Anomalous electron trapping by magnetic flux tubes and electric current vortices

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We consider an electron with an anomalous magnetic moment, $g>2$, confined to a plane and interacting with a nonhomogeneous magnetic field B, and investigate the corresponding Pauli Hamiltonian. We prove a lower bound on the number of bound states for the case when B is of a compact support and the related flux is $N+\epsilon$, $\epsilon\in(0,1]$. In particular, there are at least $N+1$ bound states if B does not change sign. We also consider the situation where the magnetic field is due to a localized rotationally symmetric electric current vortex in the plane. In this case the flux is zero; there is a pair of bound states for a weak coupling, and higher orbital-momentum "spin-down" states appearing as the current strength increases.

Interaction of electrons with a localized magnetic field has been a subject of interest for a long time. It has been observed recently that a magnetic flux tube can bind particles with spin antiparallel to the field provided the latter have an anomalous magnetic moment, $g>2$. Recall that this is the case for a free electron which has $g=2.0023$. The effect was demonstrated first in simple examples [CFC, Mo], notably those of a circular tube with a homogeneous or a δ-shell field, and then extended to any rotationally invariant field $B(x)$ which is of a compact support and does not change sign [CC].

Our first aim here is to show that the last condition can be substantially weakened and the rotational–invariance requirement can be dropped altogether. We consider the standard two–dimensional Pauli electron Hamiltonian [Th],

$$H_P^{(\pm)}(A) = (-i\nabla - A(x))^2 \pm \frac{g}{2}B(x) = D^*D \pm \frac{1}{2}(g\pm 2)B(x)\,, \tag{1}$$

in natural units, $2m = \hbar = c = e = 1$; here $D := (p_1 - A_1) + i(p_2 - A_2)$ and the two signs correspond to two possible spin orientations. We are free to choose the magnetic flux direction; if it points conventionally up we will be concerned primarily with the operator $H_P^{(-)}(A)$ which describes electron with the spin antiparallel to the flux. The magnetic field $B = \partial_1 A_2 - \partial_2 A_1$ is supposed to be integrable and of a compact support Σ, with

$$F := \frac{1}{2\pi} \int_\Sigma B(x)\, d^2x = N + \epsilon\,, \tag{2}$$

where $\epsilon \in (0,1]$ and N is a non–negative integer. The quantity F, positive by assumption, is the total flux measured in the natural units $(2\pi)^{-1}$.

Recall further that by the theorem of Aharonov and Casher [AC, Th] the operator $H_P^{(-)}(A)$ without an anomalous moment, $g = 2$, has in this situation N zero energy eigenvalues. The corresponding eigenfunctions are given explicitly by

$$\chi_j(x) = e^{-\phi(x)}(x_1 + ix_2)^j, \quad j = 0, 1, \dots, N-1\,, \tag{3}$$

where

$$\phi(x) := \frac{1}{2\pi} \int_\Sigma B(y)\, \ln|x-y|\, d^2y\,. \tag{4}$$

Moreover, χ_N also solves the equation $H_P^{(-)}(A)\chi = 0$ representing a zero–energy resonance; this follows from the fact that $\chi_j(x) = o(|x|^{-F+j})$ as $|x| \to \infty$ — *cf.* [Th, Sec.7.2].

Theorem 1. Under the stated assumptions, the operator $H_P^{(-)}(A)$ has for $g > 2$ at least n_B negative eigenvalues, where n_B is the number of $j = 0, 1, \dots, N$ such that

$$\int_\Sigma B(x)\, e^{-2\phi(x)}\, r^{2j}\, d^2x \geq 0\,, \tag{5}$$

where $r := (x_1^2 + x_2^2)^{1/2}$. In particular, there are at least $n_B = N+1$ bound states if $B(x) \geq 0$.

