ON THE ISING MODEL WITH STRONGLY ANISOTROPIC EXTERNAL FIELD

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Abstract

In this paper we analyze the equilibrium phase diagram of two-dimensional ferromagnetic n.n. Ising model when the external field takes alternating signs on different rows. We show that some of zero temperature coexistence lines disappear at every positive temperature whereas one (and only one) of them persists for sufficiently low temperature.

Keywords: Ising Model , anisotropic field, phase diagram, cluster expansion

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Section 1. Introduction

This paper concerns the low temperature phase diagram of highly anisotropic Ising Models; we will mainly be concerned with the two-dimensional Ising model with alternating field.

Let N be an even integer and Λ a torus with side N (Λ is a $N \times N$ square where the opposite sides are identified).

The hamiltonian is

$$H_\Lambda(\sigma) = -\frac{J}{2} \sum_{i,j\in\Lambda:|i-j|=1} \sigma_i\sigma_j - \frac{h_1}{2} \sum_{i\in\Lambda_1} \sigma_i + \frac{h_2}{2} \sum_{i\in\Lambda_2} \sigma_i \tag{1.1}$$

where $\sigma_i \in \{-1,+1\}, \quad \sigma \in \Omega_\Lambda := \{-1,+1\}^\Lambda, \quad J > 0, \quad h_1 \geq 0, \quad h_2 \geq 0.$

Λ_1 is the union of the odd rows, whereas Λ_2 is the union of the even rows in Λ:

$$\Lambda_1 = \{x = (x_1, x_2) \in \Lambda, \qquad x_2 \text{ odd}\} \tag{1.2}$$

$$\Lambda_2 = \{x = (x_1, x_2) \in \Lambda, \qquad x_2 \text{ even}\} \tag{1.3}$$

The partition function is

$$Z = \sum_{\sigma\in\Omega_\Lambda} e^{-\beta H_\Lambda(\sigma)}. \tag{1.4}$$

The phase diagram in the h_1, h_2 plane at $T = 0$ is represented in Fig (2.1). Mainly three different configurations appear as ground states in different regions: all pluses $+\underline{1}$, all minuses $-\underline{1}$ and a staggered configuration $\pm\underline{1}$ with all pluses (all minuses) on even (odd) horizontal lines. They are separated by the coexistence lines:

- $h_1 > 2J, \ h_2 = 2J$ that we will call line a
- $h_2 > 2J, \ h_1 = 2J$ that we call b

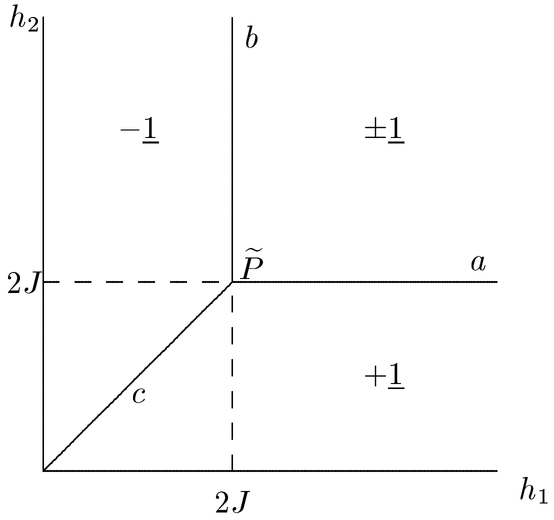


Fig. 2.1 T=0 phase diagram

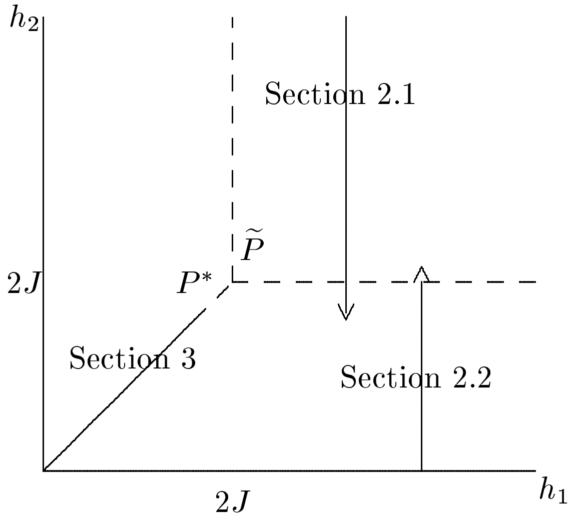


Fig. 2.2 T $>$ 0 phase diagram

- $h_1 = h_2 < 2J$ that we will call c.

  They converge in the triple point $\widetilde{P} \equiv (2J, 2J)$ (see Fig.2.1).

We will mainly analyze:

i) the vicinities of the line a ($T = 0$ coexistence line between $\pm\underline{1}$ and $+\underline{1}$), together with its symmetrical b (between $\pm\underline{1}$ and $-\underline{1}$); and

ii) the vicinities of the line c ($T = 0$ coexistence line between $+\underline{1}$ and $-\underline{1}$).

Of course we will be particularly interested in the vicinities of the triple point $\widetilde{P}$.

As we will explain, we expect that a and b zero-temperature coexistence lines will disappear at any (arbitrarily small) positive temperature $T = \frac{1}{\beta}$, whereas the c line will persist at least for sufficiently low temperature. The ending point of the c line is called $P^* = P^*(\beta)$ (see fig. 2.2). We will see that $\mathrm{dist}(P^*(\beta), \widetilde{P}) < ke^{-\beta J}$, k being suitable positive constant.

Moreover we will be able to prove uniqueness outside a small tube centered around

$\{h_1, h_2 : h_1 = h_2 < 2J - ke^{-\beta J}\}$; in particular in

$$U = \{(h_1, h_2) \quad \text{s.t.} \quad \max\{h_1, h_2\} > 2J + k' e^{-\beta J}\}. \tag{1.5}$$

The behaviour of our model depends on how much the values of h_1 and h_2 differ from $2J$. Let us call the fields h_1, h_2 "strong" resp. "weak" resp. "marginal" according to whether they are "sufficiently bigger" resp. "sufficiently smaller than" resp. "approximately equal to" the value $2J$. See sections 2.1 and 2.2 for a detailed discussion of the case when h_1 is "strong" and h_2 is arbitrary (with the emphasis, in both sections 2.1 and 2.2, on the "marginal" values of h_2).

Let us outline what kind of behaviour we expect in the different regions of parameters around the "hook" depicted in Figure 2.3:

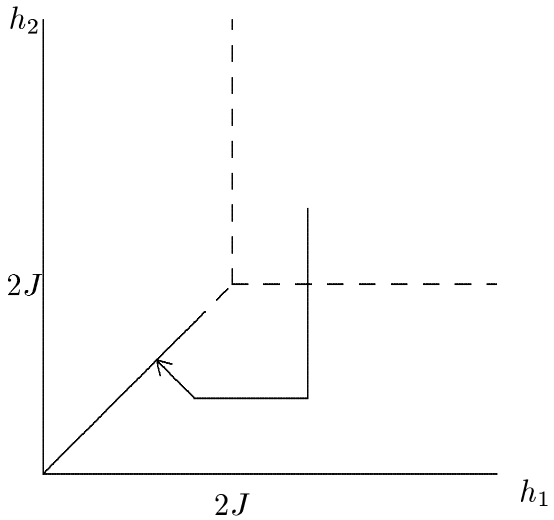


Fig. 2.3 The path in the space of parameters ("hook")

Let us start in the upper part of the hook, in some "strong" value $h_1 = h_2$. The picture observed there and around (in the region where both h_1 and h_2 are sufficiently bigger than $2J$) can be suitably described using the language of so called "rods".

These objects will be defined as segments of spins having the value opposite to the "recommended one" in the corresponding row. (Recommended by the groundstate, which has a stripped $+/-$ structure in such a case.)

In the Λ_1 lines we will thus have the "$-$ rods" (introduced and thoroughly discussed below in sections 2.1 and 2.2) and analogously we will have "+ rods" for the Λ_2 lines. These rods are rare and almost independent objects; more precisely there is some effective repulsion between the neighboring $-$ rods living in the Λ_1 lines and $+$ rods living in the Λ_2 lines.

In fact, the above region of parameters is not at all studied in detail in this paper; however its discussion is slightly easier than that of our main case of strong h_1 and arbitrary (but most interestingly marginal) h_2 which is really the most delicate one (toghether with its symmetrical: h_2 strong h_1 marginal).

So we concentrate our discussion on the latter case. Then the concept of a $+$ rod, as a "rare" event in the Λ_2 line, has to be abandoned. (In fact, we will see that plus spins will prevail in Λ_2 for sufficiently weak h_2 fields.) However, the concept of a $-$ rod in Λ_1 still plays a decisive role in our investigation.

In section 2.1 we show that we have the following picture, for h_2 reaching from above the value $2J$: For a typical configuration, one observes rare $-$ rods, denoted as r, in Λ_1 surrounded almost always by minuses above and below in what we will later call $\hat{r}$. These minuses in $\hat{r} \setminus r$ will force the remaining nearby Λ_2 spins to take mainly the minus value even for the external field h_2 being slightly below the value $2J$ (where, we recall, pure plus is already the ground state).

However, our leading strategy in 2.1, and also in 2.2, is not to look at the details of the behaviour in Λ_2 outside of the intersection of Λ_2 with the union of all the above $\hat{r}$

The approach of section 2.1 will be applicable for all strong and marginal values of h_2, but the really delicate case is the latter one. When further lowering h_2 to the marginal values around $2J$ and below we enter another region, studied in section 2.2.

The approach we use there will be applicable both to the weak and (more interestingly, too) the marginal values of h_2. Here we will have, for values h_2 sufficiently lower than $2J$, a different picture in Λ_2. This picture is quickly, but smoothly changing when h_2 reaches the marginal values. Namely, the picture one observes, for a weak field h_2, can be described as only a short range influence of the Λ_1 rods (which are always rare) on the behaviour in the Λ_2 lines.

Thus, we get mainly plus spins in the points of Λ_2 which are far from the rods. Again, we do not want to look at the details of the behaviour of the spins in the Λ_2 lines (outside of the "$-$appendices" going out of r and described in section 2.2).

So we expect a quick but smooth jump in the mean magnetizatin in the Λ_2 lines for the marginal values of h_2, with a steepest point being placed somewhere slightly below the value $h_2 = 2J$.

When further lowering the value h_2 to weak values significantly below $2J$, the minuses in Λ_2 start also to be very rare. Their segments, namely those not attached to the rods (the former will be called $-$appendices in the section 2.2) deserve also a special name. Let us call them "twigs" here.

Thus the picture for weak h_2 (well below $2J$) is the following one. Not only the rods (with minus values of spins being typically extended to the neighborhood, in Λ_2, of these rods), but also the "twigs" i.e. the segments of minuses in Λ_2, which are disconnected to rods, appear only scarcely in the Λ_2 lines. However, it is now the value $2J - h_2 > 0$ which will be "responsible" (for weak and marginal h_1) for the fact that these twigs (and rods, too) are sufficiently damped.

Thus we have essentially a low temperature gas of "legs" of two kinds: extended rods (of minuses centered in the Λ_1 lines) and segments, called twigs, of minuses "unprotected" by some rod. The latter live in the Λ_2 lines. Both these objects are rare for h_2 well below $2J$ and they allow, for h_2 sufficiently weak and at the same time sufficiently smaller than h_1 (so that the energy of the pure $+$ ground state would remain well below the energy of the pure $-$!), a description by low temperature cluster expansion.

We notice that the twigs in the Λ_2 lines are much more frequent than rods, though also sufficiently damped if h_2 is kept sufficiently weak. The rods centered in Λ_1 are rare as always (in the considered region of parameters h_1, h_2) however their control is now possible, even when lowering the value h_1 below $2J$, because it is the quantity h_2 which is held sufficiently smaller than $2J$. However, this last case is again not discussed in detail in our paper. In fact, the situation here is again a little bit less delicate than in our main case of marginal values of h_2 (and strong h_1).

Now, having kept h_2 considerably smaller than $2J$ and lowering the value of h_1 down to the value $h_2 = 2J$ and then further in direction to the line $h_1 = h_2$ we approach the region where another "local ground state" namely the pure $-$ configuration has to be also taken in account. What happens ?

When finally reaching the region of weak fields around $h_1 = h_2$, it is necessary to use another approach. In fact, the control over the large (in the horizontal direction) islands of minuses in the pure plus ground state (up to now, these islands were viewed as conglomerates of rods and twigs) is now lost. There is, in fact, a strong attraction of the rods in such a weak field regime. So yet another approach is needed.

However, we can "glue" together our twigs and (extended) rods $\bar{r}$ into connected components, and look only at the boundaries of these components. Then we come to the picture of usual Ising contours, and this turns out to be an appropriate approach in the region where h_1 and h_2 are roughly the same. Indeed, as we describe it in section 3, it is then possible to apply a variant of the usual Peierls argument.

The difficulty is that for our "segmental" contours (having a considerable energy only around the verticals) the summation over contours is a more delicate problem than in the usual isotropic case. Nevertheless we prove that at the axis $h_1 = h_2$ we have a phase transition.

We emphasise that in section 3 we consider only the case of the line $h_1 = h_2$. We completely omit the surrounding region where, of course, we expect the uniqueness. However, a rigorous proof of this fact is postponed to some future work.

Let us summarize once again what we observed along our itineration in the h_1, h_2 plane shown in figure 2.3. While the picture in the Λ_1 lines (where we observe mainly pluses!) is almost constant (namely rare $-$ rods) and changing smoothly (even analytically) during our whole path, the behaviour in the Λ_2 lines has (for a strong field $h_1 > 2J$) a smooth but quick jump (from the prevailing value $-$ to the prevailing value $+$) slightly below the value $h_2 = 2J$.

Below we give some more details on the main results of our Sections 2, 3 and 4: In section 2, we prove uniqueness in the region U of "strong" fields h_1 (or h_2). In particular we will analyze the halfline a: $h_2 = 2J, h_1 > 2J + k'e^{-\beta J}$. (Of course, by symmetry, the halfline b: $h_1 = 2J, h_2 > 2J + k'e^{-\beta J}$ can be treated in the same manner.) As we noted, that these two half-lines and their vicinity are the most difficult to deal with in the uniqueness region U (see (1.5)).

We will determine the set of values of $\vartheta > 0$ such that around the half-line $h_2 = 2J, \quad h_1 = 2J+\vartheta$, we will have uniqueness of the infinite volume Gibbs measure, decay of correlations and other typical properties of the one-phase situation. We will take $\vartheta \geq \kappa e^{-c\beta}$, $c \geq 0$, $\kappa \geq 0$ where κ will depedend on c. It will turn out that we can choose c equal the coupling constant J. The central point of our analysis will be to evaluate how close to the "triple point" $h_1 = h_2 = 2J$ we can go, still remaining in the uniqueness region. (It will appear clear that if for a given $\vartheta > 0$ we have uniqueness of the Gibbs measure, then for any $\vartheta' > \vartheta$ this property will continue to hold.)

In order to prove uniqueness we will use a perturbative approach. We will show that for $h_2 = 2J, \quad h_1 = 2J + \vartheta$, with $\vartheta = \kappa e^{-c\beta}, \ \kappa, \ c$ being a suitable positive constants, our system will exhibit a behaviour close to the one of a "reference system" where the spins on the odd rows of Λ_1 (where the external magnetic field takes the positive value $h_1 = 2J + \vartheta$) are "frozen" to the value $+1$ whereas on the even rows in Λ_2, we have a set of independent, one dimensional, zero-field systems. The heuristic argument for this is the following one: due to the excess ϑ of the external magnetic field on the odd rows in Λ_1 w.r.t. the maximal negative molecular field created by contiguous spins in the even rows in Λ_2, we have that, at very low temperature, the spins on Λ_1 will tend to be positive with overwhelming probability. Then the spins in Λ_2 will typically feel a vanishing effective field (given, on each site $x \in \Lambda_2$, by the sum of the negative external magnetic field $-2J$ plus the molecular field $2J$ generated by the two positive spins in Λ_1 nearest neighbours to x); thus on the rows of Λ_2 we will have one dimensional, zero field, Ising systems, which are independent from each other since they are separated by $+1$ rows in Λ_1. In the real situation we will certainly have some remarkable differences w.r.t. this extreme situation: the main perturbation, at any small but positive temperature, that we will observe with probability of the order $e^{-\beta J}$ will be the presence of rare segments of minus spins on the odd rows in Λ_1; they are called "rods". The possibility of the appearance of the rods on Λ_1 will break the independence of the even rows inducing a "comunication" between them.

Our strategy will be based on the use of the methods of cluster expansion. We will perturbatively treat the rods and we will evaluate the effective interaction between two contiguous rods due to the presence of finite volume zero field one dimensional Ising systems on the segments in Λ_2 between them. This effective interaction will turn out to decay exponentially fast on the scale of the correlation length of the zero-field one dimensional Ising system. This correlation length is easily seen to behave, for large β, as $e^{J\beta}$. There will be a competition between the typical density of the rods and the rate of decay of their effective interaction; this will imply a relationship between the minimal possible value for ϑ and the correlation length of the one dimensional zero field Ising model.

In Section 3 we will show coexistence in the c line $h_1 = h_2 < 2J - ke^{-\beta J}$. The heuristics for this result is the following one: on this line our model behaves similarly to the anisotropic Ising model with vertical coupling constant $J_1 = J$ and horizontal coupling constant $J_2 = J - \frac{h}{2}$. Namely a kind of a suitably adapted Peierls argument can be applied there. However when J_2 is exponentially small in β the typical Peierls contours become almost one-dimensional and the summation over the activities of contours requires some more care than usually for small J_2. (In fact, for very small $J_2 - h_2$, $h = h_2 = h_1$ we expect a second order phase transition but we are unable to prove it rigorously.)

Thus to show the existence of a phase transition via the Peierls argument, we describe the configurations in terms of contours. We first make a comparison with the above mentioned anisotropic Ising model and then we perform the sum over the corresponding contours (passing through a fixed point) in such a way to exploit the almost one-dimensionality of the contours.

Technically, the main novelty of our paper is a use of cluster expansion techniques in situations where strongly anisotropic, "segmental" polymers appear. In section 4 we give some general outlook of such polymers, expecting that in future work such a general approach could be useful. We also found a new, indeed very elementary proof of the basic cluster expansion result of Kotecký and Preiss (which was used several times throughout our paper). We explain it in section 4.1.

Section 2. Uniqueness region.

2.1. The region $\mathbf{h_1} > \mathbf{2J} + \vartheta$ and $h_2 > 2J - \delta$, $\delta\beta \leq e^{-\beta J}$, $\vartheta \gg e^{-\beta J}$.

We will start analyzing the partition function of our system enclosed in a finite volume with periodic boundary conditions. We will trasform our original system into a gas of polymers with small activity. Then, by using general methods of the theory of cluster expansion, we will be able to perform the thermodynamic limit. By similar arguments we will also be able to analyze the correlation functions and prove uniqueness of the infinite volume Gibbs measure as well as decay of truncated correlations.

