Correlations decay for Markov maps on a countable states space.

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Abstract

We estimate the decay of correlations for some Markov maps on a countable states space. A necessary and sufficient condition is given for the transfer operator to be quasi-compact on the space of locally Lipschitz functions. In the non quasi-compact case, the decay of correlations depends on the contribution to the transfer operator of the complementary of finitely many cylinders. Estimates are given for some non uniformly expanding maps and for birth-and-death processes.

Introduction

Coming from the theory of countable Markov chains, Markovian dynamics on countable states spaces also arise naturally when studying non uniformly expanding systems or non hyperbolic systems (interval maps with neutral fixed points, unimodal maps, Henon maps [Yo], [B,Y]).
Several techniques have been developed to study statistical properties of such Markov systems and especially to estimate their decay of correlations. Most of these techniques are based on the quasi-compactness of the Ruelle-Perron-Frobenius operator (or transfer operator) on a suitable Banach space and lead to exponential decay of correlations ([Bre], [Sa]). When the transfer operator does not have spectral gap, there are estimates of the decay of correlations from C. Liverani, B. Saussol and S. Vaienti ([L, S, V1]), M. Pollicott and M. Yuri ([Po,Y]) and H. Hu ([H]) for some maps with neutral fixed point and L.-S. Young ([Yo]) for towers systems. Young's strategy is very powerful if you are able to estimate the asymptotics of return times on the base of the tower.
We propose another strategy which does not require any *a priori* knowledge on return times. It is based on cones and projective metrics of G. Birkhoff ([Bi1], [Bi2]) and is especially adapted to maps with "small branches" like birth-and-death processes. It is also efficient to estimate the decay of correlations for the well known Gaspard-Wang example of interval map with neutral fixed point.

Section 1 contains the setting, the results, the basic definitions and properties of Birkhoff's cones.
Section 2 is devoted to the exponential decay of correlations. We give a necessary and sufficient condition to ensure that the transfer operator is quasi-compact on the space of locally Lipschitz functions. To this aim, we construct a cone which is strictly contracted by some iterate of the transfer operator.
In section 3 we obtain sub-exponential decay of correlations for a class of maps "without big branches at infinity" (definition page 5). In this case, the decay of correlations depends on the contribution to the transfer operator of the complementary of finitely many cylinders. The estimate is done by truncating the previous cone: $C_{N,j}$ is a cone of locally Lipschitz functions, specified only on finitely many states, it is mapped by some iterate of the transfer operator into $C_{N,j-1}$ with some contraction δ_j. The product of the δ_j gives sub-exponential decay of correlations.

In section 4, we give some explicit estimations. As far as we know, the results concerning birth-and-death like maps (section 4.1) are new. It does not seem possible to associate to these systems a tower that enjoys the properties required by L.-S. Young ([Yo]).

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1 Setting, results and basic properties of cones.

Let Σ be a sub-shift of finite type on a countable alphabet A. That is, Σ is given by $B = (b_{i,j})_{i,j\in A\times A}$ a $A\times A$ matrix of 0 and 1: $\Sigma = \{x \in A^{\mathbb{N}} \ / \ b_{x_i,x_{i+1}} = 1\}$.
On Σ, we consider the shift σ defined by: $\sigma(x_0,\ldots,x_n,\ldots) = (x_1,\ldots)$.
We will always assume that $A = \mathbb{N}$. Since the alphabet is infinite, Σ is, in general, non compact and not even locally compact. The space Σ is endowed with the product topology (on A we consider the discrete topology) which is also given by the natural distance: let $r \in]0,1[$,

$$d(x,y) = r^n \text{ iff } x_i = y_i \ i = 0\ldots n-1 \ x_n \neq y_n.$$

The product topology is generated by the cylinders: let $(a_0,\ldots,a_{k-1}) \in A^k$ and

$$[a_0,\ldots,a_{k-1}] = \{x \in \Sigma \ / \ x_i = a_i, \ i = 0,\ldots,k-1\}.$$

$[a_0,\ldots,a_{k-1}]$ is called *cylinder or k-cylinder.* So the space Σ is separable.
We will say that Σ is *irreducible* if and only if

$$\forall i,j \in A, \ \exists n \in \mathbb{N} \text{ such that } \sigma^{-n}[i] \cap [j] \neq \emptyset,$$

