October 1998

Projective Systems of Noncommutative Lattices as a Pregeometric Substratum

Giovanni Landi[1,3], Fedele Lizzi[2,3]

[1] *Dipartimento di Scienze Matematiche, Università di Trieste, P.le Europa 1, I-34127, Trieste, Europe.*
landi@univ.trieste.it

[2] *Dipartimento di Scienze Fisiche, Università di Napoli Federico II, Mostra d' Oltremare pad. 20, I-80125, Napoli, Europe.*
fedele.lizzi@na.infn.it

[3] *INFN, Sezione di Napoli, Napoli, Europe.*

Abstract

We present an approximation to topological spaces by *noncommutative* lattices. This approximation has a deep physical flavour based on the impossibility to fully localize particles in any position measurement. The original space being approximated is recovered out of a projective limit.

1 Introduction

We are going to present a *projective paradigm* for a quantum mechanical scheme of position measurements [2]. We take as a fact that any measurement procedure on a space M can involves only a *finite* number of detectors. Then, with some additional technical assumption on the nature of the detectors, to any such a system of detectors, of cardinality n, say, we shall associate a topological space P_n made of n points and endowed with a non *trivial* topology. Here nontriviality mainly means that P_n is not a Hausdorff space (for the moment we shall not linguistically distinguish between a topology on a set of point and the set itself) so that it is not possible to isolate completely its points. Were this the case, and n being finite, one can only get the trivial topology in which each point is both closed and open at the same time, so being completely *isolated*. In fact, it turn out that each space is not even T_1 but only T_0 [1]. On the one hand, the fact that P_n has only a finite number of points reflects the fact that we get only coarse information about the space M. On the other hand, the nontriviality of the topology of P_n is reflected in the non vanishing of some of its homotopy groups which exactly parallel those of M. To increase the number of detectors, so as to get more and more details of the space M, results in the construction of a *projective system of topological spaces*. The limit of the system is a T_0 topological space out of which M can be canonically identified.

Each space P_n being non Hausdorff, there is no room for $\mathbb{C}$-valued continuous functions on P_n, apart from the constant ones. The crucial and interesting fact is that there are plenty of operator-valued functions on P_n. Indeed, with any space P_n one associates a noncommutative C^*-algebra $\mathcal{A}_n$ (In fact, more than one) of operator valued functions on P_n. The space P_n itself can be identified as the space $Prim\mathcal{A}_n$ of primitive ideals of $\mathcal{A}_n$ endowed with the Jacobson topology, an ideal being called primitive if it is the kernel of an irreducible representation. Thus, each space P_n is a truly noncommutative space, and we shall call it a *noncommutative lattice* (though this is a bit of a misnomer because in reality they are not lattices in the usual sense). As we shall see, the algebras $\mathcal{A}_n$'s are approximately finite dimensional (AF) algebras, that is they can be approximated in norm by direct sums of matrix algebras. This fact allows some sort of a *second order approximation* in so that one can use matrix approximations to do calculations.

Contrary to what happens in general for noncommutative spaces, which are *characterized by the effective indiscernibility of their elements* [11, page 74], in a noncommutative lattice it is possible to discern its elements (they are indeed finite in number!) and this makes them easy to 'visualize'. However, in a noncommutative lattice there are region of nonseparability: there are points that cannot be separated from others.

Finally, we cannot resist to quote from [21]: *I heard Alain Connes say that he gets a deep hint from algebraic K-theory that the ultimate non-commutative algebra might be of the nature of the discrete C^*-algebras considered by logician.* We believe that the algebras that we present in this paper are exactly of such a nature.

[1] We recall that a topological space is called T_1 if any point of the space is closed. The space is called T_0 if for any two distinct points of it there is an open neighbourhood of one of the points which does not contain the other.

2 The Topological Approximation

The idea of a 'discrete substratum' underpinning the 'continuum' is somewhat spread among physicists. With particular emphasis this idea has been pushed by R. Sorkin who, in [26], assumes that the substratum be a *finitary* (see later) topological space which maintains some of the topological information of the continuum. It turns out that the finitary topology can be equivalently described in terms of a partial order. This partial order has been alternatively interpreted as determining the causal structure in the approach to quantum gravity of [6]. Recently, finitary topological spaces have been interpreted as noncommutative lattices and noncommutative geometry has been used to construct quantum mechanical and lattice field theory models, on them [2, 3].

Given a suitable covering of a topological space M, by identifying any two points of M which cannot be 'distinguished' by the sets in the covering, one constructs a lattice with a finite (or in general a countable) number of points. Such a lattice, with the quotient topology, becomes a T_0-space which turns out to be the structure space (or equivalently, the space of primitive ideals) of a postliminal [2] approximately finite dimensional (AF) algebra. Therefore, the lattice is truly a noncommutative space.

We will have as starting point the fact that it effectively impossible to localize (at a geometric point) the position of a particle. Detectors in actual physical situation have always a finite range. Let us suppose we are about to measure the position of a particle which moves on a circle, of radius one say, $S^1 = \{0 \leq \varphi \leq 2\pi, \text{ mod } 2\pi\}$. Our *'detectors'* will be taken to be (possibly overlapping) open subsets of S^1 with some mechanism which switches on the detector when the particle is in the corresponding open set. The number of detectors must be clearly finite and, as an example, we take them to consist of the following three open subsets whose union covers S^1,

$$U_1 = \{-\frac{1}{3}\pi < \varphi < \frac{2}{3}\pi\}, \quad U_2 = \{\frac{1}{3}\pi < \varphi < \frac{4}{3}\pi\}, \quad U_3 = \{\pi < \varphi < 2\pi\}. \tag{2.1}$$

Now, if two detectors, U_1 and U_2 say, are on, we will know that the particle is in the intersection $U_1 \cap U_2$ although we will be unable to distinguish any two points in this intersection. The same will be true for the other two intersections. Furthermore, if only one detector, U_1 say, is on, we can infer the presence of the particle in the *closed* subset of S^1 given by $U_1 \setminus \{U_1 \cap U_2 \bigcup U_1 \cap U_3\}$ but again we will be unable to distinguish any two points in this closed set. The same will be true for the other two closed sets of similar type. Summing up, if we have only the three detectors (2.1), we are forced to identify the points which cannot be distinguished and S^1 will be represented by a collection of six points $P = \{\alpha, \beta, \gamma, a, b, c\}$ which correspond to the following identifications

$$\begin{aligned} U_1 \cap U_3 = \{\frac{5}{3}\pi < \varphi < 2\pi\} &\rightarrow \alpha, \\ U_1 \cap U_2 = \{\frac{1}{3}\pi < \varphi < \frac{2}{3}\pi\} &\rightarrow \beta, \\ U_2 \cap U_3 = \{\pi < \varphi < \frac{4}{3}\pi\} &\rightarrow \gamma, \end{aligned} \tag{2.2}$$

[2] It is a general fact that for a postliminal algebra, irreducible representations are completely characterized by their kernels [23].

$$
\begin{aligned}
U_1 \setminus \{U_1 \cap U_2 \bigcup U_1 \cap U_3\} &= \{0 \leq \varphi \leq \frac{1}{3}\pi\} &\rightarrow& \quad a, \\
U_2 \setminus \{U_2 \cap U_1 \bigcup U_2 \cap U_3\} &= \{\frac{2}{3}\pi \leq \varphi \leq \pi\} &\rightarrow& \quad b, \\
U_3 \setminus \{U_3 \cap U_2 \bigcup U_3 \cap U_1\} &= \{\frac{4}{3}\pi \leq \varphi \leq \frac{5}{3}\pi\} &\rightarrow& \quad c.
\end{aligned} \tag{2.3}
$$

We can push things a bit further and keep track of the kind of set from which a point in P comes by declaring the point to be open (respectively closed) if the subset of S^1 from which it comes is open (respectively closed). Thus we endow the space P with a topology a basis of which consists by the following open (by definition) sets,

$$
\{\alpha\}, \quad \{\beta\}, \quad \{\gamma\}, \quad \{\alpha, a, \beta\}, \quad \{\beta, b, \gamma\}, \quad \{\alpha, c, \gamma\} \ . \tag{2.4}
$$

The corresponding topology on the quotient space P is the quotient topology of the one on S^1 generated by the three open sets $\{U_1, U_2, U_3\}$, by the quotient map (2.3).

In general, let us suppose that we have a topological space M together with an open covering $\mathcal{U} = \{U_\lambda\}$ which is also a topology for M, so that $\mathcal{U}$ is closed under arbitrary unions and finite intersections. We define an equivalence relation among points of M by declaring that any two points $x, y \in M$ are equivalent if every open set U_λ containing either x or y contains the other too,

$$
x \sim y \quad \text{if and only if} \quad x \in U_\lambda \Leftrightarrow y \in U_\lambda \ , \quad \forall \ U_\lambda \in \mathcal{U} \ . \tag{2.5}
$$

Thus, two points of M are identified if they cannot be distinguished by any 'detector' in the collection $\mathcal{U}$. The space $P_{\mathcal{U}}(M) =: M/\!\sim$ of equivalence classes is then given the quotient topology. If $\pi : M \rightarrow P_{\mathcal{U}}(M)$ is the natural projection, a set $U \subset P_{\mathcal{U}}(M)$ is declared to be open if and only if $\pi^{-1}(U)$ is open in the topology of M given by $\mathcal{U}$. The quotient topology is the finest one making π continuous. When M is compact, the covering $\mathcal{U}$ can be taken to be finite so that $P_{\mathcal{U}}(M)$ will consist of a finite number of points. If M is only locally compact the covering can be taken to be locally finite and each point has a neighbourhood intersected by only finitely many U_λ' s. Then the space $P_{\mathcal{U}}(M)$ will consist of a countable number of points; in the terminology of [26] $P_{\mathcal{U}}(M)$ would be a *finitary* approximation of M. If $P_{\mathcal{U}}(M)$ has N points we shall also denote it by $P_N(M)$ although this notation is incomplete since it does not keep track of the topology given on the set of N points. For the examples considered in these paper, the topology will always be given explicitly. For example, the finite space given by (2.3) is $P_6(S^1)$.

In general, $P_{\mathcal{U}}(M)$ is not Hausdorff: from (2.4) it is evident that in $P_6(S^1)$, for instance, we cannot isolate the point a from α by using open sets. It is not even a T_1-space; again, in $P_6(S^1)$ only the points a, b and c are closed while the points α, β and γ are open. In general there will be points which are neither closed nor open. However, $P_{\mathcal{U}}(M)$ is always a T_0-space, being, indeed, the T_0-quotient of M with respect to the topology $\mathcal{U}$ [26].

3 Order and Topology

What we shall show next is how the topology of any finitary T_0 topological space P can be given equivalently by means of a partial order which makes P a *partially ordered set* (or *poset* for short). Consider first the case when P is finite. Then, the collection τ of open sets (the topology on P) will be closed under arbitrary unions and arbitrary intersections. Thus, for any point $x \in P$, the intersection of all open sets containing it,

$$\Lambda(x) =: \bigcap \{U \in \tau \mid x \in U\} \ , \tag{3.6}$$

will be the smallest open set containing the point. A relation $\preceq$ is defined on P by

$$x \preceq y \ \Leftrightarrow \ \Lambda(x) \subseteq \Lambda(y) \ , \quad \forall \ x, y \in P \ . \tag{3.7}$$

Now, $x \in \Lambda(x)$ always, so that the previous definition is equivalent to

$$x \preceq y \ \Leftrightarrow \ x \in \Lambda(y) \ , \tag{3.8}$$

which can also be stated saying that

$$x \preceq y \ \Leftrightarrow \ \text{every open set containing } y \text{ also contains } x \ , \tag{3.9}$$

or, in turn, that

$$x \preceq y \ \Leftrightarrow \ y \in \overline{\{x\}} \ , \tag{3.10}$$

with $\overline{\{x\}}$ the closure of the one point set $\{x\}$. Another equivalent definition can be given by saying that $x \preceq y$ if and only if the constant sequence $(x, x, x, \cdots)$ converges to y. It is worth noticing that in a T_0-space the limit of a sequence need not be unique so that the constant sequence $(x, x, x, \cdots)$ may converge to more than one point.
¿From (3.7) it is clear that the relation $\preceq$ is reflexive, and transitive. Furthermore, since P is a T_0-space, for any two distinct points $x, y \in P$, there is at least one open set containing x, say, and not y. This, together with (3.9), implies that the relation $\preceq$ is symmetric as well, $x \preceq y \ , \ y \preceq x \ \Rightarrow \ x = y$. Summing up, we see that a T_0 topology on a finite space P determines a reflexive, antisymmetric and transitive relation, namely a *partial order*. Conversely, given a partial order $\preceq$ on the set P, one produces a topology on P by taking as a basis for it the finite collection of 'open' sets defined by

$$\Lambda(x) =: \{y \in P \mid y \preceq x\} \ , \quad \forall \ x \in P \ . \tag{3.11}$$

Thus, a subset $W \subset P$ will be open if and only if it is the union of sets of the form (3.11), that is, if and only if $x \in W$ and $y \preceq x \ \Rightarrow \ y \in W$. Indeed, the smallest open set containing W is given by $\Lambda(W) = \bigcup_{x \in W} \Lambda(x)$, and W is open if and only if $W = \Lambda(W)$. The resulting topological space is clearly T_0 by the antisymmetry of the order relation.