We will show, and this will be the main result of Section 2, that our partition function can be expressed as

$$Z = \tilde{Z}\Xi \tag{2.1}$$

where $\tilde{Z}$ is a simple term representing the partition function of our reference system whereas Ξ is the partition function (Lemma 2.2) of a gas of polymers with small activities (Theorem 2.8).

Let us start our computation of the partition function.

We first introduce the approach convenient to describe the region above the line a given by: $h_1 > 2J,\ \ 2J = h_2$; it can be pushed also a little under the line. Indeed we have the limitation that $\delta = 2J - h_2$ is either negative or, if positive, it has to be less than $\frac{1}{8\beta}\varepsilon$ where

$$\varepsilon = e^{-\beta J}, \tag{2.2}$$

and $\vartheta = h_1 - 2J$. The staggered configuration $\sigma^{\pm}$ is given by

$$\sigma^{\pm}(x) = \begin{cases} +1 & x \in \Lambda_1 \\ -1 & x \in \Lambda_2 \end{cases} \tag{2.3}$$

We write

$$Z = \sum_{\sigma \in \Omega_\Lambda} e^{-\beta(H(\sigma) - H(\sigma\pm))} \qquad (2.4);$$

indeed in the region of parameters that we are presently considering, we take as reference configuration the staggered one $\sigma^{\pm}$ so that in (2.4) we are subtracting $H(\sigma^{\pm})$. In the other region $(h_1 > 2J, h_2 < 2J)$ we will subtract $H(+\underline{1})$ (see (2.82) below).

Def. 2.1. Given $\sigma \in \Omega_\Lambda$, a maximal connected component (segment) of -1 spins in Λ_1 is called *rod*; for $\mathbf{x} \in \Lambda_1, \quad l \in N$, we denote by $r_{\mathbf{x},l}$ the rod whose extreme left point is $\mathbf{x} = (x_1, x_2)$ and whose length (cardinality) is l

$$r_{\mathbf{x},l} = \{\mathbf{y} = (y_1, y_2) \in \Lambda_1 \mid x_1 \le y_1 \le x_1 + l - 1 \quad \text{and } y_2 = x_2\} \tag{2.5}$$

We use the same symbol $r_{\mathbf{x},l}$ also to denote the union of unit cubes centered at the sites in $r_{\mathbf{x},l}$.

Def. 2.2. Two rods $r_{\mathbf{x}^1,l_1}$ and $r_{\mathbf{x}^2,l_2}$ are *incompatible* if they intersect:

$$r_{\mathbf{x}^1,l_1} \cap r_{\mathbf{x}^2,l_2} \neq \emptyset \tag{2.6}$$

We will call R_Λ the set of all possible rods (with any size and location) in Λ_1. Given $\sigma \in \Omega_\Lambda$ we denote by $L(\sigma)$ the family of coordinates of compatible rods generated by σ. We write:

$$L(\sigma) = \left\{ (\mathbf{x}^1, l_1) \dots (\mathbf{x}^n, l_n), \quad r_{\mathbf{x}^j, l_j} \text{is a rod in } \sigma \right\} \tag{2.7.}$$

Let

$$\tilde{\mathrm{L}} = \bigcup_{(\mathbf{x},l) \in L(\sigma)} r_{\mathbf{x},l}. \tag{2.8}$$

be the *support* of the set of rods $r_{\mathbf{x},l}$ such that $(\mathbf{x}, l) \in L(\sigma)$; notice that, whereas this set of rods is an element of $(\mathrm{R}_\Lambda)^n$, for some n, $\tilde{\mathrm{L}}$ is a subset of Λ_1. We denote by

$$\mathcal{L} = \{ L(\sigma) : \quad \sigma \in \Omega_\Lambda \} \tag{2.9}$$

the set of all possible families of compatible rods in Λ_1. We call $\sigma(r_{\mathbf{x},l})$ the configuration σ where the -1 spins are precisely the ones in $r_{\mathbf{x},l}$ and in Λ_2; $\zeta(r_{\mathbf{x},l})$ is the weight (or activity) associated to the rod $r_{\mathbf{x},l}$. We get

$$\zeta(r_{\mathbf{x},l}) := e^{-\beta[H(\sigma(r_{\mathbf{x},l})) - H(\sigma \pm)]} = \varepsilon^2 e^{-\vartheta l} \tag{2.10}$$

We recall that $\varepsilon = e^{-J\beta}$. In what follows we will choose $\vartheta = \kappa \varepsilon$ for some positive constant κ. We have:

$$Z_\Lambda = \sum_{n \geq 0} \sum_{L = \{(\mathbf{x}^1, l_1) \dots (\mathbf{x}^n, l_n)\} \in \mathcal{L}} \left[\prod_{j=0}^{n} \varepsilon^2 \exp(-\beta \vartheta l_j) \right] Z_{\Lambda_2}^{\{L\}} \tag{2.11}$$

where we use the convention that if $n = 0$ then $\sum_L$ is absent, while $\prod_{j=0}^{0} = 1$. $Z_{\Lambda_2}^{\{L\}}$ denotes the partition function on Λ_2 with boundary conditions given by -1 on $\widetilde{\mathrm{L}}$ and $+1$ elsewhere. Again for $n = 0$ we set $\widetilde{\mathrm{L}} = \bigcup_{j=0}^{0} r_{\mathbf{x}^j, l_j} = \emptyset$.

Def. 2.3. We call Λ_2-*enlargement* of a rod $r_{\mathbf{x},l}$, and denote it by $\hat{r}_{\mathbf{x},l}$, the set of sites immediately above and below $r_{\mathbf{x},l}$ $\big(\, \mathbf{x} = (x_1, x_2) \,\big)$:

$$\hat{r}_{\mathbf{x},l} = \{\mathbf{y} = (y_1, y_2) \ \in \Lambda_2 \quad | \qquad x_1 \leq y_1 \leq x_1 + l - 1 \quad \text{and } y_2 = x_2 \pm 1\} \tag{2.12}$$

We set

$$\bar{r}_{\mathbf{x},l} = \hat{r}_{\mathbf{x},l} \cup r_{\mathbf{x},l} \tag{2.13}$$

and call it *extended rod.*

Let us denote by $E(r_{\mathbf{x},l})$ the set of four extreme points of $\hat{r}_{\mathbf{x},l}$:

$$E(r_{\mathbf{x},l}) = \{(x_1, x_2 \pm 1); \quad (x_1 + l - 1, x_2 \pm 1)\}\,. \tag{2.14}$$

Given $\sigma \in \Omega_\Lambda$, we denote by $\mathrm{C}(\sigma)$ the set of maximal connected components (segments $\rho \subset \Lambda_2$) of $+1$ spins in Λ_2 that have non-empty intersection with some $\hat{r}_{\mathbf{x},l}$. We emphasize that for any $h_2 > 0$, if we suppose that the spins in $r_{\mathbf{x},l}$ are -1, then the typical value of the spins in $\hat{r}_{\mathbf{x},l}$ will be also minus. With this in mind we introduce the following definitions.

Def. 2.4. Given $\sigma \in \Omega_\Lambda$, we call *protuberances* and denote by p the components ρ completely contained in $\hat{r}_{\mathbf{x},l} \setminus E(\hat{r}_{\mathbf{x},l})$ for at least one $(\mathbf{x}, l) \in L(\sigma)$. We denote by $\mathrm{P}(\sigma)$ the set of all protuberances appearing in the configuration σ:

$$\mathrm{P}(\sigma) = \{\rho \in \mathrm{C}(\sigma) : \quad \exists (\mathbf{x}, l) \in L(\sigma) : \ \rho \subset \hat{r}_{\mathbf{x},l} \setminus E(\hat{r}_{\mathbf{x},l})\}\,. \tag{2.15}$$

Def. 2.5. We call *appendices* of σ and denote by a the other components ρ in $C(\sigma)$. $\mathrm{A}(\sigma)$ is the set of appendices appearing in σ

$$\mathrm{A}(\sigma) = \mathrm{C}(\sigma) \setminus \mathrm{P}(\sigma). \tag{2.16}$$

Def. 2.6. Two elements a, a' (appendices), p, p' (protuberances), a, p are *incompatible* if they intersect.

Remark 2.1. Every appendix contains at least one site in $E(r_{\mathbf{x},l})$ for some rod $r_{\mathbf{x},l}$; thus we can write, for the length $|a|$ of an appendix a:

$$|a| = l_1 + l_2$$

where $l_1 \geq 1$ is the length of the part of a contained in $\hat{r}_{\mathbf{x},l}$ for some $(\mathbf{x}, l)$ whereas $l_2 \geq 0$ is the length of the rest (part of a outside any $\hat{r}_{\mathbf{x},l}$).

We set:

$$\forall a \in \mathrm{A}(\sigma): \ |a| = l_1 + l_2 \qquad \zeta_0(a) = \varepsilon^2 (\varepsilon^2 \exp(\delta\beta))^{l_1} \exp(\delta\beta)^{l_2} \tag{2.17}$$

$$\forall p \in \mathrm{P}(\sigma): \ |p| = l \qquad \zeta_0(p) = \varepsilon^2 (\varepsilon^2 \exp(\delta\beta))^{l} \tag{2.18}$$

We recall that $\delta = 2J - h_2$ is positive below the line a, but negative above. Given $\mathrm{L} \in \mathcal{L}$, a pair of families A, P of (compatible) appendices and protuberances is called "L- compatible " if there exists $\sigma \in \Omega$ such that $\mathrm{L}(\sigma), \mathrm{A}(\sigma), \mathrm{P}(\sigma) = \mathrm{L}, \mathrm{A}, \mathrm{P}$. We denote by $\widetilde{\mathrm{A}}$, $\widetilde{\mathrm{P}} \subset \Lambda_2$ the supports of A, P, respectively:

$$\widetilde{\mathrm{A}} = \bigcup_{a \in \mathrm{A}} a \qquad \widetilde{\mathrm{P}} = \bigcup_{p \in \mathrm{P}} p$$

The set of all families of appendices and protuberances (A, P) compatible with L is denoted by

$$\big(\mathcal{A}(\mathrm{L}), \mathcal{P}(\mathrm{L})\big) = \{\mathrm{A}, \mathrm{P}: \quad a, a' \in \mathrm{A}, \ p, p' \in \mathrm{P}, \ a, p \in (\mathrm{A}, \mathrm{P}) \text{ are pairwise compatible and L-compatible}\} \tag{2.19}$$

Given a compatible set L,A we denote by $\bar{\mathrm{A}}$ the extension of $\widetilde{\mathrm{A}}$ to the nearest neighbour sites in Λ_2 not contained in any $\hat{r}_{\mathbf{x},l}$ with $(\mathbf{x},l)\in \mathrm{L}$:

$$\bar{\mathrm{A}} = \widetilde{\mathrm{A}} \cup \left\{ \mathbf{y}\in\Lambda_2 : \mathrm{dist}(\mathbf{y},\widetilde{\mathrm{A}}) = 1, \quad \mathbf{y}\cap\left(\bigcup_{(\mathbf{x},l)\in\mathrm{L}} \hat{r}_{\mathbf{x},l}\right) = \emptyset \right\} \tag{2.20}$$

if $\widetilde{\mathrm{A}}=\emptyset$ we set $\bar{\mathrm{A}}=\emptyset$. We can write:

$$Z_\Lambda = \sum_{n\geq 0} \sum_{\mathrm{L}=\{(\mathbf{x}^1,l_1)\dots(\mathbf{x}^n,l_n)\}\in\mathcal{L}} \left[\prod_{j=0}^{n} \varepsilon^2 \exp(-\beta\vartheta l_j)\,\cdot \right.$$

$$\left. \cdot \sum_{(\mathrm{P},\mathrm{A})\in(\mathcal{P}(\mathrm{L}),\mathcal{A}(\mathrm{L}))} \prod_{p\in\mathrm{P}} \zeta_0(p) \prod_{a\in\mathrm{A}} \zeta_0(a)\; Z_{\Lambda_2}^{\{\mathrm{L};\,\mathrm{A};\,\mathrm{P}\}} \right] \tag{2.21}$$

where $Z_{\Lambda_2}^{\{\mathrm{L};\,\mathrm{A};\,\mathrm{P}\}}$ is the partition function in $\Lambda_2\setminus\left[\bar{\mathrm{A}}\bigcup(\bigcup_{\mathbf{x},l\in\mathrm{L}}\hat{r}_{\mathbf{x},l})\right]$ with $+$ boundary conditions on the contiguous sites on Λ_1 and $-$ boundary conditions on the contiguous sites in Λ_2.

Then $Z_{\Lambda_2}^{\{\mathrm{L};\,\mathrm{A};\,\mathrm{P}\}}$ splits into the product of partition functions of one dimensional Ising systems with external field δ on some intervals in Λ_2 with $-$ boundary conditions on their extrema. Given L, A, P we set: $\Lambda_2\setminus\left[\bar{\mathrm{A}}\bigcup(\bigcup_{\mathbf{x},l\in\mathrm{L}}\hat{r}_{\mathbf{x},l})\right]=\bigcup_j g_j$. The g_j are disjoint intervals that lie in Λ_2 between pairs of sites which are either extreme points of some extended appendices in $\bar{\mathrm{A}}$ or extreme points of some $\hat{r}_{\mathbf{x},l}$. This latter case corresponds to saying that in $y\in E(r_{\mathbf{x},l})$ we have $\sigma(y)=-1$.

The interior sites of the intervals g_j are free and we are going to sum up over their values obtaining, in this way, the partition function of a δ-field, one dimensional Ising model on an interval of length l with -1 boundary condition. Given L, A, P we denote by $\mathrm{G}(\mathrm{L},\mathrm{A},\mathrm{P})$ the set of the above defined intervals $\{g_j\}$. In what follows we will denote by $Z_l^{\tau,\,\tau'}(\delta,-)$ the partition function of a one dimensional Ising model on the interval $\{1,\ldots,l\}$ with external field δ, and $\tau,\ \tau'$ boundary conditions when the reference configuration (zero of the energy) is $-\underline{1}$ (all minuses)

$$Z_l^{\tau,\,\tau'}(\delta,-)=\sum_{\sigma\in\{-1,+1\}^l}e^{-\beta[H(\sigma|\tau\,\tau')-H(-\underline{1}|\tau\,\tau')]}\tag{2.22}$$

where

$$H(\sigma|\tau\,\tau')=-\frac{J}{2}\tau\sigma_1+\frac{J}{2}\sum_{i=1}^{l-1}\sigma_i\sigma_{i+1}-\frac{J}{2}\tau'\sigma_l-\frac{\delta}{2}\sum_{i=1}^{l}\sigma_i\tag{2.23}$$

Proposition 2.1

There exist costants $c_1(\delta,\tau,\tau'),\ c_2(\delta,\tau,\tau')$

$$Z_l^{\tau,\tau'}(\delta,-)=c_1(\delta,\tau,\tau')\lambda_0^{l+1}\left(1+c_2(\delta,\tau,\tau')\left(\frac{\lambda_1}{\lambda_0}\right)^{l+1}\right)\tag{2.24}$$

where

$$\lambda_{0,1}=\frac{1+e^{\delta\beta}}{2}\pm\sqrt{\frac{(1-e^{\delta\beta})^2}{4}+e^{(-2J+\delta)\beta}}\tag{2.25}$$

If the external field is $\delta=0$ *and* $\tau=\tau'=-1$*, we have* $\lambda_0=1+\varepsilon,\ \lambda_1=1-\varepsilon$ *and* $c_1(0,\tau,\tau')=\frac{1}{2},\quad c_2(0,\tau,\tau')=1.$

Proof

The proposition follows from an elementary computation, based on the transfer matrix. Indeed in the case of periodic boundary condition one has $Z^{per}(\delta,-)=\mathrm{Tr}A^l(\delta,-)$ where

$$A(\delta,-)=\begin{pmatrix}e^{\beta\delta} & e^{(\frac{\delta}{2}-J)\beta}\\ e^{(\frac{\delta}{2}-J)\beta} & 1\end{pmatrix}$$

is the corresponding transfer matrix. While for a generic boundary condition τ,τ' outside $[1,\ldots,l]$ one has

$$Z_l^{\tau,\tau'}(\delta,-)=(v_\tau,A^l(\delta,-)v_{\tau'})$$

where $v_\mathcal{T}, v_{\mathcal{T}'}$ are suitable vectors. See also section 4.2 for a different derivation. $\square$

Using Proposition 2.1

$$Z_l^{--}(\delta,-) = c_1(\delta,--)\lambda_0^{l+1}\left(1 + c_2(\delta,--)\left(\frac{\lambda_1}{\lambda_0}\right)^{l+1}\right) \tag{2.26}$$

This sum over the values of the spins on the sites of $\Lambda_2 \setminus \left[\bar{\mathrm{A}} \bigcup (\bigcup_{\mathbf{x},l\in \mathrm{L}} \hat{r}_{\mathbf{x},l})\right]$ gives rise to an effective interaction between pairs of rods.

We want now to put a factor λ_0 in evidence for each site in Λ_2 in the r. h. s. of (2.21); we get:

$$Z_\Lambda = \lambda_0^{\frac{N^2}{2}} \left\{ \sum_{n\geq 0} \sum_{(\mathbf{x}^1,l_1)\dots(\mathbf{x}^n,l_n)\in\mathcal{L}} \left[\prod_{j=0}^{n} \varepsilon^2 \exp(-\beta\vartheta l_j) \left(\frac{1}{\lambda_0}\right)^{|\cup_{\mathbf{x},l}\hat{r}_{\mathbf{x},l}|} \cdot \right.\right.$$

$$\left.\left. \cdot \sum_{(\mathrm{P},\mathrm{A})\in(\mathcal{P}(\mathrm{L}),\mathcal{A}(\mathrm{L}))} \left(\frac{1}{\lambda_0}\right)^{|\bar{\mathrm{A}}\setminus\widetilde{\mathrm{A}}|} \prod_{p\in\mathrm{P}} \zeta(p) \prod_{a\in\mathrm{A}} \zeta(a) \prod_{g\in\mathrm{G}(\mathrm{L},\mathrm{A},\mathrm{P})} \zeta(g) \right] \right\} \tag{2.27}$$

where

$$\zeta(p) = \zeta_0(p)\left(\frac{1}{\lambda_0}\right)^l = \varepsilon^2\left(\frac{\varepsilon^2\exp(\delta\beta)}{\lambda_0}\right)^l \leq \varepsilon^2\left(\varepsilon^2(1-\varepsilon)\right)^l \tag{2.28}$$

$$\zeta(a) = \zeta_0(a)\left(\frac{1}{\lambda_0}\right)^{l_2} = \varepsilon^2\left(\frac{\varepsilon^2\exp(\delta\beta)}{\lambda_0}\right)^{l_1}\left(\frac{\exp(\delta\beta)}{\lambda_0}\right)^{l_2} \leq \varepsilon^2(\varepsilon^2(1-\varepsilon))^{l_1}\left(1-\frac{15}{16}\varepsilon\right)^{l_2} \tag{2.29}$$

$$\zeta(g) = c_1\lambda_0\left(1 + c_2\left(\frac{\lambda_1}{\lambda_0}\right)^{|g|+1}\right) \tag{2.30}$$

The last inequality in (2.29) comes from our choice $\delta\beta < \frac{\varepsilon}{8}$. Of course for negative δ we could have much better estimates but our emphasis is on $\delta = 0$.

