ANDERSON TRANSITIONS FOR QUASI-PERIODIC SCHRÖDINGER OPERATORS IN DIMENSION 1

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Abstract. In this paper we study the spectral properties of families of quasi-periodic Schrödinger operators on the real line in the adiabatic limit. We show that the spectrum is located in exponentially small intervals whose centers are described by Bohr-Sommerfeld like conditions. We give a sufficient condition for the existence of absolutely continuous spectrum as well as for the existence of singular spectrum. We use this condition to study the coexistence of both spectral types. This enables us to define asymptotic mobility edges and study their locus.

Résumé. Cet article est consacré à l'étude du spectre de certaines familles d'équations de Schrödinger quasi-périodiques sur l'axe réel. Nous établissons que le spectre est situé dans des intervalles exponentiellement petits centrés en des points satisfaisant des conditions de type Bohr-Sommerfeld. Nous donnons une condition suffisante pour l'existence de spectre absolument continu et ainsi que pour l'existence de spectre singulier. Nous utilisons cette condition pour étudier la coexistence des deux types spectraux. Ceci nous permet de définir les seuils de mobilités asymptotiques et de les étudier.

0. Introduction

In this paper, we continue our analysis of the family of almost periodic Schrödinger equations

$$H_{\phi,\varepsilon}\psi = -\frac{d^2}{dx^2}\psi(x) + (V(x-\phi) + W(\varepsilon x))\psi(x) = E\psi(x), \quad x \in \mathbb{R} \tag{0.1}$$

where

- V and W are not constant and periodic,

$$V(x+1) = V(x), \quad W(x+2\pi) = W(x), \quad x \in \mathbb{R}, \tag{0.2}$$

- ε is fixed positive number,
- ϕ is a real parameter indexing the equations of the family.

In this paper, we study the adiabatic limit i.e. ε is small. We choose ε so that $2\pi/\varepsilon \notin \mathbb{Q}$. Hence the potential $x \mapsto V(x-\phi) + W(\varepsilon x)$ is quasi-periodic.

During the last 20 years, quasi-periodic Schrödinger operators have been the object of many studies both rigorous and numerical (see, e.g. [21, 10] for references). Motivated by the physics of condensed matter, among the main goals was the understanding of their spectra.

One of the first questions that has been addressed is the topological nature and the location of the spectrum. The spectra of quasi-periodic Schrödinger operators often exhibit remarkable

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topological structures (Hofstadter's butterfly, Cantor structure, etc) that have been proved rigorously for only a few models (e.g. limit-periodic Schrödinger equation ([28]) or the Almost Mathieu equation and some of its perturbations [19, 25, 24]).

Another question that has drawn a lot of interest (e.g. [21, 10]) is the nature of the spectrum: it is on this question that we will focus in this paper. One of the most remarkable phenomena exhibited by the spectrum of these operators is the so-called Anderson transition or "metal-insulator" transition. It was observed numerically (e.g. [20, 21]) that, for certain quasi-periodic Schrödinger operators, the spectrum is split into energy ranges where it is purely absolutely continuous and other energy ranges where it is pure point. Roughly, from the physicist's point of view, the pure point spectrum corresponds to phases where the material is an insulator, and the absolutely continuous spectrum to phase where the material is a conductor. The transition from one of these phases to the other one is expected to be sharp and to happen at an energy called the *mobility edge*. Such transitions do not always exist; e.g. for highly symmetrical models like the Almost Mathieu equation, it is believed and proved in many cases that no such transitions happen, the spectral type being determined by a natural coupling constant associated to this equation ([22]).
As far as rigorous results go, most of them have been established in the weak coupling or large energy, or in the strong coupling regime. For example, it is known that, for general one dimensional quasi-periodic Schrödinger operators, the spectrum is absolutely continuous for sufficiently high energies ([8]). On the other hand, in [15], for a model similar to ours, it was proved that the bottom of the spectrum is pure point in the large coupling constant limit. So, different types of spectra may coexist.

In [14], we have introduced the notion of *monodromy matrix* for the family of equations (0.1) and have calculated the asymptotics of a monodromy matrix in the adiabatic limit i.e. as $\varepsilon \to 0$. In the present paper, using these results on the monodromy matrix, we study the location of the spectrum of the family (0.1) and prove the coexistence of absolutely continuous and singular spectra. We will also prove that the transitions from the regions of absolutely continuous spectrum to those of singular spectrum (and vice-versa) are asymptotically sharp as $\varepsilon \to 0$. The energies where these transitions occur are called *asymptotic mobility edges.* We give a simple description of these energies in terms of analytic objects naturally associated to equation (0.1). To the best of our knowledge, this is the first instance of quasi-periodic Schrödinger operators for which such a coexistence has been proved outside of the large or small coupling regime and sharp transitions have been found.

Let us now describe our model more precisely. We will assume that ε stays in a fixed Diophantine class i.e. that, for some fixed $\aleph > 0$,

$$\varepsilon \in \mathcal{D}_\aleph \Longleftrightarrow \forall q \in \mathbb{Z} \setminus \{0\},\ \inf_{p\in\mathbb{N}} \left| \frac{2\pi}{\varepsilon} - \frac{p}{q} \right| \geq \frac{\aleph}{|q|^3}.$$

As about the potentials V and W, we assume that

- V is a real valued function in L^2_{loc};
- the first gap in the spectrum of the periodic operator

$$(H\psi)(x) = -\frac{d^2}{dx^2}\psi(x) + V(x)\psi(x) \tag{0.3}$$

  is open;
- W is real analytic;

- W has exactly one local maximum and exactly one local minimum on its period $[0, 2\pi)$. The extrema are non degenerate.

We will impose one more condition on the functions W and V. Roughly, we will ask that, in a vicinity of the real line (depending on V), the topological structure of the set $W^{-1}(\mathbb{R}) = \{\phi \in \mathbb{C} | \operatorname{Im} W(\phi) = 0\}$ be the simplest possible. We will formulate this condition precisely later.

We study the spectrum of (0.1) near its bottom. The size of the neighborhood where we study the spectrum is determined by V and W and is independent of ε. We prove that, in that neighborhood, the spectrum is situated in exponentially small (in ε) intervals such that the distances between any two neighboring such intervals are of order ε. Moreover there exists a real analytic function $\Delta S(E)$ (independent of ε) such that the subintervals where $\Delta S(E) < 0$ contain only singular spectrum, and the subintervals where $\Delta S(E) > 0$ contain absolutely continuous spectrum; actually in the latter subintervals, most of the spectrum is absolutely continuous. Finally, we study general properties of the function $\Delta S(E)$. For this, we introduce an additional parameter (coupling constant) α into equation (0.1). The new potential takes the form $V(\cdot - \phi) + \alpha W(\varepsilon \cdot)$. For small α, ΔS takes positive values for certain energies; hence, the bottom of the spectrum of the operator $H_{\phi,\varepsilon}$ still contains absolutely continuous spectrum. On contrary, for large α, the function ΔS becomes negative, and thus, the bottom of the spectrum of the unperturbed operator becomes singular, the absolutely continuous spectrum disappears. For intermediate values of α, there are always one or several asymptotic mobility edges i.e. points where the function ΔS vanishes. These are the points where the transitions from absolutely continuous to singular spectrum (or vice versa) occur.

To illustrate this, we now briefly describe situations where we have computed the asymptotic mobility edges numerically. We take the simplest example of V i.e we take V to be a one-gap potential associated to the periodic spectrum $[E_1, E_2] \cup [E_3, +\infty)$. In our case $E_1 = 0$, $E_2 = 8$. The value of E_3 is computed so that the potential V do have period 1. Such a potential is unique (see e.g. [27] and the references therein). The potential W is taken from the family

$$W_c(x) = \frac{(\cos x + c)^3 - c^3 - 3c}{3c^2 + 1}, \quad |c| > 1.$$

Let us note that, for any c, $-1 \leq W_c \leq 1$ and that $W_{+\infty}(x) = \cos(x)$.
In Fig. 1, we represented the *phase diagram* i.e. the curve $\mathcal{S}_0 = \{(\alpha, E) : \Delta S(E, \alpha) = 0\}$ of the asymptotic mobility edges (for the potential $V(\cdot - \phi) + \alpha W_c(\varepsilon \cdot)$) for 4 different values of the parameter c. The spectrum is represented vertically and the parameter α horizontally. The oblique rectangle delimits the interval of energy where we study the spectrum of (0.1). It comprises the bottom of this spectrum. α_0 (resp. α_1) is the value of α at which mobility edges start (resp. stop) to exist (in our energy interval). So, we get quite different pictures:

- for $c = +\infty$: in this case, when one turns the coupling constant α on, the lowest edge of the spectrum starts to "localize" first, though not immediately. When $\alpha > \alpha^*$, all the spectrum is singular.
- for $c = 6$: for $\alpha < \alpha_0$, the spectrum is absolutely continuous. For $\alpha > \alpha_0$, there exist two mobility edges starting from inside the spectrum; one of them is propagating towards the center of the band and the other one towards the edges. For $\alpha > \alpha_2$, the edge of the band is "localized" but one has to wait until $\alpha > \alpha^*$ to have the whole spectrum (in our observation window) "localized".
- for $c = 4$: the picture is similar to the one obtained for $c = 6$ except that the center of the band becomes "localized" before the edge.

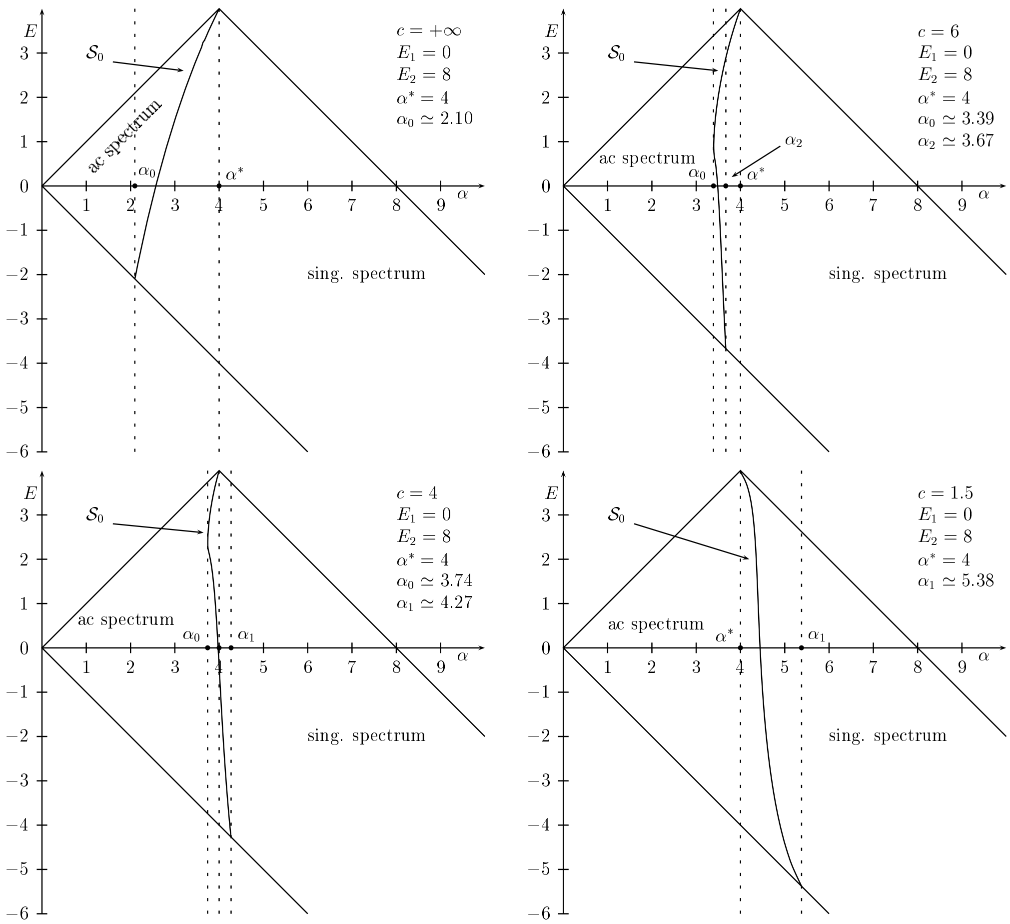

FIGURE 1. Examples of phase diagrams

- for $c = 1.5$: in this case, it is the very interior of the spectrum ($E = 4$ is the center of the unperturbed first band) that starts to "localize" first and then "localization" propagates to the edge.

That such phenomena may occur was already noted numerically ([20]).

Remark 0.1. In the description given above, we used the word "localized" very loosely to denote singular spectrum. For c and α fixed, the domain marked by "ac spectrum" (resp. "sing.spectrum") in Fig. 1 contains absolutely continuous (resp. singular) spectrum".

1. THE MAIN RESULTS

1.1. **The main objects.** We now define the main objects needed to formulate our statements. We will also describe the interval where we study the spectrum and formulate the topological condition on the function W which was only mentioned in the introduction.

1.1.1. *Periodic Schrödinger operator.* Consider the periodic Schrödinger operator (0.3). The spectrum of H on $L^2(\mathbb{R})$ is absolutely continuous and consists of intervals $[E_1, E_2]$, $[E_3, E_4]$,

…, $[E_{2n+1}, E_{2n+2}]$, …, of the real axis such that

$$E_1 < E_2 \le E_3 < E_4 \ldots E_{2n} \le E_{2n+1} < E_{2n+2} \le \ldots,$$
$$E_n \to +\infty, \quad n \to +\infty.$$

The above intervals are called the *spectral zones or bands*, and the open intervals (E_2, E_3), (E_4, E_5), …, (E_{2n}, E_{2n+1}), …, are called the *spectral gaps*.

Denote the *Bloch quasi-momentum* associated to (0.3) by $k(E)$, and the *dispersion law* by $\mathbf{E}(\kappa)$ (see section 2.3.1). The quasi-momentum is a multi-valued analytic function. It has its branching points at $(E_n)_{n\in\mathbb{N}}$. Along the spectral zones, $k(E)$ is real, and along the gaps, it takes complex values so that between E_{2n} and E_{2n+1}, $n \in \mathbb{N}$, one has $\operatorname{Re} k(E) = \pi n$, $\operatorname{Im} k(E) > 0$.

If $E_{2n} < E_{2n+1}$, one says that the n-th gap is open. So, one of our conditions on the function V means that

$$E_2 < E_3.$$

1.1.2. *The part of the spectrum we study.* Let W_+ (resp. W_-) denote the maximum (resp. the minimum) of W on the real axis. Obviously, the spectrum of $H_{\phi,\varepsilon}$ is contained in $[E_1+W_-, +\infty)$. We will study it near its bottom that is for energies

$$E \in J_\delta, \quad J_\delta = [E_1 + W_-, \inf(E_1 + W_+, E_2 + W_-) - \delta]. \tag{1.1}$$

Here δ is a positive number kept fixed as $\varepsilon \to 0$.

1.1.3. *Complex momentum.* For the asymptotic study of the family (0.1), we use the function $\kappa(\varphi)$ defined by the relation

$$\mathbf{E}(\kappa) + W(\varphi) = E,$$

or by

$$\kappa(\varphi) = k(E - W(\varphi)).$$

We call it *the complex momentum.* This function is also a multi-valued analytic function and its branching points are given by the equation

$$E - W(\varphi) = E_n, \; n \in \mathbb{N}.$$

1.1.4. *The topological condition on the set $W^{-1}(\mathbb{R})$.* As W is real analytic, the set $W^{-1}(\mathbb{R})$ consists of the real line and of complex branches symmetric with respect to the real line. Recall that W has only two non-degenerate extrema on each of its periods. Therefore, there are two complex branches beginning at each of the extrema. Without loss of generality, we shall assume that, on the period $[0, 2\pi)$, W reaches its maximum at $\varphi = 0$ and its minimum at $\varphi = \varphi_0$. Then, in at least a small vicinity of the real line, the set $W^{-1}(\mathbb{R})$ looks as in Fig. 2 (it is represented by the thick lines). This means that the complex branches are open curves beginning at the extrema, and for any of the branches, the projection to the imaginary axis is a smooth one-to-one map. In a larger vicinity of the real line these two properties may be violated; but, for example, for $W(\varphi) = \cos\varphi$, they stay true in the whole complex plane. If these properties are valid in a vicinity of the real line, we say that the set $W^{-1}(\mathbb{R})$ has the *"simplest" structure* in this neighborhood.

The branching points of the complex momentum lie on the set $W^{-1}(\mathbb{R})$. For E satisfying the condition (1.1), the branching points defined by $E - W(\varphi) = E_1$ are real. All the others are situated on the complex branches of $W^{-1}(\mathbb{R})$ beginning at the minima of W. In Fig. 2, we show an example of the branching points (denoted by $\varphi_j^\pm$).

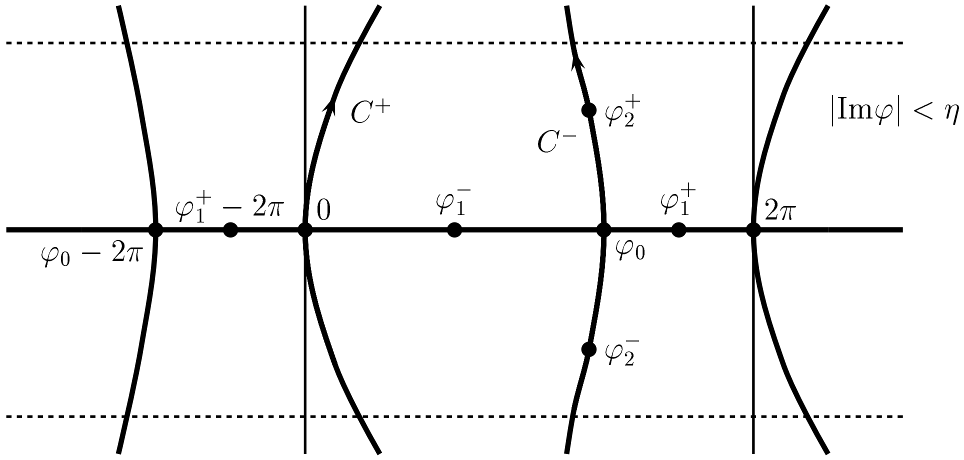

FIGURE 2. $W^{-1}(\mathbb{R})$ and some branching points

Our last condition on the function W (it is the condition we only outlined in the introduction) is that there is a vicinity of the real line, say $\mathcal{V} \subset \mathbb{C}$, such that

- W is analytic in $\mathcal{V}$;
- the set W^{-1} has the "simplest" structure in $\mathcal{V}$;
- for all $E \in J_\delta$, the branching points defined by $E - W(\varphi) = E_2$ lie in $\mathcal{V}$.

Remark 1.1. Actually the results of the present paper are valid under less restrictive assumptions on V and W. In particular the topological assumptions made above can be relaxed (see [11, 12]).