It is easy to express the closure operation in terms of the partial order. From (3.10), the closure $V(x) = \overline{\{x\}}$, of the one point set $\{x\}$ is given by

$$V(x) =: \{y \in P \mid x \preceq y\} \ , \quad \forall \ x \in P \ . \tag{3.12}$$

A subset $W \subset P$ will be closed if and only if $x \in W$ and $x \preceq y \ \Rightarrow \ y \in W$. Indeed, the closure of W is given by $V(W) = \bigcup_{x \in W} V(x)$, and W is closed if and only if $W = V(W)$.

If one relaxes the condition of finiteness of the space P, there is still an equivalence between topology and partial order for any T_0 topological space which has the additional property that every intersection of open sets is an open set (or equivalently, that every union of closed sets is a closed set), so that the sets (3.6) are all open and provide a basis

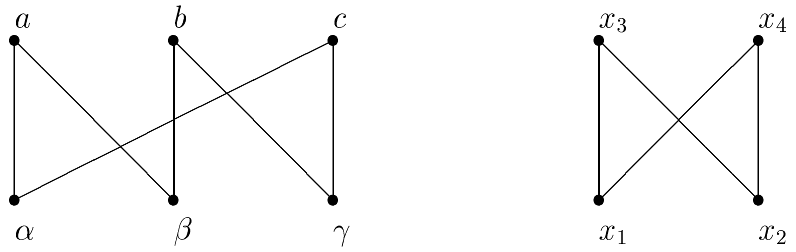

Figure 1: The Hasse diagrams for $P_6(S^1)$ and for $P_4(S^1)$

for the topology [1, 7]. This would be the case if P were a finitary approximation of a (locally compact) topological space M, obtained then from a locally finite covering of M.

A pictorial representation of the topology of a poset is obtained by constructing the associated *Hasse diagram*: one arranges the points of the poset at different levels and connects them by the rules : ($x \prec y$ will indicate that x precedes y while $x \neq y$)

1. if $x \prec y$, then x is at a lower level than y;
2. if $x \prec y$ and there is no z such that $x \prec z \prec y$, then x is at the level immediately below y and these two points are connected by a link.

Figure 1 shows the Hasse diagram for $P_6(S^1)$ whose basis of open sets is in (2.4) and for $P_4(S^1)$. For the former, the partial order reads $\alpha \prec a$, $\alpha \prec c$, $\beta \prec a$, $\beta \prec b$, $\gamma \prec b$, $\gamma \prec c$. The latter is a four point approximation of S^1 obtained from a covering consisting of two intersecting open sets. The partial order reads $x_1 \prec x_3, \ x_1 \prec x_4, \ x_2 \prec x_3, \ x_2 \prec x_4$.
In Fig. 1, (and in general, in any Hasse diagram the smallest open set containing any point x consists of all points which are below the given one, x, and can be connected to it by a series of links. For example, for $P_4(S^1)$, we have as the minimal open sets,

$$\Lambda(x_1) = \{x_1\} \ , \quad \Lambda(x_2) = \{x_2\} \ , \quad \Lambda(x_3) = \{x_1, x_2, x_3\} \ , \quad \Lambda(x_4) = \{x_1, x_2, x_4\} \ , \tag{3.13}$$

which are a basis for the topology of $P_4(S^1)$.
The generic finitary poset $P(\mathbb{R})$ associated with the real line $\mathbb{R}$ is shown in Fig. 2. The corresponding projection $\pi : \mathbb{R} \to P(\mathbb{R})$ is given by

$$\begin{array}{rcl} U_i \cap U_{i+1} & \longrightarrow & x_i \ , \quad i \in \mathbb{Z} \ , \\ U_{i+1} \setminus \{U_i \cap U_{i+1} \bigcup U_{i+1} \cap U_{i+2}\} & \longrightarrow & y_i \ , \quad i \in \mathbb{Z} \ . \end{array} \tag{3.14}$$

A basis for the quotient topology is provided by the collection of all open sets of the form

$$\Lambda(x_i) = \{x_i\} \ , \quad \Lambda(y_i) = \{x_i, y_i, x_{i+1}\} \ , \quad i \in \mathbb{Z} \ . \tag{3.15}$$

Figure 3 shows the Hasse diagram for the six-point poset $P_6(S^2)$ of the two dimensional sphere, coming from a covering with four open sets, which was derived in [26]. A basis for its topology is given by

$$\Lambda(x_1) = \{x_1\} \ , \quad \Lambda(x_2) = \{x_2\} \ , \quad \Lambda(x_3) = \{x_1, x_2, x_3\} \ , \quad \Lambda(x_4) = \{x_1, x_2, x_4\} \ ,$$

$$\Lambda(x_5) = \{x_1, x_2, x_3, x_4, x_5\} \ , \quad \Lambda(x_6) = \{x_1, x_2, x_3, x_4, x_6\} \ . \tag{3.16}$$

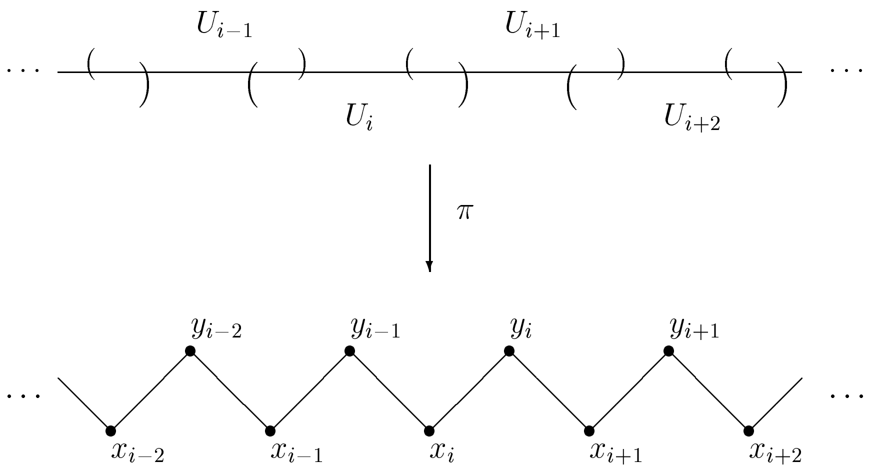

Figure 2: The finitary poset of the line $\mathbb{R}$

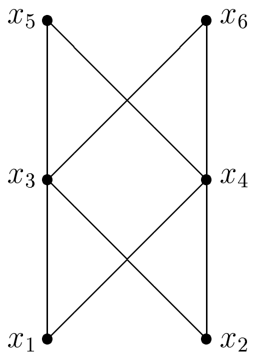

Figure 3: The Hasse diagram for the poset $P_6(S^2)$

The top two points are closed, the bottom two points are open and the intermediate ones are neither closed nor open.

One key feature of noncommutative lattices is that, although composed of a finite number of elements, not all of the topological information of the original set has disappeared. For example, one can prove that for the first homotopy group, $\pi_1(P_N(S^1)) = \mathbb{Z} = \pi(S^1)$ whenever $N \geq 4$ [26].

4 The Reconstruction of the Approximated Space

We shall now briefly describe how the topological space being approximated can be recovered 'in the limit' by considering a sequence of finer and finer coverings, the appropriate framework being that of projective (or inverse) systems of topological spaces [26].

Let us consider a topological space M and a sequence $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ of finer and finer coverings, that is of coverings such that

$$\mathcal{U}_i \subseteq \tau(\mathcal{U}_{i+1}) \ , \tag{4.17}$$

where $\tau(\mathcal{U})$ is the topology generated by the covering $\mathcal{U}$. Here we are relaxing the harmless assumption made in Sect. 2 that each $\mathcal{U}$ is already a subtopology, namely that $\mathcal{U} = \tau(\mathcal{U})$.

In Sect. 2 we have associated with each covering $\mathcal{U}_i$ a T_0-topological space P_i and a continuous surjection

$$\pi_i : M \to P_i \ . \tag{4.18}$$

We now construct a *projective system of spaces P_i together with continuous maps*

$$\pi_{ij} : P_j \to P_i \ , \tag{4.19}$$

defined whenever $i \leq j$ and such that

$$\pi_i = \pi_{ij} \circ \pi_j \ . \tag{4.20}$$

These maps are uniquely defined by the fact that the spaces P_i's are T_0 and that the map π_i is continuous with respect to $\tau(\mathcal{U}_j)$ whenever $i \leq j$. Indeed, if U is open in P_i, then $\pi_i^{(-1)}(U)$ is open in the $\mathcal{U}_i$-topology by definition, thus it is also open in the finer $\mathcal{U}_j$-topology and π_i is continuous in $\tau(\mathcal{U}_j)$. Furthermore, uniqueness also implies the compatibility conditions

$$\pi_{ij} \circ \pi_{jk} = \pi_{ik} \ , \tag{4.21}$$

whenever $i \leq j \leq k$. Indeed, the map π_{ij} is the solution (by definition it is then unique) of a universal mapping problem for maps relating T_0-spaces [26]. ¿From the surjectivity of the maps π_i's and the relation (4.20), it follows that all maps π_{ij} are surjective.
The projective system of topological spaces together with continuous maps $\{P_i, \pi_{ij}\}_{i,j \in \mathbb{N}}$ has a unique *projective limit*, i.e. a topological space P_∞, together with continuous maps

$$\pi_{i\infty} : P_\infty \to P_i \ , \tag{4.22}$$

such that

$$\pi_{ij} \circ \pi_{j\infty} = \pi_{i\infty} \ , \tag{4.23}$$

whenever $i \leq j$. The space P_∞ and the maps π_{ij} can be constructed explicitly. An element $x \in P_\infty$ is an arbitrary coherent sequence of elements $x_i \in P_i$,

$$x = (x_i)_{i \in \mathbb{N}} \ , \ x_i \in P_i \mid \exists \ N_0 \quad \text{s.t.} \quad x_i = \pi_{i,i+1}(x_{i+1}) \ , \quad \forall \ i \geq N_0 \ . \tag{4.24}$$

As for the map $\pi_{i\infty}$, it is simply defined by

$$\pi_{i\infty}(x) = x_i \ . \tag{4.25}$$

The space P_∞ is made into a T_0 topological space by endowing it with the weakest topology making all maps $\pi_{i\infty}$ continuous: a basis for it is given by the sets $\pi_{i\infty}^{(-1)}(U)$, for all open sets $U \subset P_i$. The projective system and its limit are depicted in Fig. 4.
It turns out that the limit space P_∞ is *bigger* than the starting space M and that the latter is contained as a dense subspace. Furthermore, M can be characterized as the set of all *closed* points of $P_{i\infty}$. First of all, we also get a unique (by universality) continuous map

$$\pi_\infty : M \to P_\infty \ , \tag{4.26}$$

which satisfies

$$\pi_i = \pi_{i\infty} \circ \pi_\infty \ , \quad \forall \ i \in \mathbb{N} \ . \tag{4.27}$$

$$
\begin{tikzcd}
M \arrow[rr, "\pi_\infty"] \arrow[ddr, "\pi_j"] \arrow[dddr, "\pi_i"'] & & P_\infty \arrow[ddl, "\pi_{j\infty}"'] \arrow[dddl, "\pi_{i\infty}"] \\
& \vdots \arrow[d] & \\
& P_j \arrow[d, "\pi_{ij}"] & \\
& P_i &
\end{tikzcd}
$$

Figure 4: The projective system of topological spaces with continuous maps which approximates the space M

The map π_∞ is the 'limit' of the maps π_i. However, while the latter are surjective, under mild hypothesis, the former turns out to be *injective.* We have, indeed, the following results whose proof is in [26, 22].

Proposition 4.1 *The image $\pi_\infty(M)$ is dense in P_∞.*

Proposition 4.2 *Let M be T_0 and the collection $\{\mathcal{U}_i\}$ of coverings be such that for every $m \in M$ and every neighbourhood $N \ni m$, there exists an index i and an element $U \in \tau(\mathcal{U}_i)$ such that $m \in U \subset N$. Then, the map π_∞ is injective.*

In a sense, the second condition in the previous Proposition just says that the covering $\mathcal{U}_i$ contains 'enough small open sets', a condition one would expect in the process of recovering M by a refinement of the coverings.