Let us call *"leg"* an element that can be a rod, a protuberance or an appendix.

Remark 2.2. Notice that the number of factors $(\frac{1}{\lambda_0})$ that we get in (2.24) depends on the overlapping between the $\hat{r}_{\mathbf{x},l}$; this induces an (effective) interaction between rods; a similar effect takes place also for the other kinds of legs, thus, it makes sense to give the following definitions.

Def. 2.7. Two compatible rods $r_{\mathbf{x}^1,l_1}$, $r_{\mathbf{x}^2,l_2}$ are *interacting* if the intersection between their Λ_2-enlargements $\hat{r}_{\mathbf{x}^1,l_1}$, $\hat{r}_{\mathbf{x}^2,l_2}$ is non-empty: $\hat{r}_{\mathbf{x}^1,l_1} \cap \hat{r}_{\mathbf{x}^2,l_2} \neq \emptyset$.

Def. 2.8. A compatible pair given by a rod r and an appendix a (or a protuberance p) is *interacting* if the intersection between the appendix a (or the protuberance p) and the Λ_2-enlargement of the rod r is non-empty: $\hat{r}_{\mathbf{x}^1,l_1} \cap a \neq \emptyset$.

Def. 2.9. Two compatible appendices a, a' are *interacting* if there exist a site $\mathbf{x} \in a$ and a site $\mathbf{y} \in a'$ such that $d(\mathbf{x},\mathbf{y}) \leq 2$

We use symbol $\mathcal{I}$ to denote interaction; for example $r\mathcal{I}a$ means that the rod r interacts with the appendix a.

We now develop the product $\prod_{g\in \mathrm{G(L,A,P)}} \frac{\lambda_0}{2}\left(1+\left(\frac{\lambda_1}{\lambda_0}\right)^{|g|+1}\right)$. A single term of the development will correspond to a choice of the term 1 or $\left(\frac{\lambda_1}{\lambda_0}\right)^{|g|+1}$ for each $g \in \mathrm{G(L,A,P)}$.

Given a term of the above development we will call *bond* and denote by b those among the intervals g which hold the term $\left(\frac{\lambda_1}{\lambda_0}\right)^{|g|+1}$. The activity of a bond b is

$$\zeta(b) = c_2(\delta,--)\left(\frac{\lambda_1}{\lambda_0}\right)^{|b|+1} \leq (1-\varepsilon+o(\varepsilon))^{|b|+1} \leq (1-\varepsilon+o(\varepsilon))^{|b|+1} \leq \left(1-\frac{7}{4}\varepsilon\right)^{|b|+1} \tag{2.31}$$

Notice that $\zeta(b)$ has an expression partially similar to $\zeta(a)$ but without the prefactor

ε^2. Geometrically a bond is a generic interval in $\mathrm{G(L,A,P)}$; we can write:

$$Z_\Lambda = \lambda_0^{\frac{N^2}{2}} \left\{ \sum_{n\geq 0} \sum_{(\mathbf{x}^1,l_1)\dots(\mathbf{x}^n,l_n)\in\mathcal{L}} \left[\prod_{j=0}^{n} \varepsilon^2 \exp(-\beta\vartheta l_j) \left(\frac{1}{\lambda_0}\right)^{|\cup_{\mathbf{x},l}\hat{r}_{\mathbf{x},l}|} \cdot \right.\right.$$

$$\cdot \sum_{(\mathrm{P,A})\in(\mathcal{P}(\mathrm{L}),\mathcal{A}(\mathrm{L}))} \left(\frac{1}{\lambda_0}\right)^{|\bar{\mathrm{A}}\setminus\widetilde{\mathrm{A}}|} \prod_{p\in\mathrm{P}} \zeta(p) \prod_{a\in\mathrm{A}} \zeta(a)\lambda_0 c_1(\delta,--)^{|\mathrm{G(L,P,A)}|} \cdot$$

$$\left.\left. \cdot \sum_{\mathrm{B}\subset\mathrm{G(L,P,A)}} \prod_{b\in\mathrm{B}} c_2(\delta,--)\left(\frac{\lambda_1}{\lambda_0}\right)^{|b|+1} \right] \right\} \tag{2.32}$$

Def. 2.10. A rod r and a bond b are *interacting* if there exist a site $x\in r$ and a site $y\in b$ such that $\mathrm{d}(x,y)=\sqrt{2}$.

Def. 2.11. An appendix a and a bond b are *interacting* if there exist a site $x\in r$ and a site $y\in b$ such that $\mathrm{d}(x,y)=2$.

The above notions of interactions will be now used to define particular collections of legs and bonds.

Def. 2.12. Given a compatible set L, A, P, a *"conglomerate"* γ is a maximal connected component of interacting legs.

We are now ready to give the basic definition of polymer

Def. 2.13. Given a compatible set L, A, P, and a set B of bonds in G(L,A,P), a *polymer* Γ is a maximal component of interacting rods, appendices, protuberances and bonds.

Def. 2.14. For each rod we can define four corners: at the top to the left, at the top to the right at the bottom to the left and at the bottom to the right.

We call "vertex" a generic site belonging to a leg or to a bond; the support $\widetilde{\Gamma}$ of a polymer Γ is the union of its vertices.

Given a rod $r_{\mathbf{x},l}$, we denote by $\mathrm{C}(r_{\mathbf{x},l})$ the set of its (four) corners c. To any c we can associate a site $y(c) \in E(r_{\mathbf{x},l})$: the one which is contiguous to c; this correspondence is one to one when $l \geq 2$.

Given (L, A) we denote by $\bar{\mathrm{C}}(\mathrm{L}, \mathrm{A})$ the set of corners c belonging to a rod in L such that the corresponding $y(c)$ is connected to some interval g or belongs to some appendix $a \in \mathrm{A}(\sigma)$ connected to some g.

Given a conglomerate γ we call

$$\mathrm{L}(\gamma) = \{(\mathbf{x}, l) : \quad r_{\mathbf{x},l} \in \gamma\}$$

similarly, by $\mathrm{A}(\gamma)$, $\mathrm{P}(\gamma)$, we denote the set of appendices, or protuberances, respectively, belonging to γ. We have

$$\zeta(\gamma) = \prod_{(\mathbf{x},l)\in \mathrm{L}(\gamma)} \left[\zeta(r_{x,l}) \left(\sqrt{\lambda_0 c_1(\delta, --)} \right)^{|\mathrm{C}(r_{\mathbf{x},l}) \cap \bar{\mathrm{C}}(\mathrm{L},\mathrm{A})|} \right] \left(\frac{1}{\lambda_0} \right)^{|\cup_{\mathbf{x},l} \hat{r}_{\mathbf{x},l}|} \cdot$$

$$\cdot \qquad \left(\frac{1}{\lambda_0} \right)^{|\bar{\mathrm{A}} \setminus \widetilde{\mathrm{A}}|} \prod_{p\in \mathrm{P}(\gamma)} \zeta(p) \prod_{a \in \mathrm{A}(\gamma)} \zeta(a) \tag{2.33}$$

Remark 2.3. A conglomerate is nothing but a polymer Γ containing no bonds.

Given a polymer Γ, containing the conglomerates $\gamma_1, \ldots, \gamma_n$ and the bonds b_1, $\ldots$, b_m, the *weight* of Γ is given by

$$\zeta(\Gamma) = \prod_{i=1}^{n} \zeta(\gamma_i) \prod_{j=1}^{m} \zeta(b_j).$$

Using the above definitions of polymer and interaction we are now able to express our partition function as the one of a gas of polymers. We summarize our result in the following:

Lemma 2.2

Z_Λ can be expressed as:

$$Z_\Lambda = \lambda_0^{\frac{N^2}{2}} \left(1 + \sum_{n\geq 1} \sum_{\Gamma_1 \ldots \Gamma_n} \zeta(\Gamma_1)\ldots\zeta(\Gamma_n) \right) \tag{2.34}$$

where, the sum on the r.h.s. of the (2.34) runs over all families $\Gamma_1, \ldots, \Gamma_n$ of polymers in Λ such that for any i, j, Γ_i do not interact with Γ_j.

We will now look at a polymer Γ as a graph whose nodes are conglomerates and whose connections are given by bonds. Notice that every bond b has to be connected at each one of its two extremes to a rod or to an appendix belonging to some conglomerate. It may happen that a bond connects two vertices of the same conglomerate . In this case we say that we have an internal bond. Notice that, the polymers Γ, seen as graphs composed by conglomerates and bonds, do not have in general a tree structure, since they can contain loops. In particular every internal bond gives rise to a loop. It turns out that the difficult part in the computation of the sum over all polymers (passing throught a fixed point) comes from the bonds. In order to be able to control the contribution of the bonds, we reorganize the sum over the polymers in such a way to associate to every bond one conglomerate, whose weight is so small to compensate the (large) result of the sum over the bonds. To achieve this purpose it happens that it is useful to introduce a tree structure.

Lemma 2.3.

Given a polymer Γ we can find a tree $T \subset \Gamma$ (which is in general not unique) obtained from Γ by removing some bonds in Γ.

Proof

We will use a hierarchical construction. We choose a vertex v belonging to a conglomerate $\gamma_1 \subset \Gamma$. Then we consider all conglomerates that are connected to γ_1 by some bonds and we denote them by $\left\{\gamma_1^2 \dots \gamma_{k_2}^2\right\}$. They constitute the "second generation". We can iteratively define the subsequent generations. Given the j-th generation $\left\{\gamma_1^j \dots \gamma_{k_j}^j\right\}$, we define the $(j+1)$-th generation by taking all conglomerates that are connected at least by a bond to some conglomerate belonging to the j-th generation but not connected to any conglomerate belonging to some previous generation.

This hierarchical description provides a criterion to remove the bonds in order to obtain a tree T out of Γ. All bonds that connect the first with the second generation belong to the tree. Starting from $j = 2$, we consider the conglomerates of the j-th generation and we remove from Γ all internal bonds and all bonds that connect pairs of conglomerates both belonging to the same j-th generation. In order to get a tree structure, we want that any conglomerate of the $(j+1)$-st generation is connected by a single bond to a single conglomerate of the j-th generation. So if in Γ there is more than one bond connecting a given conglomerate γ_k^{j+1} (of the $(j+1)$-st generation) to $\cup_m \gamma_m^j$ (the set of conglomerates of the j-th generation), we keep in T the first one in lexicografic order (of the first extremum) and remove all the others. It is clear that, continuing in this way, we end up with a tree. $\square$

Given a polymer Γ, we call $\mathcal{T}(\Gamma)$ the tree T obtained via the above construction.

Remark 2.4. We know that the tree T contains all conglomerates of Γ, so the possible positions of the removed bonds is fixed.

Given a tree T consider:

1. any pair of corners of rods (belonging to the same or to two conglomerates) of T which lie contiguously on the same line of Λ_2 and that may be directly connected

by a bond (without appendices in between); or

2. any pair of corners of rods (belonging to the same or to two conglomerates) of T which lie contiguously on the same line of Λ_1 with one or two appendices emerging from them and such that a bond can connect the appendix to the corner or the two appendices.

A graph Γ such that $\mathcal{T}(\Gamma) = T$ can be obtained by deciding whether or not we put a bond (directly or via one or two appendices) between the above pairs of corners of rods. We can think to have introduced a dicotomic variable τ taking the value 1 if we add the bond, and zero if we do not add the bond.

Let us denote by $n_r(\gamma)$ the number of rods in γ.

Lemma 2.4.

$$\sum_{\Gamma:\,\mathcal{T}(\Gamma)=T} \zeta(\Gamma) \le \zeta(T) \prod_{\gamma\in T} 3^{4n_r(\gamma)}. \tag{2.35}$$

Proof

There are at most $2^{4n_r(\gamma)}$ choices of the set of bonds that we need to add to the tree in order to obtain a graph Γ such that $\mathcal{T}(\Gamma) = T$.

We have,

$$\sum_{\Gamma:\,\mathcal{T}(\Gamma)=T} \zeta(\Gamma) = \zeta(T) \sum_{\Gamma:\,\mathcal{T}(\Gamma)=T} \zeta(\Gamma\setminus T) \le \zeta(T) \prod_{\gamma\in T} \left(1 + 2^{4n_r(\gamma)} \max_b \zeta(b)\right) \le$$

$$\le \zeta(T) \prod_{\gamma\in T} 3^{4n_r(\gamma)} \prod_{\gamma\in T} \frac{1}{3^{4n_r(\gamma)}} \left(1 + 2^{4n_r(\gamma)} (1 - \frac{7}{4}\varepsilon)\right) \le \zeta(T) \prod_{\gamma\in T} 3^{4n_r(\gamma)}. \tag{2.36}$$

$\square$

Given $t > 1$, we introduce the modified weigth $\zeta_t(\gamma)$ of $\zeta(\gamma)$ (see (2.33)) given by

$$\zeta_t(\gamma) = \prod_{(\mathbf{x},l)\in \mathrm{L}(\gamma)} \left[t\,\zeta(r_{x,l}) \left(\sqrt{\lambda_0} c_1(\delta, --)\right)^{|\mathrm{C}(r_{\mathbf{x},l})\cap \bar{\mathrm{C}}(\mathrm{L},\mathrm{A})|} \left(\frac{1}{\lambda_0}\right)^{|\cup_{\mathbf{x},l} \hat{r}_{\mathbf{x},l}|} \right.\,.$$

$$\cdot \qquad \left(\frac{1}{\lambda_0}\right)^{|\bar{\mathrm{A}}\setminus\widetilde{\mathrm{A}}|} \prod_{p\in\mathrm{P}(\gamma)} \zeta(p) \prod_{a\in\mathrm{A}(\gamma)} \zeta(a)\Bigg] \tag{2.37}$$

This ''worsening" parameter t is introduced in order to subsequently control some entropic factors; the ''worsened" activity ζ_t represents an upper bound for the result of some partial sum. This will appear clear during the proof of the Theorem 2.6 (see, for instance (2.57) below). Notice that in (2.37) we multiplied the weight $\zeta(r)$ by a factor t and we left the other weights unchanged. (One has $\zeta(\gamma) = \zeta_1(\gamma)$) In the following Lemma 2.5 we perform the main step (and simplified version) of the Theorem 2.6.

Lemma 2.5. *For every $t : 0 < t < \frac{\vartheta}{10\varepsilon}$ and site $x \in \Lambda$ we have for ε sufficiently small*

$$\sum_{\gamma:\widetilde{\gamma}\ni x} \zeta_t(\gamma) \leq 16\ t\ (1+\varepsilon^2)^4 \frac{\varepsilon^2}{\vartheta} \tag{2.38}$$

Proof

First we compute the sum over all conglomerates containing a vertex x belonging to Λ_1 and then we extend this result to the vertices in Λ_2.

We remark that, given a site $x \in \Lambda_1$, a conglomerate γ can contain x only if there exists a rod $r \in \gamma$ containing x as one of its vertices. To compute this sum we use a hierarchic construction. Notice that we do not have a tree structure inside the conglomerates and now we only have legs (and no bonds): we simply use a method of sum which is reminiscent of a tree organization of the connections inside the graph given by a conglomerate. We take the rod that contains x and we call it r_0: it is the zeroth generation. We construct the 1-st generation taking all the legs that interact with r_0. So, by iteration, we construct all generations: if we have the j-th generation we construct the $(j+1)$-th generation by taking all the legs that are interacting with the legs of the j-th generation, but not interacting with the legs belonging to the previous generations. Let us suppose that the number of generations is M.

In the zero generation we have only r_0. In the subsequent j-th generations, $j > 0$ we have

$$a_1^j \dots a_{h_j}^j \quad \text{appendices;} \quad p_1^j \dots p_{k_j}^j \quad \text{protuberances;} \quad r_1^j \dots r_{n_j}^j \quad \text{rods.}$$

We denote by $C(r_i^{(h)})$ the set of four corners of the i-th rod of the h-th generation.

$$\begin{aligned}
&\mathcal{C}_0(r) = C(r_0); \qquad \widetilde{\mathrm{R}}_0 = r_0; \qquad \mathcal{C}_1(r) = \bigcup_{i=1}^{n_1} C(r_i^{(1)}); \qquad \widetilde{\mathrm{R}}_1 = \bigcup_{i=1}^{n_1} r_i^{(1)}; \\
&\widetilde{\mathrm{A}}_1 = \bigcup_{i=1}^{h_1} a_i^{(1)} \qquad \widetilde{\mathrm{P}}_1 = \bigcup_{i=1}^{k_1} p_i^{(1)} \qquad \dots \quad \mathcal{C}_M(r) = \bigcup_{i=1}^{n_M} C(r_i^{(1)}); \\
&\widetilde{\mathrm{R}}_M = \bigcup_{i=1}^{n_M} r_i^{(M)} \qquad \widetilde{\mathrm{A}}_M = \bigcup_{i=1}^{h_M} a_i^{(M)} \qquad \widetilde{\mathrm{P}}_M = \bigcup_{i=1}^{k_M} p_i^{(M)}
\end{aligned} \tag{2.39}$$

We want to estimate, for β sufficently large, the weight of a conglomerate γ as a product of independent weights associated to the single constituents of γ (rods, appendices, protuberances). We introduce the quantity $\hat{\zeta}_t(r, \mathrm{L}, \mathrm{A})$ defined by:

$$\zeta_t(r)\ t\ \left(\sqrt{\lambda_0 c_1(\delta, --)}\right)^{|\mathrm{C}(r_{\mathbf{x},l}) \cap \bar{\mathrm{C}}(\mathrm{L},\mathrm{A})|} =: \frac{1}{2}\ \hat{\zeta}_t(r, \mathrm{L}, \mathrm{A}) \left([e^{-\vartheta/3}]^2\right)^l \left(\frac{1}{1+\varepsilon^2}\right)^4 \tag{2.40}$$

¿From (2.10), (2.2) we get

$$\hat{\zeta}_t(r, \mathrm{L}, \mathrm{A}) \le \widetilde{\zeta}(r) := \varepsilon^2\ t\ \left(e^{-\vartheta/3}\right)^l 2(1+\varepsilon^2)^4 \left(\frac{\lambda_0}{2}\right)^2 \le \varepsilon^2\ t\ \left(e^{-\vartheta/3}\right)^l 2(1+\varepsilon^2)^4. \tag{2.41}$$

The factor $(1+\varepsilon^2)^{-4} \approx 1$ is introduced to control the sum over the appendices (see (2.49) below). Let a be an appendix, we introduce $\widetilde{\zeta}(a)$ defined by

$$\zeta(a) =: \frac{1}{2}\ \widetilde{\zeta}(a) \left[1 - \frac{1}{8}\varepsilon\right]^{2l} \tag{2.42}$$

¿From (2.29) and (2.42) we get

$$\widetilde{\zeta}(a) \leq \varepsilon^2 \left(\varepsilon^2 \left(1 - \frac{11}{16}\varepsilon\right)\right)^{l_1} \; 2 \left[1 - \frac{10}{16}\varepsilon\right]^{l_2} . \tag{2.43}$$