1.1.5. *the phase and action integrals.* The main objects needed to describe the spectrum of a typical equation of the family (0.1) in J_δ will be the actions and phase. They can be defined in terms of real component Γ of the iso-energy curve

$$\mathbf{E}(\kappa) + W(\varphi) = E, \tag{1.2}$$

where $\mathbf{E}(\kappa)$ is the dispersion law associated to the periodic Schrödinger equation (0.3). Note that the role of the real component of the iso-energy curve for adiabatic problems was pointed out by V. Buslaev (for a list of references see [14]).
For $E \in J_\delta$, the curve Γ looks roughly like in Fig. 3 (the dotted contours are complex). We define the *action integrals* S and S_1 and the *phase* Φ as

$$S(E) = -i\int_{\varphi_1^+-2\pi}^{\varphi_1^-} \kappa d\varphi, \quad S_1(E) = -i\int_{\kappa_1}^{2\pi-\kappa_1} (\varphi-\varphi_0)d\kappa, \quad \Phi(E) = \int_{\varphi_1^-}^{\varphi_1^+} \kappa d\varphi. \tag{1.3}$$

Here, the integrations are understood along the contours $\gamma_{1,2,3}$ in the iso-energy curve (1.2) depicted above (see section 2.3.2 for more details). For $E \in J_\delta$, one has

$$S(E) > 0, \quad S_1(E) > 0, \quad \Phi(E) \geq 0 \quad \text{and} \quad \Phi'(E) > 0.$$

Here, Φ' denotes the derivative of Φ with respect to E. Notice also that $\Phi(E_1 + W_-) = 0$.
The three functions S, S_1 and Φ can be also defined in terms of the contour integrals taken around branching points of the complex momentum.

1.2. **Main results.** We are now ready to state our results precisely.

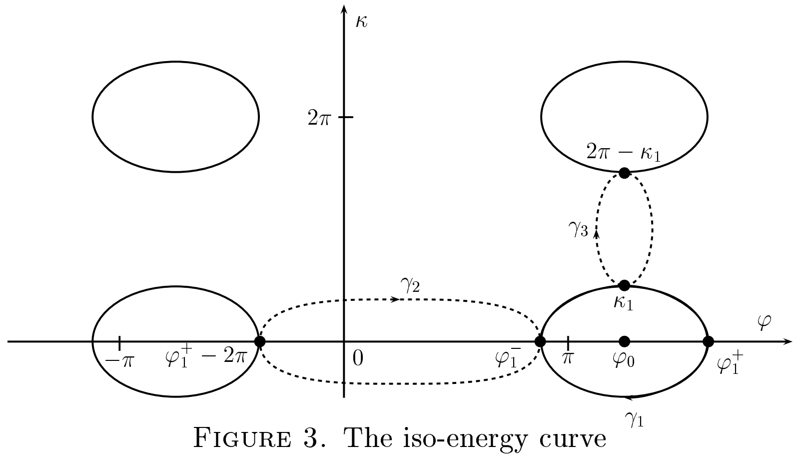

FIGURE 3. The iso-energy curve

1.2.1. *Location of the spectrum.* We first state a result on the location of the spectrum in J_δ. In J_δ, consider the points $E^{(l)}$, $l \in \mathbb{N}$, defined by

$$\frac{1}{\varepsilon}\Phi(E^{(l)}) = \pi/2 + \pi l, l \in \mathbb{N}. \tag{1.4}$$

The number of these points is finite; we denote it by $L+1$. For sufficiently small ε, the distances between these points satisfy the inequalities

$$c_1\varepsilon \leq E^{(l)} - E^{(l-1)} \leq c_2\varepsilon,\ l = 0, \dots, L, \tag{1.5}$$

where c_1 and c_2 are two positive constant independent of ε. We then prove

Theorem 1.1. *Fix δ a positive constant. Assume that V and W satisfy the assumptions described in sections 0 and 1.1. Then, for ε positive and sufficiently small, there exists a collection of intervals $I_l \subset J_\delta$, $l = 0, 1 \dots L$, such that the spectrum of equation* (0.1) *in J_δ is contained in the union $\cup_{l=0}^{L} I_l$, and*

- *the interval I_l lies in an $o(\varepsilon)$-vicinity of $E^{(l)}$,*
- *the measure of I_l is given by*

$$|I_l| = \frac{2\varepsilon}{\Phi'(E^{(l)})F(1/2+l)}\left(e^{-\frac{1}{\varepsilon}S(E^{(l)})} + e^{-\frac{1}{\varepsilon}S_1(E^{(l)})}\right)(1+o(1)), \tag{1.6}$$

$$\text{where } F(\zeta) = \frac{1}{\sqrt{\pi}}\Gamma(\zeta+1/2)e^{-\zeta\ln\zeta+\zeta},\ \zeta > 0. \tag{1.7}$$

Moreover, if $dN_\varepsilon(E)$ denotes the density of states measure of $H_{\phi,\varepsilon}$ at the energy E, then

$$\int_{I_l} dN_\varepsilon(E) = \frac{\varepsilon}{2\pi}. \tag{1.8}$$

Condition (1.4) is the quantization condition that, for our model, replaces the usual Bohr-Sommerfeld condition. Note that the intervals I_l are exponentially small and separated by a distance of order $O(\varepsilon)$. Formula (1.8) tells us that each of these intervals does contain some spectrum.

At last we note that Theorem 1.1 can be proved without any number theoretical assumptions on ε.

1.2.2. *The nature of the spectrum.* Our aim is to study the spectral properties of (0.1). It is well known that the spectrum of $H_{\phi,\varepsilon}$ does not depend on ϕ nor does its absolutely continuous spectrum (see e.g. [2, 26]). We will denote these sets respectively by Σ and $\Sigma_{\rm ac}$. Hence, the singular spectrum of $H_{\phi,\varepsilon}$ (i.e. the union of the singular continuous and of the pure point spectrum) does not depend on ϕ either; we denote it by $\Sigma_{\rm sing}$. Note that, in general, the pure point and the singular continuous spectrum depend on ϕ (see e.g. [22, 26]). The sets Σ, $\Sigma_{\rm sing}$ and $\Sigma_{\rm ac}$ depend on ε but we do not recall this dependence explicitly to alleviate the notations.

The function that will tell us the nature of the spectrum of (0.1) is

$$\Delta S(E) := S_1(E) - S(E).$$

For a fixed $a > 0$, define the sets

$$J_a^- = \{E \in J_\delta;\ \Delta S(E) < -a\} \text{ and } J_a^+ = \{E \in J_\delta;\ \Delta S(E) > a\}.$$

If $J_a^+ \neq \emptyset$ (resp. $J_a^- \neq \emptyset$), then, for sufficiently small ε, the number of intervals I_l lying in J_a^+ (resp. in J_a^-) is of order $O(1/\varepsilon)$. We prove

Theorem 1.2. *Fix $\aleph > 0$ and $\delta > 0$. Assume that V and W satisfy the assumptions described in sections 0 and 1.1. Then, there exist $C_0 = C_0(\aleph, \delta) > 0$ and $\varepsilon_0 = \varepsilon_0(\aleph, \delta) > 0$ such that, for $\varepsilon \in (0, \varepsilon_0) \cap \mathcal{D}_\aleph$, we have*

- *each of the intervals $I_l \subset J_{C_0\varepsilon}^-$ contains only singular spectrum (i.e. singular continuous or pure point spectrum),*
- *each of the intervals $I_l \subset J_{C_0\varepsilon}^+$ contains absolutely continuous spectrum. Moreover, for $C \geq C_0$ and any $I_l \subset J_{C\varepsilon}^+$, one has*

$$|I_l \cap \Sigma_{ac}|/|I_l| = 1 + o(1),$$

  where
  - *$|\cdot|$ denotes the Lebesgue measure,*
  - *$o(1) \to 0$ when $C \to +\infty$, $\varepsilon \to 0$ and $C\varepsilon$ remains bounded.*

Remark 1.2. When considering large values of C, we have to assume that ε is small enough so that one still can find some of the intervals I_l inside $J_{C\varepsilon}^+$.

First, let us say that we expect the whole spectrum in $J_{C\varepsilon}^+$ to be absolutely continuous; why we have to exclude some very small sets will be discussed in section 1.3.
Second, we also expect the singular spectrum in $J_{C\varepsilon}^-$ to be pure point for a.e. ϕ; what makes this tricky to prove with our approach will also be seen in section 1.3.
In Theorem 1.2, we have to exclude $C_0\varepsilon$-vicinities of the asymptotic mobility edges. In view of the remarks made above, these are the intervals of energy where we expect the actual sharp transitions to happen i.e. the true mobility edges to lie.

At last, taking these remarks into consideration, let us reformulate our results in more physical terms. Essentially, Theorem 1.2 tells us that, if, at energy E, we have

1. if $S_1 < S$, i.e. if the interaction (or tunneling) between two nearest vertical real components of iso-energy surface (1.2) (see Fig. 3) is larger than the interaction between two nearest horizontal real components of (1.2) then we get pure point spectrum i.e. localization in position. This corresponds to delocalization in momentum.
2. if $S_1 > S$, i.e. if the interaction between two nearest horizontal real components (1.2) is larger than the interaction between two nearest vertical real components of (1.2) then we get absolutely continuous spectrum i.e. delocalization in position. This corresponds to localization in momentum.

This is a version of the Aubry-André duality principle that is local in energy ([1]).

1.2.3. *The phase diagram.* We will now check that the transitions, possible in view of Theorem 1.2, actually occur. Therefore, we will study the dependence of $\Delta S(E) = S_1(E) - S(E)$ as on the energy E so on the potential W. We replace W by αW and discuss the behavior of $\Delta S(E)$ as a function E and α. Throughout this discussion, V and W are kept fixed and we assume that they satisfy the assumptions described in sections 0 and 1.1. To underline the dependence on α, let us slightly change our notations and write $S(\alpha, E) = S(E)$, $S_1(\alpha, E) = S_1(E)$ and $\Delta S(\alpha, E) = \Delta S(E)$. To simplify the discussion, we will assume that $W_- < 0 < W_+$.

To be able to use the results of sections 1.2.1 and 1.2.2, V and αW have to satisfy the assumptions described in sections 0 and 1.1 for some E in $[E_1 + \alpha W_-, \inf(E_1 + W_+, E_2 + W_-)$. This imposes the following conditions on α and E: let $\mathcal{V}$ be the largest neighborhood of the real axis where W is analytic and $W^{-1}(\mathbb{R})$ has the "simplest" structure, and let W^- be the infimum of W on $W^{-1}(\mathbb{R}) \cap \mathcal{V}$. Of course, $W^- < W_-$, and it may happen that $W^- = -\infty$. If we define

$$(1.9) \quad \mathcal{S} = \{(\alpha, E);\ 0 < \alpha,\ \alpha W^- + E_2 < E,\ E_1 + \alpha W_- \leq E \leq \inf(E_2 + \alpha W_-, E_1 + \alpha W_+)\},$$

then, for $(\alpha, E) \in \mathcal{S}$, V and αW satisfy the assumptions described in sections 0 and 1.1 for energy E.

Remark 1.3. In this paper, we do not analyze the uniformity of our asymptotic constructions in α. However, it appears that they are uniform in α only if $\alpha \in [\alpha_1, \alpha_2]$, where α_1 and α_2 are two positive constants kept fixed as $\varepsilon \to 0$; our constructions stop to be uniform as for $\alpha \to 0$ so for $\alpha \to \infty$.

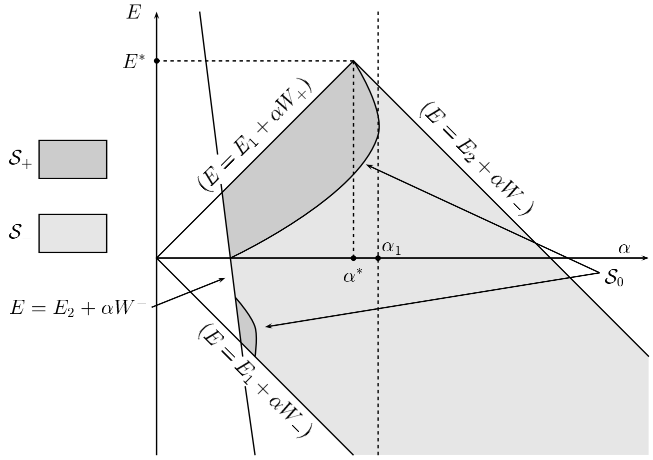

FIGURE 4. An example of phase diagram when $W^- > -\infty$

We can apply our results for all $(\alpha, E) \in \mathcal{S}$. We define

$$\mathcal{S}_\pm = \{(\alpha, E) \in \mathcal{S};\ \Delta S(\alpha, E) \gtrless 0\} \text{ and } \mathcal{S}_0 = \{(\alpha, E) \in \mathcal{S};\ \Delta S(\alpha, E) = 0\}.$$

and

$$\alpha^* := \frac{E_2 - E_1}{W_+ - W_-} \text{ and } E^* := \frac{E_2 W_+ - E_1 W_-}{W_+ - W_-}.$$

$\mathcal{S}_+$, $\mathcal{S}_0$ and $\mathcal{S}_-$ form a partition of $\mathcal{S}$. One has the following

Theorem 1.3. *Assume that V and W satisfy the assumptions described in sections 0 and 1.1. Then*

- *$\mathcal{S}_0$ is a non-empty piecewise analytic curve containing one piece starting at (α^*, E^*);*
- *$\mathcal{S}_+$ is not empty;*
- *$\mathcal{S}_-$ is connected and there exists $\alpha_1 > \alpha^* > 0$ such that $\mathcal{S} \cap \{\alpha > \alpha_1\} \subset \mathcal{S}_-$.*

Remark 1.4. Another characteristic property of $\mathcal{S}_0$ (that will prove very valuable in the proof of Theorem 1.3) is that, for any $t \in [W_-, W_+]$, $\mathcal{S}_0$ intersects each straight half-line $E = E_1 + t\alpha$ $(\alpha > 0)$ at a single point at most.

The topological structure of the phase diagram one gets from Theorem 1.3 is essentially the one given in Fig. 4. One can compute the slope of $\mathcal{S}_0$ at (α^*, E^*). Therefore, we introduce M_1 (resp. M_2) the effective mass for $k(E)$ at E_1 (resp. E_2) (see the end of section 8 for a definition). We define

$$c_{\text{eff}} = \sqrt{\frac{M_2}{M_1}\left|\frac{W''(0)}{W''(\varphi_0)}\right|}. \tag{1.10}$$

At the point (α^*, E^*), $\mathcal{S}_0$ has the slope

$$\frac{d\alpha}{dE} = \frac{1 - c_{\text{eff}}}{W_+ - c_{\text{eff}} W_-}.$$

Hence, we see that, if $c_{\text{eff}} < 1$ then the picture of the phase diagram that one gets near (α^*, E^*) is roughly the same as in Fig. 1 in the case when $c = +\infty$. On the other hand, if $c_{\text{eff}} > 1$, we get a picture similar to Fig. 1 in the case when $c = 1.5$. I.e. we have singular spectrum appearing in the center of the band of absolutely continuous spectrum of the initial periodic operator.

Remark 1.5. Note that, for the potential W_c defined in the introduction, for $c \neq +\infty$, $W_c^- > -\infty$. It can can even be close to -1. So that strictly speaking, the curves shown in Fig. 1 are only proved to be correct above the half-line $E = E_2 + \alpha W_c^-$ Nevertheless, the assumptions made on the topology of $W^{-1}(\mathbb{R})$ can be relaxed. Thus one can extend the results presented here to the whole domain enclosed by the axis $E = E_1 + \alpha W_-$, $E = E_1 + \alpha W_+$ and $E = E_2 + \alpha W_-$ (see [12]).

In some cases $W^- = -\infty$. This happens e.g. when W is entire and when the set $W^{-1}(\mathbb{R})$ has the " simplest" structure on the whole complex plane (e.g. if $W = \cos\varphi$). But also in other cases, e.g. for $W(\varphi) = \dfrac{1}{2 + \cos\varphi}$: in this case, $W^{-1}(\mathbb{R})$ has the same structure as in the case of $\cos\varphi$ but W has poles on the axes $\operatorname{Im}\varphi = \pi$. So that the branching points $(\varphi_j^{\pm})_j$ accumulate near these poles. When $W^- = -\infty$ one can say more on the sets $\mathcal{S}_0$ and $\mathcal{S}_+$; namely one has

Theorem 1.4. *Assume that V and W satisfy the assumptions described in sections 0 and 1.1. If $W^- = -\infty$, in addition to Theorem 1.3, we know that*

- *$\mathcal{S}_0$ is analytic, connected and joining the point (α^*, E^*) to some point on the half-line $E = E_1 + \alpha W_-$ $(\alpha > 0)$.*
- *$\mathcal{S}_+$ is connected and there exists $\alpha_0 > 0$ such that $\mathcal{S} \cap \{\alpha < \alpha_0\} \subset \mathcal{S}_+$.*

In this case, the picture of the phase diagram one gets from Theorem 1.4 is essentially the one given in Fig. 5.

From Theorems 1.4 and 1.2, we deduce that, for small α, the spectrum stays (essentially) absolutely continuous. This can be considered as an analogue of one of the results of [8]; indeed, there it is proven that, for a small quasi-periodic perturbation of $-\dfrac{d^2}{dx^2}$, the spectrum is purely absolutely continuous; and here, we are dealing with small perturbations of general periodic operators.

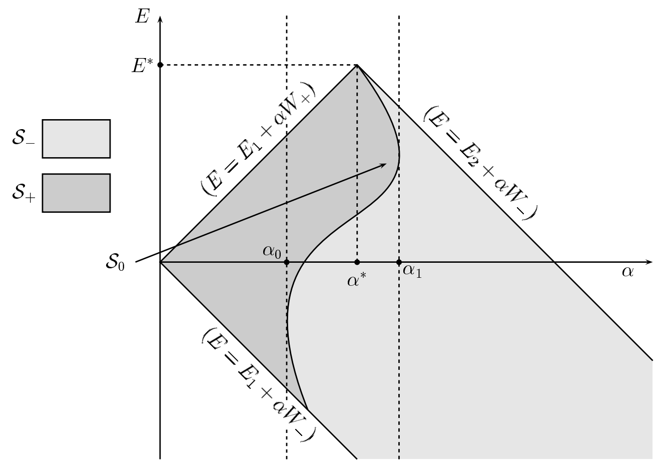

FIGURE 5. An example of phase diagram when $W^- = -\infty$

1.3. **The methods of the paper.** The main idea of our analysis is to reduce the family of equations (0.1) to an effective Hamiltonian. This reduction is done for every energy in J_δ. The effective Hamiltonian one gets is a second order finite difference equation; it depends of E. Up to exponentially small terms, it is given by

$$(u_{n+1} + u_{n-1}) + 2\lambda(E)\cos(nh + \phi)u_n = \mathcal{E}(E)u_n. \tag{1.11}$$

Note that, in this equation, the coupling constant λ and the effective spectral parameter $\mathcal{E}$ depend on one another. This tells this equation from the Almost Mathieu equation which was the subject of many studies during the last 20 years (see e.g. [7, 28, 29, 25, 22] for references).