As alluded to before, there is a nice characterization of the points of M (or better still of $\pi_\infty(M)$) as the set of all closed points of P_∞.

Proposition 4.3 *Let M be T_1 and let the collection $\{\mathcal{U}_i\}$ of coverings fulfil the 'fineness' condition of Proposition 4.2. Let each covering $\mathcal{U}_i$ consist only of sets which are bounded (have compact closure). Then $\pi_\infty : M \to P_\infty$ embeds M in P_∞ as the subspace of closed points.*

As for the extra points of P_∞, one can prove that for any extra $y \in P_\infty$, there exists an $x \in \pi_\infty(M)$ to which y is 'infinitely close'. Indeed, P_∞ can be turned into a poset by defining a partial order relation as follows

$$x \preceq_\infty y \quad \Leftrightarrow \quad x_i \preceq y_i \ , \quad \forall \, i \ , \tag{4.28}$$

where the coherent sequences $x = (x_i)$ and $y = (y_i)$ are any two elements of P_∞. In fact, one could directly construct P_∞ as the projective limit of a projective system of posets by defining a partial order on the coherent sequences as in (4.28).

Then one can characterize $\pi_\infty(M)$ as the *set of maximal elements of* P_∞, with respect to the order $\preceq_\infty$. Given any such maximal element x, the points of P_∞ which are infinitely close to x are all (non maximal) points which converge to x, namely all (not maximal) $y \in P_\infty$ such that $y \preceq_\infty x$. In P_∞, these points y cannot be separated from the corresponding x. By identifying points in P_∞ which cannot be separated one recovers M. The interpretation that emerges is that the top points of a poset $P(M)$ (which are always closed) approximate the points of M and give all of M in the limit. The rôle of the remaining points is to 'glue' the top points together so as to produce a topologically nontrivial approximation to M. They also give the extra points in the limit.

In [5] a somewhat different interpretation of the approximation and of the limiting procedure in terms of simplicial decompositions has been proposed.

5 Noncommutative Lattices

It turns out that any (finite) poset P is the structure space $\widehat{\mathcal{A}}$ (the space of irreducible representations) of a noncommutative C^*-algebra $\mathcal{A}$ of operator valued functions which then plays the rôle of the algebra of continuous functions on P. It is worth noticing that, a poset P being non Hausdorff, there cannot be 'enough' $\mathbb{C}$-valued continuous functions on P since the latter separate points. For instance, on the poset of Fig. 1 or Fig. 3 the only $\mathbb{C}$-valued continuous functions are the constant ones. In fact, the previous statement is true for each connected component of any poset.

Indeed, there is a complete classification of all separable C^*-algebras with a finite dual [4]. Given any finite T_0-space P, it is possible to construct a C^*-algebra $\mathcal{A}(P,d)$ of operators on a separable Hilbert space $\mathcal{H}(P,d)$ which satisfies $\widehat{\mathcal{A}(P,d)} = P$. Here d is a function on P with values in $\mathbb{N} \cup \infty$ which is called a *defector*. Thus there is more than one algebra with the same structure space. We refer to [4, 19] for the actual construction of the algebras together with extensions to countable posets. Here, we shall instead describe a more general class of algebras, namely approximately finite dimensional ones, a subclass of which is associated with posets. As the name suggests, these algebras can be approximated by finite dimensional algebras, a fact which has been used in the construction of physical models on posets as we shall describe in Sect. 6.

Before we proceed, we mention that if a separable C^*-algebra has a finite dual than it is postliminal [4]. As alluded to already, for any such algebra $\mathcal{A}$, irreducible representations are completely characterized by their kernels so that the space of irreducible representations is homeomorphic with the space $Prim\mathcal{A}$ of primitive ideals. Furthermore, the Jacobson topology on $Prim\mathcal{A}$ is equivalent to the partial order defined by the inclusion of ideals. This fact in a sense 'closes the circle' making any poset, when thought of as $Prim\mathcal{A}$ of a noncommutative algebra $\mathcal{A}$, a truly noncommutative space or, rather, a *noncommutative lattice.*

5.1 AF-Algebras

In this Section we shall describe approximately finite dimensional algebras following [8]. A general algebra of this sort may have a rather complicated ideal structure and a complicated primitive ideal structure. As mentioned before, for applications to posets only a special subclass is selected.

Definition 5.1 *A C^*-algebra $\mathcal{A}$ is said to be approximately finite dimensional (AF) if there exists an increasing sequence*

$$\mathcal{A}_0 \overset{I_0}{\hookrightarrow} \mathcal{A}_1 \overset{I_1}{\hookrightarrow} \mathcal{A}_2 \overset{I_2}{\hookrightarrow} \cdots \overset{I_{n-1}}{\hookrightarrow} \mathcal{A}_n \overset{I_n}{\hookrightarrow} \cdots \tag{5.29}$$

of finite dimensional C^*-subalgebras of $\mathcal{A}$, such that $\mathcal{A}$ is the norm closure of $\bigcup_n \mathcal{A}_n$, $\mathcal{A} = \overline{\bigcup_n \mathcal{A}_n}$. The maps I_n are injective *-morphisms.

The algebra $\mathcal{A}$ is the *inductive* (or *direct*) *limit* of the *inductive system* $\{\mathcal{A}_n, I_n\}_{n\in\mathbb{N}}$ of algebras [27]. As a set, $\bigcup_n \mathcal{A}_n$ is made of coherent sequences,

$$\bigcup_n \mathcal{A}_n = \{a = (a_n)_{n\in\mathbb{N}} \ , a_n \in \mathcal{A}_n \mid \exists N_0 \ , a_{n+1} = I_n(a_n) \ , \forall \ n > N_0\}. \tag{5.30}$$

Now the sequence $(||a_n||_{\mathcal{A}_n})_{n\in\mathbb{N}}$ is eventually decreasing since $||a_{n+1}|| \leq ||a_n||$ (the maps I_n are norm decreasing) and therefore convergent. One writes for the norm on $\mathcal{A}$,

$$||(a_n)_{n\in\mathbb{N}}|| = \lim_{n\to\infty} ||a_n||_{\mathcal{A}_n} \ . \tag{5.31}$$

Since the maps I_n are injective, the expression (5.31) gives a true norm directly and not simply a seminorm and there is no need to quotient out the zero norm elements.

We shall assume that the algebra $\mathcal{A}$ has a unit $\mathbb{I}$. If $\mathcal{A}$ and $\mathcal{A}_n$ are as before, then $\mathcal{A}_n + \mathbb{C}\mathbb{I}$ is clearly a finite dimensional C^*-subalgebra of $\mathcal{A}$ and moreover, $\mathcal{A}_n \subset \mathcal{A}_n + \mathbb{C}\mathbb{I} \subset \mathcal{A}_{n+1} + \mathbb{C}\mathbb{I}$. We may thus assume that each $\mathcal{A}_n$ contains the unit $\mathbb{I}$ and that the maps I_n are unital.

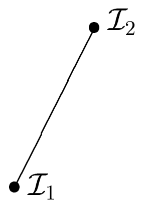

Figure 5: The two point poset of the interval

Example 5.1 Let $\mathcal{H}$ be an infinite dimensional (separable) Hilbert space. The algebra

$$\mathcal{A} = \mathcal{K}(\mathcal{H}) + \mathbb{C}\mathbb{I}_{\mathcal{H}} \ , \tag{5.32}$$

with $\mathcal{K}(\mathcal{H})$ the algebra of compact operators, is an AF-algebra [8]. The approximating algebras are given by

$$\mathcal{A}_n = \mathbb{M}_n(\mathbb{C}) \oplus \mathbb{C} \ , \quad n > 0 \ , \tag{5.33}$$

with embedding

$$\mathbb{M}_n(\mathbb{C}) \oplus \mathbb{C} \ni (\Lambda, \lambda) \mapsto \left(\begin{Bmatrix} \Lambda & 0 \\ 0 & \lambda \end{Bmatrix}, \lambda \right) \in \mathbb{M}_{n+1}(\mathbb{C}) \oplus \mathbb{C} \ . \tag{5.34}$$

Indeed, let $\{\xi_n\}_{n\in\mathbb{N}}$ be an orthonormal basis in $\mathcal{H}$ and let $\mathcal{H}_n$ be the subspace generated by the first n basis elements, $\{\xi_1, \cdots, \xi_n\}$. With $\mathcal{P}_n$ the orthogonal projection onto $\mathcal{H}_n$, define

$$\begin{aligned} \mathcal{A}_n &= \{T \in \mathcal{B}(\mathcal{H}) \mid T(\mathbb{I} - \mathcal{P}_n) = (\mathbb{I} - \mathcal{P}_n)T \in \mathbb{C}(\mathbb{I} - \mathcal{P}_n)\} \\ &\simeq \mathcal{B}(\mathcal{H}_n) \oplus \mathbb{C} \simeq \mathbb{M}_n(\mathbb{C}) \oplus \mathbb{C} \ . \end{aligned} \tag{5.35}$$

Then $\mathcal{A}_n$ embeds in $\mathcal{A}_{n+1}$ as in (5.34). Since each $T \in \mathcal{A}_n$ is a sum of a finite rank operator and a multiple of the identity, one has that $\mathcal{A}_n \subseteq \mathcal{A} = \mathcal{K}(\mathcal{H}) + \mathbb{C}\mathbb{I}_{\mathcal{H}}$ and, in turn, $\overline{\bigcup_n \mathcal{A}_n} \subseteq \mathcal{A} = \mathcal{K}(\mathcal{H}) + \mathbb{C}\mathbb{I}_{\mathcal{H}}$. Conversely, since finite rank operators are norm dense in $\mathcal{K}(\mathcal{H})$, and finite linear combinations of strings $\{\xi_1, \cdots, \xi_n\}$ are dense in $\mathcal{H}$, one gets that $\mathcal{K}(\mathcal{H}) + \mathbb{C}\mathbb{I}_{\mathcal{H}} \subset \overline{\bigcup_n \mathcal{A}_n}$.

The algebra (5.32) has only two irreducible representations [4],

$$\begin{aligned} &\pi_1 : \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H}) \ , &&a = (k + \lambda\mathbb{I}_{\mathcal{H}}) \mapsto \pi_1(a) = a \ , \\ &\pi_2 : \mathcal{A} \longrightarrow \mathcal{B}(\mathbb{C}) \simeq \mathbb{C} \ , &&a = (k + \lambda\mathbb{I}_{\mathcal{H}}) \mapsto \pi_2(a) = \lambda \ , \end{aligned} \tag{5.36}$$

with $\lambda_1, \lambda_2 \in \mathbb{C}$ and $k \in \mathcal{K}(\mathcal{H})$; the corresponding kernels being

$$\mathcal{I}_1 =: ker(\pi_1) = \{0\} \ , \quad \mathcal{I}_2 =: ker(\pi_2) = \mathcal{K}(\mathcal{H}) \ . \tag{5.37}$$

The partial order given by the inclusions $\mathcal{I}_1 \subset \mathcal{I}_2$ produces the two point poset shown in Fig. 5. As we shall see, this space is really the fundamental building block for all posets. A comparison with the poset of the line in Fig. 2, shows that it can be thought of as a two point approximation of an interval.

