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More about the $S = 1$ relativistic oscillator*

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Abstract

Following to the lines drawn in my previous paper about the $S = 0$ relativistic oscillator I build up an oscillatorlike system which can be named as the $S = 1$ Proca oscillator. The Proca field function is obtained in the framework of the Bargmann-Wigner prescription and the interaction is introduced similarly to the $S = 1/2$ Dirac oscillator case regarded by Moshinsky and Szczepaniak. We obtained the intriguing rule of quantization: $\mathcal{E} = \hbar\omega/2$ for the parity states $(-1)^j$ and $\mathcal{E} = \pm\hbar\omega(j + 1/2)$ for the parity states $-(-1)^j$. There are no radial excitations. Finally, I apply the above-mentioned procedure to the case of the two-body relativistic oscillator.

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I. THE PROCA OSCILLATOR — PUZZLED QUANTIZATION

In this Letter I continue the study of oscillatorlike systems first undertaken by M. Moshinsky and A. Szczepaniak, Ref. [1], for the $S=1/2$ case. Extensions of this model to the case of a two-body problem and to the cases of other spins as well have been presented in Ref. [2] and in Refs. [3–5], respectively. Moreover, a detailed consideration of the $S=1/2$ case, which was presented in Ref. [6], demonstrated that this form of interaction is free of the problem known as the Klein paradox. From the formal point of view the oscillatorlike interaction can be interpreted as the interaction with a linear electric field $E^i=\kappa r^i$, ref. [7], through the term of $\sigma_{\mu\nu}F^{\mu\nu}/2$.

In my previous works, Refs. [8–11], several interesting features of oscillatorlike systems have been found. For instance, in Refs. [8,10] the possibility of oscillatorlike construct for arbitrary spin in the Dowker's formalism [12] has been proved. Some ideas providing a basis for the matrix construct of the Klein-Gordon oscillator [3] have been presented in Refs. [8,9]. In Ref. [11] the Bargmann-Wigner (BW) set of equations is considered, with an *antisymmetric* second-rank spinor being chosen as a field function. Such a description led to the Kemmer-Dirac formalism [13] for the $S=0$ particle. The interaction introduced in each of the BW equations in the form proposed by Moshinsky and Szczepaniak results in the oscillatorlike equation with the double degeneracy (in N, the principal quantum number) of the spectrum in the low frequency limit (*cf.* with Ref. [3]).

In my opinion, all the above-said encourages further investigations. The aim of this paper is to consider the $S=1$ Proca oscillator[1] with the interaction introduced in the same manner as in ref. [11]. Namely, I start from the Bargmann-Wigner equations with the non-gauge interaction obtained after the substitution $\partial_i\rightarrow\partial_i+ik\gamma^0 r^i,\ i=1,2,3$

$$\begin{cases}\left[i\gamma^\mu\partial_\mu-k\gamma^i\gamma^0 r^i-m\right]\Psi(x)=0\\ \Psi(x)\left[i(\gamma^\mu)^T\partial_\mu-k(\gamma^i\gamma^0)^T r^i-m\right]=0\end{cases}\quad . \tag{1}$$

The $S=1$ BW field function presents itself a *symmetric* spinor of the second rank (4×4 symmetric matrix); the derivative acts to the left hand in the second equation. So the field function obeys the Dirac oscillator equation in each of indices.

The symmetric wave function is expanded in the complete set of γ- matrices[2]

$$\Psi_{\{\alpha\beta\}}=\gamma^\mu_{\alpha\delta}C_{\delta\beta}A_\mu+\sigma^{\mu\nu}_{\alpha\delta}C_{\delta\beta}F_{\mu\nu}\quad . \tag{2}$$

The obtained equations for A_μ and $F_{\mu\nu}$ are the following:

$$\partial_\nu F^{\nu 0}=-\frac{m}{2}A^0+\frac{k}{2}(r^iA^i)\quad , \tag{3a}$$

$$\partial_\nu F^{\nu i}=-\frac{m}{2}A^i+\frac{k}{2}r^iA^0\quad , \tag{3b}$$

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[1] I take a liberty to name the equations obtained below as the Proca oscillator since in a free case the equations (3a-3d) are the well-known Proca equations, Ref. [14].