The reduction of the analysis of the family (0.1) to the analysis of (1.11) is based on the notion of *monodromy matrix*. For periodic Schrödinger equations, this is a classical notion [32]. It was also introduced and used for the spectral analysis of difference Schrödinger equations with periodic coefficients (see, for example [4]). It is difficult to relate a monodromy matrix to a single almost periodic Schrödinger equation. But, in [14], we have seen that the monodromy matrix is a natural notion for the family of almost periodic Schrödinger equations (0.1).

It is well known that the spectral properties of one dimensional Schrödinger equations are determined by the behavior of their solutions as $x \to \pm\infty$ (see e.g. [17]). It appears that the behavior at infinity of solutions of the input equation (0.1) "mimics" the behavior at infinity of solutions of a second order finite difference equation defined by the monodromy matrix. In the adiabatic limit, i.e. as $\varepsilon \to 0$, this equation takes the form of (1.11).

For the asymptotic analysis, we use a WKB method for adiabatic perturbations of periodic Schrödinger operators. This method is developed in [14].

The role of the model equation (1.11) for the spectral analysis of the family of almost periodic equations can be compared with the role of the Airy equation for the classical WKB constructions. In particular, for the Almost Mathieu equation, it is known that, roughly, if the coupling constant is large then the spectrum is pure point, and if the coupling constant is small then it is absolutely continuous. In our case, the coupling constant $\lambda(E)$ is given quite explicitly in terms of the action difference ΔS: one has

$$\lambda(E) \simeq e^{-\Delta S(E)/\varepsilon}.$$

This immediately explains the part played by ΔS. To get a rigorous proof of our results, we also need to take into account small perturbations of (1.11) and therefore use results on more general quasi-periodic finite difference equations (see [3, 9, 10]). These results are not tailor-suited to our needs; this essentially explains why we don't prove that all the spectrum in $J^+_{C\varepsilon}$ is absolutely continuous. What makes it difficult to prove that the singular spectrum is actually pure point (for a.e. ϕ) is that, in (1.11), both the effective potential and the effective spectral parameter $\mathcal{E}$ depend on E.

To finish this section, let us note that, in [16], one-dimensional quasi-periodic Schrödinger operators were reduced to effective Hamiltonians though in a large coupling constant regime. These effective Hamiltonians were also small perturbations of the Almost Mathieu equation (for the critical coupling constant) but, as the reduction was done in the large coupling constant regime, it did not have a particular dependence in energy. In that respect the effective Hamiltonian obtained in [16] is closer to the model studied in [15].

1.4. **The structure of the paper.** In Section 2, we recall the definition of the monodromy matrices, discuss its spectral role and describe the asymptotics of one of the monodromy matrices related to (0.1). In Section 3, we explain how to derive the effective Hamiltonian and its precise link to (0.1). In Sections 4 and 5, by means of this equation, we prove Theorem 1.1. In particular, Section 5 is devoted to the calculation of the integrated density of states. In Sections 6 and 7, we prove Theorem 1.2: Section 6 is devoted to the study of the absolutely continuous spectrum, and, in Section 7, we discuss the singular spectrum. Finally, in the last section, Section 8, we study the phase diagrams and prove Theorems 1.3 and 1.4.

Contents

2. Monodromy matrices and monodromy equation

In this section, we define the monodromy matrix (see [14]) and show that the spectral analysis of the family of equations (0.1) can be reduced to a study of solutions of a finite difference equation defined by this matrix. Then, we describe the asymptotics of a well chosen monodromy matrices in adiabatic limit.

2.1. **Monodromy matrix.** For any fixed ϕ, let $\psi_{1,2}(x,\,\phi)$ be two linearly independent solutions of equation (0.1). We say that they form a *consistent basis* if their Wronskian is independent of ϕ and if these solutions are 1-periodic in ϕ i.e.

$$\psi_{1,2}(x,\,\phi+1)=\psi_{1,2}(x,\,\phi),\quad \forall x,\phi, \tag{2.1}$$

Consider a consistent basis $(\psi_{1,2})$. In view of (0.2), the functions $\psi_{1,2}(x+2\pi/\varepsilon,\phi+2\pi/\varepsilon)$ are solutions of equation (0.1). Therefore, one can write

$$\Psi\,(x+2\pi/\varepsilon,\phi+2\pi/\varepsilon)=M\,(\phi)\,\Psi\,(x,\phi), \tag{2.2}$$

where Ψ is the vector

$$\Psi^T(x,\phi)=\left(\psi_1(x,\,\phi),\psi_2(x,\phi)\right), \tag{2.3}$$

^{T} being the symbol of transposition introduced for convenience, and $M\,(\phi)$ is a matrix with coefficients independent of x. We call this matrix *the monodromy matrix* corresponding to the consistent basis $(\psi_{1,2})$. In [14], we have seen that this is a natural generalization of the canonical definition of a monodromy matrix for a periodic Schrödinger equation.

Note that, for any consistent basis, the monodromy matrix satisfies

$$\det M\,(\phi)\equiv 1,\quad M\,(\phi+1)=M\,(\phi),\quad \forall\phi. \tag{2.4}$$

2.2. **Monodromy equation.** Set

$$h=\frac{2\pi}{\varepsilon}\bmod 1. \tag{2.5}$$

Let M be a monodromy matrix corresponding to a consistent basis $(\psi_{1,2})$. The spectral analysis of (0.1) can be reduced to the investigation of the solutions of the *monodromy equation*

$$\chi\,(\phi+h)=M\,(\phi)\,\chi\,(\phi)\qquad\forall x\in\mathbb{R}. \tag{2.6}$$

It is well known that the spectral properties of one dimensional Schrödinger equations can be completely described in terms of the behavior of its solutions as $x\to\pm\infty$ (see [17]). It appears that the behavior of solutions of (0.1) for $x\to\pm\infty$ repeats the behavior of solutions of the monodromy equation for $\phi\to\mp\infty$. Let us formulate the precise statement. Therefore, consider the restriction

$$\chi_n=\chi(\phi+nh) \tag{2.7}$$

of a solution χ of (2.6) to the lattice

$$\phi_n=\phi+nh,\ n\in\mathbb{Z}. \tag{2.8}$$

It satisfies the equation

$$\chi_{n+1}=M(\phi+nh)\chi_n,\quad\forall n. \tag{2.9}$$

We call this equation *lattice monodromy equation.* In the sequel, we identify the vector solutions of the lattice equation with functions $F:\mathbb{Z}\to\mathbb{C}^2$; we denote the values of F by F_n. One has

Theorem 2.1. *Let $\psi_{1,2}$ be consistent basis solutions of* (0.1) *and let M be the corresponding monodromy matrix. Fix $\phi = \phi_0 \in \mathbb{R}$. Then, for any vector solution F of the equation* (2.9)*, there exists a unique solution f of* (0.1) *such that*

$$\left(\begin{array}{c} f(x+2\pi n/\varepsilon) \\ f'(x+2\pi n/\varepsilon) \end{array}\right) = \Psi(x, \phi_0 - nh)\ \sigma\ F_{-n}, \quad \forall x \in \mathbb{R},\ n \in \mathbb{Z}, \tag{2.10}$$

where

$$\Psi = \left(\begin{array}{cc} \psi_1 & \psi_2 \\ \frac{d\psi_1}{dx} & \frac{d\psi_2}{dx} \end{array}\right), \quad \sigma = \left(\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array}\right).$$

Reciprocally, for any solution f of (0.1)*, there exists a unique vector solution F of* (2.9) *satisfying* (2.10).

Since Ψ is 1-periodic in ϕ, this theorem reduces the analysis of the asymptotics as $x \to \pm\infty$ of solutions to the equations (0.1) to the analysis of the asymptotics as $n \to \mp\infty$ of solutions to the finite difference equations (2.9).

Proof of Theorem 2.1. To prove Theorem 2.1, we need to recall some elementary facts about difference equations of the form (2.6) with an unimodular matrix M (see, for example, [5]). A matrix solution of such an equation is called *fundamental* if and only if $\det \chi \equiv 1$. One can easily construct a fundamental solution. For example, on the interval $[0, h)$, we can define χ by the formula

$$\chi(\phi) = I, \quad \forall \phi \in [0, h), \tag{2.11}$$

and, outside of this interval, we continue χ by means of the relations:

$$\chi(\phi + h) = M\,(\phi)\chi(\phi), \quad \chi(\phi - h) = M^{-1}(\phi - h)\chi(\phi). \tag{2.12}$$

Note that to write the second formula, we have used the equality $\det M \equiv 1$. In result, by construction, the matrix χ is a matrix solution of the difference equation. And since

$$\det \chi(\phi + h) = \det M(\phi) \det \chi(\phi) = \det \chi(\phi), \quad \forall \phi,$$

this matrix solution is fundamental.

Now consider the restriction (2.9) of equation (2.6). Any vector solution of this restricted equation can be represented in the form

$$F_n = \chi(\phi + nh)\, p, \quad \forall n, \tag{2.13}$$

where p is a constant vector. Moreover, for any constant vector p, formula (2.13) gives a vector solution of the restricted equation.

Now, we can easily prove Theorem 2.1. The proof consists of several steps.

1. Let ψ be the vector with components ψ_1 and ψ_2 defined by (2.3), and let χ be a fundamental solution of the monodromy equation (2.6). One can easily (see [14]) check that

$$\psi\,(x + 2\pi n/\varepsilon, \phi) = \chi\,(\phi)\,\chi^{-1}(\phi - nh)\,\psi\,(x, \phi - nh). \tag{2.14}$$

Put

$$u(x, \phi) = \chi^{-1}(\phi)\,\psi(x, \phi). \tag{2.15}$$

The components of the vector u denoted by (u_1, u_2) satisfy (0.1) for any $\phi \in \mathbb{R}$. A direct calculation of their Wronskian yields $w(u_1, u_2) = \det \chi^{-1} w(\psi_1, \psi_2) = w(\psi_1, \psi_2)$; hence (u_1, u_2) are linearly independent. Now relation (2.14) takes the form

$$u\,(x + 2\pi n/\varepsilon, \phi) = u\,(x, \phi - nh). \tag{2.16}$$

This formula and the definition of u imply that

$$u(x + 2\pi n/\varepsilon, \phi) = \chi^{-1}(\phi - nh)\, \psi(x, \phi - nh). \tag{2.17}$$

2. Let

$$U = \begin{pmatrix} u_1 & u_2 \\ \frac{du_1}{dx} & \frac{du_2}{dx} \end{pmatrix}.$$

As $\det\chi \equiv 1$, formula (2.17) implies the relation

$$U(x + 2\pi n/\varepsilon, \phi) = -\Psi(x, \phi - nh)\, \sigma\chi(\phi - nh)\sigma, \tag{2.18}$$

where Ψ is defined in Theorem 2.1.

3. Let us prove the first statement of the theorem. Fix $\phi = \phi_0$. Let F be a vector solution of (2.9) with $\phi = \phi_0$. Represent it in the form (2.13). Denote the components of the vector p in this representation by p_1 and p_2. The function

$$f(x) = -p_2 u_1(x, \phi_0) + p_1 u_2(x, \phi_0) \tag{2.19}$$

satisfies equation (0.1) with $\phi = \phi_0$. Applying the left and the right hand sides of (2.18) to the vector σp, we get the relation (2.10). This proves the first statement.

4. To prove the second statement, we represent the given solution f as a linear combination of the linearly independent solutions $u_1(x, \phi_0)$ and $u_2(x, \phi_0)$ as in (2.19). Then we denote by p the vector $(p_1, p_2)^T$ and notice that $F_n = \chi(\phi + hn)p$ is a solution of (2.9). Using relation (2.18) again, we complete the proof of Theorem 2.1.

□

2.3. **Monodromy matrix in the adiabatic limit.** We will now describe the asymptotics of the monodromy matrix for (0.1) in the adiabatic limit. To formulate the results, we have to introduce several important objects.

2.3.1. *Bloch quasi-momentum.* Consider the periodic Schrödinger equation

$$-\frac{d^2}{du^2}\psi(x) + V(x) = E\psi(x), \quad x \in \mathbb{R}. \tag{2.20}$$

Its spectrum in $L^2(\mathbb{R})$ is known to consist of intervals $[E_1, E_2]$, $[E_3, E_4]$, $\dots$ $[E_{2n+1}, E_{2n+2}]$, $\dots$, of the real axis such that

$$E_1 < E_2 \leq E_3 < E_4 \dots E_{2n} \leq E_{2n+1} < E_{2n+2} \leq \dots$$

and

$$E_n \to +\infty, \quad n \to +\infty.$$

Usually, the number of gaps (E_2, E_3), (E_4, E_5), $\dots$ is infinite.

Equation (2.20) has two solutions of the form

$$\psi_\pm(x, E) = e^{\pm ik(E)x}\, p_\pm(x, E),$$

where the function p is 1-periodic in x. These solutions are called *Bloch solutions* and the function $k(E)$ is called the *Bloch quasi-momentum.*

The Bloch quasi-momentum is a multi-valued analytic function. Its branching points are of square root type and are situated at the ends of the spectral zones. In particular, one can fix a single valued analytic branch k_0 of k on the complex plane cut along the real axis from E_1 to $+\infty$. Denote the cut plan by $\mathbb{C}_0$. The branch can be fixed by the condition

$$\operatorname{Im} k_0(E) > 0, \quad E < E_1. \tag{2.21}$$

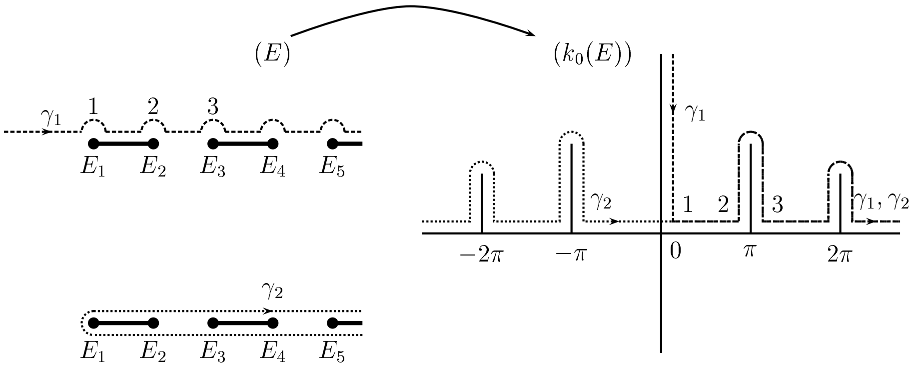

FIGURE 6. The mapping properties of k_0

The image of $\mathbb{C}_0$ by k_0 is located in the upper half of the complex plane,

$$\operatorname{Im} k_0(E) > 0, \quad E \in \mathbb{C}_0 .$$

In Figure 6, we drew several curves in $\mathbb{C}_0$ and their images under the transformation $E \mapsto k_0(E)$. The quasi-momentum $k_0(E)$ is real along the spectral zones, and, along the spectral gaps, its real part is constant; in particular, we have

$$k_0(E_1) = 0, \quad k_0(E_{2l} \pm i0) = k_0(E_{2l+1} \pm i0) = \pm \pi l, \quad l \in \mathbb{N}. \tag{2.22}$$

2.3.2. *Complex momentum and phase integrals.* We now describe the actions S and S_1 and the phase Φ in terms of contour integrals taken along paths in the complex plane of φ. Recall that the complex momentum is defined by

$$\kappa(\varphi) = k(E - W(\varphi)), \quad \varphi \in \mathbb{C}, \tag{2.23}$$

where W is the slowly varying part of the potential from (0.1).

The branching points of κ are defined by the equation

$$E - W(\varphi) = E_n, \; n \in \mathbb{N}.$$

They are located on the inverse image $W^{-1}(\mathbb{R})$ of the real line by W (see Figures 7 and 2). The set of branching points is 2π-periodic. It is symmetric with respect to the real line.
Recall that W_+ and W_- are the maximal and minimal values of W on the real line. We now describe the branching points lying in the upper half plane on the lines $C_\pm$ on $W^{-1}(\mathbb{R})$ (see Fig. 2). If $E < E_1 + W_-$, then all the branching points lie on C^-. We index them in such a way that

$$\operatorname{Im} \varphi_1^+ < \operatorname{Im} \varphi_2^+ \leq \operatorname{Im} \varphi_3^+ < \operatorname{Im} \varphi_4^+ \leq \varphi_5^+ < \dots \text{ (see Fig. 7 case 1)}$$

We note that our condition on the first gap of the periodic Schrödinger operator (0.3) means that

$$\operatorname{Im} \varphi_2^+ < \operatorname{Im} \varphi_3^+ .$$

If $E = E_1 - W_-$ then the branching points φ_1^+ and $\varphi_1^- = \overline{\varphi_1^+}$ merge into one,

$$\varphi_1^+ = \varphi_1^- = \varphi_0$$

where $\varphi_0 \in \mathbb{R}$ is the point where W reaches its minimum. If $E_1 - W_- < E < E_1 + W_+$ then the branching points φ_1^+ and φ_1^- become real; they satisfy

$$0 < \varphi_1^- < \varphi_0 < \varphi_1^+ < 2\pi \text{ (see Fig. 7 case 2)}.$$

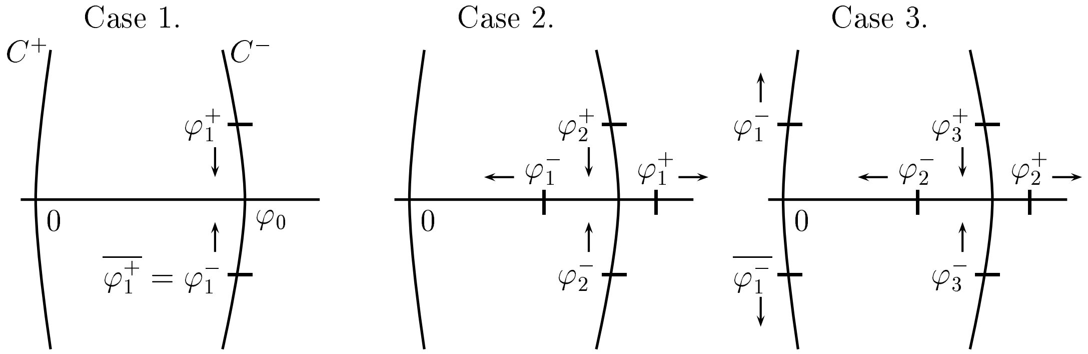

FIGURE 7. The position and movement of the branching points of κ_0

In Figure 7 case 3, we show the geometry of the turning points in the case when $\sup(E_1 + W_+, E_2 + W_-) < E < \inf(E_3 + W_-, E_2 + W_+)$. The arrows indicate the direction where the branching points are moving as E is increasing.