In general, each subalgebra $\mathcal{A}_n$ being a finite dimensional C^*-algebra, is a direct sum of matrix algebras,

$$\mathcal{A}_n = \bigoplus_{k=1}^{k_n} \mathbb{M}_{d_k^{(n)}}(\mathbb{C}) \ , \tag{5.38}$$

where $\mathbb{M}_d(\mathbb{C})$ is the algebra of $d \times d$ matrices with complex coefficients. In order to study the embedding $\mathcal{A}_1 \hookrightarrow \mathcal{A}_2$ of any two such algebras $\mathcal{A}_1 = \bigoplus_{j=1}^{n_1} \mathbb{M}_{d_j^{(1)}}(\mathbb{C})$ and $\mathcal{A}_2 = \bigoplus_{k=1}^{n_2} \mathbb{M}_{d_k^{(2)}}(\mathbb{C})$, one uses the fact that it is always possible [22] to choose bases in $\mathcal{A}_1$ and $\mathcal{A}_2$ in such a way as to identify $\mathcal{A}_1$ with a subalgebra of $\mathcal{A}_2$ having the following form

$$\mathcal{A}_1 \simeq \bigoplus_{k=1}^{n_2} \left(\bigoplus_{j=1}^{n_1} N_{kj} \mathbb{M}_{d_j^{(1)}}(\mathbb{C}) \right) \ . \tag{5.39}$$

Here, with any two nonnegative integers p, q, the symbol $p\mathbb{M}_q(\mathbb{C})$ stands for

$$p\mathbb{M}_q(\mathbb{C}) \simeq \mathbb{M}_q(\mathbb{C}) \otimes_{\mathbb{C}} \mathbb{I}_p \ , \tag{5.40}$$

and one identifies $\bigoplus_{j=1}^{n_1} N_{kj} \mathbb{M}_{d_j^{(1)}}(\mathbb{C})$ with a subalgebra of $\mathbb{M}_{d_k^{(2)}}(\mathbb{C})$. The nonnegative integers N_{kj} satisfy the condition

$$\sum_{j=1}^{n_1} N_{kj} d_j^{(1)} = d_k^{(2)} \ . \tag{5.41}$$

One says that the algebra $\mathbb{M}_{d_j^{(1)}}(\mathbb{C})$ is *partially embedded* in $\mathbb{M}_{d_k^{(2)}}(\mathbb{C})$ with *multiplicity* N_{kj}. A useful way of representing the algebras $\mathcal{A}_1$, $\mathcal{A}_2$ and the embedding $\mathcal{A}_1 \hookrightarrow \mathcal{A}_2$ is by means of a diagram, the so called *Bratteli diagram* [8], which can be constructed out of the dimensions $d_j^{(1)}$, $j = 1, \ldots, n_1$ and $d_k^{(2)}$, $k = 1, \ldots, n_2$, of the diagonal blocks of the two algebras and out of the numbers N_{kj} that describe the partial embeddings. One draws two horizontal rows of vertices, the top (bottom) one representing $\mathcal{A}_1$ ($\mathcal{A}_2$) and consisting of n_1 (n_2) vertices, one for each block which are labelled by the corresponding dimensions $d_1^{(1)}, \ldots, d_{n_1}^{(1)}$ ($d_1^{(2)}, \ldots, d_{n_2}^{(2)}$). Then, for each $j = 1, \ldots, n_1$ and $k = 1, \ldots, n_2$, the relation $d_j^{(1)} \searrow^{N_{kj}} d_k^{(2)}$ denotes the embeddings of $\mathbb{M}_{d_j^{(1)}}(\mathbb{C})$ in $\mathbb{M}_{d_k^{(2)}}(\mathbb{C})$ with multiplicity N_{kj}.

For any AF-algebra $\mathcal{A}$ one repeats the procedure for each level, and in this way one obtains a semi-infinite diagram, denoted by $\mathcal{D}(\mathcal{A})$ which completely defines $\mathcal{A}$ up to isomorphism. The diagram $\mathcal{D}(\mathcal{A})$ depends not only on the collection of $\mathcal{A}$'s but also on the particular sequence $\{\mathcal{A}_n\}_{n\in\mathbb{N}}$ which generates $\mathcal{A}$. However, one can obtain an algorithm which allows one to construct from a given diagram all diagrams which define AF-algebras which are isomorphic with the original one [8]. The problem of identifying the limit algebra or of determining whether or not two such limits are isomorphic can be very subtle. Elliot [17] has devised an invariant for AF-algebras in terms of the corresponding K theory which completely distinguishes among them (see also [16]). It is worth remarking that the isomorphism class of an AF-algebra $\overline{\bigcup_n \mathcal{A}_n}$ depends not only on the collection of algebras $\mathcal{A}_n$'s but also on the way they are embedded into each other.

Given a set $\mathcal{D}$ of ordered pairs $(n, k), k = 1, \cdots, k_n$, $n = 0, 1, \cdots$, with $k_0 = 1$, and a sequence $\{\searrow^p\}_{p=0,1,\ldots}$ of relations on $\mathcal{D}$, the latter is the diagram $\mathcal{D}(\mathcal{A})$ of an AF-algebras when the following conditions are satisfied,

(i) If $(n,k),(m,q) \in \mathcal{D}$ and $m = n+1$, there exists one and only one nonnegative (or equivalently, at most a positive) integer p such that $(n,k) \searrow^p (n+1,q)$.

(ii) If $m \neq n+1$, no such integer exists.

(iii) If $(n,k) \in \mathcal{D}$, there exists $q \in \{1,\cdots,n_{n+1}\}$ and a nonnegative integer p such that $(n,k) \searrow^p (n+1,q)$.

(iv) If $(n,k) \in \mathcal{D}$ and $n > 0$, there exists $q \in \{1,\cdots,n_{n-1}\}$ and a nonnegative integer p such that $(n-1,q) \searrow^p (n,k)$.

It is easy to see that the diagram of a given AF-algebra satisfies the previous conditions. Conversely, if the set $\mathcal{D}$ of ordered pairs satisfies these properties, one constructs by induction a sequence of finite dimensional C^*-algebras $\{\mathcal{A}_n\}_{n\in\mathbb{N}}$ and of injective morphisms $I_n : \mathcal{A}_n \to \mathcal{A}_{n+1}$ in such a manner so that the inductive limit $\{\mathcal{A}_n, I_n\}_{n\in\mathbb{N}}$ will have $\mathcal{D}$ as its diagram. Explicitly, one defines

$$\mathcal{A}_n = \bigoplus_{k;(n,k)\in\mathcal{D}} \mathbb{M}_{d_k^{(n)}}(\mathbb{C}) = \bigoplus_{k=1}^{k_n} \mathbb{M}_{d_k^{(n)}}(\mathbb{C}) \ , \tag{5.42}$$

and morphisms

$$\begin{aligned} I_n : \textstyle\bigoplus_{j=1}^{j_n} \mathbb{M}_{d_j^{(n)}}(\mathbb{C}) &\longrightarrow \textstyle\bigoplus_{k=1}^{k_{n+1}} \mathbb{M}_{d_k^{(n+1)}}(\mathbb{C}) \ , \\ A_1 \oplus \cdots \oplus A_{j_n} &\mapsto (\oplus_{j=1}^{j_n} N_{1j} A_j) \bigoplus \cdots \bigoplus (\oplus_{j=1}^{j_n} N_{k_{n+1}j} A_j) \ , \end{aligned} \tag{5.43}$$

where the integers N_{kj} are such that $(n,j) \searrow^{N_{kj}} (n+1,k)$ and we have used the notation (5.40). Notice that the dimension $d_k^{(n+1)}$ of the factor $\mathbb{M}_{d_k^{(n+1)}}(\mathbb{C})$ is not arbitrary but it is determined by a relation like (5.41), $d_k^{(n+1)} = \sum_{j=1}^{j_n} N_{kj} d_j^{(n)}$.

Example 5.2 An AF-algebra $\mathcal{A}$ is commutative if and only if all the factors $\mathbb{M}_{d_k^{(n)}}(\mathbb{C})$ are one dimensional, $\mathbb{M}_{d_k^{(n)}}(\mathbb{C}) \simeq \mathbb{C}$. Thus the corresponding diagram $\mathcal{D}$ has the property that for each $(n,k) \in \mathcal{D}$, $n > 0$, there is exactly one $(n-1,j) \in \mathcal{D}$ such that $(n-1,j) \searrow^1 (n,k)$.

Example 5.3 Let us consider the subalgebra $\mathcal{A}$ of the algebra $\mathcal{B}(\mathcal{H})$ of bounded operators on an infinite dimensional (separable) Hilbert space $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, given in the following manner. Let $\mathcal{P}_j$ be the projection operators on $\mathcal{H}_j$, $j = 1,2$, and $\mathcal{K}(\mathcal{H})$ the algebra of compact operators on $\mathcal{H}$. Then, the algebra $\mathcal{A}$ is

$$\mathcal{A}_\vee = \mathbb{C}\mathcal{P}_1 + \mathcal{K}(\mathcal{H}) + \mathbb{C}\mathcal{P}_2 \ . \tag{5.44}$$

The use of the symbol $\mathcal{A}_\vee$ is due to the fact that, as we shall see below, this algebra is associated with any part of the poset of the line in Fig. 2, of the form

$$\bigvee = \{y_{i-1}, x_i, y_i\} \ , \tag{5.45}$$

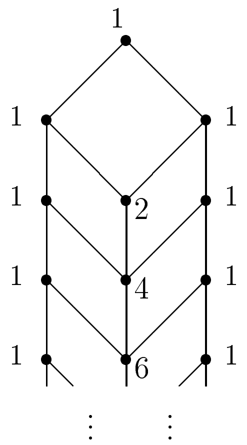

Figure 6: The Bratteli diagram of the algebra $\mathcal{A}_\vee$; the labels indicate the dimension of the corresponding matrix algebras

in the sense that this poset is identified with the space of primitive ideals of $\mathcal{A}_\vee$. The C^*-algebra (5.44) can be obtained as the inductive limit of the following sequence of finite dimensional algebras:

$$\begin{aligned} \mathcal{A}_0 &= \mathbb{M}_1(\mathbb{C}) \ , \\ \mathcal{A}_1 &= \mathbb{M}_1(\mathbb{C}) \oplus \mathbb{M}_1(\mathbb{C}) \ , \\ \mathcal{A}_2 &= \mathbb{M}_1(\mathbb{C}) \oplus \mathbb{M}_2(\mathbb{C}) \oplus \mathbb{M}_1(\mathbb{C}) \ , \\ \mathcal{A}_3 &= \mathbb{M}_1(\mathbb{C}) \oplus \mathbb{M}_4(\mathbb{C}) \oplus \mathbb{M}_1(\mathbb{C}) \ , \\ &\vdots \\ \mathcal{A}_n &= \mathbb{M}_1(\mathbb{C}) \oplus \mathbb{M}_{2n-2}(\mathbb{C}) \oplus \mathbb{M}_1(\mathbb{C}) \ , \\ &\vdots \end{aligned} \tag{5.46}$$

where, for $n \geq 1$, $\mathcal{A}_n$ is embedded in $\mathcal{A}_{n+1}$ as follows

$$\begin{aligned} &\mathbb{M}_1(\mathbb{C}) \oplus \mathbb{M}_{2n-2}(\mathbb{C}) \oplus \mathbb{M}_1(\mathbb{C}) \ \hookrightarrow \\ &\qquad \hookrightarrow \ \mathbb{M}_1(\mathbb{C}) \oplus (\mathbb{M}_1(\mathbb{C}) \oplus \mathbb{M}_{2n-2}(\mathbb{C}) \oplus \mathbb{M}_1(\mathbb{C})) \oplus \mathbb{M}_1(\mathbb{C}) \ , \end{aligned}$$

$$\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix} \mapsto \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & B & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix} , \tag{5.47}$$

for any $\lambda_1, \lambda_2 \in \mathbb{M}_1(\mathbb{C})$ and any $B \in \mathbb{M}_{2n-2}(\mathbb{C})$. The corresponding Bratteli diagram is shown in Fig. 6. The algebra (5.44) has three irreducible representations,

$$\begin{aligned} \pi_1 &: \mathcal{A}_\vee \longrightarrow \mathcal{B}(\mathcal{H}) \ , & a &= (\lambda_1\mathcal{P}_1 + k + \lambda_2\mathcal{P}_2) \mapsto \pi_1(a) = a \ , \\ \pi_2 &: \mathcal{A}_\vee \longrightarrow \mathcal{B}(\mathbb{C}) \simeq \mathbb{C} \ , & a &= (\lambda_1\mathcal{P}_1 + k + \lambda_2\mathcal{P}_2) \mapsto \pi_2(a) = \lambda_1 \ , \\ \pi_3 &: \mathcal{A}_\vee \longrightarrow \mathcal{B}(\mathbb{C}) \simeq \mathbb{C} \ , & a &= (\lambda_1\mathcal{P}_1 + k + \lambda_2\mathcal{P}_2) \mapsto \pi_3(a) = \lambda_2 \ , \end{aligned} \tag{5.48}$$

with $\lambda_1, \lambda_2 \in \mathbb{C}$ and $k \in \mathcal{K}(\mathcal{H})$. The corresponding kernels are

$$\mathcal{I}_1 = \{0\} \ , \quad \mathcal{I}_2 = \mathcal{K}(\mathcal{H}) + \mathbb{C}\mathcal{P}_2 \ , \quad \mathcal{I}_3 = \mathbb{C}\mathcal{P}_1 + \mathcal{K}(\mathcal{H}) \ . \tag{5.49}$$

The partial order given by the inclusions $\mathcal{I}_1 \subset \mathcal{I}_2$ and $\mathcal{I}_1 \subset \mathcal{I}_3$ (which is an equivalent way to provide the Jacobson topology) produces a topological space $Prim\mathcal{A}_\vee$ which is just the $\bigvee$ poset in (5.45).