[2] With taking into account symmetric properties of the field function, it is sufficiently to use only $\gamma_\mu C$ and $\sigma_{\mu\nu}C$, ref [15] in the considered case; C is used as the matrix of a charge conjugation. Cf. with the formula (4), the $S=0$ case, in Ref. [11].

$$2mF_{i0} = (\partial_i A_0 - \partial_0 A_i) + 2k\,(r^j F_i{}^j) \quad , \tag{3c}$$
$$2mF_{jk} = (\partial_j A_k - \partial_k A_j) - 2k\,(r^j F^0{}_k - r^k F^0{}_j) \quad . \tag{3d}$$

Let me introduce $E^i = F^{i0}$ and $B^i = -\frac{1}{2}\epsilon^{ijk}F^{jk}$. Then, expressing the dependence of the wave function on t as $\exp(-i\mathcal{E}t)$ one can obtain the equations ($c = \hbar = 1$):

$$\begin{cases} (\mathcal{E}+m)E^i + \sqrt{2}\epsilon^{ijk}p_j^+ B^k = \frac{i}{2}(\mathcal{E}+m)A^i - \frac{i}{\sqrt{2}}p_i^+ A^0 \\ (\mathcal{E}-m)E^i + \sqrt{2}\epsilon^{ijk}p_j^- B^k = -\frac{i}{2}(\mathcal{E}-m)A^i + \frac{i}{\sqrt{2}}p_i^- A^0 \\ 2(p_i E^i) = imA^0 - ik(r^i A^i) \\ 2m\epsilon^{ijk}B^k - 2k(r^i E^j - r^j E^i) = i(p_i A^j - p_j A^i) \quad , \end{cases} \tag{4}$$

where $\vec{p}^{\,\pm} = \frac{1}{\sqrt{2}}(\vec{p} \pm k\vec{r})$ and $\vec{p}_i = \frac{1}{i}\vec{\nabla}_i$. This set can be re-written in a more symmetric form after the substitution $D^i = E^i - \frac{i}{2}A^i$ and $F^i = E^i + \frac{i}{2}A^i = (D^i)^*$ if E^i and A^i are real quantities. In such a way one obtains

$$\begin{cases} \frac{im}{\sqrt{2}}A^0 = p_i^+ F^i + p_i^- D^i \\ mB^i = \frac{1}{\sqrt{2}}\epsilon^{ijk}\left[p_j^+ F^k - p_j^- D^k\right] \\ (\mathcal{E}+m)D^i = -\sqrt{2}\epsilon^{ijk}p_j^+ B^k - \frac{i}{\sqrt{2}}p_i^+ A^0 \\ (\mathcal{E}-m)F^i = -\sqrt{2}\epsilon^{ijk}p_j^- B^k + \frac{i}{\sqrt{2}}p_i^- A^0 \quad . \end{cases} \tag{5}$$

It is possible to eliminate A^0 and B^i on using the commutation relations $\left[p_i^+\,,\,p_j^-\right]_- = ik\delta_{ij}$ and $\left\{p_i^+ p_j^- - p_j^+ p_i^-\right\} f(\vec{r}) = k\epsilon^{ijk}\hat{L}^k f(\vec{r}) = ik(\vec{S}\vec{L})_{ij} f(\vec{r})$. The result is

$$m(\mathcal{E}+m)D^i = \left[-ik(\vec{S}\vec{L})_{ij} - \vec{p}_k^{\,+}\vec{p}_k^{\,-}\delta_{ij}\right]D^j + \left[2(\vec{S}\vec{p}^{\,+})^2_{ij} - (\vec{p}^{\,+})^2\delta_{ij}\right]F^j \quad , \tag{6a}$$
$$m(\mathcal{E}-m)F^i = \left[(\vec{p}^{\,-})^2\delta_{ij} - 2(\vec{S}\vec{p}^{\,-})^2_{ij}\right]D^j + \left[-ik(\vec{S}\vec{L})_{ij} + \vec{p}_k^{\,-}\vec{p}_k^{\,+}\delta_{ij}\right]F^j \quad , \tag{6b}$$

$\vec{S}$ are the spin-1 matrices, $\vec{L}$ is the orbital part of the angular momentum operator.

In order to carry out further decoupling, the system of equations (6a-6b) one could try to apply the procedure of ref. [11]. But, the calculations are more complicated comparing with the previous work and, moreover, it does not lead us to desirable result directly. The system is *not* decoupled after the first application of the procedure of ref. [11]. We arrive at

$$m^2(\mathcal{E}^2 - m^2)D^i = [\text{dif. op.}_1]_{ij}D^j - 2ikp_i^+ p_j^+ F^j \ , \tag{7a}$$
$$m^2(\mathcal{E}^2 - m^2)F^i = [\text{dif. op.}_2]_{ij}F^j + 2ikp_i^- p_j^- D^j \ , \tag{7b}$$

with complicated operators $\text{dif.op}_{1,2}$ on the right-hand side of the equation. On the other hand, we do not want to apply the procedure of ref. [16, p.298], because it is doubtfully that one can insert the complete set of the state vectors as in the formulas (108,109) of the cited reference between $\boldsymbol{\eta}\cdot\mathbf{S}$ and $\boldsymbol{\xi}\cdot\mathbf{S}$. Such bra- vectors as in (108a,108b) may *not* exist, e. g., in the case of low quantum numbers N and j.