Given a protuberance p, we introduce $\widetilde{\zeta}(p)$ defined by

$$\zeta(p) =: \frac{1}{2}\, \widetilde{\zeta}(p) \left(\frac{1}{2}\right)^{2|p|} \tag{2.44}$$

from (2.28) and (2.44) we get

$$\widetilde{\zeta}(p) \leq \varepsilon^2 2 \left[4\, \varepsilon^2 (1-\varepsilon)\right]^{|p|} . \tag{2.45}$$

In (2.42), (2.44), (2.40) we have extracted from the weights $\zeta(a)$, $\zeta(p)$, $t\,\zeta(r)$, a factor $\left(1 - \frac{1}{8}\varepsilon\right)^2$, $\left(\frac{1}{2}\right)^2$, $e^{-\frac{2\vartheta}{3}}$, respectively, that we associate to the vertices; this amounts to worsen (increase) the weight from ζ to $\widetilde{\zeta}$. Moreover we extract another factor $\frac{1}{2}$ that we will use to control the sum over the number of generations. In (2.46), (2.47), (2.48), we extimate the sum of $\widetilde{\zeta}$ over the various legs containing a fixed vertex. For the appendices we have:

$$\sum_{a \ni v^*} \widetilde{\zeta}(a) \leq 2\, \varepsilon^2 \sum_{l_1=1}^{\infty} \left(\varepsilon^2 \left(1 - \frac{11}{16}\varepsilon\right)\right)^{l_1} \sum_{l_2=0}^{\infty} \left(1 - \frac{5}{8}\varepsilon\right)^{l_2} \leq \frac{16}{5}\varepsilon^3 \tag{2.46}$$

For the rods:

$$\sum_{r \ni v^*} \widetilde{\zeta}(r) \leq 2\varepsilon^2 t\, (1+\varepsilon^2)^4 \sum_{l=1}^{\infty} \left(1 - \frac{1}{4}\vartheta\right)^{l} \leq 8\, t\, (1+\varepsilon^2)^4 \frac{\varepsilon^2}{\vartheta} := c_1 \frac{\varepsilon^2}{\vartheta} \tag{2.47}$$

with $c_1 = 8\; t\; (1+\varepsilon^2)^4$. Finally for the protuberances we have the following simpler analogue of (2.46)

$$\sum_{p \ni v^*} \widetilde{\zeta}(p) \leq 2\, \varepsilon^2 \sum_{l=1}^{\infty} \left[\varepsilon^2 (1-\varepsilon) 4\right]^{l} = 2\, \varepsilon^2 \frac{4\varepsilon^2(1-\varepsilon)}{1 - 4\varepsilon^2(1-\varepsilon)} \leq 8\, \varepsilon^4 \tag{2.48}$$

The idea that we use to compute $\sum_{\gamma\ni x}\zeta_t(\gamma)$ is the following: we first fix all the legs of the first $M-1$ generations and sum over the legs of the last generation (using the worsened weights $\widetilde{\zeta}$); the result of this sum is controlled by the small damping factors <1: $\left(1-\frac{1}{4}\varepsilon\right)^2$, $\left(\frac{1}{2}\right)^2$, $e^{-\frac{2\vartheta}{3}}$, that we had preliminarly extracted from the original weights ζ of the legs of the previous generation. Then we proceed iteratively by summing over the legs of the $M-1$ generation and so on. We get

$$\sum_{\gamma:\ \gamma\ni x}\zeta_t(\gamma)\ \le\ \sum_{M=0}^{\infty}\frac{1}{2}(\widetilde{\zeta}(r_0))\prod_{v\in\widetilde{\mathrm{R}}_0}\zeta_t(\gamma\setminus(\widetilde{\mathrm{A}}_{M-1}\cup\widetilde{\mathrm{P}}_{M-1}\cup\widetilde{\mathrm{R}}_{M-1}))\ \frac{1}{2}$$

$$\prod_{v\in\widetilde{\mathrm{A}}_{M-1}}\left[\left(1-\frac{1}{8}\varepsilon\right)^2\left(1+\sum_{r\mathcal{I}v}\widetilde{\zeta}(r)\right)\right]\cdot\prod_{c\in\mathcal{C}_{M-1}(r)}\left[\frac{1}{1+\varepsilon^2}\left(1+\sum_{a\ni y(c)}\widetilde{\zeta}(a)\right)\right]\cdot$$

$$\prod_{v\in\widetilde{\mathrm{R}}_{M-1}}\left[\left(1-\frac{1}{4}\vartheta\right)^2\left(1+\sum_{r\mathcal{I}v}\widetilde{\zeta}(r)+\sum_{p\mathcal{I}v}\widetilde{\zeta}(p)\right)\right]\cdot\prod_{v\in\mathcal{P}_{M-1}}\left[\left(\frac{1}{2}\right)^2\left(1+\sum_{r\mathcal{I}v}\widetilde{\zeta}(r)\right)\right] \tag{2.49}$$

Using (2.46), (2.47), (2.48), (2.49) and taking into account that a leg can interact with another leg from above or below, we get

$$\sum_{\gamma\ni x}\zeta_t(\gamma)\ \le\ \sum_{M=0}^{\infty}\frac{1}{2}(\widetilde{\zeta}(r_0))\prod_{v\in\widetilde{\mathrm{R}}_0}\zeta_t(\gamma\setminus(\widetilde{\mathrm{A}}_{M-1}\cup\widetilde{\mathrm{P}}_{M-1}\cup\widetilde{\mathrm{R}}_{M-1}))$$

$$\frac{1}{2}\Bigg\{\prod_{v\in\widetilde{\mathrm{A}}_{M-1}}\left[\left(1-\frac{1}{8}\varepsilon\right)\left(1+c_1\frac{\varepsilon^2}{\vartheta}\right)\right]^2\prod_{v\in\mathcal{P}_{M-1}}\left[\frac{1}{2}\left(1+c_1\frac{\varepsilon^2}{\vartheta}\right)\right]^2$$

$$\prod_{v\in\widetilde{\mathrm{R}}_{M-1}}\left[\left(1-\frac{1}{4}\vartheta\right)\left(1+c_1\frac{\varepsilon^2}{\vartheta}+8\,\varepsilon^4\right)\right]^2\prod_{c\in\mathcal{C}_{M-1}(r)}\left[\frac{1}{1+\varepsilon^2}\left(1+\frac{16}{5}\varepsilon^3\right)\right]\Bigg\} \tag{2.50}$$

where v^* is a fixed vertex, say the origin. If

$$c_1\frac{\varepsilon^2}{\vartheta}=8\,t(1+\varepsilon^2)^4\frac{\varepsilon^2}{\vartheta}<\frac{1}{8}\varepsilon \tag{2.51}$$

all the products enclosed in the curly brackets $\{ \quad \}$ are less than 1. Then we iterate this procedure and we obtain

$$\begin{aligned}\sum_{\gamma\ni x}\zeta_t(\gamma) \quad &\leq \sum_{M=0}^{\infty}\frac{1}{2}(\widetilde{\zeta}(r_0))\left\{\prod_{c\in\mathcal{C}_0(r)}\left[\frac{1}{1+\varepsilon^2}\left(1+16\varepsilon^3\right)\right]\cdot\right.\\ &\left.\cdot\prod_{v\in\widetilde{\mathrm{R}}_0}\left[\left(1-\frac{1}{4}\vartheta\right)\left(1+c_1\frac{\varepsilon^2}{\vartheta}+8\,\varepsilon^4\right)\right]^2\right\}\left(\frac{1}{2}\right)^{(M-1)} \leq\\ &\leq\sum_{M=0}^{\infty}\left(\frac{1}{2}\right)^M 8\,t\,(1+\varepsilon^2)^4\left(\frac{\varepsilon^2}{\vartheta}\right)\leq 16\,t\,(1+\varepsilon^2)^4\frac{\varepsilon^2}{\vartheta}\end{aligned} \tag{2.52}$$

This concludes the proof of the Lemma for $x\in\Lambda_1$. If $x\in\Lambda_2$ necessarily there exists an appendix a_0 (or a protuberance p_0), in γ which contains x; this appendix (or protuberance) has to be connected to a rod r_0 (and throught it to the rest of the conglomerate), then $\gamma = a_0\cup\{\gamma\setminus a_0\}$ (or $\gamma = p_0\cup\{\gamma\setminus p_0\}$)

$$\sum_{\gamma:\,\widetilde{\gamma}\ni x}\zeta_t(\gamma)\leq\sum_{a_0\ni x}\zeta(a_0)\,2\sum_{\gamma\ni a_0}\zeta(\gamma\setminus a_0)\leq\frac{32}{15}\varepsilon^3\left(c_1\frac{\varepsilon^2}{\vartheta}\right). \tag{2.53}$$

$$\sum_{\gamma:\,\widetilde{\gamma}\ni x}\zeta_t(\gamma)\leq\sum_{p_0\ni x}\zeta(p_0)\,2\sum_{\gamma\ni p_0}\zeta(\gamma\setminus p_0)\leq 2\varepsilon^4\left(c_1\frac{\varepsilon^2}{\vartheta}\right). \tag{2.54}$$

$\square$

Theorem 2.6.

Let $\vartheta=\kappa\varepsilon$, $\delta\beta<\frac{\varepsilon}{8}$ and suppose $\kappa>2^{12}\,21$, then there exists a constant c_2 such that for ε sufficiently small:

$$\sum_{\Gamma:\,\exists\gamma\subset\Gamma:\,\gamma\ni\bar{v}}\zeta(\Gamma)\leq c_2\frac{\varepsilon^2}{\vartheta} \tag{2.55}$$

Proof

It follows from Lemmas 2.3, 2.4 that, in order to get the Theorem, we have only to prove

$$\sum_{T:\exists\gamma\subset T:\ \gamma\ni\bar{v}} \zeta(T) \prod_{\gamma\in T} 3^{4n_r(\gamma)} \leq c_2 \frac{\varepsilon^2}{\vartheta} \tag{2.56}$$

We will use a hierarchical construction reducing the estimate of the sum on the l.h.s. of (2.56) to the estimate of a sum over trees with modified weight. We consider the conglomerate $\gamma^{(0)}$ that contains the given site $\bar{v}$, and we call it the zeroth generation. Then we consider all conglomerates $\gamma_1^{(1)} \dots \gamma_{k_1}^{(1)}$ connected by a bond to $\gamma^{(0)}$; we call them the 1-st generation. Given the j-th generation, we define the $(j+1)$-th generation taking all conglomerates $\gamma_1^{(j+1)} \dots \gamma_{k_{j+1}}^{(j+1)}$ connected by a bond to the ones of the j-th generation, but disconnected to all (up to $(j-1)$-th) previous generation.

Let N be the index of the maximal generation. We call $\mathcal{D}_0$ the set of corners of rods belonging to the conglomerate $\gamma^{(0)}$: $\mathcal{D}_0 = \cup_{r\in\gamma_0} C(r)$; we denote, for $j = 1, \dots, N$, by $\mathcal{D}_j$ the set of corners of rods belonging to the conglomerates of the j-th generation:

$$\mathcal{D}_j = \cup_{i=1}^{k_j} \cup_{r\in\gamma_i^{(j)}} C(r).$$

Remark 2.5. We want to sum over the generations from the last one to the first one. The last one (the "leaves" of the tree) is made of conglomerates; in performing the subsequent sums we exploit the particular structure of a tree: starting from the leaves and going "up" to the root of the tree we can (uniquely) associate each conglomerate to one bond. The crucial role is plaied by unseparated pairs conglomerate-bond. Indeed the sum over all bonds passing throught a given point amounts to $\frac{1}{\varepsilon}$; thus we have to use for each bond the small weight of the (previous) conglomerate to compensate the diverging term $\frac{1}{\varepsilon}$.

In order to estimate the quantity $\sum_{T\ni\bar{v}} \zeta(T) \prod_{\gamma\in T} 3^{4n_r(\gamma)}$ we introduce the modified weigths $\zeta_{t_1}(\gamma)$ with $t_1 = 3^4$. We have $\zeta_{t_1}(\gamma) = \zeta(\gamma) 3^{4n_r(\gamma)}$. Moreover we want to write the weight $\zeta_{t_1}(\gamma)$ of the conglomerate as product of some factors and $\zeta_{t_2}(\gamma)$ as follows:

$$\zeta_{t_1}(\gamma) = \left(\frac{1}{2}\right)\zeta_{t_2}(\gamma)\left(\frac{1}{1+d_4}\right)^{4n_r(\gamma)} \quad \text{with } t_2 = 3^4 2(1+d_4)^4. \tag{2.57}$$

The factor $\left(\frac{1}{2}\right)$ in (2.57) will be used to control the sum over the number N of generations whereas the factors $\left(\frac{1}{1+d_4}\right)$ appearing for each corner c of a rod in γ will be used to control the sum over the pair bond-conglomerate emerging from c (connected with $y(c)$). We have:

$$\begin{aligned}
\sum_{T\ni\bar v}\zeta(T)\prod_{\gamma\in T}3^{4n_r(\gamma)} \le \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)&\left[\sum_{\gamma_0\ni\bar v}\zeta_{t_2}(\gamma)\right]\cdot \\
\cdot\left(\frac{1}{2}\right)\prod_{c\in\mathcal{D}_0}&\left[\left(\frac{1}{1+d_4}\right)\left(1+\sum_{\gamma:\gamma\ni c}\zeta_{t_2}(\gamma)\sum_{b\mathcal{E}c}\zeta(b)\right)\right]\cdots \\
\cdot\left(\frac{1}{2}\right)\prod_{c\in\mathcal{D}_{N-1}}&\left[\left(\frac{1}{1+d_4}\right)\left(1+\sum_{\gamma:\gamma\ni c}\zeta_{t_2}(\gamma)\sum_{b\mathcal{E}c}\zeta(b)\right)\right]
\end{aligned} \tag{2.58}$$

where by $b\mathcal{E}c$ we mean that the bond b "emerges" from the corner c. We compute this preliminary sum

$$\sum_{b:\, b\mathcal{E}c}\zeta(b) \le \sum_{l=1}^{\infty}\left(1-\frac{7}{4}\varepsilon\right)^l \le \frac{1}{1-(1-\frac{7}{4}\varepsilon)} \le \left(\frac{4}{7\varepsilon}\right) \tag{2.59}$$

Using Lemma 2 with $t = t_2$ we have

$$\sum_{\gamma:\widetilde{\gamma}\ni x}\zeta_{t_2}(\gamma) \le 32\,(1+\varepsilon^2)^4(1+d_4)^4 3^4\frac{\varepsilon^2}{\vartheta} := c_3\frac{\varepsilon^2}{\vartheta} \tag{2.60}$$

$$\sum_{T\ni\bar v}\zeta(T)\prod_{\gamma\in T}3^{4n_r(\gamma)} \le \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)\sum_{\gamma_0\ni\bar v}\zeta_{t_2}(\gamma)\left(\frac{1}{2}\right)\prod_{c\in\mathcal{D}_0}\left[\left(\frac{1}{1+d_4}\right)\right.$$

$$\left(1+\sum_{\gamma:\gamma\ni c}\zeta_{t_2}(\gamma)\sum_{b\ni c}\zeta(b)\right)\Bigg]\cdots\left(\frac{1}{2}\right)\prod_{c\in\mathcal{D}_{N-1}}\left[\left(\frac{1}{1+d_4}\right)\left(1+c_3\,\frac{7}{4}\,\frac{\varepsilon}{\vartheta}\right)\right]$$

If

$$\frac{1}{1+d_4}\left(1+c_3\,\frac{7}{4}\,\frac{\varepsilon}{\vartheta}\right)<1, \tag{2.61}$$

we can extimate the quantity enclosed in the brackets [ ] by 1. A possible choice for d_4 in order to verify (2.51) (with $t=t_2$) and (2.61) is

$$d_4=\frac{1}{3};\qquad \kappa:=\frac{\vartheta}{\varepsilon}>2^{11}21(1+\varepsilon^2)^4 \tag{2.62}$$

$$\Longrightarrow\sum_{T\ni\bar{v}}\zeta(T)\prod_{\gamma\in T}3^{4n_r(\gamma)}\;\le\;\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^N c_3\frac{\varepsilon^2}{\vartheta}\le\;2\,c_3\frac{\varepsilon^2}{\vartheta}\;:=\;c_2\frac{\varepsilon^2}{\vartheta} \tag{2.63}$$

$$\text{where}\qquad c_2=2^{14}(1+\varepsilon^2)^4$$

□

In order to develop the theory of cluster expansion (see [GLMM], [K P], [D]) we need a suitable notion of incompatibility, between pairs Γ,Γ' of polymers. This new notion (which extends the one we already gave) is called *incongruousness* and denoted by the symbol ι.

Def. 2.15. Two rods which are either incompatible or interacting are said to be *incongruous*: this happens if and only if they or their Λ_2-enlargement intersect:

$$\hat{r}_1\cap\hat{r}_2\neq\emptyset\text{ or }r_1\cap r_2\neq\emptyset \tag{2.64}$$

Def. 2.16. A rod r and an appendix a are *incongruous* if there exist a site $\mathbf{x}_1\in r$ and a site $\mathbf{x}_2\in a$ such that $d(\mathbf{x}_1,\mathbf{x}_2)=1$.

Def. 2.17. A rod r and a bond b are *incongruous* if there exists a site $\mathbf{x}_1 \in r$ and a site $\mathbf{x}_2 \in b$ such that $d(\mathbf{x}_1, \mathbf{x}_2) \leq \sqrt{2}$.

Def. 2.18. An appendix a_1 (or a protuberance p_1) is *incongruous* with another appendix a_2 (or protuberance p_2) if there exist a site $\mathbf{x}_1 \in a_1$ (or a site $\mathbf{x}_1 \in p_1$) and a site $\mathbf{x}_2 \in a_2$ (or a site $\mathbf{x}_2 \in p_2$) such that $\mathbf{x}_1 = \mathbf{x}_2$ or $d(\mathbf{x}_1, \mathbf{x}_2) = 1$; or if there exists a site $x_3 \in \bar{\mathrm{A}}$ such that $d(x_1, x_3) = d(x_3, x_2) = 1$.

Def. 2.19. An appendix a (or a protuberance p) is *incongruous* with a bond b if there exist a site $\mathbf{x}_1 \in a$ (or a site $\mathbf{x}_1 \in p$) and a site $\mathbf{x}_2 \in b$ such that $\mathbf{x}_1 = \mathbf{x}_2$ or if they have horizontal distance less or equal to two

Def. 2.20 Two bonds b_1 and b_2 are *incongruous* if they intersect.

We say that a polymer Γ is incongruous with another polymer Γ' and we write $\Gamma \iota \Gamma'$ if there is an element in Γ which is incongruous with an element in Γ'.