5.2 From Noncommutative Lattices to Bratteli Diagrams (and viceversa)

From the Bratteli diagram of an AF-algebra $\mathcal{A}$ one can also obtain the (norm closed two-sided) ideals of the latter and determine which ones are primitive. On the set of such ideals the topology is then given by constructing a poset whose partial order is provided by the inclusion of ideals. Therefore, both $Prim(\mathcal{A})$ and its topology can be determined from the Bratteli diagram of $\mathcal{A}$. We refer to [22] for details. Here we shall briefly describe the reverse algorithm which allows one to construct an AF-algebra (or rather its Bratteli diagram $\mathcal{D}(\mathcal{A})$) whose primitive ideal space is a given (finitary, noncommutative) lattice P [9, 10]. We refer to [18, 19, 22] for more details and several examples.

Proposition 5.1 *Let P be a topological space with the following properties,*

(i) *The space P is T_0;*

(ii) *If $F \subset P$ is a closed set which is not the union of two proper closed subsets, then F is the closure of a one-point set;*

(iii) *The space P contains at most a countable number of closed sets;*

(iv) *If $\{F_n\}_n$ is a decreasing ($F_{n+1} \subseteq F_n$) sequence of closed subsets of P, then $\bigcap_n F_n$ is an element in $\{F_n\}_n$.*

Then, there exists an AF algebra $\mathcal{A}$ whose primitive space $Prim\mathcal{A}$ is homeomorphic to P.

Proof. The proof consists in constructing explicitly the Bratteli diagram $\mathcal{D}(\mathcal{A})$ of the algebra $\mathcal{A}$. We shall sketch the main steps while referring to [9, 10] for more details.

- Let $\{K_0, K_1, K_2, \ldots\}$ be the collection of all closed sets in the lattice P, with $K_0 = P$.

- Consider the subcollection $\mathcal{K}_n = \{K_0, K_1, \ldots, K_n\}$ and let $\mathcal{K}'_n$ be the smallest collection of (closed) sets in P containing $\mathcal{K}_n$ which is closed under union and intersection.

- Consider the algebra of sets (We recall that a non empty collection R of subsets of a set X is called an *algebra of sets* if R is closed under the operations of union, i.e. $E, F \in R \Rightarrow E \cup F \in R$, and of complement, i.e. $E \in R \Rightarrow E^c =: X \setminus E \in R$.) generated by the collection $\mathcal{K}_n$. Then, the minimal sets $\mathcal{Y}_n = \{Y_n(1), Y_n(2), \ldots, Y_n(k_n)\}$ of this algebra form a partition of P.

- Let $F_n(j)$ be the smallest set in the subcollection $\mathcal{K}'_n$ which contains $Y_n(j)$. Define $\mathcal{F}_n = \{F_n(1), F_n(2), \dots, F_n(k_n)\}$.

- As a consequence of the assumptions in the Proposition one has that

$$Y_n(k) \subseteq F_n(k) \ , \quad \bigcup_k Y_n(k) = P \ , \quad \bigcup_k F_n(k) = P \ , \tag{5.50}$$

$$Y_n(k) = F_n(k) \setminus \bigcup_{p \neq k} \{F_n(p) \mid F_n(p) \subset F_n(k)\} \ , \tag{5.51}$$

$$F_n(k) = \bigcup_p \{F_{n+1}(p) \mid F_{n+1}(p) \subseteq F_n(k)\} \ , \tag{5.52}$$

$$\text{If} \quad F \subset P \quad \text{is closed} \ , \ \exists \ n \geq 0 \ , \ \text{s.t.} \ F_n(k) = \bigcup_p \{F_n(p) \mid F_n(p) \subseteq F\} \tag{5.53}$$

- The diagram $\mathcal{D}(\mathcal{A})$ is constructed as follows.

  (1.) *The n-th level of $\mathcal{D}(\mathcal{A})$ has k_n points, one for each set $Y_n(k)$, with$k = 1, \cdots, k_n$.* Thus $\mathcal{D}(\mathcal{A})$ is the set of all ordered pairs (n, k), $k = 1, \dots, k_n$, $n = 0, 1, \dots$.

  (2.) *The point corresponding to $Y_n(k)$ at level n of the diagram is linked to the point corresponding to $Y_{n+1}(j)$ at level $n+1$, if and only if $Y_n(k) \cap F_{n+1}(j) \neq \emptyset$. The multiplicity of the embedding is always* 1.
  Thus, the partial embeddings of the diagram are given by

$$(n, k) \ \searrow^p \ (n+1, j) \ , \quad \text{with} \quad \left\{ \begin{array}{l} p = 1 \ \ \text{if} \ \ Y_n(k) \cap \ F_{n+1}(j) \neq \emptyset \ , \\ p = 0 \ \ \text{otherwise} \ . \end{array} \right. \tag{5.54}$$

That the diagram $\mathcal{D}(\mathcal{A})$ is really the diagram of an AF algebra $\mathcal{A}$, namely that conditions $(i) - (iv)$ of page 12 are satisfied, follows from the conditions (5.51)-(5.53) above.

We know that different algebras could yield the same space of primitive ideals (strong Morita equivalence). It may happen that by changing the order in which the closed sets of P are taken in the construction of the previous proposition, one produces different algebras, all having the same space of primitive ideals though, and so all producing spaces which are homeomorphic to the starting P (any two of these spaces being, a fortiori, homeomorphic).

Example 5.4 As a simple example, consider again the lattice,

$$\bigvee = \{y_{i-1}, x_i, y_i\} \equiv \{x_2, x_1, x_3\} \ . \tag{5.55}$$

This topological space contains four closed sets:

$$K_0 = \{x_2, x_1, x_3\} \ , K_1 = \{x_2\} \ , K_2 = \{x_3\} \ , K_3 = \{x_2, x_3\} = K_1 \cup K_2 \ . \tag{5.56}$$

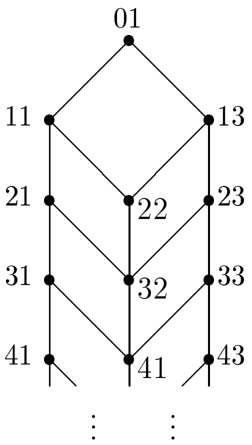

Figure 7: The Bratteli diagram associated with the poset $\bigvee$; the label nk stands for $Y_n(k)$

Thus, with the notation of Proposition 5.1, it is not difficult to check that:

$$
\begin{array}{ll}
\mathcal{K}_0 = \{K_0\} \ , & \mathcal{K}'_0 = \{K_0\} \ , \\
\mathcal{K}_1 = \{K_0, K_1\} \ , & \mathcal{K}'_1 = \{K_0, K_1\} \ , \\
\mathcal{K}_2 = \{K_0, K_1, K_2\} \ , & \mathcal{K}'_2 = \{K_0, K_1, K_2, K_3\} \ , \\
\mathcal{K}_3 = \{K_0, K_1, K_2, K_3\} \ , & \mathcal{K}'_3 = \{K_0, K_1, K_2, K_3\} \ , \\
\vdots &
\end{array}
$$

$$
\begin{array}{llll}
Y_0(1) = \{x_1, x_2, x_3\} \ , & & F_0(1) = K_0 \ , & \\
 & & & \\
Y_1(1) = \{x_2\} \ , & Y_1(2) = \{x_1, x_3\} \ , & F_1(1) = K_1 \ , & F_1(2) = K_0 \ , \\
 & & & \\
Y_2(1) = \{x_2\} \ , & Y_2(2) = \{x_1\} \ , & F_2(1) = K_1 \ , & F_2(2) = K_0 \ , \\
Y_2(3) = \{x_3\} \ , & & F_2(3) = K_2 \ , & \\
 & & & \\
Y_3(1) = \{x_2\} \ , & Y_3(2) = \{x_1\} \ , & F_3(1) = K_1 \ , & F_3(2) = K_0 \ , \\
Y_3(3) = \{x_3\} \ , & & F_3(2) = K_2 \ , & \\
\vdots & & &
\end{array}
\tag{5.57}
$$

Since $\bigvee$ has only a finite number of points (three), and hence a finite number of closed sets (four), the partition of $\bigvee$ repeats itself after the third level. Figure 7 shows the corresponding diagram, obtained through rules (1.) and (2.) in Proposition 5.1 above (on page 16). By using the fact that the first matrix algebra $\mathcal{A}_0$ is $\mathbb{C}$ and the fact that all the embeddings have multiplicity one, the diagram of Fig. 7 is seen to coincide with the diagram of Fig. 6. As we have previously said, the latter corresponds to the AF-algebra

$$
\mathcal{A}_{\vee} = \mathbb{C}\mathcal{P}_1 + \mathcal{K}(\mathcal{H}) + \mathbb{C}\mathcal{P}_2 \ , \ \mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \ . \tag{5.58}
$$

Example 5.5 Another interesting example is provided by the lattice $P_4(S^1)$ for the one-dimensional sphere in Fig. 1. This topological space contains six closed sets:

$$\begin{aligned} &K_0 = \{x_1, x_2, x_3, x_4\} \ , \ K_1 = \{x_1, x_3, x_4\} \ , \ K_2 = \{x_3\} \ , \ K_3 = \{x_4\} \ , \\ &K_4 = \{x_2, x_3, x_4\} \ , \ K_5 = \{x_3, x_4\} = K_2 \cup K_3 \ . \end{aligned} \tag{5.59}$$

Thus, with the notation of Proposition 5.1, one finds,

$$\begin{array}{ll} \mathcal{K}_0 = \{K_0\} \ , & \mathcal{K}'_0 = \{K_0\} \ , \\ \mathcal{K}_1 = \{K_0, K_1\} \ , & \mathcal{K}'_1 = \{K_0, K_1\} \ , \\ \mathcal{K}_2 = \{K_0, K_1, K_2\} \ , & \mathcal{K}'_2 = \{K_0, K_1, K_2\} \ , \\ \mathcal{K}_3 = \{K_0, K_1, K_2, K_3\} \ , & \mathcal{K}'_3 = \{K_0, K_1, K_2, K_3, K_5\} \ , \\ \mathcal{K}_4 = \{K_0, K_1, K_2, K_3, K_4\} \ , & \mathcal{K}'_4 = \{K_0, K_1, K_2, K_3, K_4, K_5\} \ , \\ \mathcal{K}_5 = \{K_0, K_1, K_2, K_3, K_4, K_5\} \ , & \mathcal{K}'_5 = \{K_0, K_1, K_2, K_3, K_4, K_5\} \ , \\ \vdots & \end{array}$$

$$\begin{array}{llll} Y_0(1) = \{x_1, x_2, x_3, x_4\} \ , & & F_0(1) = K_0 \ , & \\ & & & \\ Y_1(1) = \{x_1, x_3, x_4\} \ , & & & \\ & Y_1(2) = \{x_2\} \ , & F_1(1) = K_1 \ , & F_1(2) = K_0 \ , \\ & & & \\ Y_2(1) = \{x_3\} \ , & Y_2(2) = \{x_2\} \ , & F_2(1) = K_2 \ , & F_2(2) = K_0 \ , \\ Y_2(3) = \{x_1, x_4\} \ , & & F_2(3) = K_1 \ , & \\ & & & \\ Y_3(1) = \{x_3\} \ , & Y_3(2) = \{x_2\} \ , & F_3(1) = K_2 \ , & F_3(2) = K_0 \ , \\ Y_3(3) = \{x_1\} \ , & Y_3(4) = \{x_4\} \ , & F_3(3) = K_1 \ , & F_3(4) = K_3 \ , \\ & & & \\ Y_4(1) = \{x_3\} \ , & Y_4(2) = \{x_2\} \ , & F_4(1) = K_2 \ , & F_4(2) = K_4 \ , \\ Y_4(3) = \{x_1\} \ , & Y_4(4) = \{x_4\} \ , & F_4(3) = K_1 \ , & F_4(4) = K_3 \ , \\ & & & \\ Y_5(1) = \{x_3\} \ , & Y_5(2) = \{x_2\} \ , & F_5(1) = K_2 \ , & F_5(2) = K_4 \ , \\ Y_5(3) = \{x_1\} \ , & Y_5(4) = \{x_4\} \ , & F_5(3) = K_1 \ , & F_5(4) = K_3 \ , \\ \vdots & & & \end{array} \tag{5.60}$$