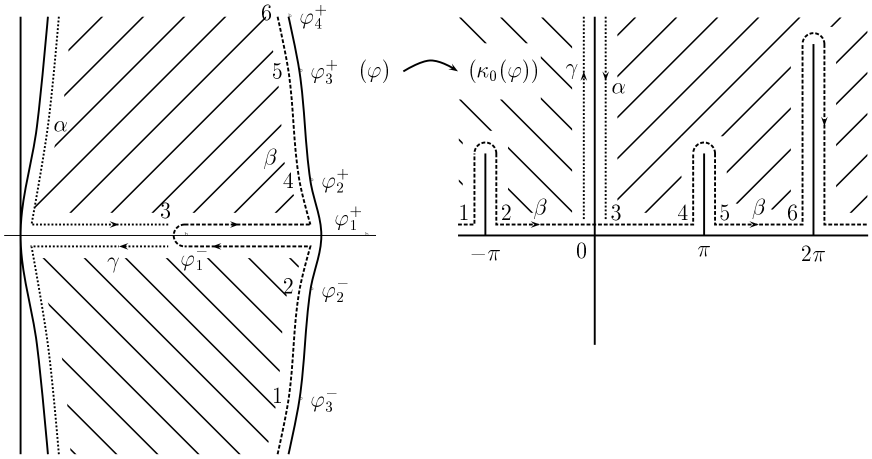

FIGURE 8. The mapping properties of κ_0

As in the introduction, we assume that

$$E_1 + W_- \leq E < \min\{E_2 + W_-, E_1 + W_+\}. \tag{2.24}$$

In this case, the branching points $\varphi_2^+, \ldots, \varphi_l^+, \ldots$ remain complex located on C^- and

$$0 < \varphi_1^- \leq \varphi_0.$$

Consider the domain bounded by C^+ and C^-. Cut it between φ_1^- and φ_0 as in Fig. 8. In this domain, fix a single valued branch of the complex momentum denoted by κ_0 by the condition $\operatorname{Im} \kappa_0(\varphi) \geq 0$. On left part of Figure 8, we show several curves and on the right part, we draw their images by κ_0.

Now, we can easily define the three contour integrals needed to describe the asymptotics of the monodromy matrix. Let

$$\Phi(E) = \frac{1}{2} \oint_{C_1} \kappa(\varphi)\, d\varphi$$

where $C_1 \subset \mathbb{C}_0$ is a contour going around the branching points φ_1^+ and φ_1^- (see Fig. 9). Here κ is the branch of the complex momentum obtained from κ_0 by analytic continuation along C_1. We call $\Phi(E)$ *the phase integral.* The above representation implies that the phase integral Φ is real analytic in E, $E < E_1 + W_+$, and that it increases monotonically from 0 to $\int_0^{2\pi} \kappa(\varphi)d\varphi$ as E increases from $E_1 + W_-$ to $E_1 + W_+$. Note that, here, $\Phi'(E) > 0$ (see Lemma 4.2). So that $\Phi(E)$ is negative if $E < E_1 + W_-$ and positive if $E > E_1 + W_-$.

Define C_3 to be the part of C_- from φ_2^- to φ_2^+ and $C_2 = [\varphi_1^+ - 2\pi, \varphi_1^-]$ (see Fig. 9). We can

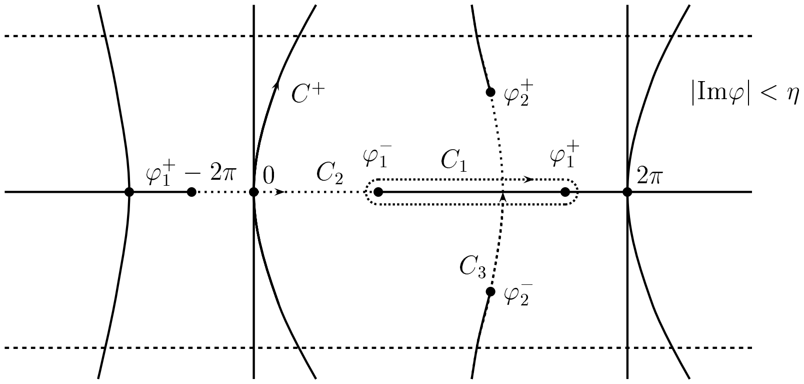

FIGURE 9. The paths C_1, C_2 and C_3

check (see [14]) that the action integrals S and S_1 (see (1.3)) can be rewritten in the following way

$$S(E) = -i \int_{C_2} \kappa(\varphi)\, d\varphi \text{ and } S_1(E) = i \int_{C_3} (\kappa(\varphi) - \pi)\, d\varphi.$$

The functions $S(E)$ and $S_1(E)$ are defined for all E satisfying (2.24); we fix the branch of κ so that, for these E, we have

$$S(E) > 0 \text{ and } S_1(E) > 0.$$

2.3.3. *Asymptotics.* Now we can describe the asymptotics of the monodromy matrix. Set

$$F(\zeta) = \frac{1}{\sqrt{\pi}} \Gamma(\zeta + 1/2) e^{-\zeta \ln \zeta + \zeta}.$$

Stirling's formula gives

$$F(\zeta) \to 1, \quad \zeta \to +\infty. \tag{2.25}$$

One has

Theorem 2.2 ([12]). *Fix a positive constant δ. Assume that V and W satisfy the assumptions described in sections 0 and 1.1. There exists two constants $C_1 > 0$ and $C_2 > 0$ such that, for sufficiently small positive ε, there exists a consistent basis such that the corresponding monodromy matrix is analytic in ϕ and in E in the domain* $\mathrm{Re}\, E \in J_\delta$, $|\mathrm{Im}\, E| \le C_1$,

$|\operatorname{Im}\phi| \leq \frac{1}{\varepsilon}(C_2 + \operatorname{Im}\varphi_2^+(E))$. *Moreover, in that domain, its coefficients admit the uniform asymptotic representations*

$$
\begin{aligned}
M_{11} &= \tfrac{1}{2}\tfrac{e^{-S/\varepsilon+o\,(1)}}{F(\Phi/\pi\varepsilon)}\left[\cos(\Phi/\varepsilon+o\,(1)) + \tfrac{e^{-S_1/\varepsilon+o\,(1)}}{F(\Phi/\pi\varepsilon)}\cos(s+o\,(1))\right],\\
M_{12} &= \left[-\sin(\Phi/\varepsilon+o\,(1)) + \tfrac{e^{-S_1/\varepsilon+o\,(1)}}{F(\Phi/\pi\varepsilon)}\sin(s+o\,(1))\right](1+o\,(1)),\\
M_{21} &= \left[\sin(\Phi/\varepsilon+o\,(1)) + \tfrac{e^{-S_1/\varepsilon+o\,(1)}}{F(\Phi/\pi\varepsilon)}\sin(s+o\,(1))\right](1+o\,(1)),\\
M_{22} &= 2\,e^{S/\varepsilon+o\,(1)}\,F(\Phi/\pi\varepsilon)\left[\cos(\Phi/\varepsilon+o\,(1)) - \tfrac{e^{-S_1/\varepsilon+o\,(1)}}{F(\Phi/\pi\varepsilon)}\cos(s+o\,(1))\right],
\end{aligned}
\tag{2.26}
$$

$$s = 2\pi\phi + s_0,$$

where

- $s_0 = s_0(E)$ *is a real valued function of* E *satisfying*
$$s_0, \frac{ds_0}{dE} = O(1),$$
- *the functions* $o\,(1)$ *are real analytic in* E *and* ϕ *and they satisfy*
$$\frac{do\,(1)}{dE} = o(1),\ \frac{do\,(1)}{d\phi} = o(1).$$

All these estimates are uniform in the domain described above.

We will not give a complete proof of Theorem 2.2 here: it will be found in [12].

Remark 2.1. In the asymptotic formulae (2.26), it looks like, in the same asymptotic formula, we keep terms that are exponentially small with respect to others. For example, in the formula

$$M_{21} \sim \sin(\Phi/\varepsilon) + \frac{e^{-S_1/\varepsilon}}{F(\Phi/\pi\varepsilon)}\sin s,$$

when $\operatorname{Im} E = \operatorname{Im}\phi = 0$, the second term is negligible with respect to the first one. However, if $\operatorname{Im}\phi > \operatorname{Im}\varphi_2^+(E)/\varepsilon$, then it is the second term that becomes the main one. Indeed

$$
\begin{aligned}
\left|e^{-S_1(E)/\varepsilon}\sin s\right| \sim \left|e^{-S_1(E)/\varepsilon-2i\pi\phi}\right| &= \left|e^{-\frac{i}{\varepsilon}\int_{\varphi_2^-}^{\varphi_2^+}(\kappa(\varphi)-\pi)d\varphi-\frac{2\pi i}{\varepsilon}(\varphi_2^+-\varphi_0)}\right|\cdot\left|e^{2i\pi(\varphi_2^+/\varepsilon-\phi)}\right|\\
&= \left|e^{-\frac{i}{\varepsilon}\int_{\varphi_2^-}^{\varphi_2^+}\kappa(\varphi)d\varphi}\right|\cdot\left|e^{2i\pi(\varphi_2^+/\varepsilon-\phi)}\right|.
\end{aligned}
\tag{2.27}
$$

and, since along C^-, between φ_2^- and φ_2^+, we have $\pi > \kappa(\varphi) > 0$, both terms in (2.27) are exponentially large.

In Theorem 2.2, we have described the asymptotics assuming that $E \geq E_1 + W_-$, since the case $E < E_1 + W_-$ is of no interest for the spectral analysis of (0.1). Note also that if

$$E \geq E_1 + W_- + C, \tag{2.28}$$

where $C > 0$ is any fixed number, then the above asymptotic formulae can be simplified. Indeed, in view of (2.25), in all these formulae, we can replace F just by 1.

The proof of Theorem 2.2 is based on the complex WKB method for adiabatic problems developed in [14]. In that paper, we have proved analogous asymptotic formulae under the condition (2.28) in the case when $W(\varphi) = \cos\varphi$. However, the asymptotic calculations were made in quite general terms. It is the structure of the set of branching points of the complex momentum and its conformal properties that were important for the analysis. Our assumptions on the function W (see the introduction) are precisely the assumptions guaranteeing the needed properties of the complex momentum. So, when condition (2.28) holds, the calculations made in [14] remain correct under our assumptions on W.

We also note that the monodromy matrix described in [14] differs from the one described in Theorem 2.2 by a transformation $M(\phi) \to T(\phi+h)M(\phi)T(\phi)^{-1}$ where $T(\phi)$ is a matrix relating the consistent basis $(\tilde{\psi}_{1,2})$ described in [14] to the consistent basis used for Theorem 2.2 and defined by

$$\psi_{1,2} = \frac{1}{2}\left(\tilde{\psi}_{1,2} + \overline{\tilde{\psi}_{1,2}}\right).$$

We need to introduce this basis in order to get a monodromy matrix with real analytic coefficients (see [12]).

If $E \sim E_1 + W_-$, then there is a pair of branching points of the complex momentum which are close one to another and to the real axis. These branching points make an important contribution to the asymptotic formulae for the monodromy matrix and lead to the appearance of the function $F(\Phi/\pi\varepsilon)$. The problem of two close branching points is a standard problem of the semi-classical analysis. The precise computations handling that problem will be published elsewhere along with the derivation of different models related to other interesting branching point configurations (see [13]). These configurations are the ones arising when studying (0.1) for energies E larger than $\inf(E_2 + W_-, E_1 + W_+)$ i.e. when condition (2.24) is violated.

3. Equivalence of the family of equations (0.1) and of a family of selfadjoint equations on $\ell^2(\mathbb{Z})$.

To study the spectrum of (0.1), we reduce the lattice monodromy equation to a finite difference Schrödinger equation. In the adiabatic limit, this equation takes a canonical form, and, for its analysis, one can apply well known theorems based on the KAM theory approach. The reduction can be done under very weak conditions on the coefficients of the monodromy matrix

$$a = M_{11}, \quad b = M_{12}, \quad c = M_{21}, \quad d = M_{22}.$$

So, first, we describe the general idea and then, we apply it to (0.1) in the adiabatic limit.

3.1. The general scheme.

Here, we assume only that

- (H1): the coefficients of the monodromy matrix are real analytic in ϕ;
- (H2): a, c, d and $1/c$ be bounded;
- (H3): the number $h/2\pi$ Diophantine.

Under these assumptions, we transform the monodromy equation (2.6) into the discrete Schrödinger equation

$$\psi_{k+1} + \psi_{k-1} + p(kh+\phi, E)\psi_k = \mathcal{E}(E)\psi_k, \quad k \in \mathbb{Z}, \tag{3.1}$$

where p is a real analytic 1-periodic function of ϕ. To describe the relation between the input family of equations (0.1) and the Schrödinger equation, it will be convenient to introduce a new notion, the notion of *$\mathcal{S}$-equivalence*.

Definition 3.1. We say that the family of equations (0.1) is $\mathcal{S}$*-equivalent* to a family of equations of the form (3.1) if, for any fixed $\phi \in \mathbb{R}$, there is a one to one correspondence between the set of L^2_{loc}-solutions ψ of (0.1) and the set of solutions $(\psi_k)_k$ of (3.1) such that

$$\left(\begin{array}{c} \psi(x+2\pi n/\varepsilon) \\ \psi'(x+2\pi n/\varepsilon) \end{array}\right) = \mathcal{S}(x, \phi - nh) \left(\begin{array}{c} \psi_{-n+1} \\ \psi_{-n} \end{array}\right), \quad x \in [0, 2\pi/\varepsilon],\ \phi \in \mathbb{R}, \tag{3.2}$$

where $\mathcal{S} \in SL(2,\mathbb{R})$ is a matrix bounded in x and ϕ.

So, our aim is to prove that, under the conditions (H1) – (H3), the family (0.1) is $\mathcal{S}$-equivalent to a family of finite difference Schrödinger equations of the form (3.1). This will be done in two steps that we describe now.

3.1.1. *Second component of a vector solution of the monodromy equation.* We begin by reducing the monodromy equation which is a system of two difference equations to one second order difference equation. One can easily check

Lemma 3.1. *Let $c(\phi) \neq 0$. A vector function $\chi(\phi) \in \mathbb{C}^2$ satisfies equation (2.6) if and only if it can be represented in the form*

$$\chi(\phi) = \left(\begin{array}{c} \frac{g(\phi+h) - d(\phi) g(\phi)}{c(\phi)} \\ g(\phi) \end{array}\right), \tag{3.3}$$

where g is a solution of the equation

$$g(\phi+h) + \rho(\phi) g(\phi-h) = v(\phi)\, g(\phi) \tag{3.4}$$

with

$$\rho(\phi) = \frac{c(\phi)}{c(\phi-h)}, \quad v(\phi) = a(\phi-h)\rho(\phi) + d(\phi). \tag{3.5}$$

In fact, Lemma 3.1 says that g is the second component of a vector solution of (2.6) if and only if it satisfies (3.4). One checks the lemma by straightforward calculations. Omitting the elementary details, we note that one uses only two properties of the monodromy matrix: the hypothesis $c \neq 0$ and the equality $\det M \equiv 1$. Note that the coefficients of (3.4) are 1-periodic together with the monodromy matrix.

3.1.2. *Homological equation.* To perform the second step of the reduction, we have to solve a first order difference equation of the form

$$\theta(\phi+h) - \theta_2(\phi-h) = l(\phi). \tag{3.6}$$

So, let us recall some elementary results on such equations. If we assume that

- (A1): l is a 1-periodic function analytic in a vicinity of the real line,
- (A2): the shift parameter h is a Diophantine number i.e. if for positive constants C and N, one has

$$\inf_{p\in\mathbb{Z}} |h - p/q| \geq C/q^N, \quad \forall q \in \mathbb{N}, \tag{3.7}$$

- (A3): the mean value of l equals to zero,

then the *homological equation* (3.6) admits a 1-periodic solution. This solution can be easily written down in terms of the Fourier coefficients l_m of the function l:

$$\theta_0(\phi) = \frac{1}{2i} \sum_{m=-\infty}^{\infty} \frac{l_m}{\sin hm}\, e^{im\phi}.$$

This series is converging in the strip of analyticity of l since the Fourier coefficients l_m are exponentially decaying and h satisfies the estimate (3.7).

The general solution of equation (3.6) admits the representation $\theta_0(\phi) + \theta_1(\phi)$ where θ_1 is an arbitrary $2h$-periodic function. So, if there exists a 1-periodic solution, then it is unique. Note that if l is real analytic, then the periodic solution is also real analytic.

3.1.3. *Passage to the family of finite difference equations.* Let

$$l(\phi) = \ln \rho(\phi) = \ln \frac{c(\phi)}{c(\phi - h)}. \tag{3.8}$$

In this formula, fix the branch of the logarithm so that l be real analytic. Obviously,

$$\int_0^1 l(\phi)\, d\phi = \int_0^1 \ln c(\phi)\, d\phi - \int_0^1 \ln c(\phi - h)\, d\phi = 0.$$

Hence l has mean value zero. Denote by θ the periodic solution of the homological equation for that l. Define

$$\mathcal{E}(E) = \int_0^1 q(\phi) d\phi, \quad p(\phi) = \mathcal{E}(E) - q(\phi), \tag{3.9}$$

where

$$q(\phi) = e^{\theta(\phi) - \theta(\phi + h)} v\,(\phi). \tag{3.10}$$

Clearly, p is a real analytic, 1-periodic function. One has

Proposition 3.1. *Let a monodromy matrix for the family of almost periodic differential Schrödinger equations* (0.1) *satisfy the assumptions (H1), (H2) and let ε be in the Diophantine class $D_\aleph$. Then the family* (0.1) *is $\mathcal{S}$-equivalent to the family of finite difference Schrödinger equations* (3.1) *with the potential p defined in* (3.9).

Proof. Consider equation (3.4) for the second component of a vector solution of the lattice monodromy equation. Represent its solution ψ in the form

$$\psi_k = e^{\theta(\phi + hk)} \tilde{\psi}_k.$$

Clearly, $\tilde{\psi}_k$ satisfies equation (3.1). Now, the $\mathcal{S}$-equivalence immediately follows from Theorem 2.1 and Lemma 3.1. □

3.2. **The adiabatic case.** Here, we describe the family of Schrödinger equation (3.1) in the adiabatic limit. For this, we start with the monodromy matrix described in Theorem 2.2. To formulate the result, we need a definition.

Definition 3.2. Pick $\aleph > 0$, h belongs to a given Diophantine class $\tilde{D}_\aleph$ if and only if

$$\forall q \in \mathbb{Z} \setminus \{0\}, \; \inf_{p \in \mathbb{N}} \left| h - \frac{p}{q} \right| \geq \frac{\aleph}{|q|^3}.$$

Note that if ε and h are related by (2.5), then $h \in \tilde{D}_\aleph$ if and only if $\varepsilon \in D_\aleph$.