Following Kotecký and Preiss we want to look at a polymer model with a new weight function given by $\zeta(\Gamma)e^{\alpha(\Gamma)+d(\Gamma)}$ where $\alpha(\Gamma), d(\Gamma)$ are positive functions that we take of the form

$$\alpha(\Gamma) = \sum_{v \in \Gamma} \alpha(v) \qquad\qquad d(\Gamma) = \sum_{v \in \Gamma} d(v)$$

where $d(v) = \frac{1}{16}\varepsilon \quad \forall v \in a, b, p, r$ and $\alpha(v)$ is given by

$$\alpha(v) = \begin{cases} \frac{1}{16}\varepsilon & \text{if } v \in a,\ b,\ p \\ c & \text{if } v \in r \text{ and has two corners} \\ \frac{1}{16}\varepsilon & \text{if } v \in r \end{cases} \tag{2.65}$$

The criterion of choice in (2.65) for the different elements a, b, p, r is the following one: the smaller is the weight of a vertex v the larger is the term $\alpha(v)$. Indeed the ending points of a rod (made by a pair of corners) carries a small factor of the order of ε so that we can attribute to them a factor $\alpha = c$ whereas in the others cases we set $\alpha = \frac{1}{16}\varepsilon$.

Corollary 2.7.

Let $c > 0, \vartheta = \kappa\varepsilon$, $\delta\beta < \frac{\varepsilon}{8}$ and suppose $\kappa > 21\ 2^{12}\, e^{2c}$, then, for ε sufficiently small:

$$\sum_{\Gamma:\ \exists\gamma\subset\Gamma:\ \gamma\ni v_1} \zeta(\Gamma) e^{\alpha(\Gamma)+d(\Gamma)} \leq c_4 \frac{\varepsilon^2}{\vartheta} \tag{2.66}$$

where $c_4 = 2^{15} e^{2c}$

Proof

To get the result we easly adapt the argument of proof of Theorem 2.6 to the new weights. We use the following estimates valid for sufficiently small ε. From (2.10) and (2.65) we have

$$\zeta(r) e^{\alpha(r)+d(r)} = \varepsilon^2 e^{2\,c} e^{-(\vartheta - \frac{1}{8}\varepsilon)|r|} = \varepsilon^2 e^{2\,c} e^{-\bar{\vartheta}|r|} \tag{2.67}$$

with $\bar{\vartheta} = \vartheta - \frac{1}{8}\varepsilon$. From (2.31) and (2.65) we have ($|b| = l$):

$$\zeta(b) e^{\frac{1}{8}\varepsilon|b|} \leq \left[e^{\frac{1}{8}\varepsilon}\left(1 - 2\varepsilon + o(\varepsilon)\right)\right]^l \leq \left(1 - \frac{7}{4}\varepsilon\right)^l \tag{2.68}$$

so that we get the same result as the one in equation (2.59). From (2.29) and (2.65) we have ($|a| = l_1 + l_2$):

$$\zeta(a) e^{\frac{1}{8}\varepsilon|a|} = \varepsilon^2 \left(\frac{\varepsilon^2 e^{\delta\beta} e^{\frac{1}{8}\varepsilon}}{\lambda_0}\right)^{l_1} \left(\frac{e^{\frac{1}{8}\varepsilon}}{\lambda_0}\right)^{l_2} \leq \varepsilon^2 \left[\varepsilon^2\left(1 - \frac{13}{16}\varepsilon\right)\right]^{l_1} \left(1 - \frac{6}{8}\varepsilon\right)^{l_2}. \tag{2.69}$$

¿From (2.18) and (2.65), we get

$$\zeta(p)e^{\frac{1}{8}\varepsilon|p|} = \varepsilon^2\left(\frac{\varepsilon^2 e^{\delta\beta+\frac{1}{8}\varepsilon}}{\lambda_0}\right)^{|p|} \le \varepsilon^2\left(\varepsilon^2\left(1-\frac{1}{2}\varepsilon\right)\right)^{|p|} \tag{2.70}$$

With the same steps of Theorem 2.6 we prove (2.66) with:

$$c_4 = 2^{14}(1+\varepsilon^2)^4 e^{2c} \quad \text{and} \quad \kappa > 21\cdot 2^{11} e^{2\,c}(1+\varepsilon^2)^4 + \frac{1}{8} \tag{2.71}$$

□

Theorem 2.8.

Let $c=\frac{1}{4}$, $\vartheta = \kappa\varepsilon$, $\delta\beta < \frac{\varepsilon}{8}$ and suppose $\kappa > 2^{19}\, e^{\frac{1}{2}}$, then, for β sufficiently large:

$$\sum_{\Gamma \iota \Gamma'} \zeta(\Gamma')e^{\alpha(\Gamma')+d(\Gamma')} \le \alpha(\Gamma) \tag{2.72}$$

Proof

A sufficient condition for (2.72) is that for any $v_1 \in \Gamma$ which is incongruous with $v_2 \in \Gamma'$, there exists a vertex $\widetilde{v}_1 \in \Gamma$ ($\widetilde{v}_1$ can coincide with v_1) which is incongruous with $\widetilde{v}_2$ (which, in turn, can coincide with v_2) still belonging to Γ', and such that

$$\sum_{\Gamma' : \widetilde{v}_1 \iota \Gamma'} \zeta(\Gamma')e^{\alpha(\Gamma')+d(\Gamma')} \le \alpha(\widetilde{v}_1) \tag{2.73}$$

Case 1 $v_1 \in \Lambda_1$, so v_1 belongs to a rod r_1 of Γ and it is incompatible with a site v_2 of Γ'. The site $v_2 \in \Gamma'$ belongs to

$a)$ a rod $r' \in \Gamma'$

$b)$ an appendix $a' \in \Gamma'$

$c)$ a protuberance $p' \in \Gamma'$,

for all these cases v_2 belongs to a conglomerate of Γ'; in this case we can take $\tilde{v}_1 = v_1$, $\tilde{v}_2 = v_2$ and we can apply Corollary 2.7. We have

$$\sum_{\Gamma' :\, \Gamma' \iota \tilde{v}_1} \zeta(\Gamma')e^{\alpha(\Gamma')+d(\Gamma')} = \sum_{\Gamma' : \tilde{v}_2 \in \Gamma'} \zeta(\Gamma')e^{\alpha(\Gamma')+d(\Gamma')} \le c_4\frac{\varepsilon^2}{\vartheta} < \alpha(v_2) \tag{2.74}$$

A first condition is then:

$$c_4 \frac{\varepsilon^2}{\bar{\vartheta}} < \frac{1}{16}\varepsilon. \tag{2.75}$$

$d)$ The site $v_2 \in \Gamma'$ belongs to a bond $b' \in \Gamma'$. If there exists $\tilde{v}_1 \in \Gamma$ which is incongruous with Γ', in the sense specified in one of the above cases 1) $a)$ 1) $b)$ 1) $c)$ this means that this Γ' is considered in $\sum_{\Gamma':\Gamma'\iota\tilde{v}_1}$. If all incongruesness between Γ and Γ' are of this type 1) $d)$, necessarily, we have a situation like the one depicted in the following figure 2.4.

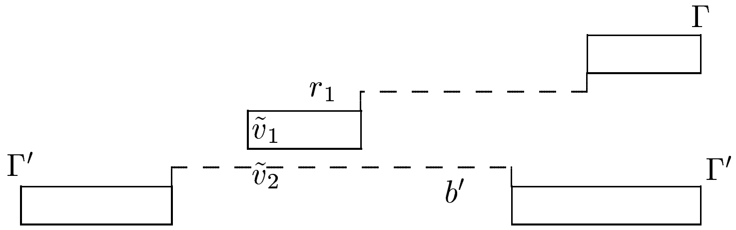


Fig. 2.4

The lenth of the bond $b' \in \Gamma'$ is larger than the lenth of the rod r_1 and the two minus-sites at the extrema of the bond have empty intersection with $\tilde{r}_1$. In this case we choose $\tilde{v}_1 \in r_1 \in \Gamma$ as the left-most site of the rod r_1, so $\alpha(\tilde{v}_1) = \frac{1}{8}\varepsilon + 2c$. This implies that there exists another site $\tilde{v}_2$ belonging to $b' \in \Gamma'$ which is incongruous with $\tilde{v}_1$. We call b_1 the part of b' that is lying on the left of $\tilde{v}_2$, b_2 the part of b' that is lying on the right; Γ_1 the part of Γ' that is touching b_1, and x_- the site in Γ_1 connecting b_1 with Γ_1. Likewise we call b_2 the rest of b': $b_2 = b' \setminus \{b_1 \cup \tilde{v}_2\}$, Γ_2 the part of Γ' that is connected with b_2 on the right and $x_+ \in \Gamma_2$ the site that is connected with b_2. See fig.2.4.

We have:

$$\sum_{\Gamma':\tilde{v}_1 \iota \Gamma'} \zeta(\Gamma') e^{\alpha(\Gamma')+d(\Gamma)} = \sum_{\Gamma':\tilde{v}_2 \in \Gamma'} \zeta(\Gamma') e^{\alpha(\Gamma')} \leq \sum_{|b_1|=1}^{\infty} \left(1 - \frac{7}{4}\varepsilon\right)^{|b_1|} \cdot$$

$$\sum_{\Gamma_1 : \Gamma_1 \ni x_-} \zeta(\Gamma_1) e^{\alpha(\Gamma_1)+d(\Gamma_1)} \cdot \sum_{|b_2|=1}^{\infty} \left(1 - \frac{7}{4}\varepsilon\right)^{|b_2|} \sum_{\Gamma_2 : \Gamma_2 \ni x_+} \zeta(\Gamma_2) e^{\alpha(\Gamma_2)+d(\Gamma_2)} \tag{2.76}$$

Using Corollary 2.7 and (2.59)

$$\text{r.h.s. of (2.76)} \leq \left(\frac{4}{7\varepsilon} c_4 \frac{\varepsilon^2}{\vartheta}\right)^2 = \left(\frac{4}{7} c_4 \frac{\varepsilon}{\vartheta}\right)^2 < \frac{1}{16}\varepsilon + c \tag{2.77}$$

A second condition is

$$\left(\frac{4}{7} c_4 \frac{\varepsilon}{\vartheta}\right)^2 < c \tag{2.78}$$

Case 2 $v_1 \in \Lambda_2$, so v_1 belongs to an appendix a_1 of Γ and it is incompatible with a site v_2 of Γ'. The site $v_2 \in \Gamma'$ belongs to

a) a rod $r' \in \Gamma'$

b) an appendix $a' \in \Gamma'$

c) a protuberance $p' \in \Gamma'$,

these cases are treated like case 1 a), b), c): (2.75) and (2.78).

d) the site $v_2 \in b' \in \Gamma'$. In this case there must be another kind of incongrouesness, that we have seen in the previous cases.

Case 3 $v_1 \in \Lambda_2$, so v_1 belongs to a protuberance p_1 of Γ; the conditions are the same as the ones of the case 2.

Case 4 $v_1 \in \Lambda_2$, so v_1 belongs to an appendix a_1 of Γ and there is incongruesness with a site v_2 of Γ'. The site $v_2 \in \Gamma'$ belongs to

a) a rod $r' \in \Gamma'$

b) an appendix $a' \in \Gamma'$

$c)$ a protuberance $p' \in \Gamma'$,

Again these cases can be treated like 1 a), b), c): (2.75) and (2.78).

$d)$ the site $v_2 \in b' \in \Gamma'$ there must be another kind of incongruesness, that we have seen in the previous cases.

In order to verify estimates (2.75) and (2.78) we have to impose conditions on κ

$$16\, c_4 \frac{\varepsilon}{\vartheta} < 1 \iff \kappa > 16\, c_4 + \frac{1}{8}. \tag{2.79}$$

$$\left(\frac{4}{7} c_4 \frac{\varepsilon}{\vartheta}\right)^2 < c \iff \kappa > \frac{4}{7}\frac{c_4}{\sqrt{c}} + \frac{1}{8} \tag{2.80}$$

Both these conditions are verified for $c = \frac{1}{4}$ if

$$\kappa > 2^{18}(1+\varepsilon^2)^4 e^{\frac{1}{2}} + \frac{1}{8} \tag{2.81}$$

the proof is concluded. □

2.2. The region $\mathbf{h_1 > 2J + \vartheta}$ and $\mathbf{h_2 < 2J + \delta'}$, $\delta'\beta \le \mathbf{e^{-\beta J}}$, $\vartheta \gg \mathbf{e^{-\beta J}}$.

In the region $h_1 > 2J$ and $h_2 < 2J$, due to the fact that the external fields dominates w.r.t. the molecular field, the + spins are everywhere typically preferred. We take the zero of the energy in the configuration $\underline{+1}$ (all spins +1). It will become clear that on the line $h_1 = 2J$ both approaches will be acceptable. We will also see that the regions of applicability of the approaches overlap in a strip containing the line $h_2 = 2J$. We write:

$$Z = \sum_{\sigma \in \Omega_\Lambda} e^{-\beta[H(\sigma) - H(\underline{+1})]}. \tag{2.82}$$

Also in the region $h_1 > 2J$ and $h_2 < 2J$ we want to write the partition function of our system as the product of the partition function of a reference system $\widetilde{Z}$ times the partition function of a gas of polymers with small activity:

$$Z = \widetilde{Z}\,\widetilde{\widetilde{\Xi}}. \tag{2.83}$$

This gas of polymers will be different w.r.t. the previous one.

Remark 2.6.

Let us now heuristically outline how the geometrical constructions of sections 2.1 and 2.2 "coexist" around the line $h_2 = 2J$. First we emphasise that the concept of rod remains the same and for its succesful application we require that h_1 would be "strong" enough i.e. that $\vartheta = h_1 - 2J$ should not be too small.

Let us describe what kind of behaviour of the rods we expect and how we analyze it in the complementary approaches of sections 2.1 and 2.2.

We want to prove that rods are rarely appearing, with a probability of the order ε^2, and have a typical length of the order $1/\vartheta$. Crossing the line $h_2 = 2J$ does not produce abrupt changes or non-analyticity of these quantities. On the other hand, if we cross the $h_2 = 2J$ line from above we expect a rather quick, but analytic, shift of the mean magnetization in the Λ_2 lines from values almost -1 to almost $+1$.

But this is not a phase transition (like in dimensions 3 and more, see ref. [HZ]); we can say that in our approach we are able to subordinate the quickly changing behaviour of the Λ_2 lines to the (almost constant) one of the Λ_1 lines (namely to the behaviour of the rods there) if $h_1 > 2J$ is kept "strong enough".

Needless to say, the concept of a rod works as well in the Λ_2 lines if the field h_2 is sufficiently strong and so we have two possible alternate approaches if both $h_1 - 2J$ and $h_2 - 2J$ are sufficiently bigger than zero. In fact in such a region where both h_1 and h_2 are strong we have a third alternative approach to the problem. Namely, one could work with the rods defined simultaneously both in the even and odd lines.

This is the most appropriate approach in such a case but we omit the details here and concentrate only on our most interesting and difficult case above when $h_2 \approx 2J$ and $h_1 > 2J + \vartheta$ (or $h_1 \approx 2J$ and $h_2 > 2J + \vartheta$) .

As we already noted in the introduction, in the case when both h_1 and h_2 are marginal the concept of a rod is not usable in general.

Given a rod r, consider the extended rod $\bar{r} = r \cup r' \cup r''$ where r' resp. r'' is just the vertical shift of r by value 1 resp. -1. (See (2.12),(2.13) where we wrote $\hat{r} = r' \cup r''$.)

Let us emphasise that given a rod r (of minuses in Λ_1) a typical configuration appearing in r' is given by -1 spins everywhere in r', and in r''.

A) Let us call this an "antiferromagnetic" (from the point of view of the rest of the line to which r belongs) behaviour. This behaviour is typical both for the "weak" and "strong" fields h_2 and it is even more frequent in the situation of a strong h_2 field. This ideal picture is of course affected by the presence of the impurities inside of $\hat{r}$, called "protuberances" (which, indeed, appear with a very small probability, in both regimes).

B) There is however also another possibility, having a small probability but requiring a careful estimate (and being slightly more frequent in the case of a weak field h_2 studied in this section). Namely, we have in mind the situation when there is a segment of pluses in the r' line (or r'' line) but not contained inside r', i.e. a + appendix in the sense of previous section.

These + appendices of section 2.1 intersect, typically in one extremal point only, the segment r'. This will be called an "extremal protuberance" now and the whole situation is dealt with, in section 2.2, in the following different way: the values of spins outside r' are simply "forgotten" and we look only for the partition function of the “rest”. (To avoid confusion, let us stress once again that the notion of a $-$appendix defined in this section appears in quite a different context than the $+$appendices of section 2.1!)

Maybe it is instructive to compare the methods that we are using to deal with these two cases A) and B) in (1) the previous section and (2) the present one.

(1) In the "strong field h_2" situation (that of the previous section) we applied the following approach. In the case A) of absence of a $+$appendix, we decided "not to ask what happens outside r' " and we expanded the partition function Z^{--} in the connected regions in Λ_2 starting immediately at the ends of r'. (This expansion lead to the notion of “bond”.) These connected regions with $-$ boundaries have their other endpoints either in an extreme point of some other r'resp. r'' or, possibly, at the boundary of some $+$appendix.

The case B) of a nonempty $+$appendix has a small probability but is rather subtle. Namely, $+$appendices, once they appear, tend to be quite large (in comparison to the, much more damped, protuberances) especially if we try to shift our analysis from the $h_2 = 2J$ line slightly also to the region of a weak h_2 field.

2) Now, in the present, “weak field h_2” situation the rods are the same as before but otherwise we are doing, in a sense, an opposite construction. Namely, it is in the case of a "ferromagnetic" behaviour (if we have some pluses in r' touching its boundary; namely the case B)) where we “cut our interest on whether the observed pluses at the boundary of r' continue also outside of r'”. The segments where we expand Z^{++} (leading to bonds) defined is the situation (2) are $+$ ones (compared to segments, with boundary condition $-$, of the previous section). They will start immediately at the end of r', provided that the latter carries the value $+$. Otherwise, we will look for the continuation of such a row of minuses intersecting r' also when

it is leaving r'. The next interval where we expand Z^{++} will then start just at the end of such a $-$ appendix.

So the partition functions Z^{++} (to be expanded and leading to bonds) are now taken on intervals starting either (less typically) directly in the $+$ boundary points of r' or (much more typically!) touching the $-$ appendices just mentioned. Contrary to the concept of a (strong h_2 field) $+$appendix, the appearance of $-$appendices in the weak h_2 field situation is a typical phenomenon accompanying almost always the appearance of a rod.

The analysis of these $-$ appendices is again a rather subtle one, being possible essentially only in the $h_2 < 2J$ regime (with an overlap with the the $h_2 > 2J$ marginal regime).

Notice that it is important to take the correct choice of Z^{--} resp. Z^{++} in the two respective (strong field and weak field h_2) cases; otherwise the coefficient c_1 in (2.25) (and analogously for the $+$ situation) decreases to zero with $h_2 - 2J$ becoming negative (resp. positive), making (2.34) a not easily controllable object.

To summarize, around the line $h_2 = 2J$ we have developed two competing approaches proving uniqueness and analyticity; each of these approaches works easily in one of the regions $h_2 > 2J$ resp. $h_2 < 2J$. With some care, it can be extended also slightly inside of the other region.