Since there are a finite number of points (four), and hence a finite number of closed sets (six), the partition of $P_4(S^1)$ repeats itself after the fourth level. The corresponding Bratteli diagram is exhibited in Fig. 8. The ideal $\{0\}$ is not primitive. The algebra is given by

$$\begin{aligned} \mathcal{A}_0 &= \mathbb{M}_1(\mathbb{C}) \ , \\ \mathcal{A}_1 &= \mathbb{M}_1(\mathbb{C}) \oplus \mathbb{M}_1(\mathbb{C}) \ , \\ \mathcal{A}_2 &= \mathbb{M}_1(\mathbb{C}) \oplus \mathbb{M}_2(\mathbb{C}) \oplus \mathbb{M}_1(\mathbb{C}) \ , \\ \mathcal{A}_3 &= \mathbb{M}_1(\mathbb{C}) \oplus \mathbb{M}_4(\mathbb{C}) \oplus \mathbb{M}_2(\mathbb{C}) \oplus \mathbb{M}_1(\mathbb{C}) \ , \\ \mathcal{A}_4 &= \mathbb{M}_1(\mathbb{C}) \oplus \mathbb{M}_6(\mathbb{C}) \oplus \mathbb{M}_4(\mathbb{C}) \oplus \mathbb{M}_1(\mathbb{C}) \ , \\ &\ \vdots \\ \mathcal{A}_n &= \mathbb{M}_1(\mathbb{C}) \oplus \mathbb{M}_{2n-2}(\mathbb{C}) \oplus \mathbb{M}_{2n-4}(\mathbb{C}) \oplus \mathbb{M}_1(\mathbb{C}) \ , \\ &\ \vdots \end{aligned} \tag{5.61}$$

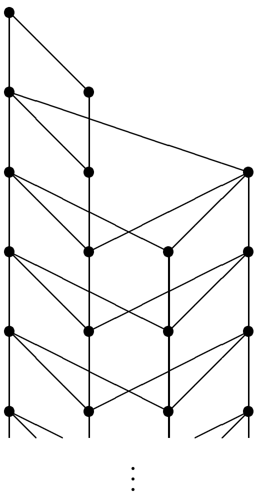

Figure 8: The Bratteli diagram for the circle poset $P_4(S^1)$

where, for $n > 2$, $\mathcal{A}_n$ is embedded in $\mathcal{A}_{n+1}$ as follows

$$
\begin{bmatrix}
\lambda_1 & & & \\
& B & & \\
& & C & \\
& & & \lambda_2
\end{bmatrix}
\mapsto
\begin{bmatrix}
\lambda_1 & & & & & & & \\
& \lambda_1 & 0 & 0 & & & & \\
& 0 & B & 0 & & & & \\
& 0 & 0 & \lambda_2 & & & & \\
& & & & \lambda_1 & 0 & 0 & \\
& & & & 0 & C & 0 & \\
& & & & 0 & 0 & \lambda_2 & \\
& & & & & & & \lambda_2
\end{bmatrix} , \tag{5.62}
$$

with $\lambda_1, \lambda_2 \in \mathbb{M}_1(\mathbb{C})$, $B \in \mathbb{M}_{2n-2}(\mathbb{C})$ and $C \in \mathbb{M}_{2n-4}(\mathbb{C})$; elements which are not shown are equal to zero. The algebra limit $\mathcal{A}_{P_4(S^1)}$ can be realized explicitly as a subalgebra of bounded operators on an infinite dimensional Hilbert space $\mathcal{H}$ naturally associated with the poset $P_4(S^1)$. Firstly, to any *link* $(x_i, x_j), x_i \succ x_j$, of the poset one associates a Hilbert space $\mathcal{H}_{ij}$; for the case at hand, one has four Hilbert spaces, $\mathcal{H}_{31}, \mathcal{H}_{32}, \mathcal{H}_{41}, \mathcal{H}_{42}$. Then, since all links are at the same level, $\mathcal{H}$ is just given by the direct sum

$$\mathcal{H} = \mathcal{H}_{31} \oplus \mathcal{H}_{32} \oplus \mathcal{H}_{41} \oplus \mathcal{H}_{42} \ . \tag{5.63}$$

The algebra $\mathcal{A}_{P_4(S^1)}$ is given by [19],

$$\mathcal{A}_{P_4(S^1)} = \mathbb{C}\mathcal{P}_{\mathcal{H}_{31} \oplus \mathcal{H}_{32}} + \mathcal{K}_{\mathcal{H}_{31} \oplus \mathcal{H}_{41}} + \mathcal{K}_{\mathcal{H}_{32} \oplus \mathcal{H}_{42}} + \mathbb{C}\mathcal{P}_{\mathcal{H}_{41} \oplus \mathcal{H}_{42}} \ . \tag{5.64}$$

Here $\mathcal{K}$ denotes compact operators and $\mathcal{P}$ orthogonal projection. The algebra (5.64) has four irreducible representations. Any element $a \in \mathcal{A}_{P_4(S^1)}$ is of the form

$$a = \lambda \mathcal{P}_{3,12} + k_{34,1} + k_{34,2} + \mu \mathcal{P}_{4,12} \ , \tag{5.65}$$

with $\lambda, \mu \in \mathbb{C}$, $k_{34,1} \in \mathcal{K}_{\mathcal{H}_{31} \oplus \mathcal{H}_{41}}$ and $k_{34,2} \in \mathcal{K}_{\mathcal{H}_{32} \oplus \mathcal{H}_{42}}$. The representations are,

$$
\begin{array}{lll}
\pi_1 : \mathcal{A}_{P_4(S^1)} \longrightarrow \mathcal{B}(\mathcal{H}) \ , & a \mapsto \pi_1(a) = \lambda \mathcal{P}_{3,12} + k_{34,1} + \mu \mathcal{P}_{4,12} \ , \\
\pi_2 : \mathcal{A}_{P_4(S^1)} \longrightarrow \mathcal{B}(\mathcal{H}) \ , & a \mapsto \pi_2(a) = \lambda \mathcal{P}_{3,12} + k_{34,2} + \mu \mathcal{P}_{4,12} \ , \\
\pi_3 : \mathcal{A}_{P_4(S^1)} \longrightarrow \mathcal{B}(\mathbb{C}) \simeq \mathbb{C} \ , & a \mapsto \pi_3(a) = \lambda \ , \\
\pi_4 : \mathcal{A}_{P_4(S^1)} \longrightarrow \mathcal{B}(\mathbb{C}) \simeq \mathbb{C} \ , & a \mapsto \pi_4(a) = \mu \ ,
\end{array} \tag{5.66}
$$

with corresponding kernels,

$$
\begin{aligned}
&\mathcal{I}_1 = \mathcal{K}_{\mathcal{H}_{32}\oplus\mathcal{H}_{42}} \ , \quad \mathcal{I}_2 = \mathcal{K}_{\mathcal{H}_{31}\oplus\mathcal{H}_{41}} \ , \\
&\mathcal{I}_3 = \mathcal{K}_{\mathcal{H}_{31}\oplus\mathcal{H}_{41}} + \mathcal{K}_{\mathcal{H}_{32}\oplus\mathcal{H}_{42}} + \mathbb{C}\mathcal{P}_{\mathcal{H}_{41}\oplus\mathcal{H}_{42}} \ , \\
&\mathcal{I}_4 = \mathbb{C}\mathcal{P}_{\mathcal{H}_{31}\oplus\mathcal{H}_{32}} + \mathcal{K}_{\mathcal{H}_{31}\oplus\mathcal{H}_{41}} + \mathcal{K}_{\mathcal{H}_{32}\oplus\mathcal{H}_{42}} \ .
\end{aligned} \tag{5.67}
$$

The partial order given by the inclusions $\mathcal{I}_1 \subset \mathcal{I}_3$, $\mathcal{I}_1 \subset \mathcal{I}_4$ and $\mathcal{I}_2 \subset \mathcal{I}_3$, $\mathcal{I}_2 \subset \mathcal{I}_4$ produces a topological space $Prim\mathcal{A}_{P_4(S^1)}$ which is just the circle poset in Fig. 1.

5.3 The general case

In fact, by looking at the previous examples a bit more carefully one can infer the algorithm by which one goes from a (finite) poset P to the corresponding Bratteli diagram $\mathcal{D}(\mathcal{A}_P)$. Let $(x_1, \cdots, x_N)$ be the points of P and for $k = 1, \cdots, N$, let $S_k =: \overline{\{x_k\}}$ be the smallest closed subset of P containing the point x_j. Then, the Bratteli diagram repeats itself after level N and the partition $Y_n(k)$ of Proposition 5.1 is just given by

$$Y_n(k) = Y_{n+1}(k) = \{x_k\} \ , \quad k = 1, \ldots, N \ , \quad \forall\, n \geq N \ . \tag{5.68}$$

As for the associated $F_n(k)$, from level $N+1$ on, they are given by the S_k,

$$F_n(k) = F_{n+1}(k) = S_k \ , \ k = 1, \ldots, N \ , \quad \forall\, n \geq N+1 \ . \tag{5.69}$$

In the diagram $\mathcal{D}(\mathcal{A}_P)$, for any $n \geq N$, $(n,k) \searrow (n+1,j)$ if and only if $\{x_k\} \bigcap S_j \neq \emptyset$, that is if and only if $x_k \in S_j$.
We also sketch the algorithm used to construct the algebra limit $\mathcal{A}_P$ determined by the Bratteli diagram $\mathcal{D}(\mathcal{A}_P)$ (This algebra is really defined only modulo Morita equivalence). [4, 19]. The idea is to associate to the poset P an infinite dimensional separable Hilbert space $\mathcal{H}(P)$ out of tensor products and direct sums of infinite dimensional (separable) Hilbert spaces $\mathcal{H}_{ij}$ associated with each link $(x_i, x_j), x_i \succ x_j$, in the poset. (The Hilbert spaces could all be taken to be the same. The label is there just to distinguish among them.) Then for each point $x \in P$ there is a subspace $\mathcal{H}(x) \subset \mathcal{H}(P)$ and an algebra $\mathcal{B}(x)$ of bounded operators acting on $\mathcal{H}(x)$. The algebra $\mathcal{A}_P$ is the one generated by all the $\mathcal{B}(x)$ as x varies in P. In fact, the algebra $\mathcal{B}(x)$ can be made to act on the whole of $\mathcal{H}(P)$ by defining its action on the complement of $\mathcal{H}(x)$ to be zero. Consider any *maximal chain* C_α in P: $C_\alpha = \{x_\alpha, \ldots, x_2, x_1 \mid x_j \succ x_{j-1}\}$ for any maximal point $x_\alpha \in P$. To this chain one associates the Hilbert space

$$\mathcal{H}(C_\alpha) = \mathcal{H}_{\alpha,\alpha-1} \otimes \cdots \otimes \mathcal{H}_{3,2} \otimes \mathcal{H}_{2,1} \ . \tag{5.70}$$

By taking the direct sum over all maximal chains, one gets the Hilbert space $\mathcal{H}(P)$,

$$\mathcal{H}(P) = \bigoplus_\alpha \mathcal{H}(C_\alpha) \ . \tag{5.71}$$

The subspace $\mathcal{H}(x) \subset \mathcal{H}(P)$ associated with any point $x \in P$ is constructed in a similar way by restricting the sum to all maximal chains containing the point x. It can be split into two parts,

$$\mathcal{H}(x) = \mathcal{H}(x)^u \otimes \mathcal{H}(x)^d \ , \tag{5.72}$$

with,

$$\begin{array}{ll} \mathcal{H}(x)^u = \mathcal{H}(P_x^u) \ , & P_x^u = \{y \in P \mid y \succeq x\} \ , \\ \mathcal{H}(x)^d = \mathcal{H}(P_x^d) \ , & P_x^d = \{y \in P \mid y \preceq x\} \ . \end{array} \tag{5.73}$$

Here $\mathcal{H}(P_x^u)$ and $\mathcal{H}(P_x^d)$ are constructed as in (5.71); also, $\mathcal{H}(x)^u = \mathbb{C}$ if x is a maximal point and $\mathcal{H}(x)^d = \mathbb{C}$ if x is a minimal point. Consider now the algebra $\mathcal{B}(x)$ of bounded operators on $\mathcal{H}(x)$ given by

$$\mathcal{B}(x) = \mathcal{K}(\mathcal{H}(x)^u) \otimes \mathbb{C}\mathcal{P}(\mathcal{H}(x)^d) \simeq \mathcal{K}(\mathcal{H}(x)^u) \otimes \mathcal{P}(\mathcal{H}(x)^d) \ . \tag{5.74}$$

As before, $\mathcal{K}$ denotes compact operators and $\mathcal{P}$ orthogonal projection. We see that $\mathcal{B}(x)$ acts by compact operators on the Hilbert space $\mathcal{H}(x)^u$ determined by the points which follow x and by multiples of the identity on the Hilbert space $\mathcal{H}(x)^d$ determined by the points which precede x. These algebras satisfy the rules: $\mathcal{B}(x)\mathcal{B}(y) \subset \mathcal{B}(x)$ if $x \preceq y$ and $\mathcal{B}(x)\mathcal{B}(y) = 0$ if x and y are not comparable. As already mentioned, the algebra $\mathcal{A}(P)$ of the poset P is the algebra of bounded operators on $\mathcal{H}(P)$ generated by all $\mathcal{B}(x)$ as x varies over P. It can be shown that $\mathcal{A}(P)$ has a space of primitive ideals which is homeomorphic to the poset P [4, 19]. We refer to [18, 19] for additional details and examples.

