Scaling Variables and Stability of Hyperbolic Fronts

Th. Gallay and G. Raugel

CNRS et Université de Paris-Sud
URA 760, Analyse Numérique et EDP
Bâtiment 425
F-91405 Orsay Cedex, France
Thierry.Gallay@math.u-psud.fr
Genevieve.Raugel@math.u-psud.fr

Abstract. We consider the damped hyperbolic equation

$$\varepsilon u_{tt} + u_t \;=\; u_{xx} + F(u) \;, \quad x \in \mathbf{R} \;, \quad t \geq 0 \;, \tag{1}$$

where ε is a positive, not necessarily small parameter. We assume that $F(0) = F(1) = 0$ and that F is concave on the interval $[0, 1]$. Under these hypotheses, Eq.(1) has a family of monotone travelling wave solutions (or propagating fronts) connecting the equilibria $u = 0$ and $u = 1$. This family is indexed by a parameter $c \geq c_*$ related to the speed of the front. In the critical case $c = c_*$, we prove that the travelling wave is asymptotically stable with respect to perturbations in a weighted Sobolev space. In addition, we show that the perturbations decay to zero like $t^{-3/2}$ as $t \to +\infty$ and approach a universal self-similar profile, which is independent of ε, F and of the initial data. In particular, our solutions behave for large times like those of the parabolic equation obtained by setting $\varepsilon = 0$ in Eq.(1). The proof of our results relies on careful energy estimates for the equation (1) rewritten in self-similar variables $x/\sqrt{t}$, $\log t$.

Keywords : damped wave equation, travelling wave, stability, asymptotic behavior, self-similar variables

AMS classification codes (1991) : 35B40, 35B35, 35B30, 35L05, 35C20

1. Introduction

In this paper, we study the asymptotic stability of travelling wave solutions to nonlinear damped hyperbolic equations on the real line. Besides describing the propagation of voltage along nonlinear transmission lines, these equations have been proposed as mathematical models for spreading and interacting particles [DO], [Ha2], [Ha3]. In the latter context, they provide an alternative to the reaction-diffusion systems which are very common in chemistry and biology, especially in genetics and population dynamics [Mu]. The two classes of models differ by the choice of the stochastic process describing the spatial spread of the individuals: instead of Brownian motion, the damped hyperbolic equations are based on a more realistic velocity jump process which takes into account the inertia of the particles [Go], [Kac], [Za]. Since this process is asymptotically diffusive, the long-time behavior of the solutions is expected to be essentially parabolic [GR2].

We study here the simple case of a scalar equation with a nonlinearity of "monostable" type. To be specific, we consider the equation

$$\varepsilon U_{\mathrm{tt}} + U_{\mathrm{t}} \;=\; U_{\mathrm{xx}} + \mathcal{F}(U)\;, \tag{1.1}$$

where $\mathrm{x} \in \mathbf{R}$, $\mathrm{t} \ge 0$, and ε is a positive, *not necessarily small* parameter. We assume that the nonlinearity $\mathcal{F} \in \mathcal{C}^2(\mathbf{R})$ satisfies

$$\mathcal{F}(0) = \mathcal{F}(1) = 0\;,\quad \mathcal{F}'(0) > 0\;,\quad \mathcal{F}'(1) < 0\;,\quad \mathcal{F}''(U) \le 0 \;\text{ for }\; U \in [0,1]\;. \tag{1.2}$$

In particular, $U = 1$ is a stable equilibrium of Eq.(1.1), and $U = 0$ is unstable. A typical nonlinearity satisfying (1.2) is $\mathcal{F}(U) = U - U^m$, with $m \ge 2$.

Under the assumptions (1.2), Eq.(1.1) has monotone travelling wave solutions (or propagating fronts) connecting the equilibrium states $U = 1$ and $U = 0$ [Ha1], [GR1]. Indeed, choosing $c > 0$ and setting $U(\mathrm{x},\mathrm{t}) = h(\sqrt{1+\varepsilon c^2}\mathrm{x} - c\mathrm{t})$, we obtain for h the ordinary differential equation

$$h''(\xi) + ch'(\xi) + \mathcal{F}(h(\xi)) \;=\; 0\;,\quad \xi \in \mathbf{R}\;. \tag{1.3}$$

Eq.(1.3) is known to have a strictly decreasing solution satisfying $h(-\infty) = 1$ and $h(+\infty) = 0$ if and only if $c \ge c_* = 2\sqrt{\mathcal{F}'(0)}$ [KPP], [AW]. This solution is unique up to translations in the variable ξ. Thus, Eq.(1.1) has a family of monotone travelling waves indexed by the speed parameter $c \geq c_*$. Note that the actual speed of the wave is not c, but $c/\sqrt{1+\varepsilon c^2}$, a quantity which is bounded by $1/\sqrt{\varepsilon}$ for all $c \geq c_*$.

In an earlier paper [GR1], we investigated the stability of the travelling waves of (1.1) in the case where $\mathcal{F}(U) = U - U^2$. In particular, we showed that, for all $\varepsilon > 0$ and all $c \geq c_*$, the front h is asymptotically stable with respect to small perturbations in a weighted Sobolev space (with exponential weight). This local stability result holds in fact for all nonlinearities satisfying (1.2), see [GR3]. In addition, if $\varepsilon > 0$ is sufficiently small, we proved in [GR1] that the front h is stable with respect to large perturbations, provided some positivity conditions are fulfilled. This global stability property relies on the hyperbolic Maximum Principle, and can also be extended to more general nonlinearities [GR3]. Finally, we showed in all cases that the perturbations converge uniformly to zero faster than $\mathrm{t}^{-1/4}$ as $\mathrm{t} \to +\infty$.

When $\varepsilon \to 0$, Eq.(1.1) reduces to the semilinear parabolic equation $U_{\mathrm{t}} = U_{\mathrm{xx}} + \mathcal{F}(U)$ which has been intensively studied since the pioneering works of Fisher [Fi] and Kolmogorov, Petrovskii and Piskunov [KPP]. Using the parabolic Maximum Principle and probabilistic techniques, the convergence of a large class of solutions to travelling waves has been established [AW], [Br]. In the more general context of parabolic systems, a local stability analysis of the waves has been initiated by Sattinger [Sa] and extended by many authors [Ki], [EW], [Kap], [BK1], [RK], using resolvent estimates, energy functionals and renormalization techniques. In the critical case $c = c_*$, it has been proved by one of us [Ga] that the perturbations of the front decay to zero like $\mathrm{t}^{-3/2}$ as $\mathrm{t} \to +\infty$ and approach a universal self-similar profile. The aim of the present paper is precisely to extend this detailed convergence result to the hyperbolic case $\varepsilon > 0$. Together with earlier results from [GR1], [GR3], this will provide a fairly complete picture of the stability properties of the travelling waves of Eq.(1.1).

To study the stability of the critical front h with $c = c_*$, it is convenient to go to a moving frame using the change of variables $U(\mathrm{x},\mathrm{t}) = V(\sqrt{1+\varepsilon c_*^2}\,\mathrm{x} - c_*\mathrm{t}, \mathrm{t})$. The equation for V is

$$\varepsilon V_{\mathrm{tt}} + V_{\mathrm{t}} - 2\varepsilon c_* V_{\xi\mathrm{t}} \;=\; V_{\xi\xi} + c_* V_\xi + \mathcal{F}(V)\ , \tag{1.4}$$

where $\xi = \sqrt{1+\varepsilon c_*^2}\,\mathrm{x} - c_*\mathrm{t}$. By construction, h is a stationary solution of (1.4). Following [Ki], [Ga], we consider perturbed solutions of the form

$$V(\xi,\mathrm{t}) \;=\; h(\xi) + w(\xi,\mathrm{t}) \;\equiv\; h(\xi) + h'(\xi) W\left(\xi, \frac{\mathrm{t}}{1+\varepsilon c_*^2}\right)\ . \tag{1.5}$$

The reason for this Ansatz is that the function $W(\xi,\tau)$ defined by (1.5) becomes asymptotically self-similar as $\mathrm{t} \to +\infty$, while the actual perturbation $w(\xi,\mathrm{t})$ does not, see Corollary 1.3 below. Remark that W is well-defined, since $h'(\xi) < 0$ for all $\xi \in \mathbf{R}$. For notational convenience, we rescale the time variable t by setting $\tau = \mathrm{t}/(1{+}\varepsilon c_*^2)$. The equation for W then reads

$$\eta W_{\tau\tau} + (1-\nu\gamma(\xi))W_\tau - 2\nu W_{\xi\tau} \;=\; W_{\xi\xi} + \gamma(\xi)W_\xi + h'(\xi)W^2\mathcal{N}(h(\xi),h'(\xi)W)\;, \tag{1.6}$$

where

$$\eta \;=\; \frac{\varepsilon}{(1+\varepsilon c_*^2)^2}\;, \quad \nu \;=\; \frac{\varepsilon c_*}{1+\varepsilon c_*^2}\;, \quad \gamma(\xi) \;=\; c_* + 2\frac{h''(\xi)}{h'(\xi)}\;, \tag{1.7}$$

and

$$\mathcal{N}(a,b) \;=\; \int_0^1 (1{-}s)\mathcal{F}''(a+sb)\,\mathrm{d}s \;=\; \frac{1}{b^2}\bigl(\mathcal{F}(a+b)-\mathcal{F}(a)-b\mathcal{F}'(a)\bigr)\;. \tag{1.8}$$

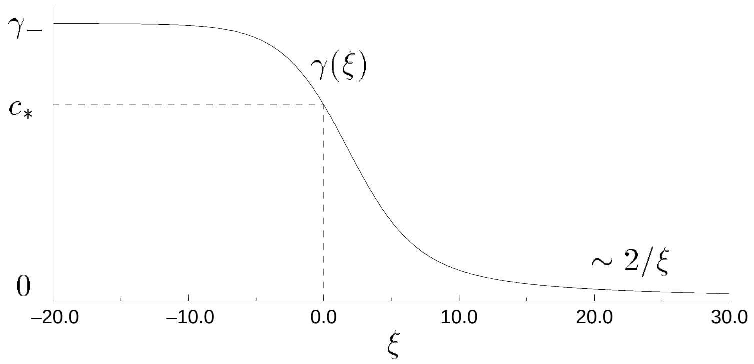

Fig. 1: The function $\gamma(\xi)$ in the case where $\mathcal{F}(U) = U - U^2$ (hence $c_* = 2$, $\gamma_- = 2\sqrt{2}$).

Before analyzing the solutions of (1.6), we briefly comment on the definitions (1.7). We first remark that there is no loss of generality in assuming $\varepsilon = 1$ in Eq.(1.1), since $(\varepsilon,\mathcal{F})$ can be transformed into $(1,\varepsilon\mathcal{F})$ by rescaling x and t. However, we find more convenient to fix the nonlinearity $\mathcal{F}$ and to consider ε as a free parameter. Then $c_* > 0$ is fixed, and η,ν are functions of ε only. These expressions are not independent, since $\nu^2 + \eta = \nu/c_*$. Observe also that η,ν are uniformly bounded for all $\varepsilon > 0$, and converge to zero as $\varepsilon \to 0$. We now list the properties of the "drift" $\gamma(\xi)$ which will be crucial for our analysis. From [Sa], [AW], we know that the front h (with $c = c_*$) satisfies

$$h(\xi) \;=\; \begin{cases} 1 - a_3\mathrm{e}^{\kappa\xi} + \mathcal{O}(\mathrm{e}^{2\kappa\xi}) & \text{as } \xi \to -\infty\;, \\ (a_1\xi{+}a_2)\mathrm{e}^{-c_*\xi/2} + \mathcal{O}(\xi^2\mathrm{e}^{-c_*\xi}) & \text{as } \xi \to +\infty\;, \end{cases} \tag{1.9}$$

where $a_1, a_3 > 0$, $a_2 \in \mathbf{R}$, and $\kappa = \frac{1}{2}(-c_* + \sqrt{c_*^2 - 4\mathcal{F}'(1)}) > 0$. Using (1.9) and similar asymptotic expansions for the derivatives h', h'', we obtain

$$\gamma(\xi) \;=\; \begin{cases} \gamma_- + \mathcal{O}(\mathrm{e}^{\kappa\xi}) & \text{as } \xi \to -\infty \;, \\ 2/(\xi+\xi_0) + \mathcal{O}(\xi \mathrm{e}^{-c_*\xi/2}) & \text{as } \xi \to +\infty \;, \end{cases} \tag{1.10}$$

where $\gamma_- = c_* + 2\kappa = 2\sqrt{\mathcal{F}'(0) - \mathcal{F}'(1)}$ and $\xi_0 = (a_2/a_1 - 2/c_*)$. It also follows from (1.3), (1.7) that

$$\gamma'(\xi) \;=\; -\frac{1}{2}\gamma(\xi)^2 + 2\big(\mathcal{F}'(0) - \mathcal{F}'(h(\xi))\big) \;, \quad \xi \in \mathbf{R} \;. \tag{1.11}$$

Together with (1.2), this equation implies that $-\frac{1}{2}\gamma(\xi)^2 \le \gamma'(\xi) \le 0$ for all $\xi \in \mathbf{R}$. Indeed, the lower bound on $\gamma'(\xi)$ is obvious, and the upper bound follows from the inequality $\gamma''(\xi) + \gamma(\xi)\gamma'(\xi) \le 0$ obtained by differentiating (1.11). In fact, we even have $\gamma'(\xi) < 0$ whenever $\gamma(\xi) < \gamma_-$. Replacing thus $h(\xi)$ by a translate, we may (and will always) assume that $\gamma(0) = c_*$, *i.e.* $h''(0) = 0$, see Fig. 1. This amounts to fixing the origin in the moving frame.

To study the behavior of the solutions W of (1.6), we use the *scaling variables* or *self-similar variables* defined by

$$x \;=\; \frac{\xi}{\sqrt{\tau+\tau_0}} \;, \quad t = \log(\tau+\tau_0) \;, \tag{1.12}$$

where $\tau_0 > 0$ will be fixed later. These variables have been widely used to investigate the long time behavior of solutions to parabolic equations, in particular to prove convergence to self-similar solutions [Kav], [EZ], [GV], [EKM], [BK2], [Wa], [GM]. Although the scaling (1.12) is parabolic in essence, we have shown in [GR2] that self-similar variables are also a powerful tool in the realm of damped hyperbolic equations. The reason is that the long-time behavior of such systems is often determined by simpler parabolic equations, see [HL], [Ni], [GR2] for specific examples of this phenomenon. In our case, the result of [Ga] in the parabolic limit $\varepsilon = 0$ suggests that $W(\xi,\tau)$ should behave like $\tau^{-3/2}\varphi^*(\xi/\sqrt{\tau})$ as $\tau \to +\infty$, where φ^* is given by (1.20) below. Thus, following the method developped in [GR2], we define rescaled functions u and v by

$$u(x,t) \;=\; \mathrm{e}^{3t/2} W(x\mathrm{e}^{t/2}, \mathrm{e}^t - \tau_0) \;, \quad v(x,t) \;=\; \mathrm{e}^{5t/2} W_\tau(x\mathrm{e}^{t/2}, \mathrm{e}^t - \tau_0) \;, \tag{1.13}$$

or equivalently

$$\begin{aligned} W(\xi,\tau) \;&=\; \frac{1}{(\tau+\tau_0)^{3/2}}\, u\left(\frac{\xi}{\sqrt{\tau+\tau_0}}, \log(\tau+\tau_0)\right) \;, \\ W_\tau(\xi,\tau) \;&=\; \frac{1}{(\tau+\tau_0)^{5/2}}\, v\left(\frac{\xi}{\sqrt{\tau+\tau_0}}, \log(\tau+\tau_0)\right) \;. \end{aligned} \tag{1.14}$$

Then the functions $u(x,t), v(x,t)$ satisfy the system

$$
\begin{aligned}
& u_t - \frac{x}{2}u_x - \frac{3}{2}u \;=\; v \;, \\
& \eta e^{-t}\big(v_t - \frac{x}{2}v_x - \frac{5}{2}v\big) + (1-\nu\gamma(xe^{t/2}))v - 2\nu e^{-t/2}v_x \;= \\
& \qquad u_{xx} + e^{t/2}\gamma(xe^{t/2})u_x + e^{-t/2}h'(xe^{t/2})u(x,t)^2 N(x,t) \;,
\end{aligned}
\tag{1.15}
$$

where $x \in \mathbf{R}$, $t \ge t_0 = \log\tau_0$, and $N(x,t) = \mathcal{N}(h(xe^{t/2}), e^{-3t/2}h'(xe^{t/2})u(x,t))$.

We next introduce the function spaces in which we shall study the solutions of (1.15). For $t \ge 0$, $k \in \mathbf{N}$, we denote by L^2_t, H^k_t the weighted Lebesgue and Sobolev spaces defined by the norms

$$
\begin{aligned}
\|u\|^2_{\mathrm{L}^2_t} \;&=\; \int_{-\infty}^0 e^{2\kappa x e^{t/2}}|u(x)|^2\,\mathrm{d}x + \int_0^\infty (1{+}x)^6|u(x)|^2\,\mathrm{d}x \;, \\
\|u\|^2_{\mathrm{H}^k_t} \;&=\; \sum_{i=0}^k \|\partial_x^i u\|^2_{\mathrm{L}^2_t} \;,
\end{aligned}
\tag{1.16}
$$

where κ appears in (1.9). Our basic space will be the product $\mathrm{Z}_t = \mathrm{H}^1_t \times \mathrm{L}^2_t$ equipped with the standard norm $\|(u,v)\|_{\mathrm{Z}_t} = (\|u\|^2_{\mathrm{H}^1_t} + \|v\|^2_{\mathrm{L}^2_t})^{1/2}$. In order to state results which are uniform in ε as $\varepsilon \to 0$, it is convenient to introduce also the quadratic form

$$
\Phi_\eta(t,u,v) \;=\; \|u\|^2_{\mathrm{H}^1_t} + \eta e^{-t}\|v\|^2_{\mathrm{L}^2_t} \;. \tag{1.17}
$$

From (1.13), (1.14), we see that $(u,v) \in \mathrm{Z}_t$ if and only if $(W, W_\tau) \in \mathrm{Z}_0 = \mathrm{H}^1_0 \times \mathrm{L}^2_0$. Moreover, since $h', h'' = \mathcal{O}(e^{\kappa\xi})$ as $\xi \to -\infty$ and $h', h'' = \mathcal{O}(\xi e^{-c_*\xi/2})$ as $\xi \to +\infty$, it is easy to verify that $(W, W_\tau) \in \mathrm{Z}_0$ if and only if the actual perturbation $w = h'W$ satisfies

$$
\int_{-\infty}^0 (w^2 + w_\xi^2 + w_{\mathrm{t}}^2)\,\mathrm{d}\xi + \int_0^\infty (1+\xi)^4 e^{c_*\xi}(w^2 + w_\xi^2 + w_{\mathrm{t}}^2)\,\mathrm{d}\xi \;<\; \infty \;. \tag{1.18}
$$

The comparison of (1.9), (1.18) reveals that the perturbations we consider decay to zero slightly faster than the front h itself as $\xi \to +\infty$. This is a necessary condition for stability, because the equilibrium state $U = 0$ of (1.1) is linearly unstable [Sa]. In particular, small translations of the front h are *not* allowed as perturbations.

Since our function space Z_t depends on time, we have to specify what we mean by a "solution of (1.15) in Z_t". As the system (1.15) has been obtained from the simpler equation (1.6) through the change of variables (1.13), the following definition is very natural:

Definition 1.1. Let $t_2 > t_1 \geq t_0$, and let $\tau_i = \mathrm{e}^{t_i} - \tau_0$ for $i = 1, 2$. We say that "$(u, v) \in \mathcal{C}([t_1, t_2], \mathrm{Z}_t)$ is a solution of the system (1.15)" if there exists a (mild) solution $(W, W_\tau) \in \mathcal{C}([\tau_1, \tau_2], \mathrm{Z}_0)$ of (1.6) such that the relations (1.13), (1.14) hold.

In particular, if $(u, v) \in \mathcal{C}([t_1, t_2], \mathrm{Z}_t)$ is a solution of (1.15), then $(u(t), v(t)) \in \mathrm{Z}_t$ for all $t \in [t_1, t_2]$. However, the continuity of (u, v) with respect to t has to be understood as the continuity in Z_0 of the functions (W, W_τ) defined by (1.14). In Proposition 2.2 below, we shall show that the Cauchy problem for (1.15) in Z_t is locally well-posed.

Before stating our main result, we explain its content in a heuristic way. Taking formally the limit $t \to +\infty$ in (1.15) and using (1.10), we see that u satisfies the linear parabolic equation

$$u_t = \mathcal{L}_\infty u \stackrel{\mathrm{def}}{=} u_{xx} + \Big(\frac{x}{2} + \frac{2}{x}\Big) u_x + \frac{3}{2} u \quad \text{if } x > 0 \;, \quad u_x = 0 \quad \text{if } x \leq 0 \;. \tag{1.19}$$

Therefore, it is reasonable to expect that the long-time behavior of the solutions of (1.15) is determined by the spectral properties of the operator $\mathcal{L}_\infty$ on $\mathbf{R}_+$, with Neumann boundary condition at $x = 0$. Now, as is easily verified, this limiting operator is just the image under the scaling (1.14) of the radially symmetric Laplace operator in three dimensions. Indeed, if u and W are related through (1.14), the equation $u_t = \mathcal{L}_\infty u$ is equivalent to $W_\tau = W_{\xi\xi} + (2/\xi) W_\xi$, $\xi > 0$. This crucial observation explains the factor $(\tau+\tau_0)^{-3/2}$ in (1.14), and allows to compute exactly the spectrum of $\mathcal{L}_\infty$ in various function spaces, see [GR2, Appendix A]. For instance, in the space $\mathrm{H}^1(\mathbf{R}_+, (1+x)^6 \mathrm{d}x)$, the spectrum of $\mathcal{L}_\infty$ consists of a simple, isolated eigenvalue at $\lambda = 0$, and of "continuous" spectrum filling the half-plane $\{\lambda \in \mathbf{C} \,|\, \mathrm{Re}\, \lambda \leq -1/4\}$. The eigenfunction corresponding to $\lambda = 0$ is the Gaussian $\mathrm{e}^{-x^2/4}$. Therefore, we expect that the solution $u(x, t)$ of (1.15) converges as $t \to +\infty$ to $\alpha \varphi^*(x)$ for some $\alpha \in \mathbf{R}$, where

$$\varphi^*(x) = \frac{1}{\sqrt{4\pi}} \begin{cases} 1 & \text{if } x < 0 \;, \\ \mathrm{e}^{-x^2/4} & \text{if } x \geq 0 \;. \end{cases} \tag{1.20}$$

This function is normalized so that $\int_0^\infty x^2 \varphi^*(x) \,\mathrm{d}x = 1$. Since $v = u_t - \frac{x}{2} u_x - \frac{3}{2} u$, we also expect that $v(x, t)$ converges to $\alpha \psi^*(x)$, where $\psi^* = -\frac{x}{2} \varphi^*_x - \frac{3}{2} \varphi^*$. It is crucial to note that Eq.(1.19) is independent of ε: this explains why the solutions of (1.6), hence of (1.1), behave for large times like those of the corresponding parabolic equations.

Our main result shows that these heuristic considerations are indeed correct:

Theorem 1.2. *Assume that the nonlinearity $\mathcal{F}$ satisfies (1.2), and let $\varepsilon > 0$. There exist $t_0 > 0$, $\delta_0 > 0$ and $C > 0$ such that, for all initial data $(u_0, v_0) \in \mathrm{Z}_{t_0}$ with $\Phi_\eta(t_0, u_0, v_0) \le \delta_0^2$, the system (1.15) has a unique solution $(u,v) \in \mathcal{C}([t_0,+\infty), \mathrm{Z}_t)$ satisfying $(u(t_0), v(t_0)) = (u_0, v_0)$. In addition, there exists $\alpha^* \in \mathbf{R}$ such that, for all $t \ge t_0$,*

$$\begin{aligned}\|u(t) - \alpha^*\varphi^*\|^2_{\mathrm{H}^1_t} + \eta \mathrm{e}^{-t}\|v(t) - \alpha^*\psi^*\|^2_{\mathrm{L}^2_t} + \int_{t_0}^t \mathrm{e}^{-(t-s)/2}\|v(s) - \alpha^*\psi^*\|^2_{\mathrm{L}^2_s}\,\mathrm{d}s& \\ \le C(1+t)^2\mathrm{e}^{-t/2}\Phi_\eta(t_0,u_0,v_0)&\;.\end{aligned} \tag{1.21}$$

Remarks.

1. In the proof of Theorem 1.2, we shall take for convenience the parameter $t_0 = \log(\tau_0)$ large enough, but this choice is irrelevant since, as reflected in Corollary 1.3 below, the results for the original equation (1.1) are not affected.

2. The estimate (1.21) shows in particular that the solution $u(t)$ converges to $\alpha^*\varphi^*$ like $t\mathrm{e}^{-t/4}$ as $t \to +\infty$. As was already mentioned, the decay rate $\mathrm{e}^{-t/4}$ corresponds to the spectral gap of the linear operator $\mathcal{L}_\infty$ in $\mathrm{H}^1(\mathbf{R}_+, (1+x)^6\,\mathrm{d}x)$, and is thus optimal in our function space. The same argument suggests that this rate could be improved up to $\mathrm{e}^{-t/2}$ at the expense of assuming a faster decay of u, v as $x \to +\infty$, as in [Ga].

3. Theorem 1.2 does not give a satisfactory estimate of the term $\|v(t) - \alpha^*\psi^*\|^2_{\mathrm{L}^2_t}$. If ε is sufficiently small, using three additional pairs of functionals as in Section 3, one can show that $\int_0^\infty (x+x^2)|v(x,t) - \alpha^*\psi^*(x)|^2\,\mathrm{d}x$ decays at least like $(1+t)^2\mathrm{e}^{-t/2}$ and that $\int_{-\infty}^0 \mathrm{e}^{2\kappa x \mathrm{e}^{t/2}}|v(x,t) - \alpha^*\psi^*(x)|^2\,\mathrm{d}x + \int_0^\infty |v(x,t) - \alpha^*\psi^*(x)|^2\,\mathrm{d}x$ is bounded by a polynomial in t. Since these estimates are probably not optimal and were obtained for small ε only, the corresponding calculations will not be given here.

4. Given $\varepsilon_0 > 0$ and a nonlinearity $\mathcal{F}$ satisfying (1.2), it is straightforward to verify that all the statements in the sequel (and their proofs) hold uniformly in ε for $\varepsilon \in (0, \varepsilon_0]$. In particular, the constants t_0, δ_0, C appearing in Theorem 1.2 are independent of ε for $\varepsilon \in (0, \varepsilon_0]$. As a consequence, taking the limit $\varepsilon \to 0$ in (1.21), we obtain a local stability result for the travelling waves of the parabolic equation (1.1) with $\varepsilon = 0$. Except for the use of slightly different function spaces, this result coincides with Theorem 1.1 of [Ga].

Combining Theorem 1.2 and Lemma 2.4 below, we obtain in particular the following convergence result for the perturbation in the original variables:

Corollary 1.3. *Under the assumptions of Theorem 1.2, the following estimate holds:*

$$\sup_{\xi\in\mathbf{R}}(1+\mathrm{e}^{-\kappa\xi})^{-1}\left|W(\xi,\tau)-\frac{\alpha^*}{\tau^{3/2}}\,\varphi^*\Big(\frac{\xi}{\sqrt{\tau}}\Big)\right| \;=\; \mathcal{O}(\tau^{-7/4}\log\tau)\;,$$

as $\tau\to+\infty$, where $W(\xi,\tau)$ is given by (1.14). Equivalently,

$$\sup_{\xi\in\mathbf{R}}\left(1+\frac{\mathrm{e}^{c_*\xi/2}}{1+|\xi|}\right)\left|w(\xi,\mathrm{t})-\frac{\alpha}{\mathrm{t}^{3/2}}\,h'(\xi)\varphi^*\Big(\frac{\xi\sqrt{1+\varepsilon c_*^2}}{\sqrt{\mathrm{t}}}\Big)\right| \;=\; \mathcal{O}(\mathrm{t}^{-7/4}\log\mathrm{t})\;,$$

as $\mathrm{t}\to+\infty$, where $\alpha=\alpha^*(1+\varepsilon c_*^2)^{3/2}$ and $w(\xi,\mathrm{t})$ is given by (1.5).

The rest of this paper is devoted to the proof of Theorem 1.2, which is organized as follows. First of all, we prove that the Cauchy problem for Eq.(1.15) is locally well-posed in the space Z_t, in the sense of Definition 1.1. Then, in Section 2.1, we decompose the solutions (u,v) of (1.15) using an approximate spectral projection of the time-dependent operator $\mathcal{L}_t$ defined in (2.3) below. The first term in this decomposition is one-dimensional and converges to $\alpha^*(\varphi^*,\psi^*)$ as $t\to+\infty$. The remainder (f,g) satisfies an evolution system similar to (1.15), with additional terms which are estimated in Section 2.2. The core of the proof is Section 3, where the evolution of (f,g) in Z_t is controlled using a hierarchy of energy functionals. As in [GR2], some of these quantities are constructed in terms of the primitives (F,G) rather than the functions (f,g) themselves. Finally, the results are summarized in the short Section 4.

Although the proof we present here is certainly not simple, we believe that our approach is systematic and very well adapted to study the long-time asymptotics in a large class of dissipative systems. As a matter of fact, the present proof follows exactly the same lines as in [GR2], although the problems considered are significantly different. When compared with other accurate techniques, such as the Renormalization Group used in [BK1] and [Ga], our method shows at least two advantages. First, we do not need precise estimates of the resolvent of the linearized operator around the travelling wave (although some spectral information is used to construct our energy functionals). This substantial simplification is especially interesting in the perspective of possible applications to higher-dimensional problems, where standard tools like the Evans function are not available. Next, while most of our effort is devoted to controlling the linear terms in (1.15), the nonlinearities are naturally incorporated into the scheme and do not require any extra argument. In the present case, the factor $\mathrm{e}^{-t/2}$ in front of the last term in (1.15) clearly shows that the nonlinearity is irrelevant for the long-time behavior, provided the solution $u(t)$ stays globally bounded. On the other hand, a minor drawback of our approach is the introduction of non-autonomous systems and time-dependent function spaces through the change of variables (1.13). We shall avoid this difficulty by returning to the original variables to show that the Cauchy problem for (1.15) is locally well-posed and to prove that our energy functionals are differentiable in time.

Notation. In the sequel, we denote by C a generic positive constant which may differ from place to place, while numbered constants $C_i, K_i, \ldots$ keep the same value throughout the paper.

2. Preliminaries

We begin with a local existence result for the solutions W of (1.6) in the function space $\mathrm{Z}_0 = \mathrm{H}^1_0 \times \mathrm{L}^2_0$. We recall that H^1_0, L^2_0 are defined by the norms (1.16) with $t = 0$.

Lemma 2.1. *Let $\eta > 0$ and $\delta > 0$. There exists $\hat{\tau} > 0$ such that, for all initial data $(W_0, \dot{W}_0) \in \mathrm{Z}_0$ with $\|(W_0, \dot{W}_0)\|_{\mathrm{Z}_0} \le \delta$, Eq.(1.6) has a unique (mild) solution $W \in \mathcal{C}([0,\hat{\tau}], \mathrm{H}^1_0) \cap \mathcal{C}^1([0,\hat{\tau}], \mathrm{L}^2_0)$ satisfying $(W(0), W_\tau(0)) = (W_0, \dot{W}_0)$. The solution (W, W_τ) depends continuously on the initial data in Z_0, uniformly in $\tau \in [0,\hat{\tau}]$. In addition, if $(W_0, \dot{W}_0) \in \mathrm{H}^2_0 \times \mathrm{H}^1_0$, then $W \in \mathcal{C}([0,\hat{\tau}], \mathrm{H}^2_0) \cap \mathcal{C}^1([0,\hat{\tau}], \mathrm{H}^1_0) \cap \mathcal{C}^2([0,\hat{\tau}], \mathrm{L}^2_0)$ is a classical solution of Eq.(1.6) in L^2_0.*

Proof. Let $q \in \mathcal{C}^\infty(\mathbf{R})$ be a positive function satisfying $q(\xi) = \mathrm{e}^{-\kappa\xi}$ for $\xi \le 0$ and $q(\xi) = \xi^{-3}$ for $\xi \ge 1$. Setting $W(\xi,\tau) = q(\xi)\omega(\xi,\tau)$ in (1.6), we obtain for ω the equation

$$\eta\omega_{\tau\tau} - 2\nu\omega_{\xi\tau} = \omega_{\xi\xi} + \mathcal{M}(\omega, \omega_\xi, \omega_\tau) \;, \tag{2.1}$$

where

$$\begin{aligned} \mathcal{M}(\omega,\omega_\xi,\omega_\tau) \;=\; &-\left(1 - \nu\gamma - 2\nu\frac{q'}{q}\right)\omega_\tau + \left(\gamma + \frac{2q'}{q}\right)\omega_\xi + \left(\gamma\frac{q'}{q} + \frac{q''}{q}\right)\omega \\ &+ h'q\omega^2\mathcal{N}(h, h'q\omega) \;. \end{aligned} \tag{2.2}$$

Since the functions γ, q'/q, q''/q and $h'q$ are all bounded, and since the nonlinearity $\mathcal{F}$ in (1.1) is $\mathcal{C}^2$, it is straightforward to verify that the map $\mathcal{M} : \mathrm{H}^1(\mathbf{R}) \times \mathrm{L}^2(\mathbf{R}) \to \mathrm{L}^2(\mathbf{R})$ defined by $(\omega, \omega_\tau) \mapsto \mathcal{M}(\omega, \omega_\xi, \omega_\tau)$ is locally Lipschitz, uniformly on bounded subsets. Therefore, by a classical result [CH], the Cauchy problem for (2.1) is locally well-posed in $\mathrm{H}^1 \times \mathrm{L}^2$. More precisely, for any $r > 0$, there exists $\hat{\tau} > 0$ such that, for all initial data $(\omega_0, \dot{\omega}_0) \in \mathrm{H}^1 \times \mathrm{L}^2$ with $\|(\omega_0, \dot{\omega}_0)\|_{\mathrm{H}^1 \times \mathrm{L}^2} \le r$, (2.1) has a unique (mild) solution $\omega \in \mathcal{C}([0,\hat{\tau}], \mathrm{H}^1) \cap \mathcal{C}^1([0,\hat{\tau}], \mathrm{L}^2)$ satisfying $(\omega(0), \omega_\tau(0)) = (\omega_0, \dot{\omega}_0)$. This solution depends continuously on the initial data in $\mathrm{H}^1 \times \mathrm{L}^2$, uniformly in $\tau \in [0, \hat{\tau}]$. Moreover, if $(\omega_0, \dot{\omega}_0) \in \mathrm{H}^2 \times \mathrm{H}^1$, then $\omega \in \mathcal{C}([0,\hat{\tau}], \mathrm{H}^2) \cap \mathcal{C}^1([0,\hat{\tau}], \mathrm{H}^1) \cap \mathcal{C}^2([0,\hat{\tau}], \mathrm{L}^2)$ is a classical solution of Eq.(2.1). Thus, returning to the original function $W = q\omega$ and using the fact that

$$C^{-1}\|(\omega, \omega_\tau)\|_{\mathrm{H}^1 \times \mathrm{L}^2} \le \|(W, W_\tau)\|_{\mathrm{Z}_0} \le C\|(\omega, \omega_\tau)\|_{\mathrm{H}^1 \times \mathrm{L}^2} ,$$

for some $C \ge 1$, we obtain the desired result, if $r = C\delta$. This concludes the proof of Lemma 2.1. □

As a consequence of Definition 1.1 and Lemma 2.1, we obtain the following existence result for the solution (u, v) of (1.15):

Proposition 2.2. *Let $\eta > 0$, $\delta_1 > 0$, $t_2 > t_0$. There exists $T > 0$ such that, for all $t_1 \in [t_0, t_2]$ and all $(u_1, v_1) \in \mathrm{Z}_{t_1}$ satisfying $\Phi_\eta(t_1, u_1, v_1) \le \delta_1^2$, the system (1.15) has a unique solution $(u, v) \in \mathcal{C}([t_1, t_1 + T], \mathrm{Z}_t)$ with initial data $(u(t_1), v(t_1)) = (u_1, v_1)$.*

Remark. In particular, Proposition 2.2 implies that, if $(u, v) \in \mathcal{C}([t_0, t_*), \mathrm{Z}_t)$ is a maximal solution of (1.15) and if $\Phi_\eta(t, u(t), v(t)) \le \delta_1^2$ for all $t \in [t_0, t_*)$, then actually $t_* = +\infty$, *i.e.* the solution (u, v) is globally defined.

Proof. Given $t_1 \in [t_0, t_2]$ and $(u_1, v_1) \in \mathrm{Z}_{t_1}$ satisfying $\Phi_\eta(t_1, u_1, v_1) \le \delta_1^2$, we define

$$W_1(\xi) = \mathrm{e}^{-3t_1/2} u_1(\xi \mathrm{e}^{-t_1/2}) , \quad \dot{W}_1(\xi) = \mathrm{e}^{-5t_1/2} v_1(\xi \mathrm{e}^{-t_1/2}) , \quad \xi \in \mathbf{R} .$$

Then $(W_1, \dot{W}_1) \in \mathrm{Z}_0$, and there exists a constant $C > 0$ (depending on η and t_2) such that $\|(W_1, \dot{W}_1)\|_{\mathrm{Z}_0} \le C\delta_1$. Since Eq.(1.6) is autonomous, it follows from Lemma 2.1 that there exists a time $\hat{\tau} > 0$, depending on η, $C\delta_1$ but not on $(W_1, \dot{W}_1)$, such that (1.6) has a unique (mild) solution $W \in \mathcal{C}([\mathrm{e}^{t_1}, \mathrm{e}^{t_1} + \hat{\tau}], \mathrm{H}_0^1) \cap \mathcal{C}^1([\mathrm{e}^{t_1}, \mathrm{e}^{t_1} + \hat{\tau}], \mathrm{L}_0^2)$ satisfying $W(\xi, \mathrm{e}^{t_1}) = W_1(\xi)$, $W_\tau(\xi, \mathrm{e}^{t_1}) = \dot{W}_1(\xi)$. Now, we set $T = \log(1 + \hat{\tau}\mathrm{e}^{-t_2})$, and for all $t \in [t_1, t_1 + T] \subset [t_1, \log(\mathrm{e}^{t_1} + \hat{\tau})]$ we define

$$u(x, t) = \mathrm{e}^{3t/2} W(x\mathrm{e}^{t/2}, \mathrm{e}^t) , \quad v(x, t) = \mathrm{e}^{5t/2} W_\tau(x\mathrm{e}^{t/2}, \mathrm{e}^t) .$$

By Definition 1.1, $(u, v) \in \mathcal{C}([t_1, t_1 + T], \mathrm{Z}_1)$ is a solution of (1.15) with $(u(t_1), v(t_1)) = (u_1, v_1)$, and the uniqueness of this solution follows from the uniqueness of W as a mild solution of (1.6). This concludes the proof of Proposition 2.2. □

2.1. Spectral Decomposition of the Solution

From now on, we assume that $(u,v) \in \mathcal{C}([t_0,t_1],\mathrm{Z}_t)$ is a solution of (1.15) in the sense of Proposition 2.2. Inspired by [Ga] and [GR2], we shall decompose this solution using an approximate spectral projection of the (time-dependent) linear operator

$$\mathcal{L}_t \;=\; \partial_x^2 + \Bigl(\frac{x}{2} + \mathrm{e}^{t/2}\gamma(x\mathrm{e}^{t/2})\Bigr)\,\partial_x + \frac{3}{2}\;, \tag{2.3}$$

which appears in (1.15). As is easily verified, the function φ^* defined in (1.20) is an approximate eigenfunction of $\mathcal{L}_t$, in the sense that $\|\mathcal{L}_t\varphi^*\|_{\mathrm{L}^2_t} = \mathcal{O}(\mathrm{e}^{-t/4})$ as $t \to +\infty$. The corresponding approximate spectral projection in L^2_t is given by the formula

$$u \mapsto \Bigl(\int_{\mathbf{R}} \mathrm{e}^{-t}p(x\mathrm{e}^{t/2})u(x)\,\mathrm{d}x\Bigr)\,\varphi^* \tag{2.4}$$

where $p : \mathbf{R} \to \mathbf{R}$ is the (unique) solution of the differential problem

$$p'(\xi) \;=\; \gamma(\xi)p(\xi)\;, \quad \xi \in \mathbf{R}\;, \quad \lim_{\xi\to+\infty}\frac{p(\xi)}{\xi^2} \;=\; 1\;. \tag{2.5}$$

It follows from (1.10), (2.5) that $p(\xi) > 0$ for all $\xi \in \mathbf{R}$, and $p(\xi) = \mathcal{O}(\mathrm{e}^{\gamma_-\xi})$ as $\xi \to -\infty$.

Motivated by (2.4), we introduce the functions

$$\varphi(x,t) \;=\; \frac{\varphi^*(x)}{1+\zeta(t)}\;, \quad \psi(x,t) \;=\; \varphi_t(x,t) - \frac{x}{2}\varphi_x(x,t) - \frac{3}{2}\varphi(x,t)\;, \tag{2.6}$$

where

$$\zeta(t) \;=\; \int_{\mathbf{R}} \mathrm{e}^{-t}p(x\mathrm{e}^{t/2})\varphi^*(x)\,\mathrm{d}x - 1\;. \tag{2.7}$$

We shall show in the proof of Lemma 2.5 below that $\zeta(t)$ and $\zeta'(t)$ converge to zero as $t \to +\infty$, so that $\varphi(x,t) \to \varphi^*(x)$ and $\psi(x,t) \to \psi^*(x)$. By construction, we also have

$$\int_{\mathbf{R}} \mathrm{e}^{-t}p(x\mathrm{e}^{t/2})\varphi(x,t)\,\mathrm{d}x \;=\; 1\;, \quad \int_{\mathbf{R}} \mathrm{e}^{-t}p(x\mathrm{e}^{t/2})\psi(x,t)\,\mathrm{d}x \;=\; 0\;, \quad t \ge 0\;. \tag{2.8}$$

Using these notations, we decompose the solution (u,v) of (1.15) as

$$u(x,t) \;=\; \alpha(t)\varphi(x,t) + f(x,t)\;, \quad v(x,t) \;=\; \beta(t)\varphi(x,t) + \alpha(t)\psi(x,t) + g(x,t)\;, \tag{2.9}$$

where

$$\alpha(t) \;=\; \int_{\mathbf{R}} \mathrm{e}^{-t}p(x\mathrm{e}^{t/2})u(x,t)\,\mathrm{d}x\;, \quad \beta(t) \;=\; \int_{\mathbf{R}} \mathrm{e}^{-t}p(x\mathrm{e}^{t/2})v(x,t)\,\mathrm{d}x\;. \tag{2.10}$$

In view of (2.8), (2.10), the functions f, g satisfy the "orthogonality relations"

$$\int_{\mathbf{R}} \mathrm{e}^{-t} p(x\mathrm{e}^{t/2}) f(x,t)\,\mathrm{d}x \;=\; 0\;, \quad \int_{\mathbf{R}} \mathrm{e}^{-t} p(x\mathrm{e}^{t/2}) g(x,t)\,\mathrm{d}x \;=\; 0\;. \tag{2.11}$$

We now determine the evolution equations satisfied by α, β, f, g. Our first result is:

Lemma 2.3. *If $(u,v) \in \mathcal{C}([t_0,t_1],\mathrm{Z}_t)$ is a solution of (1.15), then $\alpha \in \mathcal{C}^2([t_0,t_1])$ and*

$$\frac{\mathrm{d}}{\mathrm{d}t}\alpha(t) \;=\; \beta(t)\;, \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\eta\mathrm{e}^{-t}\beta(t) + \alpha(t)\right) \;=\; m(t)\;, \tag{2.12}$$

where

$$m(t) \;=\; \int_{\mathbf{R}} \mathrm{e}^{-t} p(x\mathrm{e}^{t/2})\left(-\nu\gamma(x\mathrm{e}^{t/2})v(x,t) + \mathrm{e}^{-t/2}h'(x\mathrm{e}^{t/2})u(x,t)^2 N(x,t)\right)\,\mathrm{d}x\;.$$

Proof. Let $\tau_1 = \mathrm{e}^{t_1} - \tau_0$, and let $W(\xi,\tau)$ be given by (1.14) for $\tau \in [0,\tau_1]$. By Definition 1.1, $W \in \mathcal{C}([0,\tau_1],\mathrm{H}^1_0) \cap \mathcal{C}^1([0,\tau_1],\mathrm{L}^2_0)$ is a (mild) solution of (1.6). Since $\alpha(t) = \int_{\mathbf{R}} p(\xi)W(\xi,\mathrm{e}^t - \tau_0)\,\mathrm{d}\xi$, it follows that $\alpha \in \mathcal{C}^1([t_0,t_1])$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\alpha(t) \;=\; \mathrm{e}^t \int_{\mathbf{R}} p(\xi)W_\tau(\xi,\mathrm{e}^t - \tau_0)\,\mathrm{d}\xi \;=\; \int_{\mathbf{R}} \mathrm{e}^{-t}p(x\mathrm{e}^{t/2})v(x,t)\,\mathrm{d}x \;=\; \beta(t)\;.$$

To prove that $\alpha \in \mathcal{C}^2([t_0,t_1])$, we first assume that $(W_0,\dot{W}_0) \equiv (W(\cdot,0),W_\tau(\cdot,0)) \in \mathrm{H}^2_0 \times \mathrm{H}^1_0$. Then, by Lemma 2.1, $W \in \mathcal{C}([0,\tau_1],\mathrm{H}^2_0) \cap \mathcal{C}^1([0,\tau_1],\mathrm{H}^1_0) \cap \mathcal{C}^2([0,\tau_1],\mathrm{L}^2_0)$ is a classical solution of (1.6), hence $\alpha \in \mathcal{C}^2([t_0,t_1])$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}(\eta\mathrm{e}^{-t}\beta(t) + \alpha(t)) \;=\; \mathrm{e}^t\int_{\mathbf{R}} p(\xi)(\eta W_{\tau\tau} + W_\tau)(\xi,\mathrm{e}^t - \tau_0)\,\mathrm{d}\xi \;\stackrel{\mathrm{def}}{=}\; m(t)\;.$$

Since $p(\eta W_{\tau\tau} + W_\tau) = (pW_\xi)_\xi + 2\nu(pW_\tau)_\xi - \nu p\gamma W_\tau + ph'W^2\mathcal{N}(h,h'W)$ by (1.6), (2.5), we find

$$\begin{aligned} m(t) \;&=\; \mathrm{e}^t\int_{\mathbf{R}} p(\xi)\left(-\nu\gamma W_\tau + h'W^2\mathcal{N}(h,h'W)\right)(\xi,\mathrm{e}^t - \tau_0)\,\mathrm{d}\xi \\ &=\; \int_{\mathbf{R}} \mathrm{e}^{-t}p(x\mathrm{e}^{t/2})\left(-\nu\gamma(x\mathrm{e}^{t/2})v(x,t) + \mathrm{e}^{-t/2}h'(x\mathrm{e}^{t/2})u(x,t)^2N(x,t)\right)\,\mathrm{d}x\;. \end{aligned}$$

For all $t \in [t_0,t_1]$, we thus have

$$\eta\mathrm{e}^{-t}\beta(t) + \alpha(t) \;=\; \eta\mathrm{e}^{-t_0}\beta(t_0) + \alpha(t_0) + \int_{t_0}^t m(s)\,\mathrm{d}s\;. \tag{2.13}$$

By Lemma 2.1, both sides of (2.13) are continuous functions of the initial data $(W_0,\dot W_0)$ in Z_0. Since (2.13) is satisfied for all $(W_0,\dot W_0)$ in the dense subspace $\mathrm{H}^2_0\times\mathrm{H}^1_0$, the equality must hold for all $(W_0,\dot W_0)\in\mathrm{Z}_0$. This shows that $\eta\mathrm{e}^{-t}\beta+\alpha\in\mathcal{C}^1([t_0,t_1])$ and that (2.12) holds. The proof of Lemma 2.3 is complete. □

It follows from (1.15), (2.9) and Lemma 2.3 that $(f,g)\in\mathcal{C}([t_0,t_1],\mathrm{Z}_t)$ is a solution (in the sense of Definition 1.1) of the system

$$\begin{aligned}&f_t-\frac{x}{2}f_x-\frac{3}{2}f \;=\; g\;,\\&\eta\mathrm{e}^{-t}\bigl(g_t-\frac{x}{2}g_x-\frac{5}{2}g\bigr)+(1-\nu\gamma(x\mathrm{e}^{t/2}))g-2\nu\mathrm{e}^{-t/2}g_x \;=\\&\qquad f_{xx}+\mathrm{e}^{t/2}\gamma(x\mathrm{e}^{t/2})f_x+r(x,t)\;,\end{aligned}\tag{2.14}$$

where

$$\begin{aligned}r(x,t)\;&=\alpha(\varphi_{xx}+\mathrm{e}^{t/2}\gamma(x\mathrm{e}^{t/2})\varphi_x-\psi)-\eta\mathrm{e}^{-t}(2\beta\psi+\alpha(\psi_t-\tfrac{x}{2}\psi_x-\tfrac{5}{2}\psi))\\&+\nu\gamma(x\mathrm{e}^{t/2})(\beta\varphi+\alpha\psi)+2\nu\mathrm{e}^{-t/2}(\beta\varphi_x+\alpha\psi_x)\\&+\mathrm{e}^{-t/2}h'(x\mathrm{e}^{t/2})u(x,t)^2N(x,t)-m(t)\varphi\;.\end{aligned}\tag{2.15}$$

Using (2.5), (2.6), (2.8) and the definition of $m(t)$ in Lemma 2.3, it is not difficult to verify that

$$\int_{\mathbf{R}}\mathrm{e}^{-t}p(x\mathrm{e}^{t/2})\bigl(r(x,t)-\nu\gamma(x\mathrm{e}^{t/2})g(x,t)\bigr)\,\mathrm{d}x \;=\; 0\;.\tag{2.16}$$

Finally, as in [GR2], it will be useful to consider also the primitives

$$F(x,t)\;=\;\int_{-\infty}^{x}\mathrm{e}^{-t}p(y\mathrm{e}^{t/2})f(y,t)\,\mathrm{d}y\;,\quad G(x,t)\;=\;\int_{-\infty}^{x}\mathrm{e}^{-t}p(y\mathrm{e}^{t/2})g(y,t)\,\mathrm{d}y\;.\tag{2.17}$$

Using (2.11) and standard inequalities (see Lemma 2.7 below and the remark at the end of this section), it is straightforward to verify that $(F,G)\in\mathcal{C}^1([t_0,t_1],\mathrm{H}^1\times\mathrm{L}^2)$ is a classical solution of the system

$$\begin{aligned}&F_t-\frac{x}{2}F_x\;=\;G\;,\\&\eta\mathrm{e}^{-t}\bigl(G_t-\frac{x}{2}G_x-G\bigr)+G-2\nu\mathrm{e}^{-t/2}G_x\;=\;F_{xx}-\mathrm{e}^{t/2}\gamma(x\mathrm{e}^{t/2})F_x+R(x,t)\;,\end{aligned}\tag{2.18}$$

where

$$R(x,t)\;=\;\int_{-\infty}^{x}\mathrm{e}^{-t}p(y\mathrm{e}^{t/2})\bigl(r(y,t)-\nu\gamma(y\mathrm{e}^{t/2})g(y,t)\bigr)\,\mathrm{d}y\;.\tag{2.19}$$

2.2. Bounds on the Nonlinear Terms

In this subsection, we assume that $(u,v) \in \mathcal{C}([t_0,t_1],\mathrm{Z}_t)$ is a solution of (1.15) satisfying the bound

$$\|u(t)\|_{\mathrm{H}^1_t} \;\le\; 1\;, \quad t \in [t_0,t_1]\;. \tag{2.20}$$

Then $u(t)$ is uniformly bounded in a weighted L^∞ space, as a consequence of the following result:

Lemma 2.4. *There exists a constant $K_0 > 0$ such that, for all $t \ge 0$ and all $w \in \mathrm{H}^1_t$,*

$$\sup_{x\le 0} \mathrm{e}^{\kappa x \mathrm{e}^{t/2}} |w(x)| + \sup_{x\ge 0} (1{+}x)^3 |w(x)| \;\le\; K_0 \|w\|_{\mathrm{H}^1_t}\;. \tag{2.21}$$

Remark. Note the crucial fact that the constant K_0 in (2.21) is independent of t.

Proof. Let $t \ge 0$ and $w \in \mathrm{H}^1_t$. By a classical inequality, there exists $C > 0$ such that

$$\sup_{x\ge 0} (1{+}x)^6 |w(x)|^2 \;\le\; C \int_0^\infty (1{+}x)^6 (w(x)^2 + w'(x)^2)\,\mathrm{d}x\;. \tag{2.22}$$

In particular, $w(0)^2 \le C\|w\|^2_{\mathrm{H}^1_t}$. On the other hand, we have for all $x < 0$:

$$\begin{aligned} \mathrm{e}^{2\kappa x \mathrm{e}^{t/2}} w(x)^2 \;&=\; w(0)^2 - \int_x^0 \mathrm{e}^{2\kappa y \mathrm{e}^{t/2}} \big(2w(y)w'(y) + 2\kappa \mathrm{e}^{t/2} w(y)^2\big)\,\mathrm{d}y \\ &\le\; w(0)^2 + \int_{-\infty}^0 \mathrm{e}^{2\kappa y \mathrm{e}^{t/2}} (w(y)^2 + w'(y)^2)\,\mathrm{d}y\;. \end{aligned} \tag{2.23}$$

Combining (2.22), (2.23), we obtain (2.21). This conclude the proof of Lemma 2.4. $\square$

In the sequel, it will be natural to control the solution (u,v) of (1.15) in terms of the functions α, β, f, g defined in (2.9), (2.10). The equivalence of the corresponding norms is the content of our next result:

Lemma 2.5. *There exists a constant $K_1 \ge 1$ such that, for all $t \ge 0$ and all $(u,v) \in \mathrm{Z}_t$,*

$$\begin{aligned} K_1^{-1} \|u\|_{\mathrm{H}^1_t} \;&\le\; |\alpha| + \|f\|_{\mathrm{H}^1_t} \;\le\; K_1 \|u\|_{\mathrm{H}^1_t}\;, \\ K_1^{-1} \|v\|_{\mathrm{L}^2_t} \;&\le\; |\alpha| + |\beta| + \|g\|_{\mathrm{L}^2_t} \;\le\; K_1 (\|u\|_{\mathrm{H}^1_t} + \|v\|_{\mathrm{L}^2_t})\;, \end{aligned} \tag{2.24}$$

where α, β are defined in (2.10) and f, g in (2.9).

Proof. From (1.10), we know that $\gamma(\xi) \to \gamma_-$ as $\xi \to -\infty$ and $\gamma(\xi) \sim 2/(\xi+\xi_0)$ as $\xi \to +\infty$. Setting $\xi_1 = -\xi_0 + 2/\gamma_-$, we decompose $\gamma(\xi)$ as $\gamma_0(\xi) + \hat\gamma(\xi)$, where

$$\gamma_0(\xi) \;=\; \begin{cases} \gamma_- & \text{if } \xi < \xi_1 \;, \\ 2/(\xi+\xi_0) & \text{if } \xi \ge \xi_1 \;. \end{cases}$$

By (1.10), the remainder $\hat\gamma(\xi)$ decays exponentially as $|\xi| \to \infty$. Thus the solution of (2.5) can be represented as

$$p(\xi) = p_0(\xi) \exp\left(-\int_\xi^\infty \hat\gamma(s)\,\mathrm{d}s\right)\;, \quad p_0(\xi) = \begin{cases} (2/\gamma_-)^2 \mathrm{e}^{\gamma_-(\xi-\xi_1)} & \text{if } \xi < \xi_1, \\ (\xi+\xi_0)^2 & \text{if } \xi \ge \xi_1. \end{cases} \tag{2.25}$$

In particular, there exists $C_0 \ge 1$ such that

$$p(\xi) \;\le\; C_0 \begin{cases} \mathrm{e}^{\gamma_-\xi} & \text{if } \xi < 0 \;, \\ (1+\xi)^2 & \text{if } \xi \ge 0 \;, \end{cases} \qquad p(\xi) \;\ge\; C_0^{-1} \begin{cases} \mathrm{e}^{\gamma_-\xi} & \text{if } \xi < 0 \;, \\ (1+\xi)^2 & \text{if } \xi \ge 0 \;. \end{cases} \tag{2.26}$$

Using (2.25) and remembering that $\int_0^\infty x^2 \varphi^*(x)\,\mathrm{d}x = 1$, we decompose the function $\zeta(t)$ defined in (2.7) as

$$\begin{aligned} \zeta(t) \;=\; & \int_{-\infty}^0 \mathrm{e}^{-t} p(x\mathrm{e}^{t/2}) \varphi^*(x)\,\mathrm{d}x + \int_0^\infty \mathrm{e}^{-t}\big(p(x\mathrm{e}^{t/2}) - p_0(x\mathrm{e}^{t/2})\big)\varphi^*(x)\,\mathrm{d}x \\ & + \int_0^\infty \big(\mathrm{e}^{-t} p_0(x\mathrm{e}^{t/2}) - x^2\big)\varphi^*(x)\,\mathrm{d}x \;=\; \zeta_1(t) + \zeta_2(t) + \zeta_3(t) \;. \end{aligned}$$

Using (1.20), we remark that

$$\zeta_1(t) = \frac{\mathrm{e}^{-3t/2}}{\sqrt{4\pi}} \int_{-\infty}^0 p(\xi)\,\mathrm{d}\xi \;, \quad \zeta_2(t) \;=\; \mathrm{e}^{-3t/2} \int_0^\infty \big(p(\xi) - p_0(\xi)\big)\varphi^*(\xi\mathrm{e}^{-t/2})\,\mathrm{d}\xi \;,$$

where $p(\xi) - p_0(\xi)$ decays exponentially to zero as $\xi \to +\infty$ due to (2.25). On the other hand, setting $\bar\xi = \max(0, \xi_1)$, we have

$$\zeta_3(t) \;=\; \mathrm{e}^{-3t/2} \int_0^{\bar\xi} \big(p_0(\xi) - (\xi+\xi_0)^2\big)\varphi^*(\xi\mathrm{e}^{-t/2})\,\mathrm{d}\xi + \int_0^\infty \big(2\xi_0 x\mathrm{e}^{-t/2} + \xi_0^2\mathrm{e}^{-t}\big)\varphi^*(x)\,\mathrm{d}x \;.$$

It follows immediately from these expressions that

$$|\zeta(t)| + |\zeta'(t)| + |\zeta''(t)| \;\le\; C_1 \mathrm{e}^{-t/2} \;, \quad (1+\zeta(t))^{-1} \;\le\; C_1 \;, \quad t \ge 0 \;, \tag{2.27}$$

for some $C_1 > 0$. As a consequence, the functions $\varphi(x,t), \psi(x,t)$ defined by (2.6) satisfy the bounds

$$\|\varphi(t)\|_{\mathrm{H}^1_t} + \|\psi(t)\|_{\mathrm{L}^2_t} \le C_2 \ , \quad t \ge 0 \ , \tag{2.28}$$

and

$$\|\varphi(t)-\varphi^*\|_{\mathrm{H}^1_t} + \|\psi(t)-\psi^*\|_{\mathrm{L}^2_t} \le C_2 \mathrm{e}^{-t/2} \ , \quad t \ge 0 \ , \tag{2.29}$$

for some $C_2 > 0$.

Now, let $t \ge 0$, $(u,v) \in \mathrm{Z}_t$, and let α, β be defined as in (2.10). In view of (2.26), we have

$$\begin{aligned} |\alpha| &\le C_0 \int_0^\infty (1{+}x)^2 |u(x)|\,\mathrm{d}x + C_0 \mathrm{e}^{-t} \int_{-\infty}^0 \mathrm{e}^{\gamma_- x \mathrm{e}^{t/2}} |u(x)|\,\mathrm{d}x \\ &\le C_0 \left(\int_0^\infty (1{+}x)^6 |u|^2\,\mathrm{d}x\right)^{1/2} + \frac{C_0 \mathrm{e}^{-5t/4}}{(\gamma_- {-}\kappa)^{1/2}} \left(\int_{-\infty}^0 \mathrm{e}^{2\kappa x \mathrm{e}^{t/2}} |u|^2\,\mathrm{d}x\right)^{1/2} , \end{aligned} \tag{2.30}$$

hence $|\alpha| \le C_3 \|u\|_{\mathrm{L}^2_t}$ for some $C_3 > 0$. Similarly, we have $|\beta| \le C_3 \|v\|_{\mathrm{L}^2_t}$. Using these bounds together with (2.9), (2.28), we obtain (2.24). This concludes the proof of Lemma 2.5. $\square$

We now estimate the remainder terms $m(t)$ and $r(x,t)$ in (2.12), (2.14).

Lemma 2.6. *There exists a constant $K_2 > 0$ such that, if $(u,v) \in \mathcal{C}([t_0,t_1],\mathrm{Z}_t)$ is a solution of (1.15) satisfying (2.20), then*

$$\|r(t)\|_{\mathrm{L}^2_t} + \mathrm{e}^{t/4} |m(t)| \le K_2 \mathrm{e}^{-t/4} \left(\alpha(t)^2 + \beta(t)^2 + \|f(t)\|^2_{\mathrm{H}^1_t} + \|g(t)\|^2_{\mathrm{L}^2_t}\right)^{1/2} , \tag{2.31}$$

for all $t \in [t_0,t_1]$, where $r(x,t)$ is defined in (2.15) and $m(t)$ in Lemma 2.3.

Proof. We first consider the function $r_1(x,t) = \varphi_{xx} + \mathrm{e}^{t/2}\gamma(x\mathrm{e}^{t/2})\varphi_x - \psi$. It follows from (1.20), (2.6) that $r_1(x,t) = (1+\zeta(t))^{-1}(\hat{r}(x,t) + \zeta'(t)\varphi(x,t))$, where

$$\hat{r}(x,t) = \begin{cases} (\mathrm{e}^{t/2}\gamma(x\mathrm{e}^{t/2}) - 2/x)\varphi^*_x & \text{if } x > 0 \ , \\ 3\varphi^*/2 & \text{if } x < 0 \ . \end{cases}$$

By (2.27), (2.28), we have $\|\zeta'(t)\varphi(t)\|_{\mathrm{L}^2_t} \le C_1 C_2 \mathrm{e}^{-t/2}$. To bound $\hat{r}(x,t)$, we observe that the function $\xi \mapsto (2 - \xi\gamma(\xi))$ belongs to $\mathrm{L}^2(\mathbf{R}_+)$ by (1.10). Since $\varphi^*_x = -(x/2)\varphi^*$ for $x > 0$, we thus find

$$\int_0^\infty (1{+}x)^6 \hat{r}(x,t)^2\,\mathrm{d}x \le \frac{\mathrm{e}^{-t/2}}{4} \left(\sup_{x\ge 0} (1{+}x)^6 \varphi^*(x)^2\right) \int_0^\infty (2 - \xi\gamma(\xi))^2\,\mathrm{d}\xi \ ,$$

$$\int_{-\infty}^0 \mathrm{e}^{2\kappa x \mathrm{e}^{t/2}} \hat{r}(x,t)^2\,\mathrm{d}x = \frac{9}{32\pi\kappa}\,\mathrm{e}^{-t/2} \ .$$

Summarizing, we obtain $\|r_1(t)\|_{\mathrm{L}^2_t} \le C_4 \mathrm{e}^{-t/4}$ for some $C_4 > 0$. Similarly, since $\gamma \in \mathrm{L}^2(\mathbf{R}_+) \cap \mathrm{L}^\infty(\mathbf{R}_-)$, we find $\|\gamma(x\mathrm{e}^{t/2})\varphi(t)\|_{\mathrm{L}^2_t} \le C_4 \mathrm{e}^{-t/4}$ and $\|\gamma(x\mathrm{e}^{t/2})\psi(t)\|_{\mathrm{L}^2_t} \le C_4 \mathrm{e}^{-t/4}$.

We next bound the non-linear term $r_2(x,t) = \mathrm{e}^{-t/2} h'(x\mathrm{e}^{t/2}) u(x,t)^2 N(x,t)$, where $N(x,t) = \mathcal{N}(h(x\mathrm{e}^{t/2}), \mathrm{e}^{-3t/2} h'(x\mathrm{e}^{t/2}) u(x,t))$. In view of (1.9), (2.20), (2.21), there exists $C_5 > 0$ such that $\sup_{x\in\mathbf{R}} |h'(x\mathrm{e}^{t/2}) u(x,t)| \le C_5$ for all $t \in [t_0, t_1]$. In particular, since $\mathcal{N} : \mathbf{R}^2 \to \mathbf{R}$ is continuous, we have $\|N(\cdot,t)\|_{\mathrm{L}^\infty} \le N_0$ for some $N_0 > 0$ and all $t \in [t_0, t_1]$. It follows that $\|r_2\|_{\mathrm{L}^2_t} \le \mathrm{e}^{-t/2} C_5 N_0 \|u(t)\|_{\mathrm{L}^2_t}$ for $t \in [t_0, t_1]$.

Finally, the function $m(t)$ defined in Lemma 2.3 can be written as $m_1(t) + m_2(t)$, where

$$m_1(t) \;=\; -\nu \int_{\mathbf{R}} \mathrm{e}^{-t} p(x\mathrm{e}^{t/2}) \gamma(x\mathrm{e}^{t/2}) v(x,t)\,\mathrm{d}x \;, \quad m_2(t) \;=\; \int_{\mathbf{R}} \mathrm{e}^{-t} p(x\mathrm{e}^{t/2}) r_2(x,t)\,\mathrm{d}x \;.$$

Proceeding as in (2.30), we find $|m_2(t)| \le C_3 \|r_2(t)\|_{\mathrm{L}^2_t} \le \mathrm{e}^{-t/2} C_3 C_5 N_0 \|u(t)\|_{\mathrm{L}^2_t}$. Moreover, since $\mathrm{e}^{-t}\gamma(x\mathrm{e}^{t/2}) p(x\mathrm{e}^{t/2}) \le C\mathrm{e}^{-t/2}(1+x)$ for $x \ge 0$, we obtain

$$|m_1(t)| \;\le\; C\nu \mathrm{e}^{-t/2} \int_0^\infty (1+x)|v(x,t)|\,\mathrm{d}x + C\nu \mathrm{e}^{-t} \int_{-\infty}^0 \mathrm{e}^{\gamma_- x\mathrm{e}^{t/2}} |v(x,t)|\,\mathrm{d}x \;,$$

hence $|m_1(t)| \le C_6 \nu \mathrm{e}^{-t/2} \|v(t)\|_{\mathrm{L}^2_t}$ for some $C_6 > 0$. Therefore, there exists $C_7 > 0$ such that

$$|m(t)| \le C_7 \mathrm{e}^{-t/2} (\|u(t)\|_{\mathrm{L}^2_t} + \|v(t)\|_{\mathrm{L}^2_t}) \;, \quad t \in [t_0, t_1] \;. \tag{2.32}$$

Summarizing our results and observing that the functions φ, φ_x, ψ, ψ_x, ψ_t, $x\psi_x$ are uniformly bounded in L^2_t by (2.6), (2.27), we see that the remainder $r(x,t)$ defined by (2.15) satisfies

$$\|r(t)\|_{\mathrm{L}^2_t} \;\le\; C_8 \mathrm{e}^{-t/4} \Big(|\alpha(t)| + |\beta(t)| + \|u(t)\|_{\mathrm{H}^1_t} + \|v(t)\|_{\mathrm{L}^2_t} \Big) \;, \quad t \in [t_0, t_1] \;, \tag{2.33}$$

for some $C_8 > 0$. Combining (2.24), (2.32), (2.33), we obtain (2.31). This concludes the proof of Lemma 2.6. $\square$

Finally, we bound the primitives F, G, R defined in (2.17), (2.19).

Lemma 2.7. *There exists a constant $K_3 > 0$ such that, for all $t \ge 0$ and all $f \in \mathrm{L}^2_t$ satisfying $\int_{\mathbf{R}} p(x\mathrm{e}^{t/2}) f(x)\,\mathrm{d}x = 0$, the following estimate holds*

$$\int_{\mathbf{R}} \left(1 + \frac{\mathrm{e}^t}{p(x\mathrm{e}^{t/2})}\right) F^2\,\mathrm{d}x \;\le\; K_3 \left(\mathrm{e}^{-2t} \int_{-\infty}^0 \mathrm{e}^{2\kappa x\mathrm{e}^{t/2}} f^2\,\mathrm{d}x + \int_0^\infty (1+x)^6 f^2\,\mathrm{d}x \right), \tag{2.34}$$

where $F(x) = \int_{-\infty}^{x} \mathrm{e}^{-t} p(y\mathrm{e}^{t/2}) f(y)\,\mathrm{d}y$.

Proof. Let $t \geq 0$ and $f \in \mathrm{L}^2_t$. We start from the identity

$$\mathrm{e}^t \int_{-\infty}^{0} \mathrm{e}^{-\gamma_- x\mathrm{e}^{t/2}} F(x)^2\,\mathrm{d}x + \frac{\mathrm{e}^{t/2}}{\gamma_-} F(0)^2 \;=\; \frac{2\mathrm{e}^{-t/2}}{\gamma_-} \int_{-\infty}^{0} \mathrm{e}^{-\gamma_- x\mathrm{e}^{t/2}} p(x\mathrm{e}^{t/2}) F(x) f(x)\,\mathrm{d}x\;,$$

which is a simple integration by parts. Applying Hölder's inequality to the right-hand side, we obtain

$$\mathrm{e}^t \int_{-\infty}^{0} \mathrm{e}^{-\gamma_- x\mathrm{e}^{t/2}} F(x)^2\,\mathrm{d}x + \frac{\mathrm{e}^{t/2}}{\gamma_-} F(0)^2 \;\leq\; \frac{4\mathrm{e}^{-2t}}{\gamma_-^2} \int_{-\infty}^{0} \mathrm{e}^{-\gamma_- x\mathrm{e}^{t/2}} p(x\mathrm{e}^{t/2})^2 f(x)^2\,\mathrm{d}x\;.$$

Using (2.26) and remembering that $\gamma_- = c + 2\kappa > 2\kappa$, we conclude that

$$\int_{-\infty}^{0} \left(1 + \frac{\mathrm{e}^t}{p(x\mathrm{e}^{t/2})}\right) F(x)^2\,\mathrm{d}x + \mathrm{e}^{t/2} F(0)^2 \;\leq\; C\mathrm{e}^{-2t} \int_{-\infty}^{0} \mathrm{e}^{2\kappa x\mathrm{e}^{t/2}} f(x)^2\,\mathrm{d}x\;, \qquad (2.35)$$

for some $C > 0$.

Since $\int_{\mathbf{R}} p(x\mathrm{e}^{t/2}) f(x)\,\mathrm{d}x = 0$, we have $F(x) = -\int_x^\infty \mathrm{e}^{-t} p(y\mathrm{e}^{t/2}) f(y)\,\mathrm{d}y$. Using (2.26) and a classical inequality of Hardy [HLP, Theorem 328], we find

$$\int_0^\infty F(x)^2\,\mathrm{d}x \;\leq\; 4\int_0^\infty \mathrm{e}^{-2t} x^2 p(x\mathrm{e}^{t/2})^2 f(x)^2\,\mathrm{d}x \;\leq\; 4C_0^2 \int_0^\infty (1{+}x)^6 f(x)^2\,\mathrm{d}x\;. \qquad (2.36)$$

On the other hand, since $F(x) = F(0) + \int_0^x \mathrm{e}^{-t} p(y\mathrm{e}^{t/2}) f(y)\,\mathrm{d}y$, we have for $x > 0$

$$\frac{\mathrm{e}^{t/2}|F(x)|}{1 + x\mathrm{e}^{t/2}} \;\leq\; \frac{\mathrm{e}^{t/2}|F(0)|}{1 + x\mathrm{e}^{t/2}} + \frac{C_0}{x}\int_0^x (1{+}y)^2 |f(y)|\,\mathrm{d}y\;. \qquad (2.37)$$

Using another form of Hardy's inequality [HLP, Theorem 327], we thus obtain

$$\int_0^\infty \frac{\mathrm{e}^t F(x)^2}{(1{+}x\mathrm{e}^{t/2})^2}\,\mathrm{d}x \;\leq\; 2\mathrm{e}^{t/2}|F(0)|^2 + 8C_0^2 \int_0^\infty (1{+}x)^4 f(x)^2\,\mathrm{d}x\;. \qquad (2.38)$$

Combining (2.35), (2.36), (2.38) and using (2.26), we arrive at (2.34). This concludes the proof of Lemma 2.7. □

Lemma 2.8. *There exists a constant* $K_4 > 0$ *such that, if* $(u,v) \in \mathcal{C}([t_0,t_1],\mathrm{Z}_t)$ *is a solution of (1.15) satisfying (2.20), then*

$$\int_{\mathbf{R}} \left(1 + \frac{\mathrm{e}^t}{p(x\mathrm{e}^{t/2})}\right) R^2\,\mathrm{d}x \;\leq\; K_4 \mathrm{e}^{-t/2} \Big(\alpha(t)^2 + \beta(t)^2 + \|f(t)\|^2_{\mathrm{H}^1_t} + \|g(t)\|^2_{\mathrm{L}^2_t}\Big)\;, \qquad (2.39)$$

for all $t \in [t_0, t_1]$, where $R(x,t)$ is defined in (2.19).

Proof. Following the proof of Lemma 2.7, we obtain as in (2.35)

$$\int_{-\infty}^{0} \left(1 + \frac{\mathrm{e}^t}{p(x\mathrm{e}^{t/2})}\right) R(x,t)^2 \,\mathrm{d}x + \mathrm{e}^{t/2} R(0,t)^2 \;\le\; C\mathrm{e}^{-2t} \int_{-\infty}^{0} \mathrm{e}^{2\kappa x \mathrm{e}^{t/2}} (r^2 + \nu^2 \gamma_-^2 g^2) \,\mathrm{d}x \;.$$

Next, remarking that $\mathrm{e}^{-t} p(x\mathrm{e}^{t/2})\gamma(x\mathrm{e}^{t/2}) \le C\mathrm{e}^{-t/2}(1{+}x)$ for $x \ge 0$, we find instead of (2.36), (2.38)

$$\int_0^\infty R(x,t)^2 \,\mathrm{d}x \;\le\; C \int_0^\infty \left((1{+}x)^6 r(x,t)^2 + \nu^2 \mathrm{e}^{-t}(1{+}x)^4 g(x,t)^2\right) \,\mathrm{d}x \;,$$

$$\int_0^\infty \frac{R(x,t)^2}{(1{+}x\mathrm{e}^{t/2})^2} \,\mathrm{d}x \;\le\; 2\mathrm{e}^{t/2} R(0,t)^2 + C \int_0^\infty \left((1{+}x)^4 r^2 + \nu^2 \mathrm{e}^{-t}(1{+}x)^2 g^2\right) \,\mathrm{d}x \;.$$

Combining these estimates, we obtain

$$\int_{\mathbf{R}} \left(1 + \frac{\mathrm{e}^t}{p(x\mathrm{e}^{t/2})}\right) R(x,t)^2 \,\mathrm{d}x \;\le\; C \left(\|r(t)\|_{\mathrm{L}^2_t}^2 + \nu^2 \mathrm{e}^{-t} \|g(t)\|_{\mathrm{L}^2_t}^2\right) \;,$$

and (2.39) follows using Lemma 2.6. This concludes the proof of Lemma 2.8. $\square$

Remark. For $t \ge 0$, $k \in \mathbf{N}$, let X_t^k be the weighted Sobolev space defined by the norm

$$\|u\|_{\mathrm{X}_t^0}^2 \;=\; \int_{-\infty}^{0} \mathrm{e}^{-\gamma_- x \mathrm{e}^{t/2}} |u(x)|^2 \,\mathrm{d}x + \int_0^\infty |u(x)|^2 \,\mathrm{d}x \;, \quad \|u\|_{\mathrm{X}_t^k}^2 \;=\; \sum_{i=0}^{k} \|\partial_x^i u\|_{\mathrm{X}_t^0}^2 \;.$$

If $(u,v) \in \mathcal{C}([t_0,t_1], \mathrm{Z}_t)$ is a solution of (1.15), it follows from Lemma 2.7 and from the definition (2.17) of F, G that $(F,G) \in \mathrm{X}_t^2 \times \mathrm{X}_t^1$ for all $t \in [t_0,t_1]$. Moreover, using a density argument as in the proof of Lemma 2.3, one verifies that $(F,G) \in \mathcal{C}^1([t_0,t_1], \mathrm{X}_t^1 \times \mathrm{X}_t^0)$ is a classical solution of (2.18). As in Definition 1.1, this means that, if

$$\tilde{F}(\xi,t) \;=\; F(\xi\mathrm{e}^{-t/2},t) \;, \quad \tilde{G}(\xi,t) \;=\; G(\xi\mathrm{e}^{-t/2},t) \;,$$

then $(\tilde{F},\tilde{G}) \in \mathcal{C}^1([t_0,t_1], \mathrm{X}_0^1 \times \mathrm{X}_0^0) \cap \mathcal{C}([t_0,t_1], \mathrm{X}_0^2 \times \mathrm{X}_0^1)$. For later use, we also note that

$$\tilde{F}_t(\xi,t) \;=\; \left(F_t - \frac{x}{2} F_x\right)(\xi\mathrm{e}^{-t/2},t) \;, \quad \tilde{G}_t(\xi,t) \;=\; \left(G_t - \frac{x}{2} G_x\right)(\xi\mathrm{e}^{-t/2},t) \;. \qquad (2.40)$$

3. Energy Estimates

As in the previous section, we assume that $(u,v) \in \mathcal{C}([t_0,t_1],\mathrm{Z}_t)$ is a solution of (1.15) satisfying the bound (2.20). To control the time behavior of the functions f, g defined in (2.9), we shall use five pairs of energy functionals.

We first introduce unweighted functionals for the primitives F, G defined in (2.18):

$$E_0(t) \;=\; \int_{\mathbf{R}} \Bigl(\frac{1}{2}F^2 + \eta\mathrm{e}^{-t}FG\Bigr)\,\mathrm{d}x\;, \quad \mathcal{E}_0(t) \;=\; \frac{1}{2}\int_{\mathbf{R}} \bigl(F_x^2 + \eta\mathrm{e}^{-t}G^2\bigr)\,\mathrm{d}x\;. \tag{3.1}$$

Lemma 3.1. *Assume that $(u,v) \in \mathcal{C}([t_0,t_1],\mathrm{Z}_t)$ is a solution of (1.15). Then E_0 and $\mathcal{E}_0$ belong to $\mathcal{C}^1([t_0,t_1])$ and*

$$\begin{aligned}
\dot{E}_0 \;&=\; -\frac{E_0}{2} + \int_{\mathbf{R}} \Bigl(-F_x^2 + \frac{\mathrm{e}^t}{2}\gamma'(x\mathrm{e}^{t/2})F^2 + \eta\mathrm{e}^{-t}G^2 - 2\nu\mathrm{e}^{-t/2}F_xG + FR\Bigr)\,\mathrm{d}x\;,\\
\dot{\mathcal{E}}_0 \;&=\frac{\mathcal{E}_0}{2} + \int_{\mathbf{R}} \Bigl(-G^2 - \mathrm{e}^{t/2}\gamma(x\mathrm{e}^{t/2})F_xG + GR\Bigr)\;,
\end{aligned}$$

for all $t \in [t_0,t_1]$, where R is defined in (2.19).

Remark. Here and in the sequel, we use the notation $\dot{E} = (\mathrm{d}E/\mathrm{d}t)$, $\dot{\mathcal{E}} = (\mathrm{d}\mathcal{E}/\mathrm{d}t)$.

Proof. Since $(F,G) \in \mathcal{C}^1([t_0,t_1],\mathrm{H}^1 \times \mathrm{L}^2)$, the functions $E_0, \mathcal{E}_0$ belong to $\mathcal{C}^1([t_0,t_1])$, and a direct calculation yields:

$$\begin{aligned}
\dot{E}_0(t) \;&=\; \int_{\mathbf{R}} \bigl(FF_t + \eta\mathrm{e}^{-t}\bigl((FG)_t - FG\bigr)\bigr)\,\mathrm{d}x\;,\\
\dot{\mathcal{E}}_0(t) \;&=\; \int_{\mathbf{R}} \bigl(-F_{xx}F_t + \eta\mathrm{e}^{-t}\bigl(GG_t - \tfrac{1}{2}G^2\bigr)\bigr)\,\mathrm{d}x\;.
\end{aligned}$$

Using the identities

$$\begin{aligned}
&FF_t + \eta\mathrm{e}^{-t}\bigl((FG)_t - \tfrac{x}{2}(FG)_x - FG - G^2\bigr)\\
&\quad= FF_{xx} + \bigl(\tfrac{x}{2} - \mathrm{e}^{t/2}\gamma(x\mathrm{e}^{t/2})\bigr)FF_x + 2\nu\mathrm{e}^{-t/2}FG_x + FR\;,
\end{aligned} \tag{3.2}$$

$$\begin{aligned}
&-F_{xx}F_t + \eta\mathrm{e}^{-t}\bigl(GG_t - \tfrac{x}{2}GG_x - G^2\bigr)\\
&\quad= -G^2 - \tfrac{x}{2}F_xF_{xx} - \mathrm{e}^{t/2}\gamma(x\mathrm{e}^{t/2})GF_x + 2\nu\mathrm{e}^{-t/2}GG_x + GR\;,
\end{aligned} \tag{3.3}$$

which follow from (2.18), and integrating by parts, we obtain the desired expressions. This concludes the proof of Lemma 3.1. $\square$

We next introduce weighted functionals for the primitives F, G:

$$\begin{aligned} E_1(t) &= \int_{\mathbf{R}} \frac{\mathrm{e}^t}{p(x\mathrm{e}^{t/2})} \left(\frac{1}{2}F^2 + \eta\mathrm{e}^{-t}FG\right) \mathrm{d}x \ , \\ \mathcal{E}_1(t) &= \frac{1}{2}\int_{\mathbf{R}} \frac{\mathrm{e}^t}{p(x\mathrm{e}^{t/2})} \left(F_x^2 + \eta\mathrm{e}^{-t}G^2\right) \mathrm{d}x \ , \end{aligned} \tag{3.4}$$

where the weight p is defined in (2.5).

Lemma 3.2. *Assume that $(u,v) \in \mathcal{C}([t_0,t_1],\mathrm{Z}_t)$ is a solution of (1.15). Then E_1 and $\mathcal{E}_1$ belong to $\mathcal{C}^1([t_0,t_1])$ and*

$$\begin{aligned} \dot{E}_1 &= \frac{E_1}{2} + \int_{\mathbf{R}} \frac{\mathrm{e}^t}{p(x\mathrm{e}^{t/2})} \left(-F_x^2 + \eta\mathrm{e}^{-t}G^2 + 2\nu\mathrm{e}^{-t/2}FG_x + FR\right) \mathrm{d}x \ , \\ \dot{\mathcal{E}}_1 &= \frac{3\mathcal{E}_1}{2} + \int_{\mathbf{R}} \frac{\mathrm{e}^t}{p(x\mathrm{e}^{t/2})} \left(-G^2 + 2\nu\mathrm{e}^{-t/2}GG_x + GR\right) \mathrm{d}x \ , \end{aligned}$$

for all $t \in [t_0,t_1]$.

Proof. We remark that

$$E_1(t) = \int_{\mathbf{R}} \frac{\mathrm{e}^{t/2}}{p(\xi)} \left(\frac{1}{2}\tilde{F}^2 + \eta\mathrm{e}^{-t}\tilde{F}\tilde{G}\right) \mathrm{d}\xi \ , \quad \mathcal{E}_1(t) = \frac{1}{2}\int_{\mathbf{R}} \frac{\mathrm{e}^{3t/2}}{p(\xi)} \left(\tilde{F}_\xi^2 + \eta\mathrm{e}^{-2t}\tilde{G}^2\right) \mathrm{d}\xi \ ,$$

where $\tilde{F}(\xi,t) = F(\xi\mathrm{e}^{-t/2},t)$, $\tilde{G}(\xi,t) = G(\xi\mathrm{e}^{-t/2},t)$. Since $(\tilde{F},\tilde{G}) \in \mathcal{C}^1([t_0,t_1],\mathrm{X}_0^1 \times \mathrm{X}_0^0)$ (see the remark at the end of the previous section), it follows that $E_1, \mathcal{E}_1 \in \mathcal{C}^1([t_0,t_1])$. Using (2.40), we thus find

$$\begin{aligned} \dot{E}_1 &= \frac{E_1}{2} + \int_{\mathbf{R}} \frac{\mathrm{e}^t}{p(x\mathrm{e}^{t/2})} \left(FF_t - \tfrac{x}{2}FF_x + \eta\mathrm{e}^{-t}\big((FG)_t - \tfrac{x}{2}(FG)_x - FG\big)\right) \mathrm{d}x \ , \\ \dot{\mathcal{E}}_1 &= \frac{3\mathcal{E}_1}{2} + \int_{\mathbf{R}} \frac{\mathrm{e}^t}{p(x\mathrm{e}^{t/2})} \left(F_xF_{xt} - \tfrac{x}{2}F_xF_{xx} - \tfrac{1}{2}F_x^2 + \eta\mathrm{e}^{-t}\big(GG_t - \tfrac{x}{2}GG_x - G^2\big)\right) \mathrm{d}x \ . \end{aligned}$$

Applying the identities (3.2), (3.3) and the relation $F_t = G + \frac{x}{2}F_x$, we obtain the desired result after some integrations by parts. This concludes the proof of Lemma 3.2. $\square$

We now define positive constants A_0, B_0 by

$$A_0 = 2\left(\inf_{\xi\geq 0} p(\xi)|\gamma'(\xi)|\right)^{-1} , \quad B_0 = \left(\sup_{\xi\in\mathbf{R}} p(\xi)\gamma(\xi)^2\right)^{-1} . \tag{3.5}$$

Due to (1.10), (1.11), (2.26), these quantities are well-defined. Moreover, the inequality $|\gamma'(\xi)| \le \frac{1}{2}\gamma(\xi)^2$ implies that $A_0 \ge 4B_0 > 0$. With these notations, we introduce the functional

$$S_1(t) \;=\; A_0 E_0(t) + B_0 \mathcal{E}_0(t) + 2E_1(t) + \mathcal{E}_1(t)\ , \quad t \in [t_0, t_1]\ .$$

Proposition 3.3. *Assume that ηe^{-t_0} is sufficiently small, and that $(u,v) \in \mathcal{C}([t_0,t_1], \mathrm{Z}_t)$ is a solution of (1.15) satisfying the bound (2.20). Then $S_1 \in \mathcal{C}^1([t_0,t_1])$, $S_1(t) \ge 0$, and there exist positive constants K_5, K_6 such that, for all $t \in [t_0,t_1]$,*

$$\dot S_1(t) + \frac{1}{2}S_1(t) \;\le\; -K_5 \int_0^\infty (x^2+x^4) f^2 \,\mathrm{d}x + K_6 \mathrm{e}^{-t/4}(\|f\|_{\mathrm{L}^2_t} + \|g\|_{\mathrm{L}^2_t}) M(t)\ , \qquad (3.6)$$

where $M(t)^2 \;=\; \alpha(t)^2 + \beta(t)^2 + \|f(t)\|^2_{\mathrm{H}^1_t} + \|g(t)\|^2_{\mathrm{L}^2_t}$.

Proof. Assuming $\eta e^{-t_0} \le \min(1/2, B_0/A_0)$, one verifies that $A_0E_0(t) + B_0\mathcal{E}_0(t) \ge 0$ and $2E_1(t) + \mathcal{E}_1(t) \ge 0$ for $t \in [t_0,t_1]$. Next, we remark that $F_x = \mathrm{e}^{-t}p(x\mathrm{e}^{t/2})f$, hence $\|F_x\|_{\mathrm{L}^2} \le C\|f\|_{\mathrm{L}^2_t}$ by (1.16), (2.26). Thus, using Lemma 2.7 and Lemma 2.8, we deduce from Lemma 3.1 that

$$\dot E_0(t) + \frac{E_0(t)}{2} \;\le\; \int_{\mathbf{R}} \Bigl(-F_x^2 + \frac{\mathrm{e}^t}{2}\gamma'(x\mathrm{e}^{t/2})F^2\Bigr)\,\mathrm{d}x + C\mathrm{e}^{-t/4}(\|f\|_{\mathrm{L}^2_t} + \|g\|_{\mathrm{L}^2_t})M(t)\ .$$

Similarly, using the bound $|\mathrm{e}^{t/2}\gamma(x\mathrm{e}^{t/2})F_xG| \le \frac{1}{2}(G^2 + \mathrm{e}^t\gamma(x\mathrm{e}^{t/2})^2F_x^2)$, we obtain

$$\dot{\mathcal{E}}_0(t) + \frac{\mathcal{E}_0(t)}{2} \;\le\; \frac{1}{2}\int_{\mathbf{R}} \Bigl(-G^2 + F_x^2 + \mathrm{e}^t\gamma(x\mathrm{e}^{t/2})^2F_x^2\Bigr)\,\mathrm{d}x + C\mathrm{e}^{-t/4}\|g\|_{\mathrm{L}^2_t}M(t)\ .$$

Finally, applying Lemma 2.7 and Lemma 2.8 again, we deduce from Lemma 3.2 that

$$\begin{aligned} 2\dot E_1(t) + \dot{\mathcal{E}}_1(t) + E_1(t) + \frac{1}{2}\mathcal{E}_1(t) \;&\le\; C\mathrm{e}^{-t/4}(\|f\|_{\mathrm{L}^2_t} + \|g\|_{\mathrm{L}^2_t})M(t) \\ &+ \int_{\mathbf{R}} \frac{\mathrm{e}^t}{p(x\mathrm{e}^{t/2})}\Bigl(-F_x^2 - G^2 + F^2 + 2\nu\mathrm{e}^{-t/2}(2F+G)G_x\Bigr)\,\mathrm{d}x\ . \end{aligned}$$

The last term in the right-hand side is bounded with the help of (2.17), (2.26) and Lemma 2.7:

$$\begin{aligned} \int_{\mathbf{R}} \frac{\mathrm{e}^{t/2}}{p(x\mathrm{e}^{t/2})}|(2F+G)G_x|\,\mathrm{d}x \;&\le\; C\Bigl(\int_{\mathbf{R}} \frac{\mathrm{e}^t(F^2+G^2)}{p(x\mathrm{e}^{t/2})}\,\mathrm{d}x\Bigr)^{1/2}\Bigl(\int_{\mathbf{R}} \mathrm{e}^{-2t}p(x\mathrm{e}^{t/2})g^2\,\mathrm{d}x\Bigr)^{1/2} \\ &\le\; C\mathrm{e}^{-t/2}(\|f\|_{\mathrm{L}^2_t} + \|g\|_{\mathrm{L}^2_t})\|g\|_{\mathrm{L}^2_t}\ . \end{aligned}$$

Combining these estimates and using (2.35), (3.5) together with the inequality $\gamma'(\xi) \le 0$ for $\xi \le 0$, we obtain

$$\begin{aligned}\dot{S}_1(t) + \frac{S_1(t)}{2} \;\le\; &-\frac{B_0}{2}\int_{\mathbf{R}} (7F_x^2 + G^2)\,\mathrm{d}x - \int_{\mathbf{R}} \frac{\mathrm{e}^t}{p(x\mathrm{e}^{t/2})}\left(\frac{1}{2}F_x^2 + G^2\right)\,\mathrm{d}x \\ &+ C\mathrm{e}^{-t/4}(\|f\|_{\mathrm{L}^2_t} + \|g\|_{\mathrm{L}^2_t})M(t)\ ,\end{aligned}$$

for all $t \in [t_0, t_1]$, and (3.6) follows using (2.17), (2.26). This concludes the proof of Proposition 3.3. $\square$

In the rest of this section, we introduce three pairs of weighted functionals $E_i, \mathcal{E}_i$ ($i = 2, 3, 4$) to control the solutions (f, g) of (2.14) in the space Z_t. To each pair will correspond a different weight function $p_i : \mathbf{R} \to \mathbf{R}_+$. To define the weight p_2, we choose a smooth function $\chi_2 : \mathbf{R} \to (0, 1]$ satisfying $\chi_2(\xi) = 2\kappa/\gamma_- < 1$ for $\xi \le -1$ and $\chi_2(\xi) = 1$ for $\xi \ge 0$. We set $\gamma_2 = \chi_2\gamma$. The weight $p_2 : \mathbf{R} \to \mathbf{R}_+$ is then the (unique) solution of the differential problem

$$p_2'(\xi) \;=\; \gamma_2(\xi)p_2(\xi)\ , \quad \xi \in \mathbf{R}\ , \quad \lim_{\xi\to+\infty} \frac{p_2(\xi)}{\xi^2} \;=\; 1\ . \tag{3.7}$$

Clearly, $p_2(\xi) = p(\xi)$ for $\xi \ge 0$, and there exists $C \ge 1$ such that $C^{-1}\mathrm{e}^{2\kappa\xi} \le p_2(\xi) \le C\mathrm{e}^{2\kappa\xi}$ for $\xi \le 0$. In particular, we have for all $u \in \mathrm{L}^2_t$

$$\int_{-\infty}^0 p_2(x\mathrm{e}^{t/2})u(x)^2\,\mathrm{d}x + \int_0^\infty \mathrm{e}^{-t}p_2(x\mathrm{e}^{t/2})u(x)^2\,\mathrm{d}x \;\le\; C\|u\|^2_{\mathrm{L}^2_t}\ . \tag{3.8}$$

We now define the functionals

$$\begin{aligned}E_2(t) \;&=\; \int_{\mathbf{R}} \mathrm{e}^{-t}p_2(x\mathrm{e}^{t/2})\left(\frac{1}{2}f^2 + \eta\mathrm{e}^{-t}fg\right)\,\mathrm{d}x\ , \\ \mathcal{E}_2(t) \;&=\; \frac{1}{2}\int_{\mathbf{R}} \mathrm{e}^{-t}p_2(x\mathrm{e}^{t/2})\left(f_x^2 + \eta\mathrm{e}^{-t}g^2\right)\,\mathrm{d}x\ ,\end{aligned} \tag{3.9}$$

together with $S_2(t) = 2E_2(t) + \mathcal{E}_2(t)$.

Proposition 3.4. *Assume that $\eta\mathrm{e}^{-t_0} \le 1/8$, and that $(u, v) \in \mathcal{C}([t_0, t_1], \mathrm{Z}_t)$ is a solution of (1.15) satisfying the bound (2.20). Then $S_2 \in \mathcal{C}^1([t_0, t_1])$ and there exist positive constants K_7, K_8 such that, for all $t \in [t_0, t_1]$,*

$$S_2(t) \;\ge\; \frac{1}{4}\int_{\mathbf{R}} \mathrm{e}^{-t}p_2(x\mathrm{e}^{t/2})\big(f^2 + f_x^2 + \eta\mathrm{e}^{-t}g^2\big)\,\mathrm{d}x\ , \tag{3.10}$$

and

$$\dot{S}_2 + \frac{1}{2}S_2 \le -K_7\left(\int_0^\infty x^2(f_x^2+g^2)\,\mathrm{d}x + \int_{-\infty}^0 \mathrm{e}^{2\kappa x \mathrm{e}^{t/2}} f^2\,\mathrm{d}x\right) \\ + K_8\left(\int_0^\infty x^2 f^2\,\mathrm{d}x + \mathrm{e}^{-t/4}(\|f\|_{\mathrm{L}^2_t} + \|g\|_{\mathrm{L}^2_t})M(t)\right) . \tag{3.11}$$

Proof. Since $2\eta\mathrm{e}^{-t}|fg| \le 4\eta\mathrm{e}^{-t}f^2 + \frac{1}{4}\eta\mathrm{e}^{-t}g^2$ and $\eta\mathrm{e}^{-t} \le 1/8$, the lower bound (3.10) is obvious. To compute the time derivative of E_2, we note that

$$E_2(t) = \int_{\mathbf{R}} \mathrm{e}^{-3t/2}p_2(\xi)\left(\frac{1}{2}\tilde{f}^2 + \eta\mathrm{e}^{-t}\tilde{f}\tilde{g}\right)\mathrm{d}\xi ,$$

where $\tilde{f}(\xi,t) = f(\xi\mathrm{e}^{-t/2},t)$, $\tilde{g}(\xi,t) = g(\xi\mathrm{e}^{-t/2},t)$. If we assume that $(u(t_0),v(t_0)) \in \mathrm{H}^2_{t_0}\times\mathrm{H}^1_{t_0}$, then (as in Lemma 2.1) $(\tilde{f},\tilde{g}) \in \mathcal{C}([t_0,t_1],\mathrm{H}^2_0\times\mathrm{H}^1_0)\cap\mathcal{C}^1([t_0,t_1],\mathrm{H}^1_0\times\mathrm{L}^2_0)$ and $\tilde{f}_t(\xi,t) = (f_t - \frac{x}{2}f_x)(\xi\mathrm{e}^{-t/2},t)$, $\tilde{g}_t(\xi,t) = (g_t - \frac{x}{2}g_x)(\xi\mathrm{e}^{-t/2},t)$. A direct calculation then yields

$$\dot{E}_2(t) = \int_{\mathbf{R}} \mathrm{e}^{-t}p_2(x\mathrm{e}^{t/2})\left(ff_t - \frac{x}{2}ff_x - \frac{3}{4}f^2 + \eta\mathrm{e}^{-t}\big((fg)_t - \frac{x}{2}(fg)_x - \frac{5}{2}fg\big)\right)\mathrm{d}x .$$

Applying the identity

$$ff_t + \eta\mathrm{e}^{-t}\big((fg)_t - \tfrac{x}{2}(fg)_x - 4fg - g^2\big) \\ = ff_{xx} + (\tfrac{x}{2} + \mathrm{e}^{t/2}\gamma(x\mathrm{e}^{t/2}))ff_x + \tfrac{3}{2}f^2 + \nu\gamma(x\mathrm{e}^{t/2})fg + 2\nu\mathrm{e}^{-t/2}fg_x + fr , \tag{3.12}$$

which follows from (2.14), and integrating by parts, we obtain

$$\dot{E}_2 = \frac{3E_2}{2} + \int_{\mathbf{R}} \mathrm{e}^{-t}p_2(x\mathrm{e}^{t/2})\Big(-f_x^2 + \tfrac{1}{2}\mathrm{e}^t\Gamma_2(x\mathrm{e}^{t/2})f^2 + \eta\mathrm{e}^{-t}g^2 \\ + \nu(\gamma-2\gamma_2)(x\mathrm{e}^{t/2})fg - 2\nu\mathrm{e}^{-t/2}f_xg + fr\Big)\,\mathrm{d}x , \tag{3.13}$$

where $\Gamma_2 = \gamma_2' - \gamma' - \gamma_2(\gamma-\gamma_2)$. As is easily verified, the right-hand side of (3.13) is a continuous function of the initial data $(u(t_0),v(t_0))$ in the topology of Z_{t_0}, uniformly in $t \in [t_0,t_1]$. Therefore, using a density argument as in the proof of Lemma 2.3, we conclude that $E_2 \in \mathcal{C}^1([t_0,t_1])$ and that (3.13) holds in the general case where $(u(t_0),v(t_0)) \in \mathrm{Z}_{t_0}$ only.

In a similar way, we obtain for regular data

$$\dot{\mathcal{E}}_2 = \int_{\mathbf{R}} \mathrm{e}^{-t}p_2(x\mathrm{e}^{t/2})\Big(f_xf_{xt} - \frac{x}{2}f_xf_{xx} - \frac{3}{4}f_x^2 + \eta\mathrm{e}^{-t}\big(gg_t - \frac{x}{2}gg_x - \frac{5}{4}g^2\big)\Big)\,\mathrm{d}x .$$

Using the relation $f_t = g + \frac{x}{2}f_x + \frac{3}{2}f$ as well as the identity

$$\begin{aligned} -f_{xx}f_t + \eta e^{-t}\big(gg_t - \tfrac{x}{2}gg_x - \tfrac{5}{2}g^2\big) \;&=\; -(1-\nu\gamma(xe^{t/2}))g^2 + 2\nu e^{-t/2}gg_x \\ &\quad - \tfrac{x}{2}f_xf_{xx} - \tfrac{3}{2}ff_{xx} + e^{t/2}\gamma(xe^{t/2})f_xg + gr\ , \end{aligned} \tag{3.14}$$

which follows from (2.14), we obtain after integrating by parts

$$\dot{\mathcal{E}}_2 \;=\; \frac{5\mathcal{E}_2}{2} + \int_{\mathbf{R}} e^{-t}p_2(xe^{t/2})\Big(-g^2 + \big(\nu g^2 + e^{t/2}f_xg\big)(\gamma-\gamma_2)(xe^{t/2}) + gr\Big)\,dx\ . \tag{3.15}$$

By the same density argument, $\mathcal{E}_2 \in \mathcal{C}^1([t_0,t_1])$ and (3.15) holds for all solutions (u,v) of (1.15) in Z_t.

We now estimate the right-hand side of (3.13). Since $|(\gamma-2\gamma_2)(\xi)| \le \gamma(\xi)$ for $\xi\in\mathbf{R}$ and $e^{-t}p_2(xe^{t/2})\gamma(xe^{t/2}) \le Ce^{-t/2}(1+x)$ for $x\ge 0$, we obtain with the help of (3.8)

$$\int_{\mathbf{R}} e^{-t}p_2(xe^{t/2})|(\gamma-2\gamma_2)(xe^{t/2})fg|\,dx \;\le\; Ce^{-t/2}\|f\|_{\mathrm{L}^2_t}\|g\|_{\mathrm{L}^2_t}\ . \tag{3.16}$$

Remarking that $\Gamma_2(\xi)=0$ for $\xi\ge 0$, we deduce from (3.8), (3.13), (3.16) and Lemma 2.6 that

$$\begin{aligned} \dot{E}_2 + \frac{1}{2}E_2 \;&\le\; \int_0^\infty e^{-t}p_2(xe^{t/2})(f^2-f_x^2)\,dx + \frac{1}{2}\int_{-\infty}^0 p_2(xe^{t/2})\Gamma_2(xe^{t/2})f^2\,dx \\ &\quad + Ce^{-t/4}(\|f\|_{\mathrm{L}^2_t} + \|g\|_{\mathrm{L}^2_t})M(t)\ . \end{aligned} \tag{3.17}$$

Since $\Gamma_2(\xi)\to -2\kappa(\gamma_- - 2\kappa) = -2\kappa c_*$ as $\xi\to-\infty$, we can write $\Gamma_2(\xi) \le -\kappa c_* + \tilde{\Gamma}_2(\xi)$ for all $\xi\le 0$, where the support of $\tilde{\Gamma}_2$ is contained in a compact interval $[-A,0]$. Applying Lemma 2.4, we thus obtain

$$\begin{aligned} &\int_{-\infty}^0 p_2(xe^{t/2})\Gamma_2(xe^{t/2})f^2\,dx + \kappa c_*\int_{-\infty}^0 p_2(xe^{t/2})f^2\,dx \\ &\qquad\le\; e^{-t/2}\sup_{x\ge -Ae^{-t/2}}|f(x,t)|^2\int_{-A}^0 p_2(\xi)\tilde{\Gamma}_2(\xi)\,d\xi \;\le\; Ce^{-t/2}\|f\|^2_{\mathrm{H}^1_t}\ . \end{aligned} \tag{3.18}$$

Similarly, remarking that $\gamma_2(\xi)=\gamma(\xi)$ for $\xi\ge 0$, we deduce from (3.8), (3.15) and Lemma 2.6 that

$$\dot{\mathcal{E}}_2 + \frac{1}{2}\mathcal{E}_2 \;\le\; \int_0^\infty e^{-t}p_2(xe^{t/2})(\tfrac{3}{2}f_x^2 - g^2)\,dx + Ce^{-t/4}(\|f\|_{\mathrm{L}^2_t} + \|g\|_{\mathrm{L}^2_t})M(t)\ . \tag{3.19}$$

Combining (3.17), (3.18), (3.19) and using (2.26), (3.7), we obtain (3.11). This concludes the proof of Proposition 3.4. $\square$

The construction of our next functionals E_3, $\mathcal{E}_3$ is one of the main difficulties in the proof of Theorem 1.2. The aim is to control the quantity

$$\int_{-\infty}^{0} \mathrm{e}^{2\kappa x \mathrm{e}^{t/2}} \left(f^2 + f_x^2 + \eta \mathrm{e}^{-t} g^2\right) \,\mathrm{d}x + \int_0^{\infty} \left(f^2 + f_x^2 + \eta \mathrm{e}^{-t} g^2\right) \,\mathrm{d}x \;,$$

which is part of the norm of (f, g) in Z_t. A natural idea is to define E_3, $\mathcal{E}_3$ by the formulas (3.9) with $\mathrm{e}^{-t} p_2(x\mathrm{e}^{t/2})$ replaced by $p_3(x\mathrm{e}^{t/2})$, where $p_3(\xi) = \mathcal{O}(\mathrm{e}^{2\kappa\xi})$ as $\xi \to -\infty$ and $p_3(\xi) \to 1$ as $\xi \to +\infty$. However, we are not able to estimate properly the time derivative of these functionals without including in $\mathcal{E}_3$ an additional term of the form

$$\int_{\mathbf{R}} p_3(x\mathrm{e}^{t/2}) \lambda(x\mathrm{e}^{t/2}) \gamma(x\mathrm{e}^{t/2}) (\nu f_x^2 - \eta \mathrm{e}^{-t/2} f_x g) \,\mathrm{d}x \;,$$

see (3.25) below. With this modification, the derivative of $\mathcal{E}_3$ contains a quadratic form $Q(x,t)$ depending on the functions λ and p_3, see (3.30). As we shall show, the evolution of $E_3, \mathcal{E}_3$ can then be controlled provided $Q(x,t)$ is positive definite.

We now construct positive functions λ, p_3 so that the quadratic form $Q(x,t)$ in (3.30) is positive definite. First of all, since $\gamma_- = c_* + 2\kappa > c_*$ and $\nu c_* < 1$ by (1.7), we can introduce

$$\lambda_- \;=\; \left(\frac{\gamma_-^2}{c_*^2} - \nu\gamma_-\right)^{-1} \;>\; 0 \;. \tag{3.20}$$

For later use, we remark that

$$\lambda_-(1 - \nu c_*) \;<\; (c_*/\gamma_-)^2 < 1 \;, \quad \text{and} \quad \lambda_-\gamma_- \;<\; \nu/\eta \;. \tag{3.21}$$

Next, in view of (1.10), (1.11), we can choose $\xi_3 > 0$ sufficiently large so that

$$\gamma(\xi_3) \;<\; c_*\lambda_- \;, \quad \nu\gamma(\xi_3) \;\le\; \frac{1}{2} \;, \quad \gamma'(\xi) \;\le\; -\frac{1}{4}\gamma(\xi)^2 \quad \text{for all} \quad \xi \ge \xi_3 \;. \tag{3.22}$$

Remark that the first condition in (3.22) is automatically satisfied if $\lambda_- \ge 1$, since $\gamma(0) = c_*$ and γ is non-increasing. Now, let $\lambda : \mathbf{R} \to \mathbf{R}_+$ be a smooth, monotone function satisfying $\lambda(\xi) = \lambda_-$ if $\xi \le 0$, $\lambda(\xi) = 1$ if $\xi \ge \xi_3$, $(\lambda\gamma)'(\xi) \le 0$ for all $\xi \in \mathbf{R}$, and

$$\lambda(\xi)\big((1 - \nu\gamma(\xi))^2 + \eta\gamma(\xi)^2\big) \;\le\; 1 \;, \quad \xi \in [0, \xi_3] \;. \tag{3.23}$$

Constructing such a function λ is easy. Indeed, if $\lambda_- < 1$, the first condition in (3.22) ensures that λ can be chosen so that $(\lambda\gamma)'(\xi) \le 0$ for $\xi \in [0, \xi_3]$. On the other hand, we observe that the function

$$\Omega(\gamma) \;=\; (1 - \nu\gamma)^2 + \eta\gamma^2 \;\equiv\; 1 - 2\nu\gamma + \frac{\nu\gamma^2}{c_*}$$

is non-increasing for $\gamma \le c_*$, with $\Omega(0) = 1$ and $\Omega(c_*) = 1 - \nu c_* > 0$. Since $\gamma(\xi) \le c_*$ for $\xi \ge 0$, the condition (3.23) is obviously satisfied if $\lambda_- \le 1$. If $\lambda_- > 1$, we remark that $\lambda_-\Omega(\gamma(0)) < (c_*/\gamma_-)^2 < 1$ by (3.21), hence it is sufficient to assume that $\lambda(\xi)$ decays rapidly enough to 1 (as ξ varies from 0 to ξ_3) so that (3.23) is satisfied.

We next define the weight function p_3. Let $\chi_3 : \mathbf{R} \to (0,1]$ be a smooth function satisfying $\chi_3(\xi) = 2\kappa/\gamma_- < 1$ for $\xi \le -1$, $\chi_3(\xi) = 1$ for $\xi \in [0,\xi_3]$, and $\chi_3(\xi) = 0$ for $\xi \ge \xi_3 + 1$. We also assume that $\xi\chi_3'(\xi) \le 0$ for all $\xi \in \mathbf{R}$. We set $\gamma_3 = \chi_3\gamma$, and define the weight function $p_3 : \mathbf{R} \to \mathbf{R}_+$ as the (unique) solution of the differential problem

$$p_3'(\xi) \;=\; \gamma_3(\xi)p_3(\xi)\;, \quad \xi \in \mathbf{R}\;, \quad \lim_{\xi\to+\infty} p_3(\xi) \;=\; 1\;. \tag{3.24}$$

Clearly, there exists $C \ge 1$ such that $C^{-1} \le p_3(\xi) \le C$ for $\xi \ge 0$ and $C^{-1}\mathrm{e}^{2\kappa\xi} \le p_3(\xi) \le C\mathrm{e}^{2\kappa\xi}$ for $\xi \le 0$.

With these definitions, we now introduce the functionals

$$\begin{aligned} E_3(t) \;&=\; \int_{\mathbf{R}} p_3(x\mathrm{e}^{t/2})\left(\frac{1}{2}f^2 + \eta\mathrm{e}^{-t}fg\right)\,\mathrm{d}x\;,\\ \mathcal{E}_3(t) \;&=\; \frac{1}{2}\int_{\mathbf{R}} p_3(x\mathrm{e}^{t/2})\left(f_x^2 + \eta\mathrm{e}^{-t}g^2 + 2(\lambda\gamma)(x\mathrm{e}^{t/2})(\nu f_x^2 - \eta\mathrm{e}^{-t/2}f_x g)\right)\,\mathrm{d}x\;, \end{aligned} \tag{3.25}$$

together with $S_3(t) = KE_3(t) + \mathcal{E}_3(t)$, where $K = 3 + 4\nu\|\lambda\gamma\|_{\mathrm{L}^\infty}$.

Proposition 3.5. *Assume that $\eta\mathrm{e}^{-t_0}$ is sufficiently small, and that $(u,v) \in \mathcal{C}([t_0,t_1],\mathrm{Z}_t)$ is a solution of (1.15) satisfying the bound (2.20). Then $S_3 \in \mathcal{C}^1([t_0,t_1])$, and there exist positive constants K_9, K_{10} such that, for all $t \in [t_0,t_1]$,*

$$S_3(t) \;\ge\; \frac{1}{8}\int_{\mathbf{R}} p_3(x\mathrm{e}^{t/2})\bigl(f^2 + f_x^2 + \eta\mathrm{e}^{-t}g^2\bigr)\,\mathrm{d}x\;, \tag{3.26}$$

and

$$\begin{aligned} \dot S_3(t) + \frac{1}{2}S_3(t) \;\le&\; -K_9\int_{\mathbf{R}} p_3(x\mathrm{e}^{t/2})(g^2 + f_x^2 + \mathrm{e}^t\gamma(x\mathrm{e}^{t/2})^2 f_x^2)\,\mathrm{d}x\\ &+ K_{10}\left(\int_{-\infty}^0 \mathrm{e}^{2\kappa x\mathrm{e}^{t/2}} f^2\,\mathrm{d}x + \int_0^\infty x^2 f_x^2\,\mathrm{d}x\right)\\ &+ K_{10}\left(\mathrm{e}^{-t/4}(\|f\|_{\mathrm{H}^1_t} + \|g\|_{\mathrm{L}^2_t})M(t) + \mathrm{e}^{-t/2}M(t)^2\right)\;. \end{aligned} \tag{3.27}$$

Proof. Since $|Kfg| \le \frac{1}{8}g^2 + 2K^2f^2$ and $|\mathrm{e}^{-t/2}\lambda\gamma f_x g| \le \frac{1}{4}\mathrm{e}^{-t}g^2 + (\lambda\gamma)^2 f_x^2$, we have

$$S_3 \;\ge\; \int_{\mathbf{R}} p_3(x\mathrm{e}^{t/2})\Bigl(\bigl(\frac{K}{2} - 2K^2\eta\mathrm{e}^{-t}\bigr)f^2 + \frac{1}{8}\eta\mathrm{e}^{-t}g^2 + \frac{1}{2}f_x^2\bigl(1 + 2\nu\lambda\gamma - 2\eta\lambda^2\gamma^2\bigr)(x\mathrm{e}^{t/2})\Bigr)\,\mathrm{d}x\;.$$

Assuming that $\eta e^{-t_0} \le (8K)^{-1}$ and noting that $\nu - \eta\lambda\gamma \ge \nu - \eta\lambda_-\gamma_- > 0$ by (3.21), we obtain (3.26).

Next, proceeding as in the proof of Proposition 3.4, we show that $E_3 \in \mathcal{C}^1([t_0, t_1])$ and that

$$\begin{aligned} \dot{E}_3 \;=\; \frac{5E_3}{2} + \int_{\mathbf{R}} p_3(xe^{t/2})\Big(-f_x^2 + e^{t/2}(\gamma - \gamma_3)(xe^{t/2}) f f_x + \eta e^{-t} g^2 \\ + \nu(\gamma - 2\gamma_3)(xe^{t/2}) f g - 2\nu e^{-t/2} f_x g + f r\Big)\, dx \;, \end{aligned} \tag{3.28}$$

for $t \in [t_0, t_1]$. The analysis of $\mathcal{E}_3$ is more complicated due to the additional term $2\lambda\gamma(\nu f_x^2 - \eta e^{-t/2} f_x g)$. First, assuming that the initial data are regular, we obtain by a direct calculation

$$\begin{aligned} \dot{\mathcal{E}}_3 \;=\; \int_{\mathbf{R}} p_3(xe^{t/2})\Big((1 + 2\nu(\lambda\gamma)(xe^{t/2}))\big(f_x f_{xt} - \tfrac{x}{2} f_x f_{xx} - \tfrac{1}{4} f_x^2\big) \\ + \eta e^{-t}\big(g g_t - \tfrac{x}{2} g g_x - \tfrac{3}{4} g^2\big) - \eta e^{-t/2}(\lambda\gamma)(xe^{t/2})\big((f_x g)_t - \tfrac{x}{2}(f_x g)_x - f_x g\big)\Big)\, dx \;. \end{aligned}$$

Using the relation $f_t = g + \frac{x}{2} f_x + \frac{3}{2} f$ together with the identities (3.14) and

$$\begin{aligned} 2\nu f_x f_{xt} - \eta e^{-t/2}\big((f_x g)_t - \tfrac{x}{2}(f_x g)_x - \tfrac{9}{2} f_x g - g g_x\big) \;=\; e^{t/2}(1 - \nu\gamma(xe^{t/2})) f_x g \\ - e^{t/2} f_x f_{xx} - e^t \gamma(xe^{t/2}) f_x^2 + \nu x f_x f_{xx} + 4\nu f_x^2 - e^{t/2} f_x r \;, \end{aligned} \tag{3.29}$$

which follow from (2.14), we obtain after integrating by parts

$$\dot{\mathcal{E}}_3 \;=\; \frac{7\mathcal{E}_3}{2} + \int_{\mathbf{R}} p_3(xe^{t/2})\Big((g - e^{t/2}(\lambda\gamma)(xe^{t/2}) f_x) r - Q(x,t)[e^{t/2}\gamma(xe^{t/2}) f_x, g]\Big)\, dx \;, \tag{3.30}$$

where $Q(x,t)$ is the quadratic form defined by

$$\begin{aligned} Q(x,t)[z_1, z_2] \;&= z_1^2\big(\lambda - \tfrac{1}{2}\lambda\gamma^{-1}(\gamma_3 + \mu)\big)(xe^{t/2}) - z_1 z_2\big(1 - \chi_3 + \lambda(1 - \nu\gamma)\big)(xe^{t/2}) \\ &\quad + z_2^2\big(1 + \nu(\gamma_3 - \gamma) - \tfrac{1}{2}\eta\lambda\gamma(\gamma_3 + \mu)\big)(xe^{t/2}) \;, \quad (z_1, z_2) \in \mathbf{R}^2 \;, \end{aligned}$$

and $\mu = (\lambda\gamma)'/(\lambda\gamma) \le 0$. By density, (3.30) holds for all solutions (u, v) of (1.15) in Z_t.

Applying Lemma 2.6 and recalling that $K = 3 + 4\nu\|\lambda\gamma\|_{L^\infty}$, we deduce from (3.28), (3.30) that

$$\begin{aligned} \dot{S}_3(t) + \frac{1}{2} S_3(t) \;\le\; \int_{\mathbf{R}} p_3(xe^{t/2})\Big(-f_x^2 + \tfrac{3}{2} K f^2 - Q(x,t)[e^{t/2}\gamma(xe^{t/2}) f_x, g] \\ - e^{t/2}(\lambda\gamma)(xe^{t/2}) f_x r + \nu K(\gamma - 2\gamma_3)(xe^{t/2}) f g + K e^{t/2}(\gamma - \gamma_3)(xe^{t/2}) f f_x\Big)\, dx \\ + C e^{-t/4}(\|f\|_{H^1_t} + \|g\|_{L^2_t}) M(t) \;. \end{aligned} \tag{3.31}$$

We shall prove below that there exists $Q_0 > 0$ such that, for all $(x,t) \in \mathbf{R} \times \mathbf{R}_+$,

$$Q(x,t)[z_1,z_2] \;\ge\; Q_0(z_1^2 + z_2^2)\;, \quad (z_1,z_2) \in \mathbf{R}^2\;. \tag{3.32}$$

Assuming for a while that (3.32) holds, and using Lemma 2.6 together with the inequalities

$$K\mathrm{e}^{t/2}\gamma(1-\chi_3)ff_x - \mathrm{e}^{t/2}\lambda\gamma f_x r \;\le\; \frac{Q_0}{2}\mathrm{e}^t\gamma^2 f_x^2 + \frac{K^2}{Q_0}f^2 + \frac{1}{Q_0}\lambda^2 r^2\;,$$
$$\nu K\gamma(1-2\chi_3)fg \;\le\; \frac{Q_0}{2}g^2 + \frac{\nu^2K^2}{2Q_0}\gamma^2 f^2\;,$$

we deduce from (3.31) that

$$\begin{aligned}\dot S_3(t) + \frac{1}{2}S_3(t) \;\le\; &\int_{\mathbf{R}} p_3(x\mathrm{e}^{t/2})\Big(-f_x^2 - \frac{Q_0}{2}(g^2 + \mathrm{e}^t\gamma(x\mathrm{e}^{t/2})^2 f_x^2)\Big)\,\mathrm{d}x \\ &+ C\left(\int_{\mathbf{R}} p_3(x\mathrm{e}^{t/2})f^2\,\mathrm{d}x + \mathrm{e}^{-t/2}M(t)^2 + \mathrm{e}^{-t/4}(\|f\|_{\mathrm{H}^1_t} + \|g\|_{\mathrm{L}^2_t})M(t)\right)\;.\end{aligned} \tag{3.33}$$

The estimate (3.27) is then a straightforward consequence of (3.33) and of the Hardy-type inequality

$$\int_{\mathbf{R}} p_3(x\mathrm{e}^{t/2})f^2\,\mathrm{d}x \;\le\; C\left(\int_{-\infty}^0 \mathrm{e}^{2\kappa x\mathrm{e}^{t/2}}f^2\,\mathrm{d}x + \int_0^\infty x^2 f_x^2\,\mathrm{d}x\right)\;.$$

It remains to prove the property (3.32), namely

$$\big(1-\chi_3+\lambda(1-\nu\gamma)\big)^2 \;<\; 4\big(\lambda - \tfrac{1}{2}\lambda\gamma^{-1}(\gamma_3+\mu)\big)\big(1+\nu(\gamma_3-\gamma) - \tfrac{1}{2}\eta\lambda\gamma(\gamma_3+\mu)\big)\;,$$

for all $\xi \in [-\infty,+\infty]$. Expanding the products in both sides, we rewrite this condition in the equivalent form

$$\begin{aligned}(1-\chi_3)^2\big(1+2\nu\lambda\gamma - \eta\lambda^2\gamma^2\big) - 2\lambda + \lambda^2\big((1-\nu\gamma)^2 + \eta\gamma^2\big)& \\ < \eta\lambda^2\mu^2 - 2\lambda^2\eta\mu(\gamma-\gamma_3) - 2\lambda\gamma^{-1}\mu(1-\nu(\gamma-\gamma_3))&\;.\end{aligned} \tag{3.34}$$

To prove (3.34), we first remark that the right-hand side is positive, since $\mu \le 0$, $\gamma - \gamma_3 \ge 0$ and $1-\nu(\gamma-\gamma_3) \ge 1-\nu c_* > 0$. We also recall that $1+2\nu\lambda\gamma - \eta\lambda^2\gamma^2 \ge 1$, since $\nu - \eta\lambda\gamma > 0$ by (3.21). We now distinguish three cases according to whether $\xi \le 0$, $\xi \in [0,\xi_3]$, or $\xi \ge \xi_3$.

1. If $\xi \le 0$, then $\lambda = \lambda_-$ and $1-\chi_3 \le c_*/\gamma_-$, hence it is sufficient to verify the stronger condition

$$\frac{c_*^2}{\gamma_-^2}\big(1+2\nu\lambda_-\gamma - \eta\lambda_-^2\gamma^2\big) - 2\lambda_- + \lambda_-^2\big((1-\nu\gamma)^2 + \eta\gamma^2\big) \;<\; 0\;, \tag{3.35}$$

for all $\gamma \in [c_*, \gamma_-]$. Let $\Psi(\gamma)$ denote the left-hand side of (3.35), considered as a function of γ. Using (3.21) and the relation $\nu^2 + \eta = \nu/c_*$, it is not difficult to verify that Ψ is convex and that

$$\Psi(\gamma_-) \;=\; -\lambda_-^2(1-\nu c_*)\left(\frac{\gamma_-^2}{c_*^2} - 1\right) \;<\; 0\;, \quad \Psi'(c_*) \;=\; \frac{2c_*^2\lambda_-}{\gamma_-^2}(\nu - \eta c_* \lambda_-) \;>\; 0\;.$$

Since $\Psi'' > 0$, it follows that $\Psi'(\gamma) \geq \Psi'(c_*) > 0$ for all $\gamma \geq c_*$, hence $\Psi(\gamma) \leq \Psi(\gamma_-) < 0$ for all $\gamma \in [c_*, \gamma_-]$, which is the desired inequality.

2. If $\xi \in [0, \xi_3]$, then $\chi_3 = 1$, hence the left-hand side of (3.34) is negative by (3.23).

3. If $\xi \geq \xi_3$, then $\lambda = 1$, $1 - \chi_3 \leq 1$, hence the left-hand side of (3.34) is bounded from above by $\nu^2\gamma^2$. Neglecting the first two terms in the right-hand side (which are positive) and noting that $\mu = \gamma'/\gamma \leq 0$, we arrive at the stronger condition

$$\nu^2\gamma(\xi)^2 \;\leq\; -2\frac{\gamma'(\xi)}{\gamma(\xi)^2}\big(1 - \nu\gamma(\xi)\big)\;, \quad \xi \geq \xi_3\;,$$

which is satisfied by assumption on ξ_3, see (3.22). This concludes the proof of Proposition 3.5. □

Finally, we introduce our last functionals

$$\begin{aligned} E_4(t) \;&=\; \int_{\mathbf{R}} \mathrm{e}^{-3t} p_4(x\mathrm{e}^{t/2})\left(\frac{1}{2}f^2 + \eta\mathrm{e}^{-t}fg\right)\mathrm{d}x\;, \\ \mathcal{E}_4(t) \;&=\; \frac{1}{2}\int_{\mathbf{R}} \mathrm{e}^{-3t} p_4(x\mathrm{e}^{t/2})\left(f_x^2 + \eta\mathrm{e}^{-t}g^2\right)\mathrm{d}x\;, \end{aligned} \tag{3.36}$$

where $p_4(\xi) = p(\xi)^3$. We set $S_4 = 2E_4 + \mathcal{E}_4$.

Proposition 3.6. *Assume that $\eta\mathrm{e}^{-t_0} \leq 1/8$ and that $(u, v) \in \mathcal{C}([t_0, t_1], \mathrm{Z}_t)$ is a solution of (1.15) satisfying the bound (2.20). Then $S_4 \in \mathcal{C}^1([t_0, t_1])$ and there exist positive constants K_{11}, K_{12} such that, for $t \in [t_0, t_1]$,*

$$S_4(t) \geq \frac{1}{4}\int_{\mathbf{R}} \mathrm{e}^{-3t} p_4(x\mathrm{e}^{t/2})\big(f^2 + f_x^2 + \eta\mathrm{e}^{-t}g^2\big)\,\mathrm{d}x\;, \tag{3.37}$$

and

$$\begin{aligned} \dot S_4(t) + \frac{1}{2}S_4(t) \;\leq\; &-K_{11}\int_0^\infty x^6(f_x^2 + g^2)\,\mathrm{d}x \\ &+ K_{12}\left(\int_0^\infty (x^4 f^2 + x^2 f_x^2)\,\mathrm{d}x + \mathrm{e}^{-t/4}(\|f\|_{\mathrm{H}^1_t} + \|g\|_{\mathrm{L}^2_t})M(t)\right)\;. \end{aligned} \tag{3.38}$$

Proof. The lower bound (3.37) is proved as in (3.10). Arguing like in the preceding propositions, we show that $E_4, \mathcal{E}_4 \in \mathcal{C}^1([t_0,t_1])$ and that

$$\begin{aligned}\dot{E}_4 &= -\frac{E_4}{2} + \int_{\mathbf{R}} \mathrm{e}^{-3t} p_4(x\mathrm{e}^{t/2})\Big(-f_x^2 - 2\mathrm{e}^{t/2}\gamma(x\mathrm{e}^{t/2}) f f_x + \eta \mathrm{e}^{-t} g^2 \\ &\qquad - 5\nu\gamma(x\mathrm{e}^{t/2}) f g - 2\nu \mathrm{e}^{-t/2} f_x g + fr\Big)\,\mathrm{d}x\ ,\end{aligned}$$

$$\dot{\mathcal{E}}_4 = \frac{\mathcal{E}_4}{2} + \int_{\mathbf{R}} \mathrm{e}^{-3t} p_4(x\mathrm{e}^{t/2})\Big(-g^2 - 2\gamma(x\mathrm{e}^{t/2})(\nu g^2 + \mathrm{e}^{t/2} f_x g) + gr\Big)\,\mathrm{d}x\ .$$

Proceeding as in (3.16) and applying Lemma 2.6, we deduce that, for $t \in [t_0,t_1]$,

$$\begin{aligned}\dot{S}_4(t) + \frac{1}{2} S_4(t) &\le C\mathrm{e}^{-t/4}(\|f\|_{\mathrm{H}^1_t} + \|g\|_{\mathrm{L}^2_t}) M(t) \\ &+ \int_0^\infty \mathrm{e}^{-3t} p_4(x\mathrm{e}^{t/2})\Big(-\tfrac{3}{2} f_x^2 - g^2 - 2\mathrm{e}^{t/2}\gamma(x\mathrm{e}^{t/2})(f_x g + 2ff_x)\Big)\,\mathrm{d}x\ .\end{aligned} \tag{3.39}$$

Since $p_4(\xi) = p(\xi)^3 \le C_0^3(1+\xi)^6$ for $\xi \ge 0$, we have for $x \ge 0$

$$\begin{aligned}&\mathrm{e}^{-3t}(\gamma p_4)(x\mathrm{e}^{t/2})|2\mathrm{e}^{t/2} f_x g + 4\mathrm{e}^{t/2} f f_x| \\ &\quad\le \mathrm{e}^{-3t} p_4(x\mathrm{e}^{t/2})\big(\tfrac{1}{2} g^2 + f_x^2 + 2\mathrm{e}^{2t}\gamma(x\mathrm{e}^{t/2})^4 f_x^2 + 8\mathrm{e}^{t}\gamma(x\mathrm{e}^{t/2})^2 f^2\big) \\ &\quad\le \mathrm{e}^{-3t} p_4(x\mathrm{e}^{t/2})\big(\tfrac{1}{2} g^2 + f_x^2\big) + C\left(\mathrm{e}^{-t}(1+x\mathrm{e}^{t/2})^2 f_x^2 + \mathrm{e}^{-2t}(1+x\mathrm{e}^{t/2})^4 f^2\right)\ ,\end{aligned}$$

for $x \ge 0$, and the estimate (3.38) follows from (3.39). This concludes the proof of Proposition 3.6. $\square$

We now summarize the decay properties of the four auxiliary functionals S_1, S_2, S_3 and S_4. To this end, we define

$$S_5(t) = B_1 S_1(t) + B_2 S_2(t) + S_3(t) + S_4(t) + \frac{1}{2}\eta \mathrm{e}^{-t}\beta(t)^2\ ,$$

where $B_2 = 1 + K_7^{-1}(K_{10} + K_{12})$ and $B_1 = 1 + K_5^{-1}(K_8 B_2 + K_{12})$. In the proof of Theorem 1.2, we shall use the following properties of $S_5(t)$:

Proposition 3.7. *There exist constants $A_1, A_3, A_4 > 0$ and $A_2 \ge 1$ such that, if $\eta \mathrm{e}^{-t_0} \le A_1$ and if $(u,v) \in \mathcal{C}([t_0,t_1],\mathrm{Z}_t)$ is a solution of (1.15) satisfying the bound (2.20), then, for all $t \in [t_0,t_1]$,*

$$A_2^{-1} S_5(t) \le \|f(t)\|^2_{\mathrm{H}^1_t} + \eta \mathrm{e}^{-t}\left(\beta(t)^2 + \|g(t)\|^2_{\mathrm{L}^2_t}\right) \le A_2 S_5(t)\ , \tag{3.40}$$

and

$$\begin{aligned}\dot{S}_5(t) + \frac{1}{2}S_5(t) \;\le\; & -A_3\left(\beta(t)^2 + \|g(t)\|^2_{\mathrm{L}^2_t} + \|f_x(t)\|^2_{\mathrm{L}^2_t}\right) \\ & + A_4 \mathrm{e}^{-t/4}\left(\|f(t)\|_{\mathrm{H}^1_t} + \|g(t)\|_{\mathrm{L}^2_t} + \mathrm{e}^{-t/4}M(t)\right)M(t)\;,\end{aligned} \tag{3.41}$$

where $M(t)^2 = \alpha(t)^2 + \beta(t)^2 + \|f(t)\|^2_{\mathrm{H}^1_t} + \|g(t)\|^2_{\mathrm{L}^2_t}$.

Proof. Since $S_1(t) \ge 0$ by Proposition 3.3, the lower bound on S_5 in (3.40) follows immediately from (3.10), (3.26), (3.37) and the properties of the weights p_2, p_3, p_4. The upper bound is proved in a similar way, using in addition Lemma 2.7 applied to F and G. On the other hand, we have by Lemma 2.3 and Lemma 2.6

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\eta\mathrm{e}^{-t}\beta(t)^2\right) + \frac{1}{4}\eta\mathrm{e}^{-t}\beta(t)^2 \;&=\; -\beta(t)^2 + \frac{3}{4}\eta\mathrm{e}^{-t}\beta(t)^2 + m(t)\beta(t) \\ &\le\; -\beta(t)^2 + C\mathrm{e}^{-t/2}M(t)^2\;.\end{aligned} \tag{3.42}$$

Combining the estimates (3.6), (3.11), (3.27), (3.38) and (3.42), we thus obtain

$$\begin{aligned}\dot{S}_5(t) + \frac{1}{2}S_5(t) \;\le\; & -K_5\int_0^\infty (x^2+x^4)f^2\,\mathrm{d}x - K_7\int_{-\infty}^0 \mathrm{e}^{2\kappa x \mathrm{e}^{t/2}} f^2\,\mathrm{d}x \\ & - \int_0^\infty (K_7x^2 + K_{11}x^6)(f_x^2+g^2)\,\mathrm{d}x - K_9\int_{\mathbf{R}} p_3(x\mathrm{e}^{t/2})(g^2+f_x^2)\,\mathrm{d}x - \beta(t)^2 \\ & + C\mathrm{e}^{-t/4}\left(\|f(t)\|_{\mathrm{H}^1_t} + \|g(t)\|_{\mathrm{L}^2_t} + \mathrm{e}^{-t/4}M(t)\right)M(t)\;,\end{aligned}$$

from which (3.41) follows using the properties of the weight p_3. This concludes the proof of Proposition 3.7. □

A useful consequence of Proposition 3.7 is:

Corollary 3.8. *There exist constants $A_5 > 0$ and $A_6 \ge 1$ such that, if $t_0 \ge A_5$ and if $(u,v) \in \mathcal{C}([t_0,t_1],\mathrm{Z}_t)$ is a solution of (1.15) satisfying the bound (2.20), then*

$$\Phi_\eta(t,u(t),v(t)) \;\equiv\; \|u(t)\|^2_{\mathrm{H}^1_t} + \eta\mathrm{e}^{-t}\|v(t)\|^2_{\mathrm{L}^2_t} \;\le\; A_6\Phi_\eta(t_0,u(t_0),v(t_0))\;, \tag{3.43}$$

for all $t \in [t_0,t_1]$.

Proof. We introduce our last functional

$$S_6(t) \;=\; \frac{1}{2}\alpha(t)^2 + \eta\mathrm{e}^{-t}\alpha(t)\beta(t) + S_5(t)\;, \quad t \in [t_0,t_1]\;.$$

In view of (3.40), if $\eta e^{-t_0} \le \min(A_1, A_2^{-1})$, there exists a constant $\tilde{C}_1 \ge 1$ such that , for $t \in [t_0, t_1]$,

$$\tilde{C}_1^{-1} S_6(t) \le \alpha(t)^2 + \|f(t)\|^2_{\mathrm{H}^1_t} + \eta e^{-t} \Big(\beta(t)^2 + \|g(t)\|^2_{\mathrm{L}^2_t}\Big) \le \tilde{C}_1 S_6(t) . \tag{3.44}$$

By Lemma 2.5, il follows that

$$\tilde{C}_2^{-1} S_6(t) \le \Phi_\eta(t, u(t), v(t)) \le \tilde{C}_2 S_6(t) , \quad t \in [t_0, t_1] , \tag{3.45}$$

for some $\tilde{C}_2 \ge 1$. Now, since $\dot{S}_6(t) = \alpha(t) m(t) + \eta e^{-t} \beta(t)^2 + \dot{S}_5(t)$ by (2.12), we deduce from (2.31) and (3.41) that

$$\dot{S}_6(t) \le -A_3(\beta(t)^2 + \|g(t)\|^2_{\mathrm{L}^2_t}) + \tilde{C}_3 e^{-t/4} M(t)^2 , \quad t \in [t_0, t_1] ,$$

for some $\tilde{C}_3 > 0$. Assuming that $\tilde{C}_3 e^{-t_0/4} \le A_3$ and using (3.44), we thus find $\dot{S}_6(t) \le \tilde{C}_1 \tilde{C}_3 e^{-t/4} S_6(t)$ for $t \in [t_0, t_1]$, hence $S_6(t) \le \tilde{C}_4 S_6(t_0)$ for $t \in [t_0, t_1]$, where $\tilde{C}_4 = \exp(4\tilde{C}_1 \tilde{C}_3)$. Combining this estimate with (3.45), we obtain (3.43). This concludes the proof of Corollary 3.8. $\square$

4. End of the Proof of Theorem 1.2

Let $t_0 \ge A_5$ and $\delta_0 \le (2A_6)^{-1/2}$, where A_5, A_6 are as in Corollary 3.8. If $(u_0, v_0) \in \mathrm{Z}_{t_0}$ satisfies $\Phi_\eta(t_0, u_0, v_0) \le \delta_0^2$, then the system (1.15) has a unique global solution $(u, v) \in \mathcal{C}([t_0, +\infty), \mathrm{Z}_t)$ with $(u(t_0), v(t_0)) = (u_0, v_0)$. Indeed, the local existence and uniqueness follow from Proposition 2.2, and Corollary 3.8 shows that $\Phi_\eta(t, u(t), v(t)) \le 1/2$ as long as the solution $(u(t), v(t))$ exists. Then Proposition 2.2, with $\delta_1 = 1/\sqrt{2}$, implies that the solution $(u(t), v(t))$ is globally defined.

It remains to prove the decay estimate (1.21). Since

$$\begin{aligned} A_4 e^{-t/4} \|g\|_{\mathrm{L}^2_t} M &\le \frac{A_3}{4} \|g\|^2_{\mathrm{L}^2_t} + C e^{-t/2} M^2 , \\ A_4 e^{-t/4} \|f\|_{\mathrm{H}^1_t} M &\le \frac{A_3}{4}(\beta^2 + \|g\|^2_{\mathrm{L}^2_t}) + A_4 e^{-t/4} \|f\|_{\mathrm{H}^1_t}(|\alpha| + \|f\|_{\mathrm{H}^1_t}) + C e^{-t/2} M^2 , \end{aligned}$$

it follows from (3.41) that

$$\dot{S}_5 + \frac{1}{2} S_5 \le -\frac{A_3}{2}(\beta^2 + \|g\|^2_{\mathrm{L}^2_t}) + A_4 e^{-t/4} \|f\|_{\mathrm{H}^1_t}(|\alpha| + \|f\|_{\mathrm{H}^1_t}) + \tilde{C}_5 e^{-t/2} M^2 ,$$

for some $\tilde{C}_5 > 0$. Setting $\rho_0^2 = \tilde{C}_1 \tilde{C}_2 A_6 \Phi_\eta(t_0, u_0, v_0)$, we have $\alpha(t)^2 + \|f(t)\|_{\mathrm{H}_t^1}^2 \le \rho_0^2$ by (3.43), (3.44), (3.45), and $\|f(t)\|_{\mathrm{H}_t^1}^2 \le A_2 S_5(t)$ by (3.40). Therefore, assuming that $\tilde{C}_5 \mathrm{e}^{-t_0/4} \le A_3/4$, we find

$$\dot{S}_5 + \frac{1}{2} S_5 \;\le\; -\frac{A_3}{4}(\beta^2 + \|g\|_{\mathrm{L}_t^2}^2) + \tilde{C}_6 \rho_0 \mathrm{e}^{-t/4} S_5^{1/2} + \tilde{C}_5 \rho_0^2 \mathrm{e}^{-t/2} \;, \quad t \ge t_0 \;,$$

for some $\tilde{C}_6 > 0$. Integrating this differential inequality and using the bound $S_5(t_0) \le A_2 \rho_0^2$, we obtain after a short computation

$$S_5(t) + \int_{t_0}^t \mathrm{e}^{-(t-s)/2} (\beta(s)^2 + \|g(s)\|_{\mathrm{L}_s^2}^2) \,\mathrm{d}s \;\le\; \tilde{C}_7 \rho_0^2 (1 + (t-t_0)^2) \mathrm{e}^{-(t-t_0)/2} \;, \qquad (4.1)$$

for $t \ge t_0$, where $\tilde{C}_7 > 0$ is independent of t_0 and ρ_0. In view of (3.40), this implies in particular

$$\|f(t)\|_{\mathrm{H}_t^1}^2 + \eta \mathrm{e}^{-t} (\beta(t)^2 + \|g(t)\|_{\mathrm{L}_t^2}^2) \;\le\; A_2 \tilde{C}_7 \rho_0^2 (1 + (t-t_0)^2) \mathrm{e}^{-(t-t_0)/2}, \quad t \ge t_0 \;. \quad (4.2)$$

Since $\dot{\alpha}(t) = \beta(t)$, we also deduce from (4.1), by a simple argument, that $\alpha(t)$ converges to some real number α^* as $t \to +\infty$, and that

$$|\alpha(t) - \alpha^*|^2 + \int_{t_0}^t \mathrm{e}^{-(t-s)/2} |\alpha(s) - \alpha^*|^2 \,\mathrm{d}s \;\le\; \tilde{C}_8 \rho_0^2 (1 + (t-t_0)^2) \mathrm{e}^{-(t-t_0)/2} \;, \quad t \ge t_0 \;, \; (4.3)$$

for some $\tilde{C}_8 > 0$. Finally, it follows from (2.9) that

$$\begin{aligned}
\|u(t) - \alpha^* \varphi^*\|_{\mathrm{H}_t^1} &\le \|f(t)\|_{\mathrm{H}_t^1} + |\alpha(t) - \alpha^*| \|\varphi(t)\|_{\mathrm{H}_t^1} + |\alpha^*| \|\varphi(t) - \varphi^*\|_{\mathrm{H}_t^1} \;, \\
\|v(t) - \alpha^* \psi^*\|_{\mathrm{L}_t^2} &\le \|g(t)\|_{\mathrm{L}_t^2} + |\beta(t)| \|\varphi\|_{\mathrm{L}_t^2} + |\alpha(t) - \alpha^*| \|\psi\|_{\mathrm{L}_t^2} + |\alpha^*| \|\psi(t) - \psi^*\|_{\mathrm{L}_t^2} \;,
\end{aligned}$$

hence the estimate (1.21) is a direct consequence of (2.28), (2.29), (4.1), (4.2) and (4.3). This concludes the proof of Theorem 1.2. $\square$

References

[AW] D.G. Aronson and H.F. Weinberger: Multidimensional Nonlinear Diffusion Arising in Population Genetics, Adv. in Math. **30** (1978), 33–76.

[Br] M. Bramson: *Convergence of Solutions of the Kolmogorov Equation to Travelling Waves*, Memoirs of the AMS **44**, nb. 285, Providence (1983).

[BK1] J. Bricmont and A. Kupiainen: Stability of Moving Fronts in the Ginzburg-Landau Equation, Commun. Math. Physics **159** (1994), 287–318.

[BK2] J. Bricmont and A. Kupiainen: Stable Non-Gaussian Diffusive Profiles, Nonlinear Anal., Theory Methods Appl. **26** (1996), 583–593.

[CH] Th. Cazenave and A. Haraux: *Introduction aux Problèmes d'Evolution Semilinéaires*, Mathématiques et Applications **1**, Ellipses (1990).

[DO] S.R. Dunbar and H.G. Othmer: On a Nonlinear Hyperbolic Equation Describing Transmission Lines, Cell Movement, and Branching Random Walks, in *Nonlinear Oscillations in Biology and Chemistry*, H.G. Othmer (Ed.), Lect. Notes in Biomathematics **66**, Springer (1986).

[EW] J.-P. Eckmann and C.E. Wayne: The Nonlinear Stability of Front Solutions for Parabolic Partial Differential Equations, Comm. Math. Phys. **161** (1994), 323–334.

[EKM] M. Escobedo, O. Kavian and H. Matano: Large Time Behavior of Solutions of a Dissipative Semi-linear Heat Equation, Comm. Partial Diff. Equations **20** (1995), 1427–1452.

[EZ] M. Escobedo and E. Zuazua: Large-time Behavior for Convection Diffusion Equations in $\mathbf{R}^N$, J. Funct. Anal. **100** (1991), 119–161.

[Fi] R.A. Fisher: The Advance of Advantageous Genes, Ann. of Eugenics **7** (1937), 355–369.

[GV] V.A. Galaktionov and J. L. Vazquez: Asymptotic Behaviour of Nonlinear Parabolic Equations with Critical Exponents. A Dynamical System Approach, J. Funct. Anal. **100** (1991), 435–462.

[Ga] Th. Gallay: Local Stability of Critical Fronts in Nonlinear Parabolic Partial Differential Equations, Nonlinearity **7** (1994), 741–764.

[GM] Th. Gallay and A. Mielke: Diffusive Mixing of Stable States in the Ginzburg-Landau Equation, Commun. Math. Phys. **199** (1998), 71–97.

[GR1] Th. Gallay and G. Raugel: Stability of Travelling Waves for a Damped Hyperbolic Equation, ZAMP **48** (1997), 451–479.

[GR2] Th. Gallay and G. Raugel: Scaling Variables and Asymptotic Expansions in Damped Wave Equations, to appear in J. Diff. Eqns (1998).

[GR3] Th. Gallay and G. Raugel: Stability of Propagating Fronts in Damped Hyperbolic Equations, in: Proceedings of the conference "Partial differential equations: theory and numerical solutions" held in Prague (1998), Longman, to appear.

[Go] S. Goldstein: On Diffusion by Discontinuous Movements and the Telegraph Equation, Quart. J. Mech. Appl. Math. **4** (1951), 129–156.

[Ha1] K.P. Hadeler: Hyperbolic Travelling Fronts, Proc. Edinb. Math. Soc. **31** (1988), 89–97.

[Ha2] K.P. Hadeler: Reaction Telegraph Equations and Random Walk Systems, in: *Stochastic and spatial structures of dynamical systems*, S. van Strien, S. Verduyn Lunel (eds.), Royal Acad. of the Netherlands, North Holland, Amsterdam (1996).

[Ha3] K.P. Hadeler: Reaction Transport Systems, in: *Mathematics inspired by biology*, V.Capasso, O.Diekmann (eds), CIME Lectures 1997, Florence, Springer Verlag, in print.

[HLP] G.H. Hardy, J.E. Littlewood and G. Polya: *Inequalities*, Cambridge University Press (1934), reprinted (1964).

[HL] L. Hsiao and T.-P. Liu: Convergence to Nonlinear Diffusion Waves for Solution of a System of Hyperbolic Conservation Laws with Damping, Comm. Math. Phys. **143** (1992), 599–605.

[Kac] M. Kac: A Stochastic Model Related to the Telegrapher's Equation, Rocky Mountain J. Math. **4** (1974), 497–509.

[Kap] T. Kapitula: On the Stability of Travelling Waves in Weighted L^∞ Spaces, J. Diff. Eqns **112** (1994), 179–215.

[Kav] O. Kavian: Remarks on the Large Time Behavior of a Nonlinear Diffusion Equation, Ann. Inst. Henri Poincaré **4** (1987), 423–452.

[Ki] K. Kirchgässner: On the Nonlinear Dynamics of Travelling Fronts, J. Diff. Eqns. **96** (1992), 256–278.

[KPP] A.N. Kolmogorov, I.G. Petrovskii and N.S. Piskunov: Etude de la diffusion avec croissance de la quantité de matière et son application à un problème biologique, Moscow Univ. Math. Bull. **1** (1937), 1–25.

[Mu] J.D. Murray: *Mathematical Biology* 2nd ed., Biomathematics **19**, Springer Verlag (1993).

[Ni] K. Nishihara: Convergence Rates to Nonlinear Diffusion Waves for Solutions of System of Hyperbolic Conservation Laws with Damping, J. Differential Equations **131** (1996), 171–188.

[RK] G. Raugel, K. Kirchgässner: Stability of Fronts for a KPP-System, II: The Critical Case, J. Differential Equations **146** (1998), 399–456.

[Sa] D.H. Sattinger: On the Stability of Waves of Nonlinear Parabolic Systems, Adv. Math. **22** (1976), 312–355.

[Wa] C.E. Wayne: Invariant Manifolds for Parabolic Partial Differential Equations in Unbounded Domains, Arch. Rat. Mech. Anal. **138** (1997), 279–306.

[Za] E. Zauderer: *Partial Differential Equations of Applied Mathematics*, New York, John Wiley (1983).