We have now to redefine (in comparison to sect. 2.1) some objects. Recall that a rod is a maximal connected segment of -1 spins in Λ_1. We still use the notation:

$$r_{x,l} = \{y \in \Lambda_1 \ \mid x_1 < y_1 < x_1 + l - 1 \quad \text{and } y_2 = x_2\} \tag{2.84}$$

Now the weight of a rod is:

$$\zeta(r_{x,l}) = e^{-\beta[H(\sigma(r_{x,l})) - H(\underline{+1})]} = e^{-2J\beta} e^{-2Jl\beta} e^{-h_1 l\beta} \tag{2.85}$$

We define $\hat{r}_{x,l}$ and $E(r_{x,l})$ as in definition 2.3 (equations (2.12), (2.14)).

We call $\sigma(\bar{r}_{x,l})$ the configuration in which the spins -1 are precisely the ones in the rectangle $\bar{r}_{x,l}$: the extension of $r_{x,l}$ which has been defined in (2.13). We have:

$$\zeta(\bar{r}_{x,l}) = e^{-\beta[H(\sigma(\bar{r}_{x,l}))-H(\underline{+1})]} = e^{-6J\beta}e^{-(2J-2h_2)l\beta}e^{-h_1 l\beta} = \varepsilon^6 e^{-(\delta+\vartheta)\beta l} \tag{2.86}$$

Given $\sigma \in \Omega_\Lambda$, we denote now by $\mathrm{C}(\sigma)$ the maximal connected components (segments $\rho \subset \Lambda_2$) of $+1$ spins in $\bigcup_{(x,l)\in L(\sigma)} \hat{r}_{x,l}$.

Def. 2.21. Given σ, we call protuberances and denote by p the components $\rho \in \mathrm{C}(\sigma)$ contained in $\hat{r}_{x,l} \setminus E(r_{x,l})$. We denote by $\mathrm{P}(\sigma)$ the set of all protuberances.

$$\mathrm{P}(\sigma) = \{\rho \in \mathrm{C}(\rho) : \exists\ (x,l) \in L(\sigma) : \ \ \rho \subset \hat{r}_{x,l} \setminus E(r_{x,l})\} \tag{2.87}$$

$$\zeta(p) = e^{-\beta[H(\sigma(\bar{r}_{x,l},p))-H(\sigma(\bar{r}_{x,l})]} = e^{-2J\beta}e^{-h_2 l\beta} \tag{2.88}$$

where $\sigma(\bar{r}_{x,l},p)$ is the configuration that has the sites -1 precisely in the rectangle $\sigma(\bar{r}_{x,l}) \setminus p$. However the following object will play an important rule.

Def. 2.22. We call extremal protuberances of σ and denote them by q the other components of $\mathrm{C}(\sigma)$.

$$\mathrm{Q}(\sigma) = \mathrm{C}(\sigma) \setminus \mathrm{P}(\sigma). \tag{2.89}$$

Given $\sigma \in \Omega_\Lambda$ we denote by $\mathrm{D}(\sigma)$ the maximal connected components (segments $\bar{\rho} \subset \Lambda_2$) of -1 spins that have non empty intersection with $\hat{r}_{x,l}$.

Def. 2.23. We call appendices and denote by a the part of the components of $\bar{\rho} \in \mathrm{D}(\sigma)$ lying outside any $\hat{r}_{x,l}$:

$$\mathrm{A}(\sigma) = \left\{ a = \bar{\rho} \setminus \left(\bigcup_{(x,l)\in L(\sigma)} \hat{r}_{x,l} \right) : \bar{\rho} \in \mathrm{D}(\sigma) \right\} \tag{2.90}$$

$$\zeta_0(q) = e^{-\beta[H(\sigma(\bar r_{x,l},q))-H(\sigma(\bar r_{x,l})]} = e^{-h_2 l\,\beta} \tag{2.91}$$

$$\zeta_0(a) = e^{-\beta[H(\sigma(\bar r_{x,l},a))-H(\sigma(\bar r_{x,l})]} = e^{-(2J-h_2)l\,\beta} \tag{2.92}$$

Notice that $\bar\rho$ can completely contain the upper or the lower part of some $\hat r_{x,l}$; in these cases it can happen that a is empty or that we have one $\bar\rho$ that gives rise to one or two appendices.

Remark 2.7 Thus the probability of the event that a given rod r and appendices a^{up}, a^{down} (below resp. above r) appear is of the order of ε^6. This looks to be incoherent with the result of the previous section giving a probability of ε^2 to the rod, but there is no contraddiction. Indeed the entropy associated to the four ends of the appendices a^{up}, a^{down} is of the order $\frac{1}{\varepsilon^4}$, thus the final result for the probability of a rod r is again of the order ε^2 in agreement to the argument of the previous section.

As in section 3, two elements a, a' (appendices), p, p' (protuberances), a, p are *incompatible* if they intersect. Given $\mathrm{L}\in\mathcal{L}$, a triple of families A, Q, P of (compatible) appendices and protuberances is called "L- compatible " if there exists $\sigma\in\Omega$ such that $\mathrm{L}(\sigma),\mathrm{A}(\sigma),\mathrm{Q}(\sigma),\mathrm{P}(\sigma)=\mathrm{L},\mathrm{A},\mathrm{Q},\mathrm{P}$. We denote by $\widetilde{\mathrm{A}}$, $\widetilde{\mathrm{Q}}$ $\widetilde{\mathrm{P}}\subset\Lambda_2$ the supports of A, P, respectively:

$$\widetilde{\mathrm{A}} = \bigcup_{a\in\mathrm{A}} a \qquad \widetilde{\mathrm{Q}} = \bigcup_{q\in\mathrm{Q}} q \qquad \widetilde{\mathrm{P}} = \bigcup_{p\in\mathrm{P}} p \tag{2.93}$$

Given a compatible set L,A,Q we denote by $\bar{\mathrm{A}}$ the extension of $\widetilde{\mathrm{A}}$ to the nearest neighbour sites in Λ_2 not contained in any $\hat r_{\mathbf{x},l}$ with $(\mathbf{x},l)\in\mathrm{L}$:

$$\bar{\mathrm{A}} = \widetilde{\mathrm{A}}\cup\left\{\mathbf{y}\in\Lambda_2 : \mathrm{dist}(\mathbf{y},\widetilde{\mathrm{A}})=1,\quad \mathbf{y}\cap\left(\bigcup_{(\mathbf{x},l)\in\mathrm{L}}\hat r_{\mathbf{x},l}\right)=\emptyset\right\}\cup$$

$$\cup_{r_{\mathbf{x},l}\in L}\left(\widetilde{E}(r_{\mathbf{x},l}) : \text{ such that in } \hat r_{\mathbf{x},l} \text{ there are no appendices and protuberances.}\right) \tag{2.94}$$

Now we have, for the partition function $Z(\Lambda)$ an expression analogous to the one given in equation (2.21); in place of $Z_{\Lambda_2}^{\{\mathrm{L};\,\mathrm{A};\,\mathrm{P}\}}$ we will have $Z_{\Lambda_2}^{\{\mathrm{L};\,\mathrm{A};\,\mathrm{Q}\ \mathrm{P}\}}$ namely the partition function in $\Lambda_2 \setminus \left[\bar{\mathrm{A}} \bigcup (\bigcup_{\mathbf{x},l\in \mathrm{L}} \hat{r}_{\mathbf{x},l})\right]$ with $+$ boundary conditions on the contiguous sites on Λ_1 and $+$ boundary conditions on the contiguous sites in Λ_2.

$Z_{\Lambda_2}^{\{\mathrm{L};\,\mathrm{A};\,\mathrm{Q};\,\mathrm{P}\}}$ splits into the product of partition functions of one dimensional Ising systems on some intervals in Λ_2 with field δ and $+$ boundary conditions on their extrema. Indeed, given L, A, Q, P we set: $\Lambda_2 \setminus \left[\bar{\mathrm{A}} \bigcup (\bigcup_{\mathbf{x},l\in \mathrm{L}} \hat{r}_{\mathbf{x},l})\right] = \bigcup_j g_j$. The g_j are disjoint intervals that lie in Λ_2 between pairs of sites which are either extreme points of some extended appendices in $\bar{\mathrm{A}}$ or sites of some $q \in \mathrm{Q}$. This latter case corresponds to saing that in $y \in E(r_{\mathbf{x},l})$ we have $\sigma(y) = +1$. Given L, A, Q, P we denote by $\mathrm{G}(\mathrm{L}, \mathrm{A}, \mathrm{P}, \mathrm{Q})$ the $\cup g_i$; the spin inside the intervals g_j are free and we are going to sum up over their values. We observe that

$$Z_l^{\tau,\,\tau'}(h,-) = Z_l^{-\tau,\,-\tau'}(-h,+) \tag{2.95}$$

we use the identification (2.95) to compute $Z_l^{+\,+}(\delta,+)$ with positive δ. As before we can use the Proposition 2.1.

$$Z_l^{++} = \lambda_0^{l+1} c_1(\delta,++)\left(1 + c_2(\delta,++)\left(\frac{\lambda_1}{\lambda_0}\right)^{l+1}\right) \tag{2.96}$$

$$\zeta(g) = \lambda_0 c_1(\delta,++)\left(1 + c_2(\delta,++)\left(\frac{\lambda_1}{\lambda_0}\right)^{|g|+1}\right) \tag{2.97}$$

If $\delta = e^{-c\beta}$ and $\delta < \varepsilon \iff c > J$

$$\zeta(a) = \zeta_0(a)\left(\frac{1}{\lambda_0}\right)^l = \left(\frac{e^{-(2J-2h_2)\beta}}{\lambda_0}\right)^l \le \varepsilon^{-\beta\delta l}\left(1 - \frac{3}{4}\varepsilon\right)^l \tag{2.98}$$

We define bonds as before, but now its activity is

$$\zeta(b) = (1 - 2\varepsilon + o(\varepsilon))^l \tag{2.99}$$

Notice that plus spins are preferred on Λ_1, so that the rods on Λ_1 are depressed, both for positive and negative δ. On the other hand the behavoiur on Λ_2 changes smoothly but quickly when δ changes sign. It easily follows from (2.24), (2.29) and (2.31), that when δ increases, the weights of $\zeta(a)$, $\zeta(b)$ decrease. In particular (2.98), (2.99) continue to hold.

In order to get a convergent cluster expansion we can repeat the same steps as the ones that we used in previous Section 2.1. We extend now the notion of conglomerates: they are composed of interacting extended rods, appendices and protuberances. The small factor ε^6 in the (2.86) will be used to controll the sum over the appendices. In this way it is easy to obtain the analogous of Theorem 2.6, Corollary 2.7 and Theorem 2.8.

We conclude this section by observing that from Lemma 2.2 and from theorem2.8 we can deduce, using the usual methods of the cluster expansion, uniqueness of the Gibbs measure and exponential decay of truncated correlations. It turns out that in the horizontal direction the correlation lenght is much larger (order of $\frac{1}{\varepsilon}$) than in the vertical direction (order 1).

Section 3. The coexistence on the line $\mathbf{h_1 = h_2}$

In this section we prove that on the line $h_1 = h_2 < 2J$, for sufficiently low temperature we have coexistence of two phases. The result is stated in the following theorem 3.1.

Theorem 3.1. *There exists $\gamma(\beta)$ going to zero as β goes to infinity, such that if we suppose $h_1 = h_2 = 2J - \eta$ with*

$$\eta = \eta(\beta) = \exp[-\beta(J - \gamma(\beta))]$$

then, for β sufficiently large, there are at least two Gibbs measures.

Proof

Consider a square $\Lambda \subset \mathbb{Z}^2$ centered at the origin and take minus boundary condition outside Λ. If we are able to prove that the Gibbs probability (with minus boundary conditions) that the spin at the origin is positive (that we denote by $\mu_\Lambda^-(\sigma_0 = +)$) is strictly less than $\frac{1}{2}$, uniformly in Λ, then, by symmetry, we get the result. Indeed, by symmetry, we have $\mu_\Lambda^-(\sigma_0 = +) = \mu_\Lambda^+(\sigma_0 = -)$.

We describe the configurations inside Λ via the usual Peierls contours. Given $\sigma \in \{-1,+1\}^\Lambda$, we draw, for any pair of n. n. sites (x, y), with opposite spins, a unit segment orthogonal to x, y (and joining two points of the dual lattice $\mathbb{Z}^2 + (\frac{1}{2}, \frac{1}{2})$). In this way we associate to each $\sigma \in \{-1,+1\}^\Lambda$ a set of polygonals such that for every point i of Λ^* (the set of points in the dual lattice contained in Λ) we have an even number of unit segments (0, 2, or 4) emerging from i. Let us use the Gallavotti convention: when we have 4 unit segments emerging from i we modify the "cross" by cutting in the direction North-East – South-West. (See [GMM]). Indeed there is a one to one correspondence between configurations $\sigma \in \{-1,+1\}^\Lambda$ (with minus b.c.) and the set of "compatible families" of "signed closed contours".

Let us clarify the concept of " compatible family". Given σ, consider the set $\gamma_1, \ldots, \gamma_n$ of closed, selfavoiding polygonals (=polygons) to which it gives rise. We assign to each γ_i the sign of the spins immediately interior to γ and we denote it by sign(γ). For istance the outer polygons γ (namely the ones connected to $\partial\Lambda$ by a n.n. path of minus spins) have the + sign; we call *contour* a signed polygon. By abuse of notation we continue to denote by γ the contour; namely the polygon (as a geometrical object) toghether with its sign. A "compatible family of contours" is a set $(\gamma_1, \ldots, \gamma_n)$ such that there exists a $\sigma \in \{-1,+1\}^\Lambda$ giving rise, with minus boundary conditions, to $(\gamma_1,\dots,\gamma_n)$ as the set of its signed closed polygons.

There is a natural partial order by inclusion in the set of polygons. It is immediate to see that if γ_i,γ_j are two contours making part of a compatible family: $(\gamma_1,\dots,\gamma_i,\dots,\gamma_j,\dots,\gamma_n)$ and if $\gamma_i\subset\gamma_j$ and " there is no other contour of the family" between γ_i and γ_j (namely there exists a path in Λ^* connecting γ_i to γ_j without intersecting any other member of the family) then the sign of γ_i is opposite to that of γ_j. Let $\sigma\longrightarrow(\gamma_1,\dots,\gamma_n)$ be a configuration, associated to the unique compatible family of contours to which it gives rise. The Gibbs probability of σ can be expressed in terms of factorized weights (activities) attributed to γ_i. Indeed let

$$\mu_\Lambda^-(\sigma)=\frac{\exp[-\beta[H(\sigma)-H(-\underline{1})]\,]}{\sum\limits_{\sigma'}\exp[-\beta[H(\sigma')-H(-\underline{1})]\,]}$$

where $-\underline{1}$ is the configuration with all spins minus.

We denote, by abuse of notation, by $H(\gamma_1,\dots,\gamma_n)$ the energy associated to σ when $(\gamma_1,\dots,\gamma_n)$ is precisely the family corresponding to σ. Suppose to partially order $(\gamma_1,\dots,\gamma_n)$ by inclusion so that γ_1 is an outer contour. We write

$$H((\gamma_1,\dots,\gamma_n))-H(-\underline{1})=H(\gamma_1)-H(-\underline{1})+H(\gamma_1,\gamma_2)-H(\gamma_1)+$$

$$\dots+H(\gamma_1,\dots,\gamma_n)-H(\gamma_1,\dots,\gamma_{n-1}).$$

It is easy to see that:

$$H(\gamma_1,\dots,\gamma_k)-H(\gamma_1,\dots,\gamma_{k-1})=H(\gamma_k)-H(-\mathrm{sign}(\gamma_k)).$$

We get, for $\sigma\longrightarrow(\gamma_1,\dots,\gamma_n)$

$$\mu_\Lambda(\sigma)=\frac{\zeta(\gamma_1)\dots\zeta(\gamma_n)}{\sum\limits_{(\gamma_1,\dots,\gamma_k)\subset\Lambda}\zeta(\gamma_1)\dots\zeta(\gamma_k)},\tag{3.1}$$

where the sum in the denominator of (3.1) extends to all compatible families of contours, contains the contribution 1 corresponding to the empty family and

$$\zeta(\gamma) := \exp[-\beta[H(\gamma) - H(-\mathrm{sign}(\gamma))]\,]$$

Notice that $\zeta(\gamma)$ depends in a non trivial way on the location of γ.

Let Ω_0 denote the set of compatible families $(\gamma_1, \ldots, \gamma_n)$ such that there exists at least a $\gamma_i \in (\gamma_1, \ldots, \gamma_n)$ containing the origin O in its interior. Given an element $(\gamma_1, \ldots, \gamma_n) \in \Omega_0$ we denote by $\hat{\gamma}(\gamma_1, \ldots, \gamma_n)$ the minimal (in the sense of inclusion) between the γ_i in $(\gamma_1, \ldots, \gamma_n)$ which contain the origin.

So, given $(\gamma_1, \ldots, \gamma_n) \subset \Omega_0$, either $n = 1$ and $\gamma_1 = \hat{\gamma}$ or $(\gamma_1, \ldots, \gamma_n) = (\hat{\gamma}, \gamma'_1, \ldots, \gamma'_{n-1})$, $O \in \Theta(\hat{\gamma})$ and there is no other contour containing O inside $\Theta(\hat{\gamma})$; here by $\Theta(\gamma)$ we denote the region of $\mathbb{Z}^2$ enclosed in γ.

We will prove, indeed, that:

$$\mu_\Lambda(\sigma_0 = +) << \frac{1}{2}. \tag{3.2}$$

We have

$$\mu^-_\Lambda(\sigma_0 = +) = \frac{\displaystyle\sum_{\bar{\gamma}:\ \mathrm{sign}(\bar{\gamma})=+,\Theta(\bar{\gamma})\ni 0} \zeta(\bar{\gamma}) \sum_{(\gamma^i_1,\ldots,\gamma^i_k,\gamma^e_1,\ldots,\gamma^e_l)\in\Gamma} \zeta(\gamma^i_1)\ldots\zeta(\gamma^i_k)\zeta(\gamma^e_1)\ldots\zeta(\gamma^e_1)}{\displaystyle\sum_{(\gamma_1,\ldots,\gamma_k)_{comp}} \zeta(\gamma_1)\ldots\zeta(\gamma_n)} \tag{3.3}$$

where Γ is the set of all collections of contour $\gamma^i_1, \ldots, \gamma^i_k, \gamma^e_1, \ldots, \gamma^e_l$ with $k \geq 0, l \geq 0$ such that

- $\gamma^i_1, \ldots, \gamma^i_k$ are contained in $\bar{\gamma}$ and do not contain the origin.
- $\bar{\gamma}, \gamma^i_1, \ldots, \gamma^i_k, \gamma^e_1, \ldots, \gamma^e_l$ is a compatible family and
- $\hat{\gamma}(\bar{\gamma}, \gamma^i_1, \ldots, \gamma^i_k, \gamma^e_1, \ldots, \gamma^e_l) = \bar{\gamma}$. In the r.h.s of the (3.3) by the notation "$(\gamma_1, \ldots,$ $\ldots, \gamma_n)_{comp}$" we express that $(\gamma_1, \ldots, \gamma_n)$ constitute a compatible family.

Now consider the transformation S acting on signed contours in the following way: $S\gamma = \gamma'$ where γ' is obtained from γ by translating it of one unit in the vertical direction and by simultaniously changing its sign. It is immediate that , due to the symmetry: $h_1 = h_2$, the activity is invariant under S:

$$\zeta(\gamma) = \zeta(S\gamma).$$

It is clear that the set $(S\gamma_1^i, \dots, S\gamma_k^i, \gamma_1^e, \dots, \gamma_l^e)$ constitutes a compatible family. We also call $\mathcal{S}$ the invertible transformation mapping

$$(\gamma_1^i, \dots, \gamma_k^i, \gamma_1^e, \dots, \gamma_l^e) \in \Gamma \longrightarrow (S\gamma_1^i, \dots, S\gamma_k^i, \gamma_1^e, \dots, \gamma_l^e).$$

Thus, we get

$$\mu_\Lambda^-(\sigma_0 = +) = \frac{\sum\limits_{\bar\gamma:\ \mathrm{sign}(\bar\gamma)=+, \Theta(\bar\gamma)\ni 0} \zeta(\bar\gamma) \sum\limits_{(\gamma_1,\dots,\gamma_m)\in\mathcal{S}\Gamma} \zeta(\gamma_1)\dots\zeta(\gamma_m)}{\sum\limits_{(\gamma_1,\dots,\gamma_k)_{comp}} \zeta(\gamma_1)\dots\zeta(\gamma_n)} \tag{3.4}$$

and so

$$\mu_\Lambda^-(\sigma_0 = +) \le \sum_{\bar\gamma:\mathrm{sign}(\hat\gamma)=+} \zeta(\bar\gamma) \tag{3.5}$$

where $\Theta(\bar\gamma) \ni 0$. We now want to make a comparison with another model; we want to prove that there exist $J_1(\beta),\ J_2(\beta)$

$$\sum_{\bar\gamma:\ \Theta(\bar\gamma)\ni 0\ \mathrm{sign}(\bar\gamma)=+} \zeta(\bar\gamma) \le \sum_{\gamma:\Theta(\gamma)\ni 0} \zeta_{J_1,J_2}(\gamma) \tag{3.6}$$

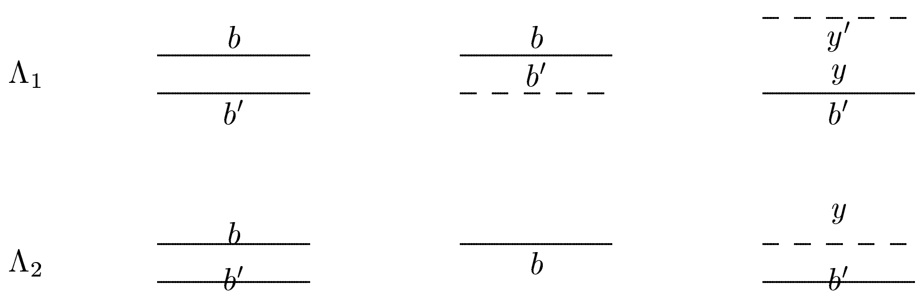
where $\zeta_{J_1,J_2}(\gamma)$ is the activity of a Peierls contour for an anisotropic n.n. two dimensional Ising Model with horizontal and vertical coupling constants equal, respectively, to $J_1,\ J_2$. Given a closed Peierls + signed contour γ, consider a horizontal unit segment b belonging to γ such that the normal to b, external to γ, is upwards. Let x be the interior site adjacent to b (x below b). Then consider the vertical line orthogonal to b in the interior of $\Theta(\gamma)$; let b' be the first unit segment encountered by this line in γ; the external normal to b' is necessarily downwards. We have two cases; either

1. the distance between b and b' is one, or,
2. the distance between b and b' is larger than one.

In the first case the energy difference w.r.t. the reference configuration $-\underline{1}$ due to the two broken bonds corresponding to b and b' and the external contribution due to the magnetic field is

$$2J - h_1 \quad \text{if } x \in \Lambda_1 \quad \text{or}$$

$$2J + h_2 \quad \text{if } x \in \Lambda_2$$



In the second case if $x \in \Lambda_1$, we associate x to b; putting toghether the contribution of the broken bond and the one of the field, the resulting energy increment is $J - h_1$. If $x \in \Lambda_2$, we consider b alone so that the energy increment is just J. Now we look at b': if y (the site adjacent from the interior to b', y above b') belongs to Λ_1, we associate to b' two sites y and $y' = y + e_2$ (e_2 vertical upwards unit vector). The corresponding energy increment is only J as the field contributions compensate, since $h_1 = h_2 = h$. If $y \in \Lambda_2$ we associate only y to b and get an energy increment $J + h_2$.

A vertical unit segment always carries an energy difference J.

It is immediate to verify that this association of the adjacent interior sites to unit segments of γ makes sense and that the remaining interior sites (the bulk) give a zero field contribution. Moreover summing the contribution to the energy of b and b' (plus possibly the suitable set of internal sites) we get a lower bound to the energy associated to each γ given by

$$J|\gamma_v| + \frac{2J-h}{2}|\gamma_h| \tag{3.7}$$

where $|\gamma_v|$ is the lenght of the vertical part of γ whereas $|\gamma_h|$ is the lenght of the horizontal part of γ.

So we get (3.6) with

$$J_1 = J, \qquad\qquad J_2 = \frac{2J-h}{2} = \frac{\eta(\beta)}{2} \tag{3.8}$$

In this way we are reduced to study the sum in the r.h.s. of (3.6).

This is the content of the following

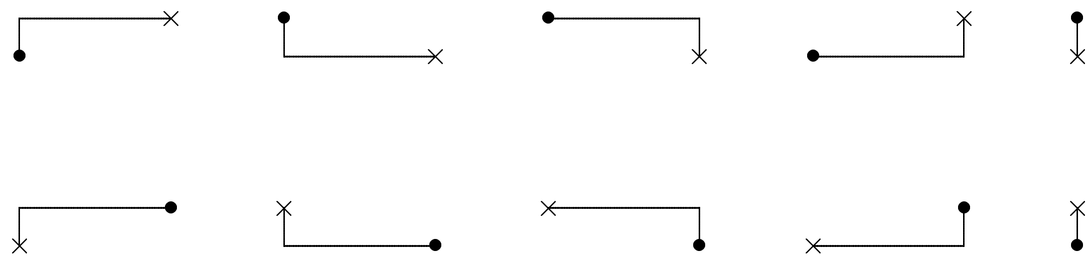
Proposition 3.2. Estimate for the $\mathbf{J_1}$, $\mathbf{J_2}$ model.

Given a J_1, J_2 model, let $p = e^{-J_2\beta}$ $\varepsilon = e^{-J_1\beta}$ with $J_2 = e^{-c\beta}$, $c \leq J_1$, we have

$$\sum_{\gamma:\Theta(\gamma)\ni 0} \zeta_{J_1,J_2}(\gamma) \quad \textit{where } \zeta_{J_1,J_2}(\gamma) = p^{|\gamma_h|}\varepsilon^{|\gamma_v|}. \tag{3.9}$$

Proof

Consider the following elementary components.



For each of them we have two differents "bricks", one has the initial point on the left (up) and another that have the right (down). In the following figure $\bullet$ rapresents the initial point and $\times$ the final point of the brick.

We say that two bricks (oriented components) S_1 and S_2 are *contiguous* and we write $S_1 \iota S_2$, if the initial point of S_2 coincides with the final point of S_1 and these two coinciding points do not belong to two horizontal unit segments.

It is easy to see that: given a closed Peierls contour γ, there exists a family of bricks such that $(S_1, \ldots, S_n)_\iota = \gamma$, where by $(S_1, \ldots, S_n)_\iota$ we express the fact that $S_1 \cup \ldots \cup S_n = \gamma$ and $S_j \iota S_{j+1} \ \forall \ j = 1, \ldots, n-1$.

The sum over all bricks having a fixed extremum is bounded by

$$\sum_{l=0}^{\infty} p^l \varepsilon = \frac{\varepsilon}{1-p} \sim \frac{e^{-J_1\beta}}{\beta e^{-c\beta}} \longrightarrow 0 \quad \text{if} \quad c = J \tag{3.10}$$

The sum over all rods of lenght l containing a given unit segment b is bounded by

$$\sum_{l_1, l_2} \varepsilon^2 p^{2l_1} p^{2l_2} \sim \frac{\varepsilon^2}{q^2} \longrightarrow 0 \text{ where } q = 1-p$$

It is easy to convince oneself, that

$$\sum_{\gamma:\Theta(\gamma)\ni 0} p^{|\gamma_h|} \varepsilon^{|\gamma_v|} \le \sum_{\gamma \ni b} |\gamma_v| p^{|\gamma_h|} \varepsilon^{|\gamma_v|} \le \sum_{\gamma \ni b} (2\varepsilon)^{|\gamma_v|} p^{|\gamma_h|} \tag{3.11}$$

We set $\varepsilon' = 2\varepsilon$; we want to prove that

$$\sum_{\gamma \ni b} \zeta'(\gamma) \le \delta(\beta) \tag{3.12}$$

where

$$\zeta'(\gamma) = (\varepsilon')^{|\gamma_v|} p^{|\gamma_h|}$$

and $\delta(\beta) \to 0$ as $\beta \to \infty$.

Let γ denote a generic closed Peierls contour; we now prove the following estimate:

$$\sum_{\gamma \ni b} \zeta'(\gamma) \leq \frac{\varepsilon'}{q} \tag{3.13}$$

where the sum in the l.h.s. of the (3.13) extends to all closed polygons containing a horizontal unit segment b .

Given a closed polygon γ we denote by $||\gamma||$ the minimal number of bricks (elementary oriented components) such that $\gamma = \cup_{i=1}^{||\gamma||} S_i$. If $\gamma \ni b$ there must exist a brick S_1 containing b; thus we have

$$\sum_{\gamma \ni b} \zeta'(\gamma) = \sum_{S_1 \ni b} \sum_{\gamma \ni S_1} \zeta'(\gamma) = \sum_{S_1 \ni b} \zeta'(S_1) \sum_{n=1}^{\infty} \sum_{\gamma:\ ||\gamma||=n+1} \zeta'(\gamma \setminus S_1) \leq \tag{3.14}$$

where $\zeta'(S_1)$ can be either ε' (vertical unit segment) or $\varepsilon' p^l$ for some suitable l (horizontal lenth of S_1)

$$\leq \sum_{S_1 \ni b} \zeta'(S_1) \sum_{n=1}^{\infty} \sum_{S_2, \ldots, S_{n+1}} \zeta'(S_2) \ldots \zeta'(S_{n+1}) \tag{3.15}$$

Consider $S_1, \ldots, S_n$ such that $S_j \iota S_{j+1} \quad \forall\, j = 1, \ldots, n$ and suppose that there exists a closed polygon γ with $||\gamma|| = n+1$ and a brick S_{n+1} such that

$$(S_1, \ldots, S_{n+1})_\iota = \gamma \tag{3.16}$$

then the pair (S_{n+1}, γ) is unique.

We have:

$$\sum_{S_1 \ni b} \zeta'(S_1) = \sum_{l_1, l_2 = 0}^{\infty} \varepsilon' p^{l_1} p^{l_2} \sim \frac{\varepsilon'}{q^2} \tag{3.17}$$

where $q = 1 - p$. Given $k \in \Lambda^*$

$$\sum_{S \iota k} \zeta'(S) \le \sum_{l=0}^{\infty} \varepsilon' p^l \sim \frac{\varepsilon'}{q} \tag{3.18}$$

where by $S\iota k$ we mean that k is the initial point of S. Now (3.14) (3.15) becomes

$$\sum_{\gamma \ni b} \zeta'(\gamma) \le \sum_{S_1 \ni b} \zeta'(S_1) \sum_{n=1}^{\infty} \varepsilon' \sum_{S_2, \dots, S_n} \zeta'(S_2) \dots \zeta'(S_n) \tag{3.19}$$

$$\le \varepsilon' \sum_{S_1 \ni b} \zeta'(S_1) \sum_{n=1}^{\infty} \left[\sum_{S \iota k} \zeta'(S) \right]^{n-1} = \varepsilon' \sum_{S_1 \ni b} \frac{1}{1 - \frac{\varepsilon'}{q}} \tag{3.20}$$

for β large is

$$\le 2 \left(\frac{\varepsilon'}{q} \right)^2 << \frac{1}{2} \qquad \text{if} \qquad c = J \tag{3.21}$$

¿From (3.5), (3.8) and (3.11) we conclude the proof of Proposition 3.2 of then the one of Theorem 3.1. $\square$

Section 4. General anisotropic polymer models

Section 4.1. New proof of Kotecký Preiss criterion

Let $\mathcal{P}$ be a finite set, its elements $P_1, \dots, P_{|\mathcal{P}|}$ are called polymers. We suppose given a binary compatibility relation. This means that in the cartesian product $\mathcal{P} \times \mathcal{P}$ we give a subset $\mathcal{V}$ called the set of compatible pairs. Two polymers which are not compatible are said to be incompatible. We write $P_1 \; c \; P_2$, $P_1 \; \iota \; P_2$ when (P_1, P_2) is a compatible, respectively, incompatible pair. We suppose given a function $w : \mathcal{P} \longrightarrow \mathbb{C}$ called activity.

The polymer partition function is:

$$Z_{\mathcal{P}} = Z_{\mathcal{P}}(w) = \sum_{(P_1,\ldots,P_n)_c : P_i \subset \mathcal{P}} \prod_{i=1}^{n} w_{P_i} \tag{4.1}$$

where the sum is over all families $(P_1, \ldots, P_n)_c$ of pairwise compatible polymers in $\mathcal{P}$. In (4.1) we use the convention that the contribution of $n = 0$ term (corresponding to the empty set) is 1. Notice that, in many applications, there is a spatial structure such that it is possible to associate a "support" $\widetilde{P}$, namely a finite subset of $\mathbb{Z}^d$, to a polymer P; for istance this is the case when considering both high and low temperature expansions for short range traslationally invariant lattice spin systems. In those cases compatibility of P_1 P_2 is just some goemetrical property of their supports like absence of overlapping; moreover we have that polymers P with $\widetilde{P}$ "well inside the bulk" of a volume Λ, have a transationally invariant activity w_P. We look at $Z_{\mathcal{P}}$ as a function of many complex variables $w_1, \ldots, w_{|\mathcal{P}|}$. In what follows we will often consider collections ρ of polymers where multiple copies are allowed. In other words a collection ρ of polymers is identified by a function n_ρ on $\mathcal{P}$ taking values on $\mathbb{Z}^+$: $n_\rho = \{n(P), \quad P \in \mathcal{P}\}$ the non negative integer $n(P)$ represents the multiplicity of the polymer P in ρ. We write $\rho = (\bar{\rho}, n_\rho)$ where $\bar{\rho}$ is the " support" of the collection ρ :

$$\bar{\rho} = \{P \in \mathcal{P} : n_\rho(P) \geq 1\} \tag{4.2}$$

The set of collections of polymers in $\mathcal{P}$ is denoted by $R(\mathcal{P})$.

Following [D] we can write the Taylor series:

$$\log Z_{\mathcal{P}} = \sum_{\rho \in R(\mathcal{P})} r_{\mathcal{P}}(\rho) \prod_{P \in \rho} w_P^{n_\rho(P)} \tag{4.3}$$

where

$$r_{\mathcal{P}}(\rho) = (n_\rho(P_1)! \ldots n_\rho(P_n)!)^{-1} \frac{\partial^{n_\rho(P_1)+\ldots+n_\rho(P_n)} \log Z_{\mathcal{P}}}{\partial^{n_\rho(P_1)} w(P_1) \ldots \partial^{n_\rho(P_n)} w(P_n)}. \tag{4.4}$$

given $\bar\rho = P_1, \dots P_n \subset \mathcal{P}$ and an activity function w, let $w^{(\bar\rho)}$ be the new activity function given

$$w^{(\bar\rho)}(P) = \begin{cases} w(P) & P \in \bar\rho \\ 0 & \text{otherwise} \end{cases} \tag{4.5}$$

We trivially have

$$Z_{\mathcal{P}}(w^{(\bar\rho)}) = Z_{\bar\rho}(w^{(\bar\rho)}) \tag{4.6}$$

and from (4.4):

$$r_{\mathcal{P}}(\rho) = r_{\bar\rho}(\rho) =: r(\rho) \tag{4.7}$$

We will call *cluster* and denote by $\mathcal{C}$ an indecomposable collection of polymers. Decomposability means that there exists a partition of $\mathcal{C}$ into two sets: $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$, such that (P_1, P_2) is a compatible pair $\forall P_1 \in \mathcal{C}_1$ and $P_2 \in \mathcal{C}_2$. The subset of $R(\mathcal{P})$ consisting in all clusters in $\mathcal{P}$ will be denoted by $G(\mathcal{P})$. Also for a cluster, polymers can appear with a multiplicity.

We have

$$\log Z_{\mathcal{P}} = \sum_{\mathcal{C} \subset G(\mathcal{P})} \Phi_{\mathcal{C}} \tag{4.8}$$

with

$$\Phi_{\mathcal{C}} = r(\mathcal{C}) \prod_{P \in \mathcal{C}} (w_P)^{n_{\mathcal{C}}(P)} \tag{4.9}$$

The important feature of (4.8) is that the sum on the r.h.s. extends to all clusters of polymers in $\mathcal{P}$. Indeed, suppose that in (4.3) it appears a decomposable collection ρ: $\bar\rho = \bar\rho_1 \cup \bar\rho_2$ with $\bar\rho_1 \neq \emptyset$, $\bar\rho_2 \neq \emptyset$, $\bar\rho_1 \cap \bar\rho_2 = \emptyset$, and every pair (P_1, P_2) with $P_1 \in \bar\rho_1$ $P_2 \in \bar\rho_2$ is compatible, we have

$$\log Z_{\bar\rho}(w) = \log Z_{\bar\rho_1}(w) + \log Z_{\bar\rho_2}(w) \tag{4.10}$$

and then we get $r(\rho) = 0$.

Our main aim, in this section, is to give a simple proof of the result in Theorem 4.1 below which is a slightly weaker version of the general result stated in [KP]. However, the proof given below will be really elementary and straightforward.

To make it as short as possible we will not try to optimize the choice of the constant $C = C(\delta)$ used below in (4.12). This constant can be pushed apparently down to the value $C = 1$ [KP] (or even lower for some special cases) with some more careful estimates. Our emphasis here is on the simplicity of all estimates.

We set

$$C = C(\delta) = \max_{x \in (0,\delta)} \left\{ \frac{-\log(1-x)}{x} \right\} = \max_{x \in (0,\delta)} \{1 + x/2 + x^2/3 + \ldots\}. \tag{4.11}$$

$C(\delta) = 1 + O(\delta)$ for small δ.

Theorem 4.1. *Assume that there are functions a and d on $\mathcal{P}$ ($a > 0$, $d \geq 0$) and $\delta > 0$ such that*

$$w_P e^{a(P)} \leq \delta \tag{4.12}$$

holds for any polymer $P \in \mathcal{P}$. Moreover, assume that for any $P \in \mathcal{P}$ we have the bound

$$\sum_{P' \iota P} |w_{P'}| e^{a(P') + d(P')} \leq \frac{a(P)}{C} \tag{4.13}$$

Then we have for any $P \in \mathcal{P}$

$$\sum_{\mathcal{C}:\ \mathcal{C} \iota P} |\Phi_{\mathcal{C}}|\ e^{\sum_{P \in \mathcal{C}} \ n_P \ d(P)} \leq a(P). \tag{4.14}$$

Note

The statement (4.14) gives an information on the decay of the terms $\Phi_{\mathcal{C}}$ in the expansion (4.8).

Proof

We will consider here only the case $d=0$. (The argument for a general d is quite analogous; just work with the weights $\tilde{w}_P = w_P e^{d(P)}$ and write only the factors $e^{d(P)}$, not $e^{n_P d(P)}$, in the estimates (4.27)-(4.31) below when inserted into (4.32).) The proof uses induction over the cardinality $|\mathcal{P}|$ of the system $\mathcal{P}$ of all available polymers. We then suppose

$$\sum_{P' \iota P} |w_{P'}| e^{a(P')} \le \frac{a(P)}{C} \tag{4.15}$$

Moreover we suppose that we already have the bound

$$\sum_{\mathcal{C} \iota P} |\Phi_{\mathcal{C}}| \le a(P) \tag{4.16}$$

when the cardinality of the set $\mathcal{P}$ is $|\mathcal{P}| = n$.

Then we want to prove (4.16) for $|\mathcal{P}| = n+1$.

In other words, from

i) the validity of (4.16) for $|\mathcal{P}| = 1$ and from

ii)the implication: [(4.15) and (4.16) for $\mathcal{P} = n$] implies [(4.16) for $\mathcal{P} = n+1$]

the theorem follows by induction.

Let us first prove (4.16) for the particular case $|\mathcal{P}| = 1$. When $|\mathcal{P}| = 1$ we have $Z_{\mathcal{P}} = 1 + z$, $z = w_P$

$$\log Z_{\mathcal{P}} = \log(1+z) = \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} z^m \qquad (4.17).$$

We would like to have the validity of (4.15). It reads in this case

$$z e^a < \frac{a}{C} : \quad \Longleftrightarrow \quad z < \frac{a}{C} e^{-a} \qquad (4.18).$$

Condition (4.16) reads

$$\sum_{m=1}^{\infty} \frac{z^m}{m} = -\log(1-z) < a \tag{4.19}$$

(the first equality above holds if $|z| < 1$). If C is given by (4.11) then from (4.19) we have:

$$-\log(1 - \frac{a}{C}e^{-a}) < ae^{-a} < a \tag{4.20}$$

Therefore (4.16) holds for $n = 1$.

Let us now prove that (4.15), together with (4.16) for $\mathcal{P} = n$, imply (4.16) for $\mathcal{P} = n + 1$. We define

$$A = Z_{\mathcal{P}\setminus P} = \sum_{(P_1,\dots,P_m)c\ \subset\mathcal{P}\ \ P\notin(P_1,\dots,P_m)} w_{P_1}\cdot\ldots\cdot w_{P_m} \tag{4.21}$$

$$B = \sum_{(P,P_1,\dots,P_m)c\ \subset\mathcal{P}\ \ P\notin(P_1,\dots,P_m)} w_{P_1}\cdot\ldots\cdot w_{P_m} \tag{4.22}$$

We set $x = w_P$. We have from (4.1)

$$Z_{\mathcal{P}} = Ax + B.$$

We have on one hand, from the equation (4.8)

I)

$$\log(Bx + A) - \log A = \sum_{m=1}^{\infty}\sum_{\mathcal{C}\ni P}^{m*}\Phi_{\mathcal{C}} \tag{4.23}$$

where $\sum_{\mathcal{C}\ni P}^{m*}\Phi_{\mathcal{C}}$ runs over the set of clusters containing P with multiplicity m.

On the other hand

II)

$$\log(Bx + A) = \log A + x\frac{A}{B} - 1/2\ \frac{x^2A^2}{B^2} + \ldots \tag{4.24}$$

$$\log(Bx + A) - \log A = \sum_{m=1}^{\infty}\frac{(w_P)^m}{m}(-1)^{m+1}\left(\frac{A}{B}\right)^m. \tag{4.25}$$

We have, from (4.8)

$$\log\frac{A}{B} = \sum_{\mathcal{C}\not\ni P}\Phi_{\mathcal{C}} - \sum_{\mathcal{C}\not\ni P:\ \mathcal{C}\ c\ P}\Phi_{\mathcal{C}} =: \sum_{\mathcal{C}\not\ni P:\ \mathcal{C}\iota P}\Phi_{\mathcal{C}}, \tag{4.26}$$

where the last sum runs over the clusters $\mathcal{C}$ not containing P but incompatible with P; from (4.25) (4.26) and identifying the corresponding powers of x from (4.23) we get:

$$\sum_{\mathcal{C}\ni P}^{m*}\Phi_{\mathcal{C}}=\frac{(w_P)^m}{m}(-1)^{m+1}\exp(m\sum_{\mathcal{C}\not\ni P:\ \mathcal{C}\iota P}\Phi_{\mathcal{C}}). \tag{4.27}$$

This is an identification of two asymptotic series.

We write, for $|\mathcal{P}|=n+1$,

$$\sum_{\mathcal{C}:\mathcal{C}\iota P}|\Phi_{\mathcal{C}}|\le\sum_{P'\iota P}\sum_{m=1}^{\infty}\sum_{\mathcal{C}:\mathcal{C}\ni P'}^{m*}|\Phi_{\mathcal{C}}|. \tag{4.28}$$

From (4.27) we get

$$\sum_{\mathcal{C}:P\in\mathcal{C}}^{m*}|\Phi_{\mathcal{C}}|\le\frac{1}{m}|(w_P)|^m\exp(m\sum_{\mathcal{C}\not\ni P:\ \mathcal{C}\iota P}|\Phi_{\mathcal{C}}|) \tag{4.29}$$

and because of the inductive hypothesis (4.16) (valid for $\mathcal{P}\setminus P$, since $|\mathcal{P}\setminus P|=n$) we have

$$\exp(m\sum_{\mathcal{C}\not\ni P:\ \mathcal{C}\iota P}|\Phi_{\mathcal{C}}|)\le\exp(ma(P)). \tag{4.30}$$

Therefore, from (4.28), (4.29) and (4.30) we get

$$\sum_{\mathcal{C}:\iota P}|\Phi_{\mathcal{C}}|\le\sum_{P'\iota P}\sum_{m=1}^{\infty}\frac{1}{m}|(w_{P'})|^m\exp(ma(P')) \tag{4.31}$$

Summing over m and using (4.11) and (4.12), we get:

$$\sum_{m=1}^{\infty}\frac{1}{m}|(w_{P'})|^m\exp(ma(P'))<Cw_{P'}e^{a(P')}. \tag{4.32}$$

From (4.31), (4.32) and (4.13) we get the result. $\square$

In the following , we study a particular example of a polymer model which was important in the previous sections of the paper (not directly in the form which is used here but with some minor modifications and generalizations).

Namely, the anisotropic nature of our Ising type models leads, rather naturally, to polymers whose weights are defined in an anisotropic way, too. We saw that these polymers (examples are rods and more generally conglomerates of rods, protuberances, appendices, bonds used in section 2.1 and 2.2 and also the Ising contours used in section 3) are composed by horizontal and vertical "segments"- and these segments carry a very different weight.

Namely, in most of our examples the energy of such objects is defined in such a way that we pay a very little price for the horizontal segments and, on the contrary, a rather big price for the vertical segments. In fact, we worked with a (rather straightforward) generalization of such a concept where **several** types of such horizontal (and vertical) segments appear (rods, protuberances, appendices, bonds).

A general observation can be made about all these examples namely that the main role of the vertical segments is to "control" the much more delicate behaviour of our polymers in the horizontal direction, and the rest of vertical part of the polymer is much easily controlled, like in the usual low temperature Ising contour systems.

Thus, vertical parts of the polymers supply the energy needed to compensate the "horizontal entropy" of these objects.

It turns out that the ideas connected with the [KP] criterion are rather useful when investigating all these models.

4.2. One dimensional polymer models

Let us start with one-dimensional examples.

Of course, in these cases there are powerful methods of expression of the partition function –namely the method of transfer matrix used in section 2.1 and 2.2 of this paper (in the expressions of partition functions Z^{++} resp. Z^{--}) and also the more general method of expansion of the generating function of the sequence of partition functions $Z[0,n]$ in volumes $[0,n]$. The latter method is briefly discussed below and we are indebted Lincoln Chayes who clarified to us this point.

It is not uninteresting to illustrate the use of the condition [KP] even in this one dimensional case. The reasons are:

1) our basic examples of anisotropic polymers constructed (in dimension $\nu = 2$) in sections 2.1 and 2.2 were, in fact, constructed from suitable one dimensional building blocks (rods, protuberances, appendices, bonds);
2) even in the context of the (above mentioned) powerful one dimensional methods of computation of partition function, the [KP] condition gives some complementary information, e.g. some lower bounds on the distance between the first two principal eigenvalues of the model.

But let us first explain the method of generating function. Let polymers in $\mathbb{Z}$ be defined as "segments" (intervals) $S = [n,m]$ with $n < m$ and let the weights w_S be translation invariant, defined as $w_{[n,m]} = w_{n-m}$ where the polymer weights w_n satisfy some condition of convergence. In fact, only very mild conditions on w_n are necessary and our condition that

$$\sum_{n\in\mathbb{Z}} n w_n e^{an} < a \tag{4.33}$$

holds for some $a > 0$ (notice that we can write it also as $\sum_{S:S\ni 0} w_S \; e^{a|S|} \le a$ which is essentially the condition (4.13)!) is unnecessarily strong for the existence of a nontrivial function f defined below: Denote by $Z_{[0,n]}$ the partition function in the volume $[0,n]$ and consider the generating functions of the complex variable z

$$g(z) = \sum_{n=1}^{\infty} w_n z^n \quad \text{and} \quad f(z) = \sum_{n=1}^{\infty} Z_{[0,n]} \; z^n . \tag{4.34}$$

Write $Z_{[0,n]}$ as Z_n and put $Z_1 = Z_2 = 1$. The recursive relation

$$Z_n = Z_{n-1} + \sum_{k=1}^{n-2} w_k \; Z_{n-k-1} \tag{4.35}$$

implies the corresponding equation between generating functions f and g namely

$$f(z) = \frac{z}{1 - z(1 + g(z))} \tag{4.36}$$

that is $f(z)$ can be extended to a meromorphic function defined at least on the convergence ball of $g(z)$. More precisely, from (4.35) we have $\sum_{n=1}^{\infty} Z_n z^n = z + z(\sum_{k=1}^{\infty} \sum_{m=1}^{\infty} w_k z^k Z_m z^m$ and this implies (4.36).

For example, for the case of $w_n = \varepsilon q^n$, $0 < q < 1$ (these weights appeared often in the previous parts of the paper) we have

$$g(z) = \sum_{n=1}^{\infty} \varepsilon (qz)^n = \frac{\varepsilon q z}{1 - qz} \tag{4.37}$$

so that $f(z)$ can be extended to a meromorphic function defined in the whole complex plane, with single poles in located in the roots of the quadratic equation $q(1-\varepsilon)z^2 - (q+1)z + 1 = 0$ i.e. in the points $z_{1,2} = \frac{(q+1) \pm \sqrt{(q-1)^2 - \varepsilon^2 q^2}}{2(1-\varepsilon)q}$.

Writing the Laurent expansion of $f(z)$ one then obtains rather precise expansions of the partition functions Z_n i.e. of the coefficients of the power series for $f(z)$. For example if $f(z)$ is meromorphic in the whole complex plane and $\lambda < 1 < \mu$ are the poles of $f(z)$ i.e. the roots of the equation

$$1 - z(1 + g(z)) = 0 \tag{4.38}$$

which are the nearest ones to the value 1 (like the values $z_1 < 1 < z_2$ in the example (4.36) above) then we have an expansion, for each natural number n

$$Z_n = a\lambda^{-n} + b\mu^{-n} + \text{ smaller terms.} \tag{4.39}$$

One can be now interested in someupper bounds for the value of the fraction $|\mu/\lambda|$ i.e. (having in mind that $\lambda \approx 1$ for small w_n) in upper bounds for the distance $\mu - \lambda$. The condition (4.33) gives some information on this, namely we can rewrite (4.33) as the condition on the derivative

$$|g'(z)| \leq a \quad \forall |z| \leq e^a. \tag{4.40}$$

Such a bound for $g'(z)$ allows to make some estimates on the localization of the poles of $f(z)$. Of course, for special examples like (4.37) one can compute exactly all these poles and this gives a much more accurate information than (4.40).

4.3. More dimensional "segmental" polymer models

The aim of this section is twofold. First, we want to give some more general, unifying commentary to what we already did in sections 2.1 and 2.2 (see the definition (2.62)). Second, this is an attempt to prepare a possible ground for a future paper dealing with phase transitions of general anisotropic models.

Having in mind a systematic use of the Pirogov Sinai theory in these problems, it will be useful to have a fairly general approach to a notion of a "segmented polymer". We expect that the framework outlined below could be adequate, with some further modifications and generalizations (taking in account also possible "volume energy terms", appearing in the "interior" of our segmented polymers) in all these investigations.

Let us define a *segment* $S_q(\mathbf{x}, l_i)$ where q is the colour of the segment; $q \in [1, \ldots, N]$ $\mathbf{x}$ is the initial point $\mathbf{x} = (x_1, x_2, \ldots, x_d) \in \mathbb{Z}^d$ such that

$$S_q(\mathbf{x}, l) = \{\mathbf{y} = (y_1, \ldots, y_d) \in \mathbb{Z}^d \quad j \neq 1 \quad y_j = x_j \quad \text{and} \quad x_1 \leq y_1 \leq x_1 + l - 1\}; \tag{4.41}$$

The support of $S_q(\mathbf{x}, l)$ is the rectangle $\widetilde{S}_q(\mathbf{x}, l) = l \times 1 \times \ldots \times 1$.

Two segments are *connected* or *incompatible* if their supports intersect; we use the symbol ι to denote incompatibility. We denote by $\zeta(S_q(\mathbf{x}, l))$ the weight of one segment. For example, it can be given by the formula

$$\zeta(S_q(\mathbf{x}, l)) := \varepsilon p^l \tag{4.42}$$

The segmental polymer is defined as a *connected* collection $P = \{S_q^k(\mathbf{x}, l)\}$ of segments: where k is an index that denotes how many times the segment $S_q(\mathbf{x}, l)$ is (repeated) in the polymer.

Connectedness of P will be defined in a usual sense of the connectedness of the support of P. We call $L_q(P)$ the set of all coordinates of the segments that are in P

$$L_q(P) = \big\{(\mathbf{x}^1, l^1) \ldots (\mathbf{x}^n, l^n) : \quad S_q(\mathbf{x}^j, l^j) \text{is a segment in } P\big\} \qquad (4.43).$$

the support $\widetilde{P}$ of a polymer P is given as:

$$\widetilde{P} := \bigcup_{(\mathbf{x}, l) \in L_q(P)} \widetilde{S}_q(\mathbf{x}, l)$$

The weight of a polymer is

$$\zeta(P) = \prod_{q=1}^{N} \prod_{(\mathbf{x}, l) \in L_q(P)} \zeta(S_q^k(\mathbf{x}, l)) = \prod_{q=1}^{N} \prod_{(\mathbf{x}, l) \in L_q(P)} \prod_{h=1}^{k} \zeta(S_q^h(\mathbf{x}, l)) \tag{4.44}$$

Proposition 4.2

Suppose that there exists a positive function a on the set of colours such that

$$\sum_{(\mathbf{x}, l): \ S_q(\mathbf{x}, l) \ni \mathbf{y}} \zeta(S_q^h(\mathbf{x}, l)) \exp(a_q |S_q^k(\mathbf{x}, l)|) < a_q \tag{4.45}$$

let:

$$a(S_{q_1}(\mathbf{x},l)) = a_{q_1} l \tag{4.46}$$

$$a(P) = \sum_{S_{q'}(\mathbf{x},l')\in P} a_{q'} l' \tag{4.47}$$

then we have

$$\sum_{P\ni\tilde{y}:\ [Col(P)]_{\tilde{y}}=\bar{q}} \zeta(P)e^{a(P)} < a_{\bar{q}} \tag{4.48}$$

Proof

We will use induction on the number of segments in P.

The base of induction holds true; indeed $|P| = 1$ means that P is just one segment; so we have (4.48) from (4.45). Now suppose that (4.48) is true for $|P| = n$, we want to prove it for $|P| = n + 1$. If $P \ni \tilde{y}$ there exists a segment $\bar{S}_{\bar{q}}(\mathbf{x}, l)$ containing $\tilde{y}$; moreover $\bar{S}_{\bar{q}}(\mathbf{x}, l)$ have to be incompatible with one or more segments in P. So we have

$$\sum_{P\in\tilde{y}:\ [Col(P)]_{\tilde{y}}=\bar{q}} \zeta(P)e^{a(P)} \leq \sum_{\bar{S}_{\bar{q}}(\mathbf{x},l)\ni\tilde{y}} \zeta(\bar{S}_{\bar{q}}(\mathbf{x},l)) \prod_{y_i\in\bar{S}_{\bar{q}}(\mathbf{x},l)} \sum_{P'\iota\mathbf{y}_i} \zeta(P')e^{a(P')} \tag{4.49}$$

where P' is a polymer in which $|P'| \leq n$ so we can apply to this last sum the inductive hypothesis. The r.h.s. of (4.49) is

$$\leq \sum_{\bar{S}_{\bar{q}}(\mathbf{x},l)\ni\tilde{y}} \zeta(\bar{S}_{\bar{q}}(\mathbf{x},l)) \prod_{y_i\in\bar{S}_{\bar{q}}(\mathbf{x},l)} \sum_{P'\iota\mathbf{y}_i} (1 + a + \frac{a^2}{2} + \ldots) \leq \tag{4.50}$$

$$\leq \sum_{\bar{S}_{\bar{q}}(\mathbf{x},l)\ni\tilde{y}} \zeta(\bar{S}_{\bar{q}}(\mathbf{x},l)) \prod_{y_i\in\bar{S}_{\bar{q}}(\mathbf{x},l)} e^{a_{\bar{q}}(\mathbf{y}_i)} \leq \sum_{\bar{S}_{\bar{q}}(\mathbf{x},l)\ni\tilde{y}} \zeta(\bar{S}_{\bar{q}}(\mathbf{x},l))e^{a_{\bar{q}}(\bar{S}_{\bar{q}}(\mathbf{x},l))} < a_{\bar{q}} \tag{4.51}$$

□

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