Fix two positive constants $s < 1$ and δ. Recall that

$$J_\delta = [E_1 + W_-, \min\{E_1 + W_+, E_2 + W_-\} - \delta].$$

In the next section, we show that, for sufficiently small ε, the spectrum of any of the equations of the family (0.1) in the interval J_δ ($\delta > 0$) can be situated only in the subintervals where $|\sin(\Phi(E)/\varepsilon)| > s$. So, we let

$$J_{s,\delta} = \{E \in J_\delta \, : |\sin(\Phi(E)/\varepsilon)| > s\}.$$

One has

Theorem 3.1. *Fix positive constants $s < 1$ and δ. Choose a Diophantine class $D_\aleph$. Suppose that V and W satisfy the assumptions described in sections 0 and 1.1.*

For sufficiently small $\varepsilon \in \mathcal{D}_\aleph$, for any $E \in J_\delta$, the family (0.1) *is $\mathcal{S}$-equivalent to a family of finite difference Schrödinger equations of the form* (3.1) *with $p(\phi, E)$ being a 1-periodic real valued function of ϕ and E, and $\mathcal{E}$ being a real valued function of E (independent of ϕ). For $E \in J_{s,\delta}$, these functions admit the following uniform asymptotic representations*

$$\begin{aligned} &p(\phi, E) = p_0(\phi, E) + p_1(\phi, E) \quad \textit{where} \\ &p_0(\phi, E) = 2\lambda(E)\cos(2\pi\phi + s_0(E)), \\ &p_1(\phi, E) = \cos(\Psi(E))O(\lambda(E)) + o(\lambda(E)), \quad \Psi(E) = \frac{1}{\varepsilon}\Phi(E) + o\,(1) \\ &\textit{and } \lambda(E) = e^{-\frac{1}{\varepsilon}(S_1(E) - S(E))} \end{aligned} \tag{3.11}$$

and

$$\mathcal{E}(E) = 2\, e^{\frac{1}{\varepsilon}S(E) + o\,(1)} F\left(\frac{\Phi(E)}{\varepsilon\pi}\right)\cos\Psi(E). \tag{3.12}$$

For $E \in J_\delta \setminus J_{s,\delta}$, $\mathcal{E}(E)$ also admits the asymptotic representation (3.12) *and $p(\phi, E)$ satisfies the estimate*

$$|p(\phi, E)| \le C \cdot \lambda(E), \;\; \phi \in \mathbb{R}.$$

The passage to (3.1) *can be chosen so that $\mathcal{E}$ and p be analytic in the domain*

$$|\mathrm{Im}\, E| \le Y_1\varepsilon, \quad ReE \in J_{s,\delta}, \quad |\mathrm{Im}\,\phi| \le Y_2, \tag{3.13}$$

where Y_1 and Y_2 are positive constants independent of ε. In this case, the above asymptotic formulae remain valid and uniform in the analyticity domain.

The functions $o\,(1)$ in these asymptotics are real analytic in E and ϕ (in the case of (3.11)*), and $d\,(o\,(1))/dE = o\,(1/\varepsilon)$ and $d\,(o\,(1))/d\phi = o\,(1)$*

Proof. Let $E \in J_{s,\delta}$. Consider the monodromy matrix M described in the Theorem 2.2. From the lattice monodromy equation defined by this matrix, we derive the finite difference Schrödinger equation (3.1) following the lines of the proof of Proposition 3.1. Let us check that the coefficients in this equation have the announced properties. By Theorem 2.2, the coefficients of the matrix M are analytic in a domain Ω_1

$$\mathrm{Re}\, E \in J_\delta, \;\; |\mathrm{Im}\, E| \le C_1, \;\; |\mathrm{Im}\,\phi| \le (C_2 + \mathrm{Im}\,\varphi_2^+(E))/\varepsilon, \tag{3.14}$$

with C_1 and C_2 independent of ε, and admit there the uniform asymptotics (2.26). The functions ρ and v defined by (3.5) are analytic in the sub-domain where $c(\phi, E) \ne 0$. The asymptotic formula for c shows that this condition is fulfilled if

$$|\sin(\Phi/\varepsilon)| > Ce^{-\mathrm{Re}\,(S_1/\varepsilon) + 2\pi|\mathrm{Im}\,\phi|},$$

and C is a positive constant independent of ε. Recalling the definitions of S_1 and Φ, we see that, for sufficiently small ε, this inequality holds for $\phi \in \mathbb{R}$ and E in the intervals $J_{s,\delta}$. Moreover, for

$E \in J_{s,\delta}$, we have

$$\rho(\phi) = \frac{c(\phi)}{c(\phi - h)} = 1 + O(e^{-S_1/\varepsilon}).$$

Clearly, all this holds also in a domain of the form (3.13). In this case, the periodic solution of the homological equation (3.6) also is close to 1; more precisely, we have

$$\theta = 1 + O(e^{-S_1/\varepsilon}).$$

Recall that $\mathcal{E}(E)$ is the zeroth Fourier coefficient of the function $q(\phi)$ and that $p(\phi) = \mathcal{E} - q(\phi)$. We can represent q in the form

$$q(\phi) = q_d(\phi) + q_a(\phi)$$

where

$$q_d(\phi) = e^{\theta(\phi)-\theta(\phi+h)} d(\phi), \quad q_a(\phi) = e^{\theta(\phi)-\theta(\phi+h)} a(\phi - h) \frac{c(\phi)}{c(\phi - h)}.$$

Begin with discussing the contribution of the term q_d to the asymptotics of $\mathcal{E}$ and p. Let us describe the Fourier coefficients of d defined by

$$d(\phi) = \sum_{l=-\infty}^{\infty} d_l e^{il(2\pi\phi + s_0(E))}.$$

Using the asymptotics of d given in Theorem 2.2, we get

$$d_0 = 2e^{S(E)/\varepsilon} F\left(\frac{\Phi(E)}{\pi\varepsilon}\right)\left(\cos\left(\frac{\Phi(E)}{\varepsilon}\right) + o(1)\right).$$

Recall that the asymptotics of d are valid and uniform in the domain (3.14). Note that, for $l \in \mathbb{Z}$, the integrals

$$d_l = \int_z^{z+1} e^{-il(2\pi\phi + s_0)} d(\phi) d\phi \tag{3.15}$$

are independent of z for z in the strip of analyticity of d. Recall also that, by the computation done in Remark 2.1, for $\operatorname{Im}\phi \gtrless \pm \operatorname{Im}\varphi_2^+/\varepsilon$, we have

$$d(\phi) = -\lambda(E) e^{\mp i(2\pi\phi + s_0) + o(1)}.$$

To estimate the other Fourier coefficients of d, we substitute this representation into (3.15) for $\operatorname{Im} z \gtrless \pm \operatorname{Im}\varphi_2^+/\varepsilon$. This easily gives the formulae

$$d_{\pm 1} = -\lambda(E)\,(1 + o\,(1)),$$

and, for (E, ϕ) in the domain (3.14),

$$d(\phi) = d_0 + d_1 e^{i(2\pi\phi + s_0(E))} + d_{-1} e^{-i(2\pi\phi + s_0(E))} + o(\lambda(E)).$$

The last term in this representation for d describes the "tail" of the Fourier series.
Now, let us come back to the function q_d. Consider it in a smaller strip of the form (3.13). Taking the asymptotic representation for θ into account, we get

$$q_d = d_0 + d_1 e^{i(2\pi\phi + s_0(E))} + d_{-1} e^{-i(2\pi\phi + s_0(E))} + o(\lambda(E)) + d_0 O(e^{-S_1/\varepsilon}). \tag{3.16}$$

We keep both the last two terms in this formula since, far from the zeros of d_0, we have $o(\lambda(E))+d_0O(e^{-S_1}) = d_0O(e^{-S_1/\varepsilon})$, and in a vicinity of the zeros, we have $o(\lambda(E)) + d_0O(e^{-S_1/\varepsilon}) = o(\lambda(E))$. Then (3.16) implies that

$$\mathcal{E}_d \equiv \int_0^1 q_d d\phi = d_0(1+o(1)) + o\left(\lambda(E)\right)$$

and that, in a domain of the form (3.13), we have

$$p_d \equiv \mathcal{E}_d - q_d = -d_1 e^{i(2\pi\phi+s_0(E))} - d_{-1}e^{-i(2\pi\phi+s_0(E))} + o(\lambda(E)) + d_0O(e^{-S_1/\varepsilon}).$$

Now, consider the term q_a. We shall need only rough estimates for a. Denote its zeroth Fourier coefficient by a_0. One easily obtains:

$$a_0 = o(1),$$

and, in a domain of the form (3.13), we have

$$a(\phi) = a_0 + O(e^{-(S+S_1)/\varepsilon}) = a_0 + o(\lambda(E)).$$

Therefore,

$$\mathcal{E}_a \equiv \int_0^1 q_a d\phi = o(1),$$

and, in a domain of the form (3.13), we have

$$p_a \equiv \mathcal{E}_a - q_a = o(\lambda(E)).$$

Combining the above results for $\mathcal{E}_{a,d}$ and $p_{a,d}$, one obtains

$$\mathcal{E} = \mathcal{E}_d + \mathcal{E}_a = d_0(1+o(1)) + o(1) + o(\lambda(E)),$$

and, in a domain of the form (3.13), we have

$$p = p_d + p_a = -d_1 e^{i(2\pi\phi+s_0(E))} - d_{-1}e^{-i(2\pi\phi+s_0(E))} + o(\lambda(E)) + d_0O(e^{-S_1/\varepsilon}).$$

The last formula can be rewritten as

$$p = -d_1 e^{i(2\pi\phi+s_0(E))} - d_{-1}e^{-i(2\pi\phi+s_0(E))} + o(\lambda(E)) + \mathcal{E}O(e^{-S_1/\varepsilon}).$$

Note that all the terms in the formulae obtained for $\mathcal{E}$ and p are analytic in (E,ϕ) in the required domain.

Taking the asymptotics of d_0 into account, we can easily transform the last representation for $\mathcal{E}$ to the form described in Theorem 3.1. Note that, for this transformation, we have to use that, for $\operatorname{Re} E \in J_{\delta,s}$, $|\operatorname{Im} E| \le C_1\varepsilon$,

$$\cos\left(\frac{\Phi(E)}{\varepsilon}\right) + o(1) = (1+o(1))\cos\left(\frac{\Phi(E)}{\varepsilon} + o(1)\right)$$

and the function $o(1)$ (in the right hand side) is analytic if the function $o(1)$ (in the left hand side) is. Finally, using the last formula for p, the asymptotics for $d_{\pm1}$ and the already proved formula for $\mathcal{E}$, we obtain the representation for p announced in Theorem 3.1. The estimates for the derivatives of the error terms $o(1)$ follow from their analyticity and the Cauchy integral representations for these derivatives.

To finish the proof, we need only to treat the case where $E \in J_\delta \setminus J_{s,\delta}$. In this case, we have $|\sin(\Phi(E)/\varepsilon)| \le s$. Now, the coefficient c of the monodromy matrix can have zeros on the real line of ϕ. We change the consistent basis $\psi_{1,2}$ defined in Theorem 2.2 to the basis

$$\tilde{\psi}_1 = \psi_1, \quad \tilde{\psi}_2 = \psi_1 + \psi_2.$$

Clearly, this new basis is also consistent. The corresponding monodromy matrix is given by the formula

$$\tilde{M} = \tau M \tau^{-1}, \quad \tau = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

For this monodromy matrix, we compute

$$\tilde{c} = -2e^{S/\varepsilon + o(1)} F\left(\frac{\Phi}{\varepsilon\pi}\right)\cos\left(\frac{\Phi}{\varepsilon} + o(1)\right), \quad \phi \in \mathbb{R}.$$

Taking this matrix instead of M, one proves the theorem in the complementary case. $\square$

4. The location of the spectrum

In this section, we show that, in the adiabatic limit, the spectrum of equations (0.1) is situated in certain exponentially small intervals. These intervals are described in terms of the actions S and S_1 and the phase integral Φ.

Our whole analysis here is based upon the $\mathcal{S}$-equivalence of the family of almost periodic equations (0.1) and the family of finite difference equations (3.1) proved in Theorem 3.1. In the sequel $p(\phi, E)$ and $\mathcal{E}(E)$ are the potential and the "new spectral parameter" described in this theorem.

We begin with checking

Lemma 4.1. *Under our assumptions on the potential of equations* (0.1), *for sufficiently small* ε, *the spectra of* $H_{\phi,\varepsilon}$ *in the interval* J_δ *may only be located in the subintervals of* J_δ *where*

$$-2 + \inf_{\phi\in\mathbb{R}} p(\phi, E) \leq \mathcal{E}(E) \leq 2 + \sup_{\phi\in\mathbb{R}} p(\phi, E). \tag{4.1}$$

Proof. The proof easily follows from Theorem 3.1. Consider one of the equations (3.1) (i.e. corresponding to a fixed value of the parameter $\phi \in \mathbb{R}$). Recall that, for two arbitrary solutions ψ^1 and ψ^2 of (3.1), their *Wronskian* is given by

$$w(\psi^1, \psi^2) = \psi^1_{m+1}\psi^2_m - \psi^2_{m+1}\psi^1_m, \quad m \in \mathbb{N}.$$

It is independent of m. Two solutions have a non-zero Wronskian if and only if they are linearly independent.

Consider, in $\ell^2(\mathbb{Z})$, the operator H_E defined by the left hand side of (3.1). It is a bounded self-adjoint operator. For a given E, its spectrum belongs to the interval

$$[-2 + \inf_{\phi\in\mathbb{R}} p(\phi, E), 2 + \sup_{\phi\in\mathbb{R}} p(\phi, E)].$$

So, if the condition (4.1) is violated, equation (3.1) has two linearly independent solutions $\psi^\pm_m$ exponentially decaying as $m \to \pm\infty$ (see, for example, [28]),

$$|\psi^\pm_m| \leq \text{Const}\, e^{\mp\gamma(E)m}.$$

By Theorem 3.1, for this ϕ, equation (0.1) has two solutions $u^\pm$ that are $\mathcal{S}$-equivalent to $\psi^\pm$.

By means of the relation (3.2) and the formula $\det S \equiv 1$, one can easily calculate the Wronskian of these two solutions:

$$w(u^+, u^-) = u^+(x)[u^-]'(x) - u^-(x)[u^+]'(x) = (\det S) w(\psi^+, \psi^-) = w(\psi^+, \psi^-) \neq 0.$$

So, $u^\pm$ are linearly independent.

Note that $u^{\pm}$ are continuous in x. So, they are locally bounded. This, the definition of the $\mathcal{S}$-equivalence and the estimates for $\psi^{\pm}$ imply that

$$|u^{\pm}(x)| \leq \mathrm{Const} e^{\pm\frac{\varepsilon\gamma(E)}{2\pi}x}.$$

So, if E does not satisfy (4.1), then the equation (0.1) has two linearly independent solutions $u^{\pm}$ exponentially decaying as $x \to \mp\infty$. This implies that E does not belong to the spectrum of this equation in $L^2(\mathbb{R})$. □

Now, we can describe the location of the spectrum. Using the asymptotic formulae for p and $\mathcal{E}$, we transform equation (4.1) into the inequality

$$|\cos(\Phi(E)/\varepsilon + o(1))| \leq \frac{e^{-S(E)/\varepsilon} + e^{-S_1(E)/\varepsilon}}{F\left(\frac{\Phi(E)}{\varepsilon\pi}\right)}(1 + o(1)). \tag{4.2}$$

Analyzing (4.2) we prove that the spectrum of (0.1) may only be located in a union of exponentially small intervals. To describe these intervals, we need to discuss the variations of the phase integral $\Phi(E)$. Recall that $\Phi(E)$ is real analytic in E in a vicinity of any point $E \in J_\delta$. Moreover, one has the

Lemma 4.2. *For any $E \in J_\delta$,*

$$\Phi'(E) > 0. \tag{4.3}$$

Proof. The function $\Phi(E)$ can be represented in the form

$$\Phi = \int_{\varphi_1^-}^{\varphi_1^+} \kappa\, d\varphi$$

where the integral is taken along the real line. Here κ is a branch of the complex momentum chosen so that $\Phi(E) > 0$ and $\varphi_1^{\pm}$ are the branching points of the complex momentum κ shown in Fig. 2 and 7. One can easily check that, for

$$E \in J_\delta, \quad E \neq E_1 + W_-, \tag{4.4}$$

one has

$$\Phi'(E) = \int_{\varphi_1^-}^{\varphi_1^+} k'\left(E - W(\varphi)\right) d\varphi. \tag{4.5}$$

Omitting the elementary details, we only note that, since k has square root branching points at $\varphi = \varphi_1^{\pm}$, this integral converges. For any $E \in J_\delta$ and $\varphi \in (\varphi_1^-, \varphi_1^+)$,

$$E_1 \leq E - W(\varphi) \leq E_2$$

But, for $E_1 < \xi < E_2$, $k'(\xi) > 0$; hence, we get the desired result for all E satisfying (4.4).

So, we need only to prove that $\Phi'(E_1 + W_-) \neq 0$. For this, we shall use the original definition of Φ in terms of a contour integral. At energy $E = E_1 + W_-$, $\varphi_1^{\pm} = \varphi_0$ where φ_0 is the point where W attains its minimum W_-. Hence

$$\Phi'(E_1 + W_-) = \oint k'(E_1 + W_- - W(\varphi))d\varphi, \tag{4.6}$$

where we integrate along a closed contour running around the point $\varphi = \varphi_0$. Recall that the minimum of W_- is non-degenerate:

$$W(\varphi) \sim W_- + w_2\,(\varphi - \varphi_0)^2, \quad w_2 > 0. \tag{4.7}$$

Recall also that, at the point $E = E_1$, $k(E)$ has a branching point of square root type:

$$k(E) \sim k_o\sqrt{E - E_1}, \quad k_o \neq 0. \tag{4.8}$$

Combining (4.7) and (4.8), we see that

$$k'\left(E_1 - W(\varphi)\right) \sim \frac{k_o}{2\sqrt{-w_2}}\,\frac{1}{\varphi - \varphi_0}, \quad \varphi \sim \varphi_0.$$

So, at the point $\varphi = \varphi_0$, the function $k'(E_1 + W_- - W(\varphi))$ has a simple pole with a non-zero residue. Thus, (4.6) implies that $\Phi'(E_1 + W_-) \neq 0$. This completes the proof. □

Now, we can easily describe the intervals where one may find the spectra of (0.1). Consider the points E being on the interval J_δ and satisfying the equation

$$\frac{1}{\varepsilon}\,\Phi(E) = \pi/2 + \pi l, \quad l \in \mathbb{Z}. \tag{4.9}$$

Recall that $\Phi(E) > 0$ for $E > E_1 + W_-$, and that $\Phi(E_1 + W_-) = 0$. In view of Lemma 4.2, there is a finite number L of values of $l = 0, 1, 2, \dots L$ such that (4.9) admits a solution in J_δ. For ε small, L is of order $1/\varepsilon$. For $l \in \{0, 1, 2, \dots L\}$, denote the unique solution of (4.9) by $E^{(l)}$. Again, in view of Lemma 4.2, the distances between these points are of order ε; more precisely, for $0 \leq l \leq L$, they satisfy (1.5).

Recall that Σ denotes the spectrum of $H_{\phi,\varepsilon}$. By means of (4.2), we shall easily prove the

Proposition 4.1. *Under the assumptions made on V and W in sections 0 and 1.1 , for sufficiently small ε, the spectra of equations* (0.1) *in the interval J_δ may only be located in subintervals I_l, $l = 0, 1, 2 \dots L$ satisfying*

- *I_l is in an $o(\varepsilon)$-vicinity of $E^{(l)}$;*
- *the length $|I_l|$ of I_l satisfies the asymptotic* (1.6).

Proof of Proposition 4.1. Assume that $E \in J_\delta$. Consider the inequality (4.2). Since $e^{-S/\varepsilon}/F$ and $e^{-S_1/\varepsilon}/F$ are exponentially small, the subset of J_δ satisfying (4.2) is contained in $J_{s,\delta}$. For $E \in J_{s,\delta}$, by Theorem 3.1, the functions $o(1)$ in left hand side of this inequality are real analytic in E, and $d\,(o\,(1))/dE = o\,(1/\varepsilon)$. Now as $\Phi' \neq 0$, the subset of J_δ satisfying (4.2) is contained in exponentially small subintervals of $J_{s,\delta}$ containing the solutions of

$$\frac{1}{\varepsilon}\Phi(E) + o\,(1) = \pi/2 + \pi l, \quad l \in \mathbb{Z}.$$

This implies the first statement in Proposition 4.1. Recalling the asymptotics and definitions of the real analytic functions $e^{-S/\varepsilon}/F$ and $e^{-S_1/\varepsilon}/F$, one sees that, up to factors $1 + o\,(1)$, these functions are constant on the exponentially small intervals. Hence (4.2) implies that the length of the subinterval situated near $E^{(l)}$ admits the asymptotic (1.6). This completes the proof of Proposition 4.1. □

Note that Proposition 4.1 implies the first half of Theorem 1.1.

5. Calculation of the integrated density of states

We now compute the integrated density of states $N(E)$ of $H_{\phi,\varepsilon}$. For the definition and properties of the integrated density of states, we refer to [28, 30]. We only recall that

- $N(E)$ is a monotonous continuous function, independent of $\phi \in \mathbb{R}$.
- Σ is the set of the points of increase of $N(E)$.

Taking Proposition 4.1 into account, the proof of Theorem 1.1 will be complete if we prove

Proposition 5.1. *Under the assumptions of Theorem 3.1, the increase of the integrated density of states on each of the intervals I_l, $l = 0, 1, 2, \ldots, L$, equals to $\varepsilon/2\pi$.*

Proposition 5.1 immediately implies

Corollary 5.1. *Under the assumptions of Theorem 3.1, we have*

$$\Sigma \cap I_l \neq \emptyset, \quad l = 0, 1, 2 \ldots L.$$

We shall now prove Proposition 5.1 using Theorem 3.1. Note, however, that as Proposition 4.1 so Proposition 5.1 can be proved without assuming that $\varepsilon/2\pi$ is a Diophantine number. Therefore one reduces the monodromy equation (2.6) to a second order finite difference equation slightly more general than (3.1).

5.0.1. *Weyl solutions, Lyapunov exponents and the integrated density of states.* Before proving Proposition 5.1, we need to recall some known results. The results described here can be found in [28, 30] Consider a family of almost periodic equations of the form (0.1) assuming that

- the potentials V and W in (0.1) are real valued functions from $L^2_{loc}(\mathbb{R})$;
- $V(x+1) = V(x), \quad W(x+2\pi) = W(x), \quad x \in \mathbb{R}$;
- $\varepsilon/2\pi$ is irrational.

For any fixed value of the spectral parameter E, $\operatorname{Im} E > 0$, there exists u^+, a solution of (0.1) such that $u^+ \in L^2(0, +\infty)$. For almost all $\phi \in \mathbb{R}$, this solution satisfies the estimate

$$\sqrt{|u^+(x, E, \phi)|^2 + |du^+(x, E, \phi)/dx|^2} = e^{-x(\gamma(E) + o(1))}, \quad x \to +\infty, \tag{5.1}$$

with a non-negative number $\gamma(E)$ independent of ϕ and x. This number is *Lyapunov exponent* of the family (0.1).

The Lyapunov exponent is the real part of a function $f(E)$ analytic in the upper half-plane $\operatorname{Im} E > 0$. The limit $-\frac{1}{\pi} \lim_{\alpha \to +0} \operatorname{Im} f(E + i\alpha)$ exists for any $E \in \mathbb{R}$. It coincides with the integrated density of states $N(E)$.

5.0.2. *Bloch solutions of difference equations.* Here, following [5], we discuss the difference equation

$$\chi(x+h) + \chi(x-h) + w(x)\chi(x) = 0, \quad \chi(x) \in \mathbb{C}, \quad x \in \mathbb{R}. \tag{5.2}$$

We assume that

(C1): w is a 1-periodic continuous function,
(C2): $w_- \equiv \min_{x \in \mathbb{R}} |w(x)| > 2$,
(C3): the increment of $\arg w(x)$ over the period of w equals to zero,
(C4): h is an irrational number.

Under these conditions, equation (5.2) has a continuous solution $\chi(x)$ satisfying the estimate

$$\chi(x) = e^{-(\Theta + o(1))x}, \quad x \to +\infty$$

with a complex number Θ independent of x and having a positive real part.

This solution can be constructed as follows. Together with (5.2), consider the difference *Ricatti* equation

$$G(x) + \frac{1}{G(x-h)} + w(x) = 0, \quad x \in \mathbb{R}. \tag{5.3}$$

This equation has a continuous 1-periodic solution represented by the continued fraction

$$G(x) = -w(x) + \cfrac{1}{-w(x-h) - \cfrac{1}{-w(x-2h) - \frac{1}{\cdots}}}. \tag{5.4}$$

Under the condition (C2), the continued fraction in (5.4) converges uniformly in $x \in \mathbb{R}$. This solution satisfies the estimates:

$$|G(x)| \geq w_-/2, \quad |G(x) - w(x)| < w_-/2, \quad x \in \mathbb{R}. \tag{5.5}$$

If w depends analytically on some parameter E, then G also is analytic in E.

Let $g(x) = \ln G(x)$, $x \in \mathbb{R}$. As w satisfies (C3), the function g is a 1-periodic continuous function of x.

Consider a solution of the equation

$$\lambda(x+h) - \lambda(x) = g(x)$$

that is continuous but not necessarily periodic in x. The function

$$\chi(x) = e^{\lambda(x)}$$

is a continuous solution of (5.2). Finally, for h satisfying (C4),

$$\lambda(x) = x(\Theta + o(1)), \quad x \to -\infty,$$

where

$$\Theta = \frac{1}{h} \int_0^1 \ln G(x)\, dx.$$

5.0.3. *The proof of Proposition 5.1.* Consider one of the intervals described in Proposition 4.1, say, I_l, $0 \leq l \leq L$. Let E^* be the zero of $\mathcal{E}(E)$ closest to $E^{(l)}$. Then

$$E^* - E^{(l)} = o(\varepsilon). \tag{5.6}$$

Fix two positive numbers c_1 and c_2 so that

$$1 < c_1 < c_2 < +\infty.$$

For $\varepsilon > 0$ small, consider the domain

$$\Omega = \{E \in \mathbb{C} \,|\, \operatorname{Im} E > 0,\ \varepsilon^{c_2} < |E - E^*| < \varepsilon^{c_1}\}.$$

Recall that $|\sin(\frac{\Phi(E^{(l)})}{\varepsilon})| = 1$. In view of (5.6) and Lemma 4.2, for sufficiently small ε, Ω is contained in one of the domains (3.13) described in Theorem 3.1.

For $E \in \Omega$, the family (0.1) is $\mathcal{S}$-equivalent to the family of finite difference equations (3.1) described in Theorem 3.1. Consider together with the finite difference equation (3.1) its continuous analog

$$\chi(x+h) + \chi(x-h) + p(x,E)\chi(x) = \mathcal{E}(E)\chi(x), \quad x \in \mathbb{R}. \tag{5.7}$$

If $\chi(x)$ satisfies this equation, then

$$\chi_k = \chi(kh + \phi) \tag{5.8}$$

is a solution of (3.1).

The asymptotic formulae for p and $\mathcal{E}$ ((3.11), (3.12)) show that, uniformly in $E \in \Omega$,

$$p(\phi, E) = 2e^{(S(E^{(l)})-S_1(E^{(l)}))/\varepsilon}\left[\cos(2\pi\phi + s_0(E)) + o(1)\right],$$
$$\mathcal{E}(E) = \frac{2}{\varepsilon}\Phi'(E^{(l)})F(l+1/2)e^{S(E^{(l)})/\varepsilon}(E-E^*)(1+o(1))$$

as $\varepsilon \to 0$. This immediately implies (C1) – (C3). Thus, we can construct a solution of (3.1) by formula (5.8) in terms of the solution χ described in the beginning of this section.

Clearly, the solution u^+ of (0.1) and the solution χ_k of (3.1) are $\mathcal{S}$-equivalent. The $\mathcal{S}$-equivalence implies that

$$\frac{2\pi}{\varepsilon}\gamma(E) = h\mathrm{Re}\,\Theta(E) = \mathrm{Re}\int_0^1 \ln G(x,E)\,dx, \quad E \in \Omega.$$

Since the Lyapunov exponent γ is the real part of the analytic function $f(E)$, and $\Theta(E)$ is analytic in E, one has

$$f(E) = \frac{\varepsilon}{2\pi}\int_0^1 \ln G(x,E)\,dx + \mathrm{Const}, \quad E \in \Omega,$$

with a constant independent of E.

Now, consider a semi circle $C \in \Omega$, beginning at a point $a < E^*$ and going to a point $b > E^*$. Note that the definition of Ω, (1.5) and (1.6) imply that a is in a gap of Σ just below I_l and b in a gap of Σ just above I_l. Hence

$$\int_{I_l} dN = N(b) - N(a) = -\frac{1}{\pi}\left(\frac{\varepsilon}{2\pi}\mathrm{Im}\int_0^1 \ln G(x,E)\,dx\Big|_C\right) = -\frac{\varepsilon}{2\pi^2}\int_0^1 \arg G(x,E)\Big|_C\,dx.$$

But, for $E \in \mathbb{R}$, together with $w(x) = p(x,E) - \mathcal{E}(E)$, the function G given by (5.4) is real; taking (5.5) into account, using the asymptotics of $\mathcal{E}$ and p in Ω, we get

$$\arg G(x,E)|_C = \arg(p(x,E) - \mathcal{E}(E))|_C = \arg(E-E^*)|_C = -\pi.$$

That is,

$$\int_{I_l} dN(E) = N(b) - N(a) = \frac{\varepsilon}{2\pi},$$

which proves Proposition 5.1. $\square$

6. The existence of absolutely continuous spectrum

Recall that, for $C_0 > 0$, we have defined

$$J^+_{C_0\varepsilon} = \{E \in J_\delta :\ S - S_1 < -C_0\varepsilon\}$$

Consider (0.1) for a fixed $\phi \in \mathbb{R}$. Recall that Σ_{ac} denotes the absolutely continuous spectrum of (0.1). One has

Proposition 6.1. *Consider the intervals I_l described in Proposition* (4.1)*. There exists $\varepsilon_0 > 0$ and $C_0 > 0$ such that, for $\varepsilon \in (0,\varepsilon_0) \cap D_\aleph$, if $I_l \subset J^+_{C_0\varepsilon}$ then*

$$I_l \cap \Sigma_{ac} \neq \emptyset.$$

Moreover, if $I_l \subset J^+_{C\varepsilon}$, then

$$|I_l \cap \Sigma_{ac}|/|I_l| \to 1$$

when $C \to \infty$, $\varepsilon \to 0$ and $C\varepsilon$ remains bounded.

To prove Proposition 6.1, we combine Theorem 3.1 with results of J. Bellissard, R. Lima and D. Testard on Bloch solutions of difference Schrödinger equations with small coupling [3]. We begin with recalling some of these results.

6.1. **Difference Schrödinger equation with small coupling.** Consider the finite difference Schrödinger equation

$$\psi_{n+1} + \psi_{n-1} + \lambda v(\phi + hn)\psi_n = E\psi_n, \quad n \in \mathbb{Z}. \tag{6.1}$$

Assume that v is a 1-periodic function analytic in the strip

$$s_r = \{\phi \in \mathbb{C} \,|\, |\mathrm{Im}\,\phi| < r\},$$

for some positive r. Let $\|v\|_r$ denote its uniform norm on s_r.

For $\theta \in \mathbb{R}$, we let

$$|||\theta||| = \inf_{m \in \mathbb{Z}} |\theta - m|.$$

Assume that

$$|||hm||| \geq f(m) \text{ for } m = 1, 2, 3, \ldots \tag{6.2}$$

where

$$f(x) = \frac{c}{1 + x^q}, \quad x \geq 0,$$

with some fixed positive c and q. Let

$$K_f = \{\theta \in (0,1) \,|\, |||\theta + nh||| \geq f(|n|),\, n \in \mathbb{Z}\}.$$

Note that $|K_f| \to 1$ as the constant c from the definition of f tends to 0 (which is possible for h kept fixed).

One has

Theorem 6.1. *There is a constant $B_{f,r} > 0$ depending only on f and r such that for all λ satisfying*

$$\lambda \, \|v\|_r < B_{f,r},$$

there exists a Lipschitz function $\mu(.) : \ \theta \in [c, 1-c] \mapsto \mathbb{C}$ such that the equation (6.1) with $E = \mu(\theta)$, $\theta \in K_f$, admits two linearly independent solutions of the form

$$\psi_n^{\pm} = g(\theta, nh + \phi) e^{\pm i\pi\theta(nh+\phi)/h}, \quad n \in \mathbb{Z}, \tag{6.3}$$

where g is a 1-periodic analytic function of ϕ and a Lipschitz function in θ. If v is real analytic, then $\mu(.) : \theta \in [c, 1-c] \to \mathbb{R}$.

One can also control the properties of the function $\mu(\theta)$.

Theorem 6.2. *The function $\mu(\theta)$ admits the asymptotics*

$$\mu(\theta) = 2\cos(\pi\theta) + O(\lambda), \quad \lambda \to 0, \tag{6.4}$$

where the error is measured in the standard Lipschitz norm. The constant in the estimate of the error depends only on the function f, the number r and the norm of the potential $\|v\|_r$. If the potential v depends analytically on a parameter, then, for any fixed $\theta \in [c, 1-c]$, the function $E(\theta)$ also depends analytically on this parameter.

In the above theorems, we just formulated results from [3] in a for us convenient form. Theorem 6.1 is Theorem 1 of [3] reformulated in terms of the solutions of the finite difference equation (6.1) introduced in Corollary 12 (of [3]). In Theorem 1, the function μ is a Lipschitz function defined only on K_f. But, by Lemma 19 (of [3]), this function can be extended to a Lipschitz function defined on the whole interval $[c, 1-c]$. This is the one we consider in Theorem 6.1.

Theorem 6.2 follows just from the proof of Theorem 1. In fact, the asymptotics described in Theorem 6.2 was systematically used in [3]. Though the analytical dependence on an additional parameter was not discussed in this paper, the result from Theorem 6.2 immediately follows from the proof of Theorem 1 (of [3]): indeed, the function μ was constructed as a converging series. If the potential depends analytically on some parameter ξ, then all the terms of this series also depend analytically on ξ. Moreover, the series defining μ converges uniformly in ξ in some set if the potential is analytic in a fixed strip s_r and satisfies the estimate $\lambda \|v\|_r < B_{f,r}$ for all ξ in this set.

6.2. **The proof of Proposition 6.1.** Under the conditions of Theorem 3.1, the family of almost periodic differential equations (0.1) is $\mathcal{S}$-equivalent to the family (3.1). Suppose that $h = 2\pi/\varepsilon$ belongs to a given Diophantine class $\tilde{D}_{\aleph}$; then h satisfies (6.2) with f corresponding to some fixed positive c and q. For $\eta > 0$, define the strip $S_\eta = \{|\mathrm{Im}\phi| < \eta\}$; then, for any $\eta > 0$, if $\varepsilon > 0$ is small enough, the potential p in (3.1) is analytic in ϕ in S_η. Moreover, the asymptotics of p implies that for any $E \in J^+_{C_0\varepsilon}$, and sufficiently small ε

$$|p(\phi)| \leq C\, e^{-C\, C_0},$$

where C denotes a positive constants independent of ϕ and ε. So, for sufficiently large C_0, the equations (3.1) satisfy the conditions of Theorem 6.1.

Consider one of the intervals $I_l \in J^+_{C\varepsilon}$. By Theorem 3.1, the potential p is analytic in E in a $O(\varepsilon)$-vicinity of this interval and p is real analytic in E and ϕ. So, the function $\mu(\theta, E)$ corresponding to (3.1) is real analytic in E.

By Theorem 6.1, for any $E \in I_l$ satisfying the relation

$$\mathcal{E}(E) = \mu(\theta, E), \tag{6.5}$$

for some $\theta \in K_f$, any of the equations (3.1) has two linearly independent solutions of the form (6.3). Let

$$I^* = \{E \in I_l \mid \mathcal{E}(E) = \mu(\theta, E) \text{ for some } \theta \in K_f\}.$$

Due to the $\mathcal{S}$-equivalence of equations (0.1) and (3.1),

$$I^* \subset \{E \in I_l \mid \gamma(E) = 0\},$$

where $\gamma(E)$ is the Lyapunov exponent for the equation family (0.1). Indeed, if, for $E \in I^*$, for equation (0.1), $\gamma(E) \neq 0$, then there exists a solution of (0.1) satisfying the estimate (5.1). Therefore, equation (3.1) also has an exponentially decaying solution. But this solution has to be a linear combination of the solution (6.3) which is impossible.

Then, by Ishii-Pastur-Kotani theorem (see [28, 7, 6]), I^* is in the essential support of the absolutely continuous measure of (0.1) for all $\phi \in \mathbb{R}$. So, to prove our proposition, it suffices to estimate the Lebesgue measure of I^*. Therefore, we use

Lemma 6.1. *For sufficiently small ε, the relation* (6.5) *uniquely defines a Lipschitz function* $E(.):\ [c,1-c]\mapsto I_l$. *This function satisfies the estimate*

$$E(\theta_1)-E(\theta_2)=\frac{1+o(1)}{\mathcal{E}'(E^{(l)})}\left(2cos(\pi\theta_1)-2\cos(\pi\theta_2)+O(\lambda)|\theta_1-\theta_2|\right) \tag{6.6}$$

where

$$\lambda(E)=e^{\frac{S(E^{(l)})-S_1(E^{(l)})}{\varepsilon}}=O(e^{-C\cdot C_0}).$$

Let us now complete the proof of Proposition 6.1. By Lemma 6.1, we have

$$I^*=E(K_f).$$

Moreover, (6.6) implies that, for almost every $\theta\in[c,1-c]$, the derivative $E'(\theta)$ exists and satisfies

$$E'(\theta)=-\frac{1+o(1)}{|\mathcal{E}'(E^{(l)})|}\left(2\pi\sin(\pi\theta)+O(\lambda)\right).$$

Hence, we compute

$$\begin{aligned}|E(K_f)|&=\int_{E(K_f)}de=\left|\int_{K_f}E'(\theta)d\theta\right|\\&\geq\frac{1+o(1)}{|\mathcal{E}'(E^{(l)})|}\int_{K_f}((2\pi\sin(\pi\theta)+O\left(\lambda\right))d\theta\\&\geq\frac{1+o(1)}{|\mathcal{E}'(E^{(l)})|}(4-2\pi(1-|K_f|)+O(\lambda)).\end{aligned}$$

Now, we note that if h satisfies (6.2) with the function f corresponding to a positive constant $c=c_0$, then it satisfies the same estimate with any $c<c_0$. By choosing c sufficiently small, we can make the number $1-K_f$ as small as needed. Then, by choosing C_0 in the definition of the interval $J^+_{C_0\varepsilon}$ large enough, we can make λ as small as required. So, if C_0 is large enough, we are still in the case of Theorem 6.1; on the other hand, $|E(K_f)|$ is as close to $\frac{4+o(1)}{|\mathcal{E}'(E^{(l)})|}$ as needed. But, this last number coincides with the leading term of the asymptotics of the length of I_l for $\varepsilon\to0$ given in (1.6). This completes the proof of Proposition 6.1.

Proof of the Lemma 6.1. It follows from Theorem 6.2 and the asymptotic formulae for $\mathcal{E}(E)$. Note that p is analytic in E in an $O(\varepsilon)$-vicinity of I_l. So, by Theorem 6.2, μ does the same. Thus, both functions $\mathcal{E}(E)$ and $\mu(\theta,E)$ are real analytic in E in an $O(\varepsilon)$-vicinity V of I_l. Fix $\theta\in[c,1-c]$. For real $E\in V$, $\mu(\theta,E)\in[-2+O(\lambda),2+O(\lambda)]$, where $\lambda=e^{(S(E^{(l)})-S_1(E^{(l)}))/\varepsilon}$, and the value of $\mathcal{E}(E)$ changes from a negative number of size $e^{S(E^{(l)})/\varepsilon}$ to a positive number of the same size. This implies the existence of a solution $E(\theta)$ of equation (6.5). To prove the uniqueness, one compares the derivatives of $\mathcal{E}$ and μ with respect to E. One estimates the derivative of μ using the following argument. In view of (6.4), in this vicinity, $\mu(\theta,E)-2\cos(\pi\theta)=O(\lambda(E^{(l)}))$. So, $\partial\mu(\theta,E)/\partial E$ can be estimated by the $O(\lambda(E^{(l)}))$ divided by the size of the vicinity, i.e.

$$\partial\mu(\theta,E)/\partial E=O\left(\lambda/\varepsilon\right). \tag{6.7}$$

By the asymptotic formulae (3.12), for $E\in I_l$, for some $C>0$ independent of ε, we have

$$|\mathcal{E}'(E)|\geq\frac{1}{C}e^{S(E^{(l)})/\varepsilon}/\varepsilon. \tag{6.8}$$

So, the function $\mathcal{E}(E)-\mu(\theta,E)$ is monotonous. This proves the uniqueness of $E(\theta)$. The estimate for $E(\theta_1)-E(\theta_2)$ follows from (6.7), (6.8) and the asymptotics of μ (6.4). $\square$

7. The absence of absolutely continuous spectrum

Here, we prove absence of the absolutely continuous spectrum on the intervals I_l when $S_1 - S < 0$ that is

Proposition 7.1. *Fix $\aleph > 0$ and $\delta > 0$. Assume that V and W satisfy the assumptions described in sections 0 and 1.1. Then, there exist $C_0 > 0$ and $\varepsilon_0 > 0$ such that, for $\varepsilon \in (0,\varepsilon_0)\cap\mathcal{D}_\aleph$, each of the intervals $I_l \subset J^-_{C_0\varepsilon}$ contains only singular spectrum (i.e. singular continuous or pure point spectrum).*

The main idea of the proof is very simple. Let ψ be a polynomially bounded solution of (0.1) for a given (E,ϕ). Then, the construction done in section 3 tells us how to construct a polynomially bounded solution of (3.1). But for energies E in the region $J^-_{C\varepsilon}$, for C large, $\lambda = e^{-(S_1-S)/\varepsilon}$ is large. Hence, by a result of Eliasson's [9], we know that for almost every ϕ, the spectrum of the operator defined by the left hand side of (3.1) is pure point. So, if ϕ is in the "good" set, by a simple extension of Eliasson's results, we see that any polynomially bounded solution of equation (3.1) has to be in $\ell^2(\mathbb{Z})$. We then use the $\mathcal{S}$-equivalence again to get that the solution of (0.1) we considered initially is square integrable. Of course, this suggests that the spectrum in the considered region is pure point. Unfortunately, the operator defined by the left hand side of (3.1) itself depends on E, and thus, the "good" set of ϕ for which its spectrum is pure point does the same. Due to the high instability of the pure point spectrum, this prevents us from proving that the spectrum is in fact pure point. But these arguments will suffice to show that the spectrum is singular.

The proof will rely on two lemmas. First

Lemma 7.1. *Under the assumptions of Theorem 3.1, there exists $\varepsilon_0 > 0$ and $C_0 > 0$ such that, for all $\varepsilon \in D_\aleph \cap (0,\varepsilon_0]$ and $E \in J^-_{C_0\varepsilon}$, and for almost every $\phi \in [0,1)$, if $(\psi)_{k\in\mathbb{Z}}$ is a polynomially bounded solution of (3.1), then $(\psi_k)_{k\in\mathbb{Z}}$ is square summable.*

Actually, our proof of Lemma 7.1 implies that $(\psi)_{k\in\mathbb{Z}}$ is of rapid decay.

Fix $\varepsilon \in D_\aleph \cap (0,\varepsilon_0)$. Recall that Σ_{ac} denotes the absolutely continuous spectrum of the family of equations (0.1). Consider the set

$$\Omega = \{(E,\phi) \in (J^-_{C_0\varepsilon}\cap\Sigma_{\text{ac}})\times[0,1);\ (0.1) \text{ admits a polynomially bounded solution}\}.$$

Then we have

Lemma 7.2. *Ω is Lebesgue-measurable.*

Let us now complete the proof of Proposition 7.1. It is well known that, for any fixed ϕ, for almost every energy E (here "almost every" refers to the spectral measure), (0.1) admits a polynomially bounded solution (see e.g. [31]). Hence, if we assume that $J^-_{C_0\varepsilon}\cap\Sigma_{\text{ac}} \neq \emptyset$, then, Ω is of positive measure. By Fubini's Theorem, we know that, for at least one energy $E \in J^-_{C_0\varepsilon}\cap\Sigma_{\text{ac}}$, there exists $\mathcal{F}$, a set of ϕ of positive measure, such that $\{E\}\times\mathcal{F}\subset\Omega$; hence, by Lemma 7.1, for $\phi\in\mathcal{F}$, (3.1) admits a non trivial square summable solution. As (0.1) is $\mathcal{S}$-equivalent to (3.1), for $\phi\in\mathcal{F}$, (0.1) admits a non trivial solution in $L^2(\mathbb{R})$. Hence, for $\phi\in\mathcal{F}$, E is an eigenvalue (of multiplicity 1) of $H_{\phi,\varepsilon}$. Moreover, $\mathcal{F}$ is of positive measure. But, standard results on ergodic operators tell us that this is impossible ([28]). Hence, $J^-_{C_0\varepsilon}\cap\Sigma_{\text{ac}}$ has to be empty. This completes the proof of Theorem 1.2. $\square$

7.1. **The proof of Lemma 7.2.** We start with the proof of Lemma 7.2. First, to prove that Ω is measurable, it is enough to prove that the set

$$\Omega' = \{(E,\phi) \in \Sigma_{\mathrm{ac}} \times [0,1);\ (0.1) \text{ admits a polynomially bounded solution}\}.$$

is measurable as $\Omega = \Omega' \cap (J^+_{C_0\varepsilon} \times [0,1))$. Denote the spectral resolution of $H_{\phi,\varepsilon}$ by $\mathcal{L}(\phi,\Delta)$ where Δ is a Borel subset of $\mathbb{R}$. $\mathcal{L}(\phi,\Delta)$ is a measurable function of ϕ ([28]). Pick $\delta > 1/2$. Then the operator $A(\phi,\Delta) = (1+x^2)^{-\delta/2}\mathcal{L}(\phi,\Delta)(1+x^2)^{-\delta/2}$ is trace-class and measurable in ϕ ([31] sections B.9 and C.5). One can represent $A(\phi,\Delta)$ in the following way ([31])

$$A(\phi,\Delta) = \int_\Delta a(\phi,E)\rho(\phi,dE)$$

where

- the trace-class operator $a(\phi,E)$ is jointly Borel-measurable in (ϕ,E).
- $\rho(\phi,\Delta) = \operatorname{tr}(A(\phi,\Delta))$ defines a spectral measure for $H_{\phi,\varepsilon}$.

Denote the distribution kernel of $a(\phi,E)$ by $a(x,y;\phi,E)$ and define

$$a^+(\phi,E) = \sup_{x,y\in\mathbb{R}} |a(x,y;\phi,E)| \text{ and } \tilde{\Omega} = \{(E,\phi) \in \Sigma_{\mathrm{ac}} \times [0,1);\ a^+(\phi,E) < +\infty\}.$$

Then a^+ and $\tilde{\Omega}$ are Lebesgue measurable. Moreover, by the proof of Lemma C.5.2 of [31], for $(E,\phi) \in \tilde{\Omega}$, $(H_{\phi,\varepsilon} - E)\psi = 0$ has a polynomially bounded solution. Hence, we have

$$\tilde{\Omega} \subset \Omega' \subset \Sigma_{\mathrm{ac}} \times [0,1).$$

On the other hand, if we endow $\Sigma_{\mathrm{ac}} \times [0,1)$ with the measure $\rho(\phi,dE)d\phi$, by Lemma C.5.2 of [31], we get that $\tilde{\Omega}$ is of total measure in $\Sigma_{\mathrm{ac}} \times [0,1)$. Hence, Ω' is measurable. This ends the proof of Lemma 7.2. $\square$

7.2. **The proof of Lemma 7.1.** We now turn to the proof of Lemma 7.1. It will heavily rely on the work done in [9]. Therefore, we will need to recall some of the results proved there. H. Eliasson considers almost periodic discrete Schrödinger operators of the type

$$L_{\phi,\lambda}u = -(u_{n+1} + u_{n-1}) + \lambda q(nh+\phi)u_n$$

acting on $\ell^2(\mathbb{Z})$ where

- $h \in \tilde{D}_\aleph$ (see definition 3.2),
- q is a 2π periodic function such that
  - q belongs to the Gevrey class

$$\sup_{x\in[0,2\pi]} |\partial^\nu q(x)| \leq C(\nu!)^2 K^\nu, \quad \forall \nu \geq 0, \tag{7.1}$$

  - for some $s \in \mathbb{N}$ and $\xi > 0$,

$$\max_{0\leq\nu\leq s} \|\partial_x^\nu(q(x+\theta) - q(\theta))\| \geq \xi, \quad \forall\theta,\ \forall x, \tag{7.2}$$

$$\max_{0\leq\nu\leq s} \|\partial_\theta^\nu(q(x+\theta) - q(\theta))\| \geq \xi|x|, \quad \forall\theta,\ \forall x. \tag{7.3}$$

For $L_{\phi,\lambda}$ defined above, he proves that there exists λ_0 (depending only on $\aleph$, ξ, s, C and K) such that, for $\lambda \geq \lambda_0$ and for almost every ϕ, the spectrum of $L_{\phi,\lambda}$ is purely punctual.
To prove this result, H. Eliasson constructs inductively a sequence of unitary matrices $(U_n)_{n\in\mathbb{N}}$ such that

- at every step n,

$$U_n^* \cdots U_1^* (L_{\phi,\lambda}) U_1 \cdots U_n = B_n + F_n$$

- B_n is block diagonal and F_n converges to 0 in norm.
- the product $U = \prod_n U_n$ converges (one has estimates on $\|U_n - I\|$ where I is the identity).

This construction is done for any ϕ and sufficiently large λ; it depends only on $\aleph$, ξ, s, C and K. The size of the blocks in B_n increases; but H. Eliasson shows that, for a set of full measure of ϕ, the blocks stop increasing after a certain step (depending on the block). Let us now state a lemma describing the $(U_n)_{n\in\mathbb{N}}$. For $\rho \in \mathbb{R}$, define $\ell^2_\rho(\mathbb{Z}) = \{(u_n)_n;\ \sum_n (1+|n|^2)^{-\rho}|u_n|^2 < +\infty\}$. $\ell^2_\rho(\mathbb{Z})$ is a Hilbert space endowed with its natural scalar product. One has $(\ell^2_\rho(\mathbb{Z}))^* = \ell^2_{-\rho}(\mathbb{Z})$.

Lemma 7.3. *For any $\rho \in \mathbb{R}$, U_n is bounded from $\ell^2_\rho(\mathbb{Z})$ to itself; one has*

$$\sum_{n\geq 1} \|U_n - I\|_{\mathcal{L}(\ell^2_\rho(\mathbb{Z}))} < +\infty.$$

This proves that, for any $\rho \in \mathbb{R}$, $U = \prod_{n\geq 1} U_n$ is bounded from $\ell^2_\rho(\mathbb{Z})$ to itself. Let us postpone the proof of this lemma for a while and complete the proof of Lemma 7.1.

Consider $p(\phi, E)$ and $\lambda(E)$ defined in Theorem 3.1 and let $L_{\phi,E,\varepsilon}$ be the operator defined in (3.1) that is

$$\left(L_{\phi,E,\varepsilon}\psi\right)_n = \psi_{n+1} + \psi_{n-1} + p(kh+\phi, E)\psi_n.$$

Define $q(\phi, E) = \lambda^{-1}(E)p(\phi, E)$. Then, as q is continuous in ε and by the asymptotics given in Theorem 3.1, there exist $\xi > 0$, $s \in \mathbb{N}$, $C > 0$ and $K > 0$ such that q satisfies (7.1), (7.2) and (7.3) uniformly for all $E \in J_\delta$ and $\varepsilon \in (0, \varepsilon_0)$ (where ε_0 is chosen small enough).

Now, for $E \in J^-_{C_0\varepsilon}$, we have $\lambda(E) = e^{-(S_1(E)-S(E))/\varepsilon} \geq C_1 e^{C_1 C_0}$ where C_1 is a positive constant. Using Eliasson's construction, we see that, for any $\aleph > 0$, there exists some $C_0 > 0$ and $\varepsilon_0 > 0$ such that, for $\varepsilon \in \mathcal{D}_\aleph \cap (0, \varepsilon_0)$ and for a.e. ϕ (depending on E), there exists U, a unitary operator on $\ell^2(\mathbb{Z})$ (that is bounded on $\ell^2_\rho(\mathbb{Z})$ for any $\rho \in \mathbb{R}$) and D a diagonal matrix such that

$$L_{\phi,E,\varepsilon} = UDU^*.$$

By Lemma 7.3, we can apply U^* to any polynomially bounded solutions of equation (3.1). Hence, if ψ is a polynomially bounded solution of (3.1), i.e. of $(L_{\phi,E,\varepsilon} - \mathcal{E}(E))u = 0$, then $U^*\psi$ is a polynomially bounded solution of $(D - \mathcal{E})u = 0$. But, D is diagonal and its eigenspaces are of finite dimension (actually, as $L_{\phi,E,\varepsilon}$ is of second order, they are one-dimensional). Hence, $\psi = \mu U \delta_k$ for some $\mu \in \mathbb{C}^* \setminus \{0\}$, $k \in \mathbb{Z}$ (here δ_k is the k-th vector of the canonical basis of $\ell^2(\mathbb{Z})$). So that $\psi \in \ell^2_\rho(\mathbb{Z})$ for any $\rho \in \mathbb{R}$ (as $\delta_k \in \ell^2_\rho(\mathbb{Z})$ for any $\rho \in \mathbb{R}$). This proves Lemma 7.1. □

Proof of Lemma 7.3. First, let us note that, using duality, it is enough to prove Lemma 7.3 for $\rho > 0$. Therefore we use the estimates on the matrix elements of $V_n = U_n - I$ that we will denote by $((V_n(k,l)))_{k,l}$ provided in [9]. Define the matrix

$$A_n = ((a_n(k,l)))_{k,l} = (((1+|k|^2)^{\rho/2} V_n(k,l)(1+|l|^2)^{-\rho/2}))_{k,l}$$

We decompose

$$A_n = \sum_{(k,l)\in\mathbb{Z}^2} a_n(k,l) E_{k,l} =: \sum_{(k,l)\in\mathbb{Z}^2} A_n(k,l)$$

where the matrices $((E_{k,l}))_{k,l}$ are defined by $E_{k,l} = ((\delta_{i,k}\delta_{j,l}))_{i,j} = |\delta_k\rangle\langle\delta_l|$.
In [9], the following estimates are given for $V_n(k,l)$

$$|V_n(p,q)| \leq (\varepsilon_n)^{1/4} e^{-r_n|p-q|} \tag{7.4}$$

where $(\varepsilon_n)_n$ and $(r_n)_n$ are sequences of positive numbers such that, for any $s > 0$ and $t > 0$, one has

$$\sum_{n\geq 1} \frac{\varepsilon_n^s}{r_n^t} < +\infty. \tag{7.5}$$

By the Cotlar-Stein Lemma (see e.g. [18]), to prove Lemma 7.3, it is enough to prove that there exists a sequence $(M_n)_{n\in\mathbb{N}}$, $M_n \geq 0$ such that

$$\sup_{p,q} \sum_{k,l} \|A_n(k,l)^* A_n(p,q)\|^{1/2} \leq M_n, \tag{7.6}$$

$$\sup_{p,q} \sum_{k,l} \|A_n(p,q) A_n(k,l)^*\|^{1/2} \leq M_n, \tag{7.7}$$

$$\text{and} \quad \sum_{n\geq 1} M_n < +\infty.$$

We will only estimate $\sup_{p,q}\sum_{k,l}\|A_n(k,l)^*A_n(p,q)\|^{1/2}$, the one in (7.7) being dealt with in exactly the same way. One computes

$$A_n(k,l)^* A_n(p,q) = \delta_{k,p}\overline{a_n(k,l)}a_n(p,q)E_{l,q}.$$

Hence, using the estimate (7.4), we compute

$$\begin{aligned}
\sum_{k,l} \|A_n(k,l)^* A_n(p,q)\|^{1/2} &= \sum_{l\in\mathbb{Z}} |a_n(p,l)a_n(p,q)|^{1/2} \\
&= (1+|p|^2)^{\rho/2}\left(|V_n(p,q)|(1+|q|^2)^{-\rho/2}\right)^{1/2} \cdot \\
&\qquad \cdot \sum_{l\in\mathbb{Z}} \left(|V_n(p,l)|(1+|l|^2)^{-\rho/2}\right)^{1/2} \\
&\qquad \left(\frac{(1+|p|^2)^{-\rho/4}}{r_n} + e^{-r_n|p|/2}|p|\right)
\end{aligned} \tag{7.8}$$

Hence, taking the supremum over p, we get

$$\sup_{p,q} \sum_{k,l} \|A_n(k,l)^* A_n(p,q)\|^{1/2} \leq C_\rho \frac{\epsilon_n^{1/4}}{r_n^{1+\rho}} =: M_n.$$

The estimate (7.5) immediately gives

$$\sum_{n\geq 1} M_n = C_\rho \sum_{n\geq 1} \frac{\epsilon_n^{1/4}}{r_n^{1+\rho}} < +\infty.$$

This ends the proof of Lemma 7.3. □

8. The phase diagram

In this section, we write W in the form αW where W is a fixed potential and α a positive parameter measuring the size of the potential. Let us briefly recall the definition of S and S_1

(see section 2.3.2). We have defined

$$S(\alpha, E) = -i \int_{\varphi_1^+ - 2\pi}^{\varphi_1^-} k_0(E - \alpha W(\varphi)) d\varphi \tag{8.1}$$

$$S_1(\alpha, E) = i \int_{\varphi_2^-}^{\varphi_2^+} [k_0(E - \alpha W(\varphi)) - \pi] d\varphi \tag{8.2}$$

where k_0 is the branch of k determined as in Fig. 6. More precisely, k_0 is first defined on the domain bounded by the lines C^+ and C^- and cut between the segment $[\phi_1^-, \phi_0]$. Then, on C^+ and C^-, outside of the cut, we define k_0 by continuity, and finally, to define the integrand in the formula for S, we continue this branch analytically along the real line up to the point $\phi_1^+ - 2\pi$.

It will be convenient to change variables $(\alpha, E) \to (\alpha, t)$ where $E = E_1 + t\alpha$. For convenience, we denote by $S(\alpha, t)$ and $S_1(\alpha, t)$ the functions obtained from $S(\alpha, E)$ and $S_1(\alpha, E)$ by this change of variables. The domain $\mathcal{S}$ defined by (1.9) becomes

$$\mathcal{K} := \left\{ (\alpha, t);\ 0 < \alpha,\ \frac{E_2 - E_1}{\alpha} + W^- < t,\ W_- \le t \le \inf\left(\frac{E_2 - E_1}{\alpha} + W_-, W_+ \right) \right\}.$$

The set $\mathcal{K}$ is represented in Fig. 10 in the case when $W^- > -\infty$.

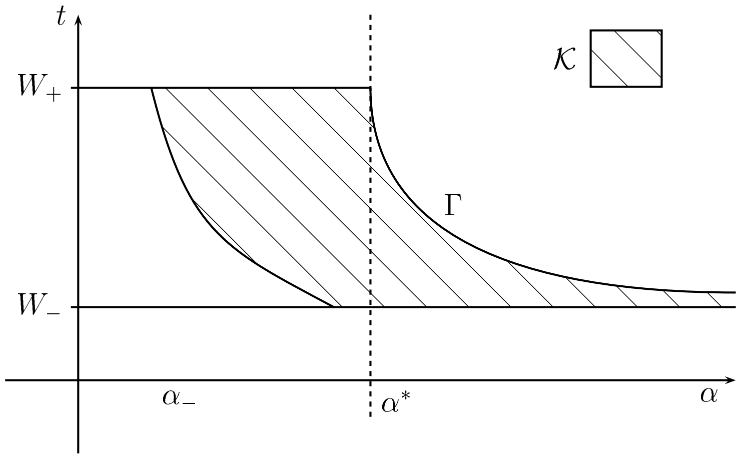

FIGURE 10. The domain $\mathcal{K}$ (when $W^- > -\infty$)

The actions $S(\alpha, t)$ and $S_1(\alpha, t)$ are analytic in (α, t) in $\dot{\mathcal{K}}$, the interior of $\mathcal{K}$. Indeed, using the analyticity of k_0, outside of its turning points (see Fig. 7), we see that

$$S(\alpha, E) = -\frac{i}{2} \int_{\gamma_1} k_0(E - \alpha W(\varphi)) d\varphi \tag{8.3}$$

$$S_1(\alpha, E) = \frac{i}{2} \int_{\gamma_2} (k_0(E - \alpha W(\varphi)) - \pi) d\varphi \tag{8.4}$$

where γ_1 and γ_2 are contours taken as on Fig. 11. Here the thick lines joining φ_1^- and $\varphi_1^+ - 2\pi$, and φ_2^- and φ_2^+ represent the cuts used to define a single valued branch of $k_0(E - \alpha W(\varphi))$. $\varphi_1^\pm$ and $\varphi_2^\pm$ depend on α and t. For any $(\alpha_0, t_0) \in \dot{\mathcal{K}}$, we can choose contours γ_1 and γ_2 independent of (α, t) in a small neighborhood of (α_0, t_0) such that S and S_1 are given by (8.3) and (8.4). Again by the analyticity of k_0, we get that S and S_1 are analytic in some open neighborhood of (α_0, t_0); this works in $\dot{\mathcal{K}}$. Hence S and S_1 are analytic in $\dot{\mathcal{K}}$. Notice that, for any fixed α, when $t \to W_-$ ($(\alpha, t) \in \dot{\mathcal{K}}$), the integration contour in the definition of S_1 gets squeezed between the

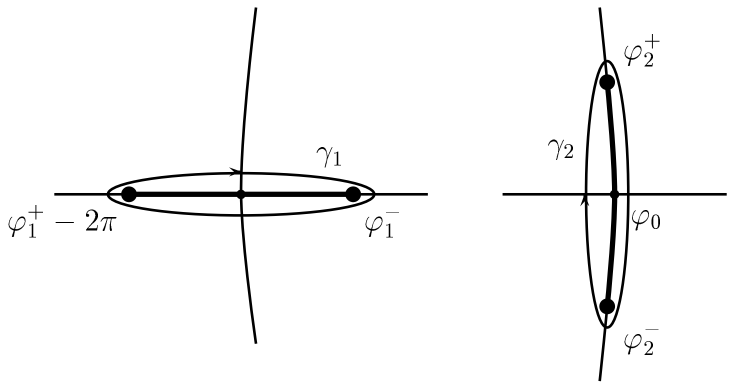

FIGURE 11. The new contours of integration

branching points φ_1^- and φ_1^+. As these points turn around φ_0 as t goes around W_-, the point $t = W_-$ is a branching point of S_1. One can easily see that this point is also a branching point of S.
One has the

Lemma 8.1. *There exists a non zero but possibly countable number, say q, of disjoint intervals $I_j = (t_j^-, t_j^+)$ ($j = 1, \dots, q$) and the same number of real analytic functions $\alpha^j : I_j \to \mathbb{R}$ such that*

$$\{(\alpha, t) \in \mathcal{K};\ S(\alpha, t) = S_1(\alpha, t)\} = \cup_{1 \le j \le q} \{(\alpha^j(t), t);\ t \in I_j\}. \tag{8.5}$$

Moreover, one has $t_1^+ = W_+$ and $\lim_{t \to W_+} \alpha^1(t) = \alpha^*$. The set $\{\Delta S(\alpha, t) < 0\}$ is connected and contains $\mathcal{K} \cap \{\alpha > \alpha_1\}$ for some $\alpha_1 > \alpha^*$.
If $W^- = -\infty$, then $q = 1$, $t_1^- = W_-$ and

$$0 < \inf_{t \in [W_-, W_+]} \alpha(t) \le \sup_{t \in [W_-, W_+]} \alpha(t) < +\infty.$$

Theorems 1.3 and 1.4 are immediate corollaries of Lemma 8.1.

Proof of Lemma 8.1. It will be a consequence of the following results. First we have

Lemma 8.2. *In $\mathcal{K} \setminus \{(\alpha^*, W_+)\}$, one has*

$$\frac{\partial S_1}{\partial \alpha} \le 0, \quad \frac{\partial S}{\partial \alpha} \ge 0, \text{ and } \frac{\partial \Delta S}{\partial \alpha} = \frac{\partial (S_1 - S)}{\partial \alpha} < 0 \tag{8.6}$$

This lemma implies that, for every $t \in [W_-, W_+]$, there exists at most one α such that $(\alpha, t) \in \mathcal{K}$ and $\Delta S(\alpha, t) = 0$.

Now let us study the behavior of ΔS at the boundary of $\mathcal{K}$. For $t \in (W_-, W_+)$, we define $\alpha_0(t) = \dfrac{E_2 - E_1}{t - W_-}$. The curve $\Gamma := \{(\alpha, t) :\ \alpha = \alpha_0(t)\}$ is one of the four curves forming the boundary of $\mathcal{K}$. As $\varphi_2^\pm$ are defined by the equation $E - \alpha W(\varphi) = E_2$, they coincide if and only if the point (α, φ) belongs to Γ. We also notice that $\varphi_1^\pm$ are always given by the equation $W(\varphi_1^\pm) = t$. Hence $\varphi_1^\pm$ are independent of α. At last, we note that, at the point (α^*, W_+), one has $\varphi_1^+ - 2\pi = \varphi_1^-$ and $\varphi_2^+ = \varphi_2^-$. This implies that $S(\alpha^*, W_+) = S_1(\alpha^*, W_+) = 0$. We then get that:

- for $t = W_+$ and $\alpha < \alpha^*$: as $\Delta S(\alpha^*, W_+) = 0$, (8.6) tells us that $\Delta S(\alpha, W_+) > 0$. So that for every $\alpha < \alpha^*$, there exists a left neighborhood of W_+, say $(W_+ - \eta, W_+]$ ($\eta = \eta(\alpha) > 0$) on which $\Delta S > 0$. By (8.6), this neighborhood is increasing as α decreases.
- on the curve Γ, when $W_- < t < W_+$, the quantity ΔS is strictly negative. Indeed, along this curve, $S_1 = 0$, and $S > 0$. The last inequality follows from the observation that $S \geq 0$ in $\mathcal{K}$ and that, as $k_0(E) > 0$ for $E_1 < E < E_2$, one has $S = 0$ if and only if $\varphi_1^- = \varphi_1^+ - 2\pi$ which is possible only for $t = W_+$.

Then by (8.6), we get that, as close as we want to (α^*, W_+), there exists $(\alpha', t') \in \dot{\mathcal{K}}$ such that $\Delta S(\alpha', t') = 0$. This, equation (8.6) and the analytic implicit function theorem ensure the existence of an analytic curve starting from (α^*, W_+) contained in $\Delta S(\alpha, t) = 0$. Moreover, by the analytic implicit function theorem, equation (8.6) tells us that the set of $t \in (W_-, W_+)$ such there exists α such that $(\alpha, t) \in \dot{\mathcal{K}}$ and $\Delta S(\alpha, t) = 0$ is open; we just saw that it is not empty. We can write it as an (at most countable) union of intervals; this, equation (8.6) and the analytic implicit function theorem ends the determination of the analytic curves described in Lemma 8.1 in the case when $W^- > -\infty$.

Let us now study what happens when $\alpha \to +\infty$ and (α, t) stays in $\mathcal{K}$. Then, as $W_- \leq t \leq W_- + (E_2 - E_1)/\alpha$ and $t - W(\varphi_2) = (E_2 - E_1)/\alpha$, we get $W(\varphi_2) \to W_-$, and, thus, $\varphi_2^\pm \to \varphi_0$; hence $S_1(\alpha, t) \to 0$ uniformly in t. On the other hand, as $S(\alpha, t)$ is non-decreasing along the lines $t = \mathit{const}$ (by Lemma 8.3), we know that, for some $\eta > 0$ and any $\alpha > \alpha^*$, $S(\alpha, t) > 2\eta$. Hence, for α large enough and $(\alpha, t) \in \mathcal{K}$, we have $\Delta S(\alpha, t) \leq -\eta$. To complete the proof of Lemma 8.1 in the case $W^- > -\infty$, we just have to notice that, along the curve $\alpha = \alpha_0(t)$ ($t \in [W_-, W_+)$), $\Delta S(\alpha, t) < 0$ and to use (8.6) to see that the set $(\alpha, t) \in \mathcal{K}$ such that $\Delta S(\alpha, t) < 0$ is connected.

In the case when $W^- = -\infty$, to prove that there exists a single analytic curve, it is enough to prove that, for any $t \in [W_-, W_+)$, there exists a unique α such that $(\alpha, t) \in \mathcal{K}$ and $\Delta S(\alpha, t) = 0$. By what has been done above, this will be established if we prove

Lemma 8.3. *There exists $\alpha_1 > 0$ such that, for $0 < \alpha \leq \alpha_1$ such that $(\alpha, t) \in \mathcal{K}$, one has $\Delta S(\alpha, t) > 0$.*

Note that this also implies that $\{\Delta S(\alpha, t) > 0\}$ is connected. This completes the proof of Lemma 8.1.

□

Proof of Lemma 8.2. Taking into account the fact that $k_0(E_1) = 0$ and $k_0(E_2) = \pi$, we compute

$$\frac{\partial S}{\partial \alpha} = -i \int_{\varphi_1^+ - 2\pi}^{\varphi_1^-} k_0'(E_1 + \alpha(t - W(\varphi)))(t - W(\varphi))d\varphi, \tag{8.7}$$

$$\frac{\partial S_1}{\partial \alpha} = i \int_{\varphi_2^-}^{\varphi_2^+} k_0'(E_1 + \alpha(t - W(\varphi)))(t - W(\varphi))d\varphi. \tag{8.8}$$

We notice that

- for $\varphi \in (\varphi_1^+ - 2\pi, \varphi_1^-) \subset \mathbb{R}$, we have $E - \alpha W(\varphi) < E_1$; hence $-ik_0'(E - \alpha W(\varphi)) < 0$ (see Fig. 6). Furthermore, the inequality $E - \alpha W(\varphi) < E_1$ is equivalent to $t - W(\varphi) < 0$. Hence, $\dfrac{\partial S}{\partial \alpha} \geq 0$. Moreover, $\dfrac{\partial S}{\partial \alpha} = 0$ if and only if $\varphi_1^+ - 2\pi = \varphi_1^-$ i.e. if and only if $t = W_+$.
- for $\varphi \in (\varphi_2^-, \varphi_2^+) \subset W^{-1}(\mathbb{R})$, we have $E_1 < E - \alpha W(\varphi) < E_2$; hence, $k_0'(E - \alpha W(\varphi)) > 0$ (see Fig. 6). On the other hand, $E_1 < E - \alpha W(\varphi)$ means that $t - W(\varphi) > 0$. Hence, $\dfrac{\partial S_1}{\partial \alpha} \leq 0$. Moreover, $\frac{\partial S_1}{\partial \alpha} = 0$ if and only if $\varphi_2^+ = \varphi_2^-$ i.e. if and only if $(\alpha, t) \in \Gamma$.

This completes the proof of Lemma 8.2.

□

Proof of Lemma 8.3. As $\alpha \to 0$, we get

- as $k_0(E) \to 0$ when $E \to E_1$, then, we get that $k(E_1 + \alpha(t - W(\varphi))) \to 0$ uniformly on $[\varphi_1^+ - 2\pi, \varphi_1^-]$ when $\alpha \to 0$. Hence $S \to 0$ as $\alpha \to 0$; moreover, this is uniform in $t \in [W_-, W_+]$ as $[\varphi_1^+ - 2\pi, \varphi_1^-]$ stays uniformly bounded.
- $\varphi_2^\pm$ is determined by the equation $\alpha W(\varphi_2^\pm) = \alpha t - (E_2 - E_1)$. Hence, there exists $c > 0$ such that, for α small enough, $|\mathrm{Im}\varphi_2^\pm| > c$. On the other hand, for any fixed c and any $|\mathrm{Im}\,\varphi| \leq c$, the integrand in the definition of S_1 tends to $-\pi$ (we recall that the integrand is always negative on the interval of integration). Hence, there exists $C > 0$ such that for all $t \in [W_-, W_+]$, we have $\lim\limits_{\alpha \to 0^+} S_1 > C$.

Hence, there exists $\alpha_1 > 0$ such that, for $0 < \alpha \leq \alpha_1$ such that $(\alpha, t) \in \mathcal{K}$, one has $\Delta S(\alpha, t) > 0$. The proof of Lemma 8.3 is complete.

□

We will end this section with a precise study of the asymptotic mobility edge in a neighborhood of the point (α^*, W_+). For $(\alpha, t) = (\alpha^*, W_+)$, we have $\varphi_1^- = 0$, $\varphi_1^+ = 2\pi$ and $\varphi_2^\pm = \varphi_0$. Performing a Taylor expansion in (α, t) near (α^*, W_+), we get

$$\begin{aligned}
\left(\varphi_1^-(\alpha,t)\right)^2 &= \frac{-2}{W''(0)}(W_+ - t)(1 + o(1)),\\
\left(2\pi - \varphi_1^+(\alpha,t)\right)^2 &= \frac{-2}{W''(0)}(W_+ - t)(1 + o(1)),\\
\left(\varphi_2^+(\alpha,t) - \varphi_0\right)^2 &= -\frac{2}{W''(\varphi_0)}\left((W_+ - t)(1 + o(1)) - (W_+ - W_-)\frac{\alpha - \alpha^*}{\alpha^*}(1 + o(1))\right).
\end{aligned}$$

On the other hand, we know that (see e.g. [23]), for $\theta > 0$ small

$$-ik(E_1 - \theta) = \sqrt{2M_1\theta}(1 + o(1)) \text{ and } -i(k(E_2 - \theta) - \pi) = i\sqrt{2M_2\theta}(1 + o(1)).$$

Here the $(M_i)_{i=1,2}$ are the effective masses attached to $(E_i)_{i=1,2}$ (see [23]). Plugging this into the definitions of S and S_1, (8.1) and (8.2), for $(\alpha, t) \in \mathcal{K}$ and close to (α^*, W_+), we get that

$$\begin{aligned}
S_1(\alpha,t) &= \frac{\pi}{2}\sqrt{\frac{2M_2}{\alpha|W''(\varphi_0)|}}\left(E_2 - E_1 - \alpha(t - W_-)\right)(1 + o(1))\\
S(\alpha,t) &= \frac{\pi}{2}\sqrt{\frac{2\alpha M_1}{|W''(0)|}}(W_+ - t)(1 + o(1)).
\end{aligned}$$

Recall that $\mathcal{S} = \{(\alpha, E);\ \Delta S(\alpha, E) = 0\}$ i.e. $\mathcal{S}$ is the locus of the asymptotic mobility edges. We define $\tilde{\mathcal{S}} = \{(\alpha, t);\ \Delta S(\alpha, t) = 0\}$. Hence, if c_{eff} is defined by (1.10), near (α^*, W_+), $\tilde{\mathcal{S}}$ is given by

$$\sqrt{\frac{M_1}{|W''(0)|}}\alpha(W_+ - t) = \sqrt{\frac{M_2}{|W''(\varphi_0)|}}\left(E_2 - E_1 - \alpha(t - W_-)\right)(1 + o(1))$$

where the remainder terms $o(1)$ are analytic in α and t.
Coming back to the variables (α, E), we get that, at the point (α^*, E^*), the curve $\mathcal{S}$ can be

parameterized by $\alpha \mapsto \alpha(E)$ and has a the slope

$$\frac{d\alpha}{dE} = \frac{1 - c_{\mathrm{eff}}}{W_+ - c_{\mathrm{eff}} W_-} \quad \text{where} \quad c_{\mathrm{eff}} = \sqrt{\frac{M_2 |W''(0)|}{M_1 |W''(\varphi_0)|}}.$$

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