Sketch of the proof: It is based on a variational argument. We employ the above mentioned zero–energy solutions to construct a family of trial functions ψ which make the quadratic form

$$(\psi, H_P^{(-)}(A)\psi) = \int_{\mathbb{R}^2} |D\psi|^2 d^2x - \frac{1}{2}(g-2) \int_{\mathbb{R}^2} B|\psi|^2 d^2x$$

negative. Specifically, we choose

$$\psi_j(x) := f_R(x)\chi_j(x) + \varepsilon h(x)\,, \tag{6}$$

where $h \in C_0^\infty(\Sigma)$ and $f_R(x) = f\left(\frac{r}{R}\right)$ for a suitable function $f : \mathbb{R}_+ \to \mathbb{R}$ such that $f(u) = 1$ for $u \leq 1$. It is then straightforward to compute the value of the energy form,

$$\begin{aligned}(\psi_j, H_P^{(-)}(A)\psi_j) \;&=\; \frac{1}{R^2}\int_{\mathbb{R}^2}\left|f'\left(\frac{r}{R}\right)\chi_j(x)\right|^2 d^2x \,+\, \varepsilon^2\int_\Sigma |(Dh)(x)|^2 d^2x \\ &-\; \frac{1}{2}(g-2)\Big\{\int_\Sigma B(x)|\chi_j(x)|^2 d^2x \\ &+\; 2\varepsilon\mathrm{Re}\int_\Sigma \bar h(x)B(x)\chi_j(x)\,d^2x \,+\, \varepsilon^2\int_\Sigma B(x)|h(x)|^2 d^2x\Big\}\,,\end{aligned}$$

where we have employed $D\chi_j=0$ and the fact that h and $f'\left(\frac{\cdot}{R}\right)$ have disjoint supports. As we have said, $\chi_j\in L^2$ for $j=0,\dots,N-1$. In this case we put $f=1$ so the first term at the *rhs* is absent. If $\int_\Sigma B|\chi_j|^2d^2x>0$ we may set also $\varepsilon=0$ to get a negative value. If B is non–negative, in particular, we obtain in this way $(\psi_j,H_P^{(-)}(A)\psi_j)<0$ for $j=0,\dots,N-1$.

For a sign–changing B the last integral might not be positive. If it is zero, a bound state still exists: it is always possible to choose h in such a way that $\mathrm{Re}\int_\Sigma \bar hB\chi_j\,d^2x\ne 0$. For small ε the linear term prevails over the quadratic ones and the form can be made negative by choosing properly the sign of ε. Finally, for $j=N$ the Aharonov–Casher solution has to be modified at large distances to produce a square integrable trial function. We choose, *e.g.*, $f\in C_0^\infty(\mathbb{R}_+)$ such that $f(u)=0$ for $u\ge 2$. Using $|\chi_N(x)|=o(r^{-\epsilon})$ we estimate the first term at the *rhs* as

$$\frac{1}{R^2}\int_{\mathbb{R}^2}\left|f'\left(\frac{r}{R}\right)\chi_j(x)\right|^2 d^2x \;\le\; C\|f'\|_\infty^2R^{-2\epsilon}$$

for a positive C. If (5) is valid, one can achieve in the same way as above that the sum of the other terms is negative; it is then sufficient to set R large enough to make the whole *rhs* negative.

We have thus constructed n_B trial functions with the desired property. They are linearly independent, since the same is true for χ_j and the latter coincides with ψ_j in $\mathcal{B}_R\setminus\Sigma$. Consequently, the ψ_j's for which the requirement (5) is satisfied span an n_B–dimensional subspace in $L^2(\mathbb{R}^2)$. $\square$

While the sufficient condition of Theorem 1 improves earlier results, it is still too restrictive. We postpone discussing how to optimize it to a subsequent paper.

The situation becomes more complicated when the total flux is zero. Here we will restrict ourselves to the particular case with a rotational symmetry; then (1) can be replaced by a family of partial wave Hamiltonians

$$H_\ell^{(\pm)} = -\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}+V_\ell^{(\pm)}(r)\,,\quad V_\ell^{(\pm)}(r):=\left(A(r)+\frac{\ell}{r}\right)^2\pm\frac{1}{2}gB(r) \tag{7}$$

on $L^2(\mathbb{R}^+,r\,dr)$. The angular component $A(r)$ of the vector potential is now related to the magnetic field by $B(r)=A'(r)+r^{-1}A(r)$.

A typical situation with a vanishing flux arises when the field is generated by an electric current vortex in the plane. The physical appeal of such a problem stems in part from the fact that local current vortices are common in transport of charged particles [EŠSF]. In the following we shall discuss this example. We assume that the current is anticlockwise, $J(x) = \lambda J(r)e_\varphi$. Here r, φ are the polar coordinates, the total current is $\lambda \int_0^\infty J(r)\,dr$, and the positive parameter λ is introduced to control the vortex "strength".

It is necessary in this case to relax the compact–support requirement on the magnetic field. We suppose that J is C^2 smooth and non–negative, $J(r) \geq 0$, and has the following asymptotic behaviour,

$$J(r) = ar^2 + \mathcal{O}(r^3) \quad \text{and} \quad J(r) = \mathcal{O}(r^{-3-\epsilon}) \tag{8}$$

for some $\epsilon > 0$ at the origin and at large distances, respectively. The corresponding vector potential is easily evaluated [Ja],

$$A(r) \,=\, 4\lambda \int_0^\infty J(r')\,\frac{r'}{r_<}\,\left[K\left(\frac{r_<^2}{r_>^2}\right) - E\left(\frac{r_<^2}{r_>^2}\right)\right]\,dr'\,, \tag{9}$$

where $K,\, E$ are the full elliptic integrals of the first and the second kind, respectively, and the usual shorthands, $r_< := \min(r,r')$ and $r_> := \max(r,r')$ are employed. In view of the regularity of J the integral is finite for every r , because $E(\zeta)$ is regular at $\zeta = 1$ and $K(\zeta)$ has a logarithmic singularity there.

Let us label the Pauli Hamiltonian (1) with the vector potential (9) and its partial–wave components (7) by the current strength λ .

Theorem 2. Under the stated assumptions, $\sigma(H_\ell^{(\pm)}(\lambda)) = [0,\infty)$ for $\ell \neq 0$, while both $H_0^{(\pm)}(\lambda)$ exhibit a bound state if λ is small enough. On the other hand, each operator $H_\ell^{(-)}(\lambda)$ has a negative eigenvalue for a sufficiently large λ .

Sketch of the proof: By the regularity of J , the effective potentials (7) are C^1 smooth and

$$V_\ell^{(\pm)}(r) \,=\, \frac{\ell^2}{r^2} \,+\, \lambda m\,\frac{2\ell \mp g}{2r^3} \,+\, \mathcal{O}(r^{-3-\epsilon})\,, \tag{10}$$

as $r \to \infty$, where $m := \pi \int_0^\infty J(r')\,r'^2 dr'$ is the dipole moment of the current for $\lambda = 1$. Consequently, the essential spectrum is not affected by the magnetic field. We rewrite the potentials into the form

$$V_\ell^{(\pm)}(r) \,=\, \left(\lambda A_1(r) + \frac{\ell}{r}\right)^2 \,\pm\, \frac{\lambda}{2}\,gB_1(r)\,, \tag{11}$$

where the indexed magnetic field refers to the value $\lambda = 1$. Since $H_\ell^{(\pm)}(\lambda)$ is nothing else than the s–wave part of the two–dimensional Schrödinger operator with the centrally symmetric potential (11), it is sufficient to find eigenvalues of the latter. If $\ell \neq 0$, the first term in (11) is below bounded by $\lambda h(r)$ for a suitably chosen positive function h of compact support. Since the second term does not contribute to $\int_0^\infty V_\ell^{(\pm)}(r)\,r\,dr$ which determines the weak–coupling behaviour, the result follows from the standard condition [Si] and the minimax principle.

While the above integral is positive in the case $\ell=0$ as well for any $\lambda\neq 0$, this fact itself need not prevent binding. A more careful Birman–Schwinger analysis up to the second order in λ is required: it shows that a weakly coupled bound state exists if

$$\int_{I\!R^2} A(x)^2\,d^2x \,+\, \frac{g^2}{8\pi}\int_{I\!R^2\times I\!R^2} B(x)\ln|x-x'|\,B(x')\,d^2x\,d^2x' \,<0\,. \tag{12}$$

Evaluating the last integral, we find that the condition is satisfied for $g>2$. This rectifies an incorrect claim made in [BEZ]; a more detailed discussion on that point will be presented in a forthcoming publication. The asymptotic behaviour of the bound state energy (in the sense of [Si]) is

$$\epsilon(\lambda)\,\approx\,-\exp\left\{-\left(\frac{\lambda^2}{8}(g^2-4)\int_{I\!R^2} A(r)^2\,r\,dr\right)^{-1}\right\} \tag{13}$$

for both spin orientations (since $g\neq 2$, the second theorem of [AC] does not apply and the degeneracy may be lifted in the next order).

On the other hand, the existence of the "antiparallel" bound states in a strong vortex follows from the behaviour of the effective potential around the origin. We have

$$A(r)\,=\,\lambda\mu r+\alpha_0(r)\,,\qquad \mu\,:=\,\int_0^\infty J(r')\,\frac{dr'}{r'}\;; \tag{14}$$

using (9) and properties of the elliptic integrals we find $\alpha_0(r)\,=\,\mathcal{O}(r^2)$. This further implies

$$B(r)\,=\,2\lambda\mu+\beta_0(r)\,,\qquad \beta_0(r)\,:=\,\alpha_0'(r)+\frac{1}{r}\alpha_0(r)\,=\,\mathcal{O}(r)\,. \tag{15}$$

Consider the case $\ell=0$. We substitute to (7) from (14,15) and employ the rescaled variable $u:=r\sqrt{\lambda}$. In this way $H_0^{(\ell)}$ is unitarily equivalent to the operator λA_λ, where $A_\lambda=A_0+W_\lambda$ on $L^2(I\!R_+,u\,du)$ with

$$A_0\,:=\,-\frac{d^2}{du^2}\,-\,\frac{1}{u}\,\frac{d}{du}\,-g\mu+\mu^2u^2 \tag{16}$$

and

$$W_\lambda(u)\,:=\,2\sqrt{\lambda}\mu\,u\alpha_0\left(\frac{u}{\sqrt{\lambda}}\right)\,+\,\lambda\,\alpha_0^2\left(\frac{u}{\sqrt{\lambda}}\right)\,-\,\frac{1}{2}\,g\,\beta_0\left(\frac{u}{\sqrt{\lambda}}\right)\,. \tag{17}$$

The limit $\lambda\to\infty$ changes the spectrum substantially; we have $\sigma_{ess}(A_\lambda)\,=\,\sigma_{ess}(\lambda A_\lambda)\,=\,[0,\infty)$ for any $\lambda>0$, while A_0 as the s–wave part of the two–dimensional harmonic oscillator has a purely discrete spectrum. Nevertheless, one can justify the use of the asymptotic perturbation theory for stable (*i.e.*, negative) eigenvalues of A_0; the fact that $W_\lambda \to 0$ pointwise together with the resolvent identity imply $A_\lambda \to A_0$ in the strong resolvent sense as $\lambda \to \infty$ [BEZ]. In that case there is a family of $\nu_n(\lambda) \in \sigma(A_\lambda)$ to any $\nu_n \in \sigma_p(A_0)$ such that $\nu_n(\lambda) \to \nu_n$ [Ka]. The spectrum of A_0 is given explicitly by

$$\nu_n = \mu\,(4n+2-g)\,, \qquad n = 0,1,\dots\,, \tag{18}$$

so ν_0 is stable for $g>2$ and A_λ has a negative eigenvalue for λ large enough. The analogous argument applies to the case $\ell \neq 0$, where the potential in (16) is replaced by $\mu^2u^2 + \ell^2r^{-2} + \mu(2\ell-g)$, and one looks for negative eigenvalues among $\nu_{n,\ell} = \mu\,(4n+2(|\ell|+\ell)+2-g)$. The critical λ at which the eigenvalue emerges from the continuum is naturally ℓ–dependent. $\square$

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