F}_\pm(\lambda) f. \tag{5.1}$$

Integrating (3.17) over $\mathbb{R}_+$, and using the spectral theorem, we find that

$$||F_\pm f||^2 = \int_0^\infty ||\mathcal{F}_\pm(\lambda) f||^2_{L^2(\mathbb{S}^{d-1})}\, d\lambda = \int_0^\infty \frac{d(E(\lambda)f,f)}{d\lambda} d\lambda = ||Pf||^2, \quad P = E(0,\infty),$$

for any $f \in B$. Hence the operator $F_\pm$ extends by continuity to a bounded operator on the entire space $\mathcal{H}$. This operator is isometric on the absolutely continuous subspace $\mathcal{H}^{(a)} = P\mathcal{H}$ of the operator H and vanishes on its orthogonal complement, i.e.,

$$F_\pm^* F_\pm = P. \tag{5.2}$$

Now, denote by Λ multiplication by the independent variable λ in the space $\tilde{\mathcal{H}}$ and by $E_\Lambda(X)$ its spectral projection acting as multiplication by the characteristic function of the Borel set $X \cap \mathbb{R}_+$. We shall establish the intertwining property

$$F_\pm E_\Lambda(X) = E_\Lambda(X) F_\pm. \tag{5.3}$$

We need only to check that

$$(F_\pm H f, \tilde{g}) = (F_\pm f, \Lambda \tilde{g})$$

for $f \in \mathfrak{S}$ and $\tilde{g}(\lambda) = \zeta(\lambda) a$, where $\zeta \in C_0^\infty(\mathbb{R}_+)$ and $a \in L^2(\mathbb{S}^{d-1})$, i.e., to verify that

$$\int_0^\infty \langle Hf, \mathcal{F}_\pm^*(\lambda) a\rangle \zeta(\lambda)\, d\lambda = \int_0^\infty \lambda \langle f, \mathcal{F}_\pm^*(\lambda) a\rangle \zeta(\lambda)\, d\lambda.$$

This equality is a direct consequence of Proposition 3.9. Let us finally show that $R(F_\pm) = \mathcal{H}$.

Lemma 5.1 *The operator* (5.1) *satisfies*

$$F_\pm F_\pm^* = I. \tag{5.4}$$

Proof – According to (5.2), (5.3), for any $f \in \mathcal{H}$ and any $X \subset \mathbb{R}_+$,

$$(E_\Lambda(X) F_\pm F_\pm^* \tilde{g}, F_\pm f) = (E_\Lambda(X)\tilde{g}, F_\pm f). \tag{5.5}$$

Let $\tilde{h} = F_\pm F_\pm^* \tilde{g}$. If $f \in B$, relation (5.5) is equivalent to

$$\int_X (\tilde{h}(\lambda), \mathcal{F}_\pm(\lambda) f)_{L^2(\mathbb{S}^{d-1})}\, d\lambda = \int_X (\tilde{g}(\lambda), \mathcal{F}_\pm(\lambda) f)_{L^2(\mathbb{S}^{d-1})}\, d\lambda.$$

Since $f \in B$ is arbitrary, this involves

$$\int_X \mathcal{F}_\pm^*(\lambda)(\tilde{h}(\lambda) - \tilde{g}(\lambda))\, d\lambda = 0.$$

and thus, the set X being arbitrary,

$$\mathcal{F}_\pm^*(\lambda)(\tilde{h}(\lambda) - \tilde{g}(\lambda)) = 0,$$

for almost all $\lambda \in \mathbb{R}_+$. By Proposition 4.5, the kernel of $\mathcal{F}_\pm^*(\lambda)$ is trivial, hence $\tilde{h}(\lambda) = \tilde{g}(\lambda)$ for almost all $\lambda \in \mathbb{R}_+$. This implies (5.4). $\square$

Let us summarize the results obtained.

Theorem 5.2 *For every $f \in B$ the function $\mathcal{F}_\pm(\lambda)f$ belongs to the space $\tilde{\mathcal{H}}$. The operator $F_\pm$ defined by* (5.1) *extends by continuity to a bounded operator on the entire space $\mathcal{H}$ and satisfies relations* (5.2), (5.3) *and* (5.4).