5.4 Recovering the Algebra

In Sect. 4 we have described how to recover a topological space M in the limit, by considering a sequence of finer and finer coverings of M. We constructed a projective system of finitary topological spaces and continuous maps $\{P_i, \pi_{ij}\}_{i,j\in\mathbb{N}}$ associated with the coverings; the maps $\pi_{ij} : P_j \to P_i \ , \ j \geq i$, being continuous surjections. The limit of the system is a topological space P_∞, in which M is embedded as the subspace of closed points. On each point m of (the image of) M there is a fibre of 'extra points'; the latter are all points of P_∞ which 'cannot be separated' by m.

¿From a dual point of view we get a *inductive system of algebras and homomorphisms* $\{\mathcal{A}_i, \phi_{ij}\}_{i,j\in\mathbb{N}}$; the maps $\phi_{ij} : \mathcal{A}_i \to \mathcal{A}_j \ , \ j \geq i$, being injective homeomorphisms. The system has a unique *inductive limit* $\mathcal{A}^\infty$. Each algebra $\mathcal{A}_i$ is such that $\widehat{\mathcal{A}}_i = P_i$ and is associated with P_i as described previously, $\mathcal{A}_i = \mathcal{A}(P_i)$. The map ϕ_{ij} is a 'suitable pullback' of the corresponding surjection π_{ij}. The limit space P_∞ is the structure space of the limit algebra $\mathcal{A}^\infty$, $P_\infty = \widehat{\mathcal{A}}^\infty$. And, finally the algebra $C(M)$ of continuous functions on M can be identified with the *center* of $\mathcal{A}^\infty$.

We also get a inductive system of Hilbert spaces together with isometries $\{\mathcal{H}_i, \tau_{ij}\}_{i,j\in\mathbb{N}}$; the maps $\tau_{ij} : \mathcal{H}_i \to \mathcal{H}_j \ , \ j \geq i$, being injective isometries onto the image. The system has a unique *inductive limit* $\mathcal{H}^\infty$. Each Hilbert space $\mathcal{H}_i$ is associated with the space P_i as in (5.71), $\mathcal{H}_i = \mathcal{H}(P_i)$, the algebra $\mathcal{A}_i$ being the corresponding subalgebra of bounded operators. The maps τ_{ij} are constructed out of the corresponding ϕ_{ij}. The limit Hilbert space $\mathcal{H}^\infty$ is associated with the space P_∞ as in (5.71), $\mathcal{H}^\infty = \mathcal{H}(P_\infty)$, the algebra $\mathcal{A}^\infty$ again being the corresponding subalgebra of bounded operators. And, finally, the Hilbert space $L^2(M)$ of square integrable functions is 'contained' in $\mathcal{H}^\infty : \mathcal{H}^\infty = L^2(M) \oplus_\alpha \mathcal{H}_\alpha$, the sum being on the 'extra points' in P_∞.

All of the previous is described in great details in [5]. Here we only make a few additional remarks. By improving the approximation (by increasing the number of 'detectors')

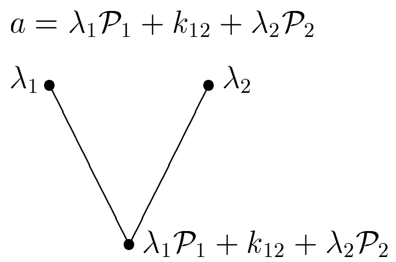

Figure 9: A function over the lattice $\bigvee$

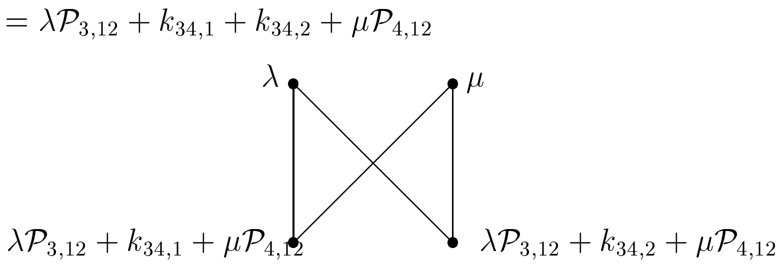

Figure 10: A function over the lattice $P_4(S^1)$

one gets a noncommutative lattice whose Hasse diagram has a bigger number of points and links. The associated Hilbert space gets 'more refined' : one may think of a *unique (and the same)* Hilbert space which is being refined while being split by means of tensor products and direct sums. In the limit the information is enough to recover completely the Hilbert space (in fact, to recover more than it). Further considerations along these lines and possible applications to quantum mechanics will have to await another occasion.

5.5 Operator Valued Functions on Noncommutative Lattices

Much in the same way as it happens for the commutative algebras [23], elements of a noncommutative C^*-algebra whose primitive spectrum $Prim\mathcal{A}$ is a noncommutative lattice can be realized as operator-valued functions on $Prim\mathcal{A}$. The value of $a \in \mathcal{A}$ at the 'point' $\mathcal{I} \in Prim\mathcal{A}$ is just the image of a under the representation $\pi_{\mathcal{I}}$ associated with $\mathcal{I}$ and such that $\ker(\pi_{\mathcal{I}}) = \mathcal{I}$,

$$a(\mathcal{I}) = \pi_{\mathcal{I}}(a) \simeq a/\mathcal{I} \ , \quad \forall \ a \in \mathcal{A}, \ \mathcal{I} \in Prim\mathcal{A} \ . \tag{5.75}$$

All this is shown pictorially in Figs. 9, 10 and 11 for the $\bigvee$ lattice, a circle lattice and a lattice Y, respectively. As it is evident in those Figures, the values of a function at points which cannot be separated by the topology differ by a compact operator. This is an illustration of the fact that compact operators play the rôle of 'infinitesimals' as is discussed at length in [11]. Furthermore, while in Figs. 9 and 10 we have only 'infinitesimals of first order', for the three level lattice of Fig. 11 we have both infinitesimals of first order, like $k_{34,2}$, and infinitesimals of second order, like $k_{34,21}$.

In fact [11], the correct way of thinking of any noncommutative C^*-algebra $\mathcal{A}$ is as the module of sections of the 'rank one trivial vector bundle' over the associated

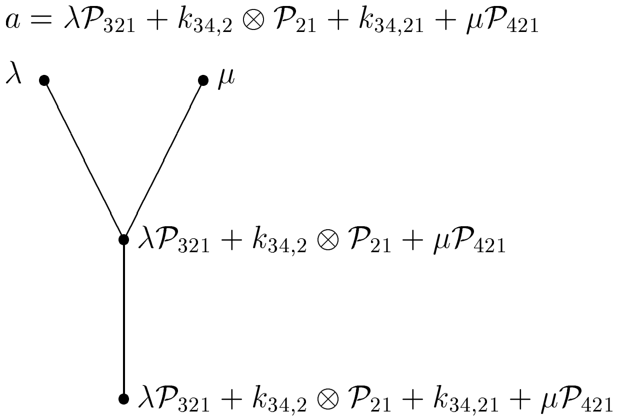

Figure 11: A function over the lattice Y

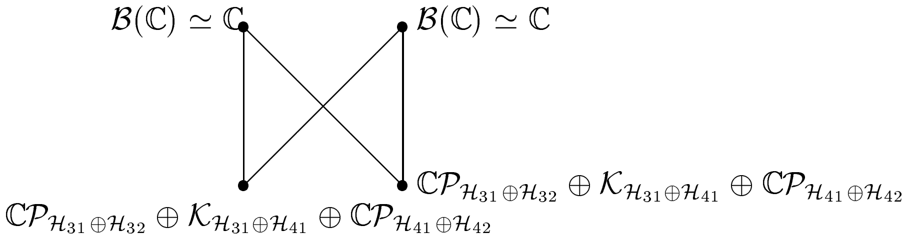

Figure 12: The fibres of the trivial bundle over the lattice $P_4(S^1)$

noncommutative space. For the kind of noncommutative lattices we are interested in, it is possible to explicitly construct the bundle over the lattice. Such bundles are examples of *bundles of C^*-algebras*, the fibre over any point $\mathcal{I} \in Prim\mathcal{A}$ being just the algebra of bounded operators $\pi_{\mathcal{I}}(\mathcal{A}) \subset \mathcal{B}(\mathcal{H}_{\mathcal{I}})$, with $\mathcal{H}_{\mathcal{I}}$ the representation space. The Hilbert space and the algebra are given explicitly by the Hilbert space in (5.72) and the algebra in (5.74) respectively, by taking for x the point $\mathcal{I}$. (At the same time, one is also constructing a bundle of Hilbert spaces.) It is also possible to endow the total space with a topology in such a manner that elements of $\mathcal{A}$ are realized as continuous sections. Figure 12 shows the trivial bundle over the lattice $P_4(S^1)$.

6 θ-Angles on Noncommutative Lattices

As a very simple example of a quantum mechanical system which can be studied with the techniques of noncommutative geometry on noncommutative lattices, we shall construct the θ-quantization of a particle on a lattice for the circle. We shall do so by constructing an appropriate 'line bundle' with a connection. We refer to [2] and [3] for more details and additional field theoretical examples. In particular, in [3] Wilson's actions for gauge and fermionic fields and analogues of topological and Chern-Simons actions were derived.

The real line $\mathbb{R}^1$ is the universal covering space of the circle S^1, and the fundamental group $\pi_1(S^1) = \mathbb{Z}$ acts on $\mathbb{R}^1$ by translation $\mathbb{R}^1 \ni x \to x + N$, $N \in \mathbb{Z}$. The quotient space

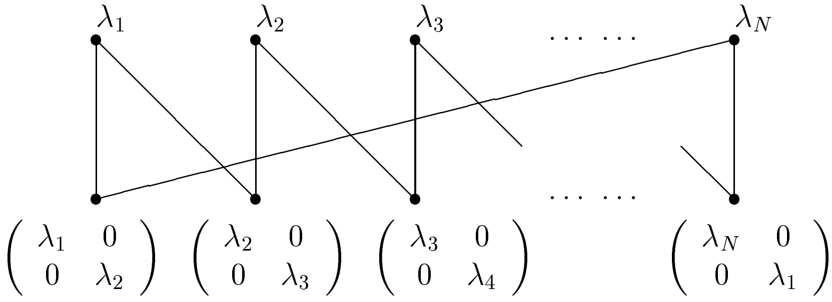

Figure 13: $P_{2N}(S^1)$ for the approximate algebra $\mathcal{C}(\mathcal{A})$

of this action is S^1 and the projection : $\mathbb{R}^1 \to S^1$ is given by $\mathbb{R}^1 \ni x \to e^{i2\pi x} \in S^1$. The domain of a typical Hamiltonian H for a particle on S^1 need not consist of functions on S^1. Rather it can be obtained from functions ψ_θ on $\mathbb{R}^1$ transforming under an irreducible representation of $\pi(S^1) = \mathbb{Z}$, $\rho_\theta : N \to e^{iN\theta}$ according to $\psi_\theta(x+N) = e^{iN\theta}\psi_\theta(x)$. The domain $D_\theta(H)$ for a typical Hamiltonian H then consists of these ψ_θ restricted to a fundamental domain $0 \leq x \leq 1$ for the action of $\mathbb{Z}$, and subject to a differentiability requirement:

$$D_\theta(H) = \{\psi_\theta : \psi_\theta(1) = e^{i\theta}\psi_\theta(0) \ ; \ \ \frac{d\psi_\theta(1)}{dx} = e^{i\theta}\frac{d\psi_\theta(0)}{dx}\} \ . \tag{6.76}$$

In addition, $H\psi_\theta$ must be square integrable. One obtains a distinct quantization, called θ-quantization, for each choice of $e^{i\theta}$.

Equivalently, wave functions can be taken to be single-valued functions on S^1 while adding a ‘gauge potential’ term to the Hamiltonian. To be more precise, one constructs a line bundle over S^1 with a connection one-form given by $i\theta dx$. If the Hamiltonian with the domain (6.76) is $-d^2/dx^2$, then the Hamiltonian with the domain $D_0(h)$ consisting of single valued wave functions is $-(d/dx + i\theta)^2$.

There are similar quantization possibilities for a noncommutative lattice for the circle as well [2]. One constructs the algebraic analogue of the trivial bundle on the lattice endowed with a gauge connection which is such that the corresponding Laplacian has an approximate spectrum reproducing the ‘continuum’ one in the limit.