or, equivalently if and only if σ is topologically transitive. We will say that Σ is *aperiodic* if and only if

$$\forall i,j \in A, \ \exists n_0 \in \mathbb{N} \text{ such that } \forall n \geq n_0 \ \sigma^{-n}[i] \cap [j] \neq \emptyset,$$

or, equivalently if and only if σ is topologically mixing.
We will denote by $C_u(\Sigma)$ the Banach space of uniformly continuous and bounded functions on Σ.
We will say that a function f on Σ is *uniformly locally Lipschitz (or u.l.L)* if there exists some constant $C > 0$ such that for any x and y in the same 1-cylinder,

$$|f(x) - f(y)| \leq Cd(x,y),$$

we will denote by $K(f)$ the smallest positive number satisfying this property, it will be called Lipschitz constant of f. Let L be the space of u.l.L. functions that are bounded on Σ. For $f \in L$, let $\|f\| = \max(K(f), \|f\|_\infty)$, this defines a norm on L which turns L into a Banach space.
Let $\mathcal{F}$ be the borelian sigma-algebra on Σ, m be a borelian probability on Σ whose support is Σ and Φ a u.l.L. function. According to thermodynamic formalism we will call Φ a *potential.* We will always assume that Φ satisfies the following assumptions.

Standing assumptions on the potential (SA).

1. Φ is a u.l.L. function,

2. $\sup_{x\in\Sigma}\sum_{\sigma y=x} e^{\Phi(y)} < \infty$, so the transfer operator $\mathcal{L}_\Phi$ associated to Φ is well defined and acts on $C_u(\Sigma)$: for $f \in C_u(\Sigma)$, $x \in \Sigma$

$$\mathcal{L}_\Phi f(x) = \sum_{\sigma y=x} e^{\Phi(y)} f(y),$$

3. the measure m is a conformal measure for $\mathcal{L}_\Phi$, that is, for any $f \in C_u(\Sigma)$,

$$\int \mathcal{L}_\Phi f dm = \int f dm.$$

Remark 1.1 The fact that we assume that the conformal measure is given may seem strange. Indeed, on one hand, under suitable assumptions, such a measure can be constructed and satisfies a variational principle (see [Sa]). On the other hand, one may think that Σ represents the coding of some geometric dynamic for which there are a natural potential and a natural conformal measure.

Using the fact that Φ is u.l.L., an easy computation leads to the following *Bounded Distortion property for $\mathcal{L}_\Phi$*:
there exists $C > 0$ such that for any x and y in the same 1-cylinder, for $n \in \mathbb{N}$,

$$\mathcal{L}_\Phi^n \mathbf{1}(x) \leq \mathcal{L}_\Phi^n \mathbf{1}(y) e^{Cd(x,y)}. \tag{\textbf{BD}}$$

The fact that m is a conformal measure for $\mathcal{L}_\Phi$ implies that if $\mu = hm$ with $h \in C_u(\Sigma)$ then, μ is σ-invariant if and only if h is a fixed point for $\mathcal{L}_\Phi$. It suffices to remark that for f and g in $C_u(\Sigma)$, $m(g\circ\sigma\cdot f) = m(\mathcal{L}_\Phi(g\circ\sigma\cdot f))$ because m is conformal and by definition of $\mathcal{L}_\Phi$, $\mathcal{L}_\Phi(g\circ\sigma\cdot f) = g\mathcal{L}_\Phi f$, so we have $m(g\circ\sigma\cdot f) = m(g\mathcal{L}_\Phi f)$.

We expect that the mixing properties of such a measure are related with the spectral properties of $\mathcal{L}_\Phi$. To be more precise, let us assume that there exists a fixed point $h \in C_u(\Sigma)$ for $\mathcal{L}_\Phi$ which is normalized $(m(h) = 1)$ and let $\mu = hm$. For $f \in C_u(\Sigma)$ and $g \in L^1(m)$, the *correlations* of f and g measure the lack of independence between f and $g \circ \sigma^n$ with respect to the invariant measure μ: for $n \in \mathbb{N}$,

$$c_n(f,g) = \left| \int f(g\circ\sigma^n)d\mu - \int f d\mu \int g d\mu \right|.$$

The measure μ is *mixing* if and only if the coefficients $c_n(f,g)$ go to zero for any $f \in C_u(\Sigma)$ and $g \in L^1(m)$. In this case, estimates on the speed of convergence to zero of $c_n(f,g)$ or equivalently estimates on the *decay of correlations* may lead to the Central Limit Theorem (see [Li3]) and to the determination of asymptotic laws for entrance times (see [G, S] and [Sau]). The following trivial computation relates the decay of correlations to the asymptotic behavior of the iterates of $\mathcal{L}_\Phi$:

$$c_n(f,g) = \left| \int [\mathcal{L}_\Phi^n(fh) - hm(fh)] g dm \right| \tag{1.1}$$

so that if $\mathcal{L}_\Phi^n f \to hm(f)$ in some reasonable way then μ is mixing and estimates on the speed of this convergence would precise the decay of correlations.
Let us state our main results.

1.1 Results.

Under the following additional hypothesis:

$$\exists M > 0 \text{ such that } \forall n \in \mathbb{N}\ \|\mathcal{L}_\Phi^n \mathbf{1}\|_\infty \le M. \tag{\textbf{K}}$$

we have the following result.

Theorem 1.1 *Let Σ be aperiodic, Φ satisfying (**SA**) and (**K**), then there exists a fixed point $h \in L$ for $\mathcal{L}_\Phi$, $h > 0$ on Σ, $m(h) = 1$, this fixed point is unique up to multiplication by a constant. Moreover, we have the following convergence for $f \in C_u(\Sigma)$:*

$$\mathcal{L}_\Phi^n f \overset{n\to\infty}{\longrightarrow} hm(f), \tag{1.2}$$

this convergence is uniform on the compact subsets of Σ and takes place in $L^1(m)$.

The proof of this theorem follows Sarig's proof (theorem 4 in [Sa]) excepted for some details in the construction of h. Let us give the outline of the arguments.
From (**K**) and (**BD**), we deduce that the sequence $(\mathcal{L}_\Phi^n \mathbf{1})_{n\in\mathbb{N}}$ is an equicontinuous and bounded sequence of elements of L. Let

$$Q_n = \frac{1}{n}\sum_{p=0}^{n-1} \mathcal{L}_\Phi^p \mathbf{1}.$$

Ascoli's theorem on separable sets implies that the sequence $(Q_n)_{n\in\mathbb{N}}$ admits an accumulation point for the topology of uniform convergence on compact sets and for the $L^1(m)$ topology by Lebesgue's dominated convergence theorem. Let h be such an accumulation point. Using (**BD**) and (**K**), we get that h belongs to L. Using that m is a conformal measure whose support is Σ, we get that h is a fixed point for $\mathcal{L}_\Phi$ which is non zero because Lebesgue's dominated convergence theorem implies that $m(h) = 1$. Now, if $h(x) = 0$ for some $x \in \Sigma$, since $h \ge 0$, for any $n \in \mathbb{N}$ and any $z \in \{y \ / \ \sigma^n y = x\}$, $h(z) = 0$. Since Σ is irreducible (indeed it is aperiodic), this set is dense in Σ, so that $h \equiv 0$ since it is continuous. This contradicts the fact that h is non zero. So, $h > 0$. The rest of the proof follows Sarig's proof and uses some general arguments on Markov dynamics from [A,D,U]. △

In fact, we have a more precise description of the spectrum of $\mathcal{L}_\Phi$. Such a description is usual in quasi-compactness setting. Our purpose is to give estimates on the decay of correlations so we will omit the proof of the following result which may be found in [Ma]. Let us just remark that (**K**) implies that the spectral radius of $\mathcal{L}_\Phi$ on the space $C_u(\Sigma)$ is less or equal than 1.

Theorem 1.2 *Let Σ be irreducible, Φ satisfies (**SA**) and (**K**) then 1 is a simple eigenvalue for $\mathcal{L}_\Phi$ acting on L, if h is the normalized eigenfunction then h is strictly positive on Σ and the invariant measure $\mu = hm$ is ergodic. Moreover, $\mathcal{L}_\Phi$ has only finitely many eigenvalues of modulus 1, there are all simple. If Σ is aperiodic then 1 is the only eigenvalue of maximal modulus and the invariant measure $\mu = hm$ is mixing. We have the convergence for $f \in C_u(\Sigma)$*

$$\mathcal{L}_\Phi^n f \overset{n\to\infty}{\longrightarrow} \pi(f),$$

uniformly on compact subsets of Σ and in $L^1(m)$, where π is the spectral projection on the finite dimensional space associated to the eigenvalues of modulus 1. In particular, if Σ is aperiodic then

$$\mathcal{L}_\Phi^n f \overset{n\to\infty}{\longrightarrow} hm(f).$$

From now on, we assume that Σ is aperiodic and Φ satisfies (**K**) and (**SA**). The normalized fixed point (given by theorem 1.1) for $\mathcal{L}_\Phi$ will always be denoted by h and μ will be the invariant measure $\mu = hm$. We will give additional conditions under which the speed of convergence in (1.2) can be estimated.
We will say that Φ satisfies (**Exp1**) if

$$\exists k_1,\ \exists n_1 \text{ such that } \forall k > k_1, \exists \rho_k < 1 \text{ such that } \forall n > n_1, \sup_{x\in[n]} \mathcal{L}_\Phi^k \mathbf{1}(x) \le \rho_k. \tag{\textbf{Exp1}}$$

(**Exp1**) means that the cylinders close to infinity do not contribute too much to the transfer operator, in fact their contribution is assumed to be uniformly strictly smaller than one. This condition is sufficient to guaranty exponential decay of correlations for observables in L (see theorem 1.3 below).
The system (Σ, σ) is *without big branches at infinity* if it exists $K \in \mathbb{N}$ such that for any $n \in \mathbb{N}$, for $x \in [n]$, σx belongs to $[p]$ with $p \ge n - K$. In other words, the matrix which defines Σ has the following form:

$$\begin{array}{c} 0 \\ \vdots \\ K \\ \\ \\ \\ \\ \end{array}
\begin{pmatrix}
* & \cdots & \cdots & \cdots & \cdots \\
\vdots & & & & \\
* & & & & \\
0 & * & & & \\
\vdots & \ddots & \ddots & & \\
0 & & 0 & * & \\
\vdots & & & \ddots & \ddots
\end{pmatrix}$$

with $* \in \{0, 1\}$. If n_1 and $N \ge n_1$ are fixed integers, let us note

$$\delta'_{k,j} := \sup\{\mathcal{L}_\Phi^k \mathbf{1}(x) \ /\ x \in [n],\ N \le n \le N + kKj\},\ j \ge 0,\ k \in \mathbb{N}.$$

We will say that Φ satisfies (**S-Exp1**) if there exists $n_1 \in \mathbb{N}$ such that for $N \ge n_1$, there exists $k_1(N)$ such that for $k \ge k_1$, there exists $R(k)$, $0 < R(k) < N + kK$ and

$$\delta'_{k,j} \le \left(1 - \frac{R(k)}{N + Kkj}\right)^\alpha, \alpha > 0,\ \forall j \ge 0. \tag{\textbf{S-Exp1}}$$

(**S-Exp1**) means that the contribution to the transfer operator of the cylinders close to infinity is strictly smaller than one but not uniformly. Under this condition and the assumption that Σ has no big branches at infinity, we can estimate the decay of correlation for observables in L (see theorem 1.4 below).
Before stating our main results, let us remark that:

$$(\textbf{Exp1}) \Rightarrow (\textbf{S-Exp1}) \Rightarrow (\textbf{K}).$$

The first implication is trivial: take $\delta'_{k,j} = \rho_k$ for all $j \in \mathbb{N}$. Let us prove the second implication. Let $N \ge n_1$ be fixed and let $k_1 = k_1(N)$, (**S-Exp1**) implies that for $n \ge N$, $k \ge k_1$,

$$\sup_{x\in[n]} \mathcal{L}_\Phi^k \mathbf{1}(x) \le 1,$$

moreover, since we always assume that $\|\mathcal{L}_\Phi \mathbf{1}\|_\infty < \infty$ (**SA**), we have

$$\sup_{k<k_1} \|\mathcal{L}_\Phi^k \mathbf{1}\|_\infty < \infty,$$

finally, let n be smaller than N, the Bounded Distortion property implies

$$\begin{aligned}
\sup_{x\in[n]} \mathcal{L}_\Phi^k \mathbf{1}(x) &\leq \mathrm{Ct}\ \frac{1}{m([n])}\int_{[n]} \mathcal{L}_\Phi^k \mathbf{1} dm \\
&\leq \mathrm{Ct}\ \frac{1}{m([n])}\int \mathcal{L}_\Phi^k \mathbf{1} dm \\
&\leq \mathrm{Ct}\ \sup_{n\leq N} \frac{1}{m([n])} \quad \text{using the fact that } m \text{ is conformal.}
\end{aligned}$$

So we have proven that there exists some $M > 0$ such that for any $n \in \mathbb{N}$, $\|\mathcal{L}_\Phi^k \mathbf{1}\|_\infty \leq M$, which is (**K**). △

Finally, let us recall that two potential Ψ_1 and Ψ_2 are *cohomologous* if there exists a positive function $v \in C_u(\Sigma)$ such that $\Psi_1 = \Psi_2 + v - v \circ \sigma$. The function v is called *change of potential.*

Theorem 1.3 *Let Σ be aperiodic. If Φ satisfies (***SA***) and (***Exp1***) then $\mathcal{L}_\Phi$ is quasi-compact on L so we have the following exponential decay of correlations: there exist $0 < \gamma < 1$ and $C > 0$ such that for $f \in L$, $g \in L^1(m)$,*

$$c_n(f,g) \leq C\ \gamma^n \|f\|\ \|g\|_{L^1}. \tag{1.3}$$

*Moreover, (***Exp1***) is a necessary condition for quasi-compactness in the following sense:*
*let Φ verify (***K***) and (***SA***) then $\mathcal{L}_\Phi$ is quasi-compact on L if and only if there exists Ψ cohomologous to Φ with a change of potential in L and bounded away from zero such that $\mathcal{L}_\Psi$ satisfies (***Exp1***).*

Let us recall that a linear operator P on a Banach space B is *quasi-compact* if there exists $0 < \Theta < 1$ such that if λ belong to the spectrum of P and $|\lambda| > s(P)\Theta$ where $s(P)$ is the spectral radius of P then λ is an eigenvalue with finite multiplicity.
The fact that (**Exp1**) is a necessary condition for quasi-compactness in the sense of theorem 1.3 is easy to see. Indeed, let us assume that $\mathcal{L}_\Phi$ is quasi-compact on L, because of theorem 1.1, we have for $f \in L$, $k \in \mathbb{N}$,

$$\|\mathcal{L}_\Phi^k f - hm(f)\| \leq C\ \gamma^k \|f\|, \tag{1.4}$$

with $C > 0$ and $0 < \gamma < 1$. Let Ψ be cohomologous to Φ with change of potential v in L and bounded away from zero. We have:

$$\mathcal{L}_\Psi^k \mathbf{1} = \frac{1}{v}\mathcal{L}_\Phi^k(v)$$

so, using (1.4) we get:

$$\begin{aligned}
\mathcal{L}_\Psi^k \mathbf{1} &= \frac{1}{v}[\mathcal{L}_\Phi^k(v) - hm(v)] + \frac{h}{v}m(v) \\
\mathcal{L}_\Psi^k \mathbf{1}(x) &\leq C\frac{\|v\|}{\inf v}\gamma^k + \frac{h}{v}(x)m(v).
\end{aligned}$$

It is always possible to find $v \in L$ (indeed, v can be chosen to be constant on 1-cylinders) such that $m(v) = 1$, $\inf v > 0$ and

$$\sup_{n\geq N}\sup_{x\in[n]} \frac{h}{v}(x) < 1$$

provided N is big enough. This conclude the proof of the necessity part of theorem 1.3. △
The rest of theorem 1.3 will be proven in section 2. This won't be done using the standard

approach consisting in the application of the Ionescu-Marinescu Tulcea theorem ([IT,M]) but we will use Birkhoff cones and projective metrics. The main advantages of this technic are: first it provides a constructive bound for the rate of convergence, second the cones are adaptable to non quasi-compact cases. Indeed in section 3 we will prove the following result.

Theorem 1.4 *Let Σ be aperiodic and without big branches at infinity, Φ satisfies (**SA**) and (**S-Exp1**). For any compact set Q of Σ, there exists a sequence $u_n(Q)$ which goes to zero when n goes to infinity such that for any $f \in L$,*

$$\|\mathcal{L}_\Phi^n f - hm(f)\|_Q \leq C(Q)\ u_n(Q)\|f\|$$

where $\|\ \|_Q$ denotes the uniform norm on Q and $C(Q)$ is a positive number depending on Q. There exists a sequence u_n which goes to zero when n goes to infinity such that for $f \in L$ and $g \in L^\infty(m)$,

$$C_n(f,g) \leq u_n\|f\|\ \|g\|_\infty.$$

Moreover, for fixed Q,

$$\frac{u_n}{u_n(Q)} < K\ \forall n \in \mathbb{N}$$

and the sequences $u_n(Q)$ and u_n depend on the product of the $\delta'_{k,j}$ and on the measure of cylinders close to infinity.

In section 4, we give large classes of examples satisfying (**Exp1**) or (**S-Exp1**) and compute explicit bounds for some birth-and-death like dynamics and non uniformly expanding maps.
We will now recall definitions and results on cones and projective metrics.

1.2 Cones and projective metrics.

The theory of cones and projective metrics of G. Birkhoff [Bi1] is a powerful tool to study linear operators. P. Ferrero and B. Schmitt [F,S 1] applied it to estimate the correlations decay for random dynamical systems. Then, this strategy had been used by many authors. Let us mention
• C. Liverani [Li1], C. Liverani, B. Saussol and S. Vaienti [L, S, V1] for one dimensional Lasota-Yorke type dynamics with finite or countable partition,
• C. Liverani [Li2] and M. Viana [V] for Anosov and Axiom A diffeomorphisms,
• V. Baladi, A. Kondah, B. Schmitt [B,K,S], T. Bogenschütz ([Bog]) and J. Buzzi ([Buz]) for random dynamical systems.
They all used Birkhoff cones to obtain exponential decay of correlations. In [K,M,S] the Birkhoff cones techniques were used in a different way to obtain sub-exponential decay of correlations. The way we use cone's techniques in section 3 follows some ideas of P. Ferrero and B. Schmitt ([F,S 2]).

Let us recall definitions and properties of cones and projective metrics (see [Li2] or [L, S, V1] for a more complete presentation).

Let B be a vector space and $C \subset B$ a cone with the following properties.

- C is convex.
- $C \cap -C = \emptyset$.
- if α_n is a sequence of real numbers such that $\alpha_n \to \alpha$ and $x - \alpha_n y \in C\ \forall n$ then $x - \alpha y \in C$. This property is called "integral closure".

For such a cone, the pseudo-metric θ_C on C is defined in the following way. Let $x, y \in C$,

$$\mu(x,y) = \inf\{\beta > 0 \text{ such that } \beta x - y \in C\},$$

$$\lambda(x,y) = \sup\{\alpha > 0 \text{ such that } y - \alpha x \in C\},$$

with the convention: $\mu(x,y) = \infty$ and $\lambda(x,y) = 0$ if the corresponding sets are empty. Let $\theta_C(x,y) = \log \frac{\mu}{\lambda}$. θ_C is called pseudo-metric because it is not necessarily finite. Moreover, it is a projective pseudo-metric: if x and x_1 are proportional then for any $y \in C$, $\theta_C(x,y) = \theta_C(x_1,y)$.

The following two results reveal the usefulness of projective metrics.
Let C and C' be two cones, P a linear operator $P : C \to C'$. Let Δ denotes the diameter of PC in C':

$$\Delta = \sup_{f,g \in C} \theta_{C'}(Pf, Pg).$$

Theorem 1.5 [Bi1] *For any f, g in C, we have:*

$$\theta_{C'}(Pf, Pg) \leq \tanh\left(\frac{\Delta}{4}\right)\theta_C(f,g).$$

This theorem implies that $P : C \to C'$ is always a contraction (in wide sense) for the projective metrics. If $\Delta < \infty$ then it is a strict contraction.
The following result relies the metric θ_C to certain norms on B. A norm $\|\ \|$ on B is a *norm adapted to* C if for f and g in B such that $f + g$ belongs to C and $f - g$ belongs to C then $\|g\| \leq \|f\|$. ρ is a *homogeneous form adapted to* C if ρ maps C to $\mathbb{R}^+$, for any $\lambda > 0$ and $f \in C$, $\rho(\lambda f) = \lambda \rho(f)$ and if $f - g \in C$ implies $\rho(g) \leq \rho(f)$.

Theorem 1.6 [Bi1], [L, S, V1] *Let C be a cone, let $\|\ \|$ and ρ be adapted to C. For any f and g in C such that $\rho(f) = \rho(g) \neq 0$ we have:*

$$\|f - g\| \leq (e^{\theta(f,g)} - 1)\min(\|f\|, \|g\|).$$

2 Quasi-compact case (proof of theorem 1.3)

In this section, we prove the first part of theorem 1.3, so we assume that Σ is aperiodic, Φ satisfies (**SA**) and (**Exp1**). As we already mentioned, (**Exp1**) implies (**K**). Let us note $\mu = hm$ where h is the normalized fixed point for $\mathcal{L}_\Phi$ given by theorem 1.1. Since Φ satisfies (**K**), h is bounded by $M = \sup_k \|\mathcal{L}_\Phi^k \mathbf{1}\|_\infty$, so for any $A \in \mathcal{F}$, $\frac{\mu(A)}{m(A)} \leq M$.
For $f \in C_u(\Sigma)$, the iterates of $\mathcal{L}_\Phi(f)$ are converging to $hm(f)$ in $L^1(m)$ (by theorem 1.1), this implies the following mixing property which will be used to obtain decay of correlations for observables in L.

$$\forall A \in \mathcal{F},\ g = \mathbf{1}_A,\ \forall f \in C_u(\Sigma),\ |m(g \circ \sigma^n f) - \mu(g)m(f)| \overset{n\to\infty}{\longrightarrow} 0. \tag{2.1}$$

Indeed, we have,

$$\begin{aligned} |m(g \circ \sigma^n \cdot f) - \mu(g)m(f)| &= \left| \int g[\mathcal{L}_\Phi^n f - h\int f dm] dm \right| \\ &\leq \|\mathcal{L}_\Phi^n f - h \int f dm\|_1. \end{aligned}$$

Let us set some notations.
For s and t fixed integers, we will denote by $\mathcal{P}_{s,t}$ the finite partition of Σ defined by:

- $\mathcal{P}_{s,t} = \mathcal{P}_1 \cup \mathcal{P}_2$
- $\mathcal{P}_1$ is the partition in s-cylinders of the set $[\![0,t]\!] := \{x \in \Sigma \ / \ x_0 \leq t\}$.
- $\mathcal{P}_2 = \{[\![0,t]\!]^c\} := \{P_2\}$.

We will denote by D_1 the diameter for the distance d of $\mathcal{P}_1$:

$$D_1 = \max\{\mathrm{diam}(P), P \in \mathcal{P}_1\} = r^s,$$

and by D_2 the measure of $\mathcal{P}_2$:

$$D_2 = m([\![0,t]\!]^c).$$

Since the measure m is finite, D_2 can be chosen as small as we want provided t is large enough.
We will also use the following conventions:

- $a = b \pm c$ means that $b - c \leq a \leq b + c$, where a, b and c are real numbers.
- If x and y belong to the same 1-cylinder, for any $k \in \mathbb{N}$ their preimages under σ^k are in bijection. If x' is a preimage of x, we will denote by y' the preimage of y belonging to the same k-cylinder.
- for any $k \in \mathbb{N}$, let g_k be defined by:

$$\forall x \in \Sigma, \ \forall f \in C_u(\Sigma), \ \mathcal{L}_\Phi^k f(x) = \sum_{\sigma^k x' = x} f(x') g_k(x').$$

Finally, let us remark that for α and α' such that $\alpha < 1 < \alpha'$, since the partition $\mathcal{P}_{s,t}$ is finite, the mixing (2.1) implies that there exists an integer k_0 such that:

$$\forall k \geq k_0, \ \forall P, P' \in \mathcal{P}_{s,t}, \ \alpha \leq \frac{m(\sigma^{-k} P \cap P')}{\mu(P) m(P')} \leq \alpha'. \tag{2.2}$$

2.1 A family of cones.

Let us begin the proof of the first part of theorem 1.3. To this aim, we will construct a cone C and an integer k such that $\mathcal{L}_\Phi^k C \subset C$ and the projective diameter of $\mathcal{L}_\Phi^k C$ in