Nevertheless, one can use another method. Namely, 1) multiplying, e. g., the first equation (6a) by $(D^i)^*$, 2) integrating over $d^3\mathbf{r}$ and 3) using identities of the hermitian conjugation, the expansion over spherical tensors and the normalization conditions the problem is solved. On this basis we derive the quantization rule for the pure imaginary $k = im\omega$

$$\mathcal{E} = -\frac{ik}{2m}\left[j(j+1) - l(l+1) - s(s+1)\right] - \frac{3ik}{2m} . \tag{8}$$

Very surprisingly, the principal quantum number does *not* present here! The energy is due to the spin-orbit interaction and the constant term, which is similar to that appeared in the Moshinsky-Szczepaniak version of the $S = 1/2$ oscillator. We remain the interpretation of this fact (as well as of Eqs. (9,10) below) for future publications. Finally, for the states of parity $(-1)^j$ one has

$$\mathcal{E} = \frac{\hbar\omega}{2} , \tag{9}$$

which can be interpreted as zero-mode oscillations (we recover $\hbar$ for visual purposes). On the other hand, for the states of parity $-(-1)^j$ one has

$$\mathcal{E} = \pm\frac{\hbar\omega}{2}(2j+1) , \tag{10}$$

i. e. the *non-relativistic* formula for the harmonic oscillator with the substitution $N \to j$ and with two signs of the energy![3]

II. THE TWO-BODY RELATIVISTIC OSCILLATOR

Now I would like to pay some attention to the case of the two-body Dirac oscillator [2]. The two-body Dirac Hamiltonian with oscillator-like interaction is given by ($m\omega = 1$ and, hence, $k = i$ here)[4]

$$\begin{aligned} &i\left(\frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2}\right)\psi = \mathcal{H}\psi = \\ &= \left[\frac{1}{\sqrt{2}}(\vec{\alpha}_1 + \vec{\alpha}_2)\cdot\vec{P} + \frac{1}{\sqrt{2}}(\vec{\alpha}_1 - \vec{\alpha}_2)\cdot\vec{p} - \frac{i}{\sqrt{2}}(\vec{\alpha}_1 - \vec{\alpha}_2)\cdot\vec{r}B + m(\beta_1 + \beta_2)\right]\psi, \end{aligned} \tag{11}$$

where the matrices are given by the direct products

$$\vec{\alpha}_1 = \begin{pmatrix} 0 & \vec{\sigma}_1 \\ \vec{\sigma}_1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1\!\!1_{2\otimes 2} & 0 \\ 0 & 1\!\!1_{2\otimes 2} \end{pmatrix}, \quad \vec{\alpha}_2 = \begin{pmatrix} 1\!\!1_{2\otimes 2} & 0 \\ 0 & 1\!\!1_{2\otimes 2} \end{pmatrix} \otimes \begin{pmatrix} 0 & \vec{\sigma}_2 \\ \vec{\sigma}_2 & 0 \end{pmatrix}, \tag{12}$$

$$B = \beta_1 \otimes \beta_2 = \begin{pmatrix} 1\!\!1_{2\otimes 2} & 0 \\ 0 & -1\!\!1_{2\otimes 2} \end{pmatrix} \otimes \begin{pmatrix} 1\!\!1_{2\otimes 2} & 0 \\ 0 & -1\!\!1_{2\otimes 2} \end{pmatrix}, \tag{13}$$

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[3]Cf. with the discussion on the page 177 in [11].

[4]The oscillatorlike system with $\sim (\vec{\alpha}_1 - \vec{\alpha}_2)\cdot\vec{r}B\Gamma_5$ has also been considered.

$$\Gamma_5 = \gamma_1^5 \otimes \gamma_2^5 = \begin{pmatrix} 0 & 1\!\!1_{2\otimes 2} \\ 1\!\!1_{2\otimes 2} & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1\!\!1_{2\otimes 2} \\ 1\!\!1_{2\otimes 2} & 0 \end{pmatrix} . \tag{14}$$

If we are in the center of mass system (c.m.s.) it is possible to equate $\vec{P}=0$. While the two-body Dirac oscillator seems not to have found considerable phenomenological applications (cf. spectra presented in [16] with the experiment), this is an interesting mathematical model.