As we have seen in Sect. 2, the algebra $\mathcal{A}$ associated with any noncommutative lattice of the circle is rather complicated and involves infinite dimensional operators on direct sums of infinite dimensional Hilbert spaces. In turn, this algebra $\mathcal{A}$, as it is AF (approximately finite dimensional), can indeed be approximated by algebras of matrices. The simplest approximation is just a commutative algebra $\mathcal{C}(\mathcal{A})$ of the form

$$\mathcal{C}(\mathcal{A}) \simeq \mathbb{C}^N = \{c = (\lambda_1, \lambda_2, \cdots, \lambda_N) \ , \lambda_i \in \mathbb{C}\} \ . \tag{6.77}$$

The algebra (6.77) can produce a noncommutative lattice with $2N$ points by considering a particular class of not necessarily irreducible representations as in Fig. 13. In that Figure, the top points correspond to the irreducible one dimensional representations

$$\pi_i : \mathcal{C}(\mathcal{A}) \to \mathbb{C} \ , \quad c \mapsto \pi_i(c) = \lambda_i \ , \quad i = 1, \cdots, N \ . \tag{6.78}$$

The bottom points correspond to the reducible two dimensional representations

$$\pi_{i+N} : \mathcal{C}(\mathcal{A}) \to \mathbb{M}_2(\mathbb{C}) \ , \quad c \mapsto \pi_{i+N}(c) = \begin{pmatrix} \lambda_i & 0 \\ 0 & \lambda_{i+1} \end{pmatrix} \ , \quad i = 1, \cdots, N \ , \tag{6.79}$$

with the additional condition that $\pi_{N+1} = \pi_1$ and $\lambda_{n+1} = \lambda_1$. The partial order, or equivalently the topology, is determined by the inclusion of the corresponding kernels as in Sect. 2.

By comparing Fig. 13 with the corresponding Fig. 10, we see that by trading $\mathcal{A}$ with $\mathcal{C}(\mathcal{A})$, all compact operators have been put to zero. A better approximation is obtained by approximating compact operators with finite dimensional matrices of increasing rank.

The finite projective module of sections $\mathcal{E}$ associated with the 'trivial line bundle' is just $\mathcal{C}(\mathcal{A})$ itself:

$$\mathcal{E} = \mathbb{C}^N = \{\eta = (\mu_1, \mu_2, \cdots, \mu_N) \ , \mu_i \in \mathbb{C}\} \ . \tag{6.80}$$

The action of $\mathcal{C}(\mathcal{A})$ on $\mathcal{E}$ is simply given by

$$\mathcal{E} \times \mathcal{C}(\mathcal{A}) \to \mathcal{E} \ , \quad (\eta, c) \mapsto \eta c = (\eta_1 \lambda_1, \eta_2 \lambda_2 \cdots \eta_N \lambda_N) \ . \tag{6.81}$$

On $\mathcal{E}$ there is a $\mathcal{C}(\mathcal{A})$-valued Hermitian structure $\langle \cdot, \cdot \rangle$,

$$\langle \eta', \eta \rangle := (\eta_1'^* \eta_1, \eta_2'^* \eta_2, \cdots, \eta_N'^* \eta_N) \ \in \ \mathcal{C}(\mathcal{A}) \ . \tag{6.82}$$

To complete the geometrical construction, in addition to the algebra and the Hilbert space we need a third element, a (generalized) Dirac operator D, which, with $\mathcal{A}$ and $\mathcal{H}$ form the so called *spectral triple*. The operator D is self adjoint, with compact resolvent and such that $[D, a]$ is bounded for a dense subset of the algebra, and it is used is in the construction of the algebra of differential forms. These are represent as differential forms as operators on $\mathcal{H}$. Define the (abstract) *universal differential algebra of forms* as the $\mathbb{Z}$-graded algebra $\Omega^*\mathcal{A} = \bigoplus_{p \geq 0} \Omega^p \mathcal{A}$ generated as follows: $\Omega^0 \mathcal{A} = \mathcal{A}$ and $\Omega^1 \mathcal{A}$ is generated by a set of abstract symbols da linear and which satisfy Leibnitz rule. Elements of $\Omega^p \mathcal{A}$ are linear combinations of elements of the form

$$\omega = a_0 da_1 \cdots da_p \tag{6.83}$$

A linear representation $\pi_D : \Omega^* \mathcal{A} \to \mathcal{B}(\mathcal{H})$ of the universal algebra of forms is defined by

$$\pi_D(a_0 da_1 \cdots da_p) = a_0 [D, a_1] \cdots [D, a_p] \tag{6.84}$$

Note, however, that $\pi_D(\omega) = 0$ does not necessarily imply $\pi_D(d\omega) = 0$. Forms ω for which this happens are called *junk forms*. They generate a $\mathbb{Z}$-graded ideal in $\Omega^* \mathcal{A}$ and have to be quotiented out [11, 22]. Then the noncommutative differential algebra is represented by the quotient space

$$\Omega_D^* \mathcal{A} = \pi_D \left[\Omega^* \mathcal{A} / (\ker \, \pi_D \oplus d \, \ker \, \pi_D) \right] \tag{6.85}$$

which we note depends explicitly on the particular choice of Dirac operator D on the Hilbert space $\mathcal{H}$. The algebra $\Omega_D^* \mathcal{A}$ determines a DeRham complex whose cohomology groups can be computed using the conventional methods. A discussion on differential calculus on finite sets can be found for example in [13].

We take $\mathbb{C}^N$ for $\mathcal{H}$ on which we represent elements of $\mathcal{C}(\mathcal{A})$ as diagonal matrices

$$\mathcal{C}(\mathcal{A}) \ni c \mapsto \mathrm{diag}(\lambda_1, \lambda_2, \ldots \lambda_N) \in \mathcal{B}(\mathbb{C}^N) \simeq \mathbb{M}_N(\mathbb{C}) \ . \tag{6.86}$$

Elements of $\mathcal{E}$ will be realized in the same manner,

$$\mathcal{E} \ni \eta \mapsto \mathrm{diag}(\eta_1, \eta_2, \ldots \eta_N) \in \mathcal{B}(\mathbb{C}^N) \simeq \mathbb{M}_N(\mathbb{C}) \ . \tag{6.87}$$

Since our triple $(\mathcal{C}(\mathcal{A}), \mathcal{H}, D)$ will be zero dimensional, the ($\mathbb{C}$-valued) scalar product associated with the Hermitian structure (6.82) will be taken to be

$$(\eta', \eta) = \sum_{j=1}^{N} \eta_j'^* \eta_j = tr\langle \eta', \eta \rangle \ , \quad \forall \ \eta', \eta \in \mathcal{E} \ . \tag{6.88}$$

By identifying $N+j$ with j, we take for the operator D, the $N \times N$ self-adjoint matrix with elements

$$D_{ij} = \frac{1}{\sqrt{2}\epsilon}(m^* \delta_{i+1,j} + m \delta_{i,j+1}) \ , \ i, j = 1, \cdots, N \ , \tag{6.89}$$

where m is any complex number of modulus one: $mm^* = 1$.

The connection 1-form ρ on the bundle $\mathcal{E}$ is the hermitian matrix with elements

$$\rho_{ij} = \frac{1}{\sqrt{2}\epsilon}(\sigma^* m^* \delta_{i+1,j} + \sigma m \delta_{i,j+1}) \ , \quad \sigma = e^{-i\theta/N} - 1 \ , \quad i, j = 1, \cdots, N \ . \tag{6.90}$$

One checks that, modulo junk forms, the curvature of ρ vanishes,

$$d\rho + \rho^2 = 0 \ . \tag{6.91}$$

It is also possible to prove that ρ is a 'pure gauge' for $\theta = 2\pi k$, with k any integer, that is that there exists a $c \in \mathcal{C}(\mathcal{A})$ such that $\rho = c^{-1}dc$. If $c = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N)$, then any such c will be given by $\lambda_1 = \lambda$, $\lambda_2 = e^{i2\pi k/N}\lambda$,..., $\lambda_j = e^{i2\pi k(j-1)/N}\lambda$,..., $\lambda_N = e^{i2\pi k(N-1)/N}\lambda$, with λ not equal to 0 (these properties are the analogues of the properties of the connection $i\theta dx$ in the 'continuum' limit).

The covariant derivative ∇_θ on $\mathcal{E}$, $\nabla_\theta : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{C}(\mathcal{A})} \Omega^1(\mathcal{C}(\mathcal{A}))$ is then given by

$$\nabla_\theta \eta = [D, \eta] + \rho\eta \ , \quad \forall \ \eta \in \mathcal{E} \ . \tag{6.92}$$

In order to define the Laplacian Δ_θ one first introduces a 'dual' operator ∇_θ^* via

$$(\nabla_\theta \eta', \nabla_\theta \eta) = (\eta', \nabla_\theta^* \nabla_\theta \eta) \ , \quad \forall \ \eta', \eta \in \mathcal{E}. \tag{6.93}$$

The Laplacian Δ_θ on $\mathcal{E}$, $\Delta_\theta : \mathcal{E} \to \mathcal{E}$, can then be defined by

$$\Delta_\theta \eta = -q(\nabla_\theta)^* \nabla_\theta \eta \ , \quad \forall \ \eta \in \mathcal{E} \ , \tag{6.94}$$

where q is the orthogonal projector on $\mathcal{E}$ for the scalar product $(\cdot, \cdot)$ in (6.88). This projection operator is readily seen to be given by

$$(qM)_{ij} = M_{ii}\delta_{ij} \ , \quad \text{no summation on } i \ , \tag{6.95}$$

with M any element in $\mathbb{M}_N(\mathbb{C})$. Hence, the action of Δ_θ on the element $\eta = (\eta_1, \cdots, \eta_N)$, $\eta_{N+1} = \eta_1$, is explicitly given by

$$\begin{aligned}
(\Delta_\theta \eta)_{ij} &= -(\nabla_\theta^* \nabla_\theta \eta)_{ii} \delta_{ij} \ , \\
-(\nabla_\theta^* \nabla_\theta \eta)_{ii} &= \left\{ -[D,[D,\eta]] - 2\rho[D,\eta] - \rho^2 \eta \right\}_{ii} \\
&= \frac{1}{\epsilon^2} \left[e^{-i\theta/N} \eta_{i-1} - 2\eta_i + e^{i\theta/N} \eta_{i+1} \right] \ ; \quad i = 1, 2, \cdots, N \ . \qquad (6.96)
\end{aligned}$$

The associated eigenvalue problem

$$\Delta_\theta \eta = \lambda \eta \ , \qquad (6.97)$$

has solutions

$$\lambda = \lambda_k = \frac{2}{\epsilon^2} \left[\cos(k + \frac{\theta}{N}) - 1 \right] \ , \qquad (6.98)$$

$$\eta = \eta^{(k)} = \mathrm{diag}(\eta_1^{(k)}, \eta_2^{(k)}, \cdots, \eta_N^{(k)}) \ , \quad k = m\frac{2\pi}{N} \ , \ m = 1, 2, \cdots, N \ , \qquad (6.99)$$

with each component $\eta_j^{(k)}$ having an expression of the form

$$\eta_j^{(k)} = A^{(k)} e^{ikj} + B^{(k)} e^{-ikj} \ , \quad A^{(k)}, B^{(k)} \in \mathbb{C} \ . \qquad (6.100)$$

The eigenvalues (6.98) are an approximation to the continuum answers $-4k^2$, $k \in \mathbb{R}$.

7 Conclusions

In this note we described a way to look at manifolds based on a coarse approximation which however retains the principal topological characteristics of the original space. The motivations for this work have been in the approximation processes natural in physics: finite size of the detectors, impossibility to probe very short distances, etc.

We would like to note also that, apart from the measurement problems, there is a scale (Planck Scale) at which the structure of space time is very likely not to be describable by the usual tools of geometry, and in this case Noncommutative Geometry seems to be the ideal tool for a more general description of spacetime. Some attempts on considering spacetime at the Planck scale as composed of noncommutative objects, finite or with a limited number of degrees of freedom, have been done for example in [20, 25, 24]. Another fruitful arena in which similar concepts play an important role, and which recently is having interesting "contaminations" with noncommutative geometry, is certainly string theory, the discussion of which will take too much room, and which we omit here.

In conclusion, to abandon the usual concepts (prejudices?) of a geometry made of separable points, lines, and complex functions is not only an important step for pure mathematics, but in a few years we may come to consider it as the most natural step imposed by physics.

Acknowledgements.
The material presented here is based on work done in collaborations with A.P. Balachandran, G. Bimonte, E. Ercolessi, G. Sparano, P. Teotonio-Sobrinho. We thank C. Buzzanca, D. Kastler and M. Rosso for their kind invitation to partecipate to this nice adventure.

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