2. This theorem is, obviously, a standard (see e.g. [6]) expansion theorem. Similarly to the short-range case [18], results of the previous section allow us to reformulate it in terms of solutions of equation (1.1) with special asymptotics at infinity. By (5.1), (5.2), the equality

$$(Pf, g) = \int_0^\infty \langle \mathcal{F}^*(\lambda)\tilde{f}_\pm(\lambda), g\rangle\, d\lambda, \quad \tilde{f}_\pm = F_\pm f, \tag{5.6}$$

holds for any $f \in \mathcal{H}$ and $g \in B$. Proposition 4.3 implies that for almost all $\lambda \in \mathbb{R}_+$ the function $u_f^{(\pm)}(\lambda) = \mathcal{F}_\pm^*(\lambda)\tilde{f}_\pm(\lambda) \in B^*$ satisfies equation (1.1) and

$$\begin{aligned} u_f^{(+)}(x,\lambda) &= -i2^{-1}\pi^{-1/2}\lambda^{-1/4}\Big[\tilde{f}(\hat{x},\lambda)w_+(x,\lambda) - (\mathcal{S}^*(\lambda)\tilde{f}(\lambda))(-\hat{x})w_-(x,\lambda)\Big] \\ &\quad + o_{av}(|x|^{-(d-1)/2}) \end{aligned} \tag{5.7}$$

$$\begin{aligned} u_f^{(-)}(\lambda)(x,\lambda) &= -i2^{-1}\pi^{-1/2}\lambda^{-1/4}\Big[(\mathcal{S}(\lambda)\tilde{f}(\lambda))(\hat{x})w_+(x,\lambda) - \tilde{f}(-\hat{x},\lambda)w_-(x,\lambda)\Big] \\ &\quad + o_{av}(|x|^{-(d-1)/2}) \end{aligned} \tag{5.8}$$

as $|x| \to \infty$. Furthermore, the following limit

$$\lim_{\rho\to\infty} \rho^{-1} \int_{|x|\le\rho} |u_f^{(\pm)}(x,\lambda)|^2\, dx = (2\pi)^{-1}\lambda^{-1/2}||\tilde{f}_\pm(\lambda)||^2_{L^2(\mathbb{S}^{d-1})} \tag{5.9}$$

exists. Now suppose additionally that $f \in E((\mu,\nu))\mathcal{H}$ for some $0 < \mu < \nu < \infty$. Then the integral (5.6) is taken only over the interval (μ,ν). Using Proposition 3.6 and (5.2), one obtains

$$\begin{aligned} \int_\mu^\nu d\lambda \int_{\mathbb{R}^d} |u_f^{(\pm)}(x,\lambda)||g(x)|dx &\le \int_\mu^\nu d\lambda ||u_f^{(\pm)}(\lambda)||_{B^*}||g||_B \\ &\le C(\mu,\nu)\int_\mu^\nu ||\tilde{f}(\lambda)||_{L^2(\mathbb{S}^{d-1})} d\lambda\, ||g||_B \\ &\le C_1(\mu,\nu)||f||\, ||g||_B. \end{aligned} \tag{5.10}$$

Since $\langle u_f^{(\pm)}(\lambda), g\rangle$ is an integral in x, (5.6) can be considered as a double integral. Thus the Fubini theorem allows us to interchange the order of integration in the variables λ and x. So we get

$$\langle f, g\rangle = \Big(\int_0^\infty u_f^{(\pm)}(\lambda)\, d\lambda, g\Big)$$

and therefore

$$f(x) = \int_0^\infty u_f^{(\pm)}(x,\lambda) d\lambda. \tag{5.11}$$

Since in (5.10), the g can be chosen as the characteristic functions of the balls $\{x \in \mathbb{R}^d : |x| \le \rho\}$, this ensures that the integral (5.11) converges (absolutely) for almost all $x \in \mathbb{R}^d$. Remark also that the integral (5.11) belongs to $\mathcal{H}$ although its integrand is from B^* only.

For arbitrary $f \in \mathcal{H}$, one has

$$Pf = s - \lim_{\mu\to 0, \nu\to\infty} f_{\mu,\nu}, \qquad f_{\mu,\nu} = E((\mu,\nu))f,$$

where $s - \lim$ means the strong limit in $\mathcal{H}$. By (5.3), $\tilde{f}_{\mu,\nu}(\lambda) = \tilde{f}(\lambda)$ if $\lambda \in (\mu,\nu)$, and $\tilde{f}_{\mu,\nu}(\lambda) = 0$ if $\lambda \notin (\mu,\nu)$. Now using (5.11) for $f_{\mu,\nu}$, we can reformulate the expansion theorem as follows

Theorem 5.3 *Let $f \in \mathcal{H}$ arbitrary, $\tilde{f}_\pm = F_\pm f$ and let $u_f^{(\pm)}(x,\lambda) = \mathcal{F}_\pm^*(\lambda)\tilde{f}_\pm(\lambda)$ be a solution of equation* (1.1). *Then*

$$Pf = s - \lim_{\mu\to 0, \nu\to\infty} \int_\mu^\nu u_f^{(\pm)}(x,\lambda) d\lambda, \tag{5.12}$$

and we have the generalized Parseval identity

$$||Pf||^2 = 2\pi \int_0^\infty \lim_{\rho\to\infty} \Big(\rho^{-1} \int_{|x|\le\rho} |u_f^{(\pm)}(x,\lambda)|^2\, dx\Big)\, \lambda^{1/2}\, d\lambda.$$

Conversely, for $\tilde{f}_\pm \in \tilde{\mathcal{H}}$ arbitrary, the right-hand side of (5.12) *is well defined, belongs to $\mathcal{H}^{(a)}$, and applying operator $F_\pm$ to this element, we recover $\tilde{f}_\pm$.*

6. TIME-DEPENDENT SCATTERING MATRIX

To find the asymptotics of $e^{-iHt}f$ as $t \to \pm\infty$, we proceed from the equality

$$e^{-iHt}f = \int_0^\infty e^{-i\lambda t}\mathcal{F}_\pm^*(\lambda)a_\pm(\lambda)d\lambda, \quad a_\pm(\lambda) = \mathcal{F}_\pm(\lambda)f,$$

following from Theorem 5.2. Recall that, for f suitably chosen, $\mathcal{F}_\pm^*(\lambda)a_\pm(\lambda)$ satisfies (3.22). The following proposition shows that the contribution of the term $R(\lambda \mp i0)g_\pm(\lambda)$ in the right-hand side of (3.22) disappears in the limit $t \to \pm\infty$. Its proof can be found in [8].

Proposition 6.1 *Let $f \in \mathcal{H}^{(a)}$. Suppose that $a_\pm(\lambda,\omega) = (F_\pm f)(\lambda,\omega) \in C^\infty(\mathbb{R}^+ \times \mathbb{S}^{d-1})$ satisfies $a_\pm(\lambda,\omega) = 0$ for $\lambda \le \mu$ and $\lambda \ge \nu$ where $0 < \mu < \nu < \infty$. Then*

$$(e^{-iHt} f)(x) = \mp i 2^{-1} \pi^{-1/2} |x|^{-(d-1)/2} \eta(x) I_\pm(x,t) + \epsilon_\pm(x,t), \tag{6.1}$$

where

$$I_\pm(x,t) = \int_0^\infty e^{-i\lambda t \pm i\varphi(x,\lambda)} \lambda^{-1/4} a_\pm(\lambda, \pm\hat{x}) d\lambda, \tag{6.2}$$

and $||\epsilon_\pm(\cdot,t)|| = o(1)$ as $t \to \pm\infty$.

If $|x| \le \alpha|t|$ or $|x| \ge \beta|t|$ for sufficiently small $\alpha = \alpha(\mu,\nu)$ and sufficienly large $\beta = \beta(\mu,\nu)$, then we can integrate (6.2) by parts

$$I_\pm(x,t) = -i \int_0^\infty e^{-i\lambda t \pm i\varphi(x,\lambda)} \partial_\lambda \Big((t \mp 2^{-1}\lambda^{-1/2}|x| \pm \Phi_\lambda(x,\lambda))^{-1} a_\pm(\lambda, \pm\hat{x}) \lambda^{-1/4} \Big) d\lambda.$$

In view of (3.2), this yields the estimate

$$|I_\pm(x,t)| \le C(|x| + |t|)^{-1}.$$

Therefore function (6.1) satisfies

$$\lim_{|t|\to\infty} \Big(\int_{|x|\le\alpha|t|} |(e^{-iHt}f)(x)|^2 dx + \int_{|x|\ge\beta|t|} |(e^{-iHt}f)(x)|^2 dx \Big) = 0.$$

In the regions $\alpha|t| \le |x| \le \beta|t|$ the asymptotics of integral (6.2) are determined by stationary points λ_s where λ_s are the solutions of equation

$$-t \pm \varphi_\lambda(x,\lambda) = 0, \tag{6.3}$$

or according to (3.3)

$$\lambda^{1/2} = (2|t|)^{-1}|x| - |t|^{-1}\Phi_\lambda(x,\lambda)\lambda^{1/2}. \tag{6.4}$$

Using again (3.2) and the method of successive approximations we obtain

Lemma 6.2 *For sufficiently large $|x|$, equation (6.4) has a unique solution $\lambda_s = \lambda_s(x,t)$ and*

$$\lambda_s(x,t) = (2|t|)^{-1}|x| + O(t^{-\delta}). \tag{6.5}$$

Set

$$\psi(x,\lambda,t) = -\lambda + |t|^{-1}\varphi(x,\lambda). \tag{6.6}$$

Applying now to integral (6.2) the stationary phase method (see e.g. [4]) we obtain that

$$I_\pm(x,t) = (2\pi)^{1/2}|t|^{-1/2}|\psi_{\lambda\lambda}|^{-1/2} e^{it\psi \pm \frac{\pi i}{4} sgn \psi_{\lambda\lambda}} a_\pm(\lambda_s(x,t), \pm\hat{x}) \lambda_s^{-1/4}(x,t) + \epsilon_\pm(x,t), \tag{6.7}$$

where

$$\psi = \psi(x,\lambda_s(x,t),t), \quad \psi_{\lambda\lambda} = \psi_{\lambda\lambda}(x,\lambda_s(x,t),t) \tag{6.8}$$

and the remainder $\epsilon_\pm(x,t)$ satisfies $||\epsilon_\pm(\cdot,t)|| = o(1)$ as $t \to \pm\infty$.

Differentiating (6.6) and taking into account (6.5), we find that

$$\psi_{\lambda\lambda}(x,\lambda_s,t) = -(4|t|)^{-1}|x|\lambda_s^{-3/2} - |t|^{-1}\Phi_{\lambda\lambda}(x,\lambda_s)) = -2|x|^{-2}|t|^2 + O(|t|^{-\delta}). \qquad (6.9)$$

In particular, $\operatorname{sgn}\psi_{\lambda\lambda} = -1$.

Let us set

$$\Omega(x,t) = t\psi(x,\lambda_s(x,t)) \pm \pi(d-3)/4, \quad \pm t > 0. \qquad (6.10)$$

Then, according again to (6.5) and (6.6),

$$\Omega(x,t) = -\lambda_s t \pm \lambda_s^{1/2}|x| \mp \Phi(x,\lambda_s) = \frac{x^2}{4t} + O(|t|^{1-\delta}). \qquad (6.11)$$

Plugging these expressions into (6.7) and taking into account (6.1), we see that

$$(e^{-iHt}f)(x) = 2^{1/2}(2it)^{-d/2}e^{i\Omega(x,t)}|(2t)^{-1}x|^{-(d-2)/2}a_\pm((4t^2)^{-1}x^2,\pm\hat{x}) + \epsilon_\pm(x,t), \quad (6.12)$$

where $||\epsilon_\pm(\cdot,t)|| = o(1)$ as $t \to \pm\infty$.

Let us check that function (6.10) satisfies the equation

$$\Omega_t + |\nabla\Omega|^2 + V_L = 0. \qquad (6.13)$$

Indeed, comparing (6.10) with (6.3) and (6.6), we see that

$$\Omega_t = -\lambda_s + t\frac{\partial\lambda_s}{\partial t} \pm \varphi_\lambda(x,\lambda_s)\frac{\partial\lambda_s}{\partial t} = -\lambda_s. \qquad (6.14)$$

Similarly,

$$\Omega_{x_i} = -t\frac{\partial\lambda_s}{\partial x_i} \pm \varphi_{x_i}(x,\lambda_s) \pm \varphi_\lambda(x,\lambda_s)\frac{\partial\lambda_s}{\partial x_i} = \pm\partial_{x_i}(x,\lambda_s). \qquad (6.15)$$

Now it follows from (6.14) and (6.15) that

$$\Omega_t(x,t) + |\nabla_x\Omega(x,t)|^2 = -\lambda_s(x,t) + |\nabla_x\varphi(x,\lambda_s(x,t))|^2.$$

Therefore (6.13) is an immediate consequence of the equation

$$|\nabla_x\varphi(x,\lambda)|^2 + V_L(x) = \lambda.$$

We summarize in the following lemma the results obtained.

Lemma 6.3 *Let $f \in \mathcal{H}^{(a)}$ be chosen as in Proposition* 6.1. *Then*

$$(e^{-iHt}f)(x) = 2^{1/2}(2it)^{-d/2}e^{i\Omega(x,t)}|(2t)^{-1}x|^{-(d-2)/2}a_\pm((4t^2)^{-1}x^2,\pm\hat{x}) + \epsilon_\pm(x,t) \quad (6.16)$$

where $\Omega(x,t)$ satisfies (1.9) *(up to a short-range term), and $||\epsilon_\pm(\cdot,t)|| = o(1)$ as $t \to \pm\infty$.*

We define now

$$(\mathbf{F}_\pm f)(\xi) = 2^{1/2}|\xi|^{-(d-2)/2}(F_\pm f)(|\xi|^2, \pm\xi/|\xi|). \qquad (6.17)$$

By Theorem 5.2, $\mathbf{F}_\pm$ is a partial isometry on $\mathcal{H}$ with the initial set $\mathcal{H}^{(a)}$ and final set $\mathcal{H}$. We also introduce a family of unitary operators by the equality

$$(U_0(t)f)(x) = e^{i\Omega(x,t)}(2it)^{-d/2}f((2t)^{-1}x), \qquad (6.18)$$

where $\Omega(x,t)$ is chosen as in Lemma 6.3. Thus Lemma 6.3 extends to an arbitrary $f \in \mathcal{H}^{(a)}$.

Proposition 6.4 *For any $f \in \mathcal{H}^{(a)}$,*

$$\lim_{t\to\infty} ||e^{-iHt}f - U_0(t)\mathbf{F}_{\pm}f|| = 0. \tag{6.19}$$

This implies the existence and completeness of the time-dependent wave operators constructed in [17].

Theorem 6.5 *The strong limits*

$$W_{\pm} = s - \lim_{t\to\pm\infty} e^{iHt}U_0(t)$$

exist and $W_{\pm} = \mathbf{F}_{\pm}^{*}$. In particular, the wave operators $W_{\pm}$ are complete, that is $R(W_{\pm}) = \mathcal{H}^{(a)}$.

Proof – For arbitrary $g \in \mathcal{H}$, set $f = \mathbf{F}_{\pm}^{*}g \in \mathcal{H}^{(a)}$ so that $g = \mathbf{F}_{\pm}f$. Then it follows from (6.19) that

$$\lim_{t\to\infty} ||\mathbf{F}_{\pm}^{*}g - e^{iHt}U_0(t)g|| = 0. \ \Box$$

Usually, the time-dependent scattering operator S is defined by $S = W_{+}^{*}W_{-}$. In a diagonal representation which reduces $H_0 = -\Delta$ to multiplication by λ, S reduces to multiplication by the operator function $\tilde{\mathcal{S}}(\lambda) : L^2(\mathbb{S}^{d-1}) \to L^2(\mathbb{S}^{d-1})$ which is called the time-dependent scattering matrix.

Proposition 6.6 *The time-dependent scattering matrix $\tilde{\mathcal{S}}(\lambda)$ coincides with the operator $\mathcal{S}(\lambda)$ defined by* (4.16).

Proof — Let us write $(\mathbf{F}_{\pm}f)(x) = (\mathbf{F}_{\pm}f)(\hat{x}, |x|)$. Then (6.19) implies

$$\tilde{\mathcal{S}}(\lambda) : (\mathbf{F}_{-}f)(\cdot, \lambda) \mapsto (\mathbf{F}_{+}f)(\cdot, \lambda),$$

According to (4.15) and (6.17), this corresponds to the definition of $\mathcal{S}(\lambda)$. $\Box$

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