Now let me apply the same procedure like that which was used for the transformation of the Bargmann-Wigner set of equations to the Proca equations (see [15,11] and what is above): the 16- component wave function of two-body Dirac equation can also be expanded in the complete set of matrices: $(\gamma^\mu C)$, $(\sigma^{\mu\nu} C)$ and C, $(\gamma^5 C)$, and $(\gamma^5\gamma^\mu C)$. The wave function is decomposed in symmetric and antisymmetric parts using the above-mentioned complete system of matrices

$$\psi = \psi_{\{\alpha\beta\}} + \psi_{[\alpha\beta]}, \tag{15}$$

where the first term is given by the formula (2) and the second term, by the formula (4) of Ref. [11]. In such a way we obtain the set of equations:[5]

$$\mathcal{E} A_0 = 0, \quad \mathcal{E} \tilde{A}_0 = -2m\tilde{\varphi}\ , \tag{16a}$$
$$\mathcal{E}\varphi = 4i\vec{p}_i^{\,-} F^{i0}\ , \tag{16b}$$
$$\mathcal{E}\tilde{\varphi} = -2m\tilde{A}_0 + 2\epsilon^{ijk}\vec{p}_i^{\,+} F^{jk}\ , \tag{16c}$$
$$\mathcal{E}\tilde{A}^i = 2i\epsilon^{ijk}\vec{p}_j^{\,\mp} A^k\ , \tag{16d}$$
$$\mathcal{E} A^i = 4imF^{0i} + 2i\epsilon^{ijk}\vec{p}_j^{\,\pm}\tilde{A}^k\ , \tag{16e}$$
$$\mathcal{E} F^{0i} = -2imA^i + 2i\vec{p}_i^{\,+}\varphi\ , \tag{16f}$$
$$\mathcal{E} F^{jk} = \epsilon^{ijk}\vec{p}_i^{\,-}\tilde{\varphi}\ . \tag{16g}$$

The signs in the set (16a - 16g) correspond to two types of Dirac oscillator-like interactions, with $\sim (\vec{\alpha}_1 - \vec{\alpha}_2)B$ and $\sim (\vec{\alpha}_1 - \vec{\alpha}_2)B\Gamma_5$, respectively.

The two-body Dirac oscillator equations in the form (16a)-(16g) can be uncoupled into the set containing only functions φ, $\tilde{\varphi}$ and $\tilde{A}_\mu$ and the another one containing only A_μ and $F_{\mu\nu}$:

$$(\mathcal{E}^2 - 8m^2)\varphi = 8(\vec{p}_i^{\,-}\vec{p}_i^{\,+})\varphi - \frac{16im}{\mathcal{E}}\,\epsilon^{ijk}\vec{p}_i^{\,-} p_j^{\pm}\tilde{A}^{\,k}\ , \tag{17a}$$
$$(\mathcal{E}^2 - 4m^2)\tilde{\varphi} = 4(\vec{p}_i^{\,+}\vec{p}_i^{\,-})\tilde{\varphi}\ , \tag{17b}$$
$$\mathcal{E}\tilde{A}_0 = -2m\tilde{\varphi}\ , \tag{17c}$$
$$(\mathcal{E}^2 - 8m^2)\tilde{A}^{\,i} = 4(\vec{p}_j^{\,\mp}\vec{p}_j^{\,\pm})\tilde{A}^{\,i} - 4(\vec{p}_j^{\,\mp}\vec{p}_i^{\,\pm})\tilde{A}^{\,j} - \frac{16im}{\mathcal{E}}\,\epsilon^{ijk}\vec{p}_j^{\,\mp}\vec{p}_k^{\,+}\varphi\ ; \tag{17d}$$

and

$$\mathcal{E} A_0 = 0\ , \tag{18a}$$
$$\mathcal{E}^2 A^i = 4im\mathcal{E} F^{0i} + 4(\vec{p}_j^{\,\pm}\vec{p}_j^{\,\mp})A^i - 4(\vec{p}_j^{\,\pm}\vec{p}_i^{\,\mp})A^j\ , \tag{18b}$$
$$\mathcal{E}^2 F^{0i} = -2im\mathcal{E} A^i - 8(\vec{p}_i^{\,+}\vec{p}_j^{\,-})F^{0j}\ , \tag{18c}$$
$$(\mathcal{E}^2 - 4m^2)F^{jk} = 2\epsilon^{ijk}\epsilon^{lmn}(\vec{p}_i^{\,-}\vec{p}_l^{\,+})F^{mn}\ . \tag{18d}$$

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[5]We corrected here the misprints in the signs of the equations of ref. [8].

This fact proves the Dirac oscillator interaction, like the case when we introduce the (self-) interaction with the transverse 4-vector potential into the Proca equation (or, equivalently, into the Bargmann-Wigner equations), does not mix $S = 1$ and $S = 0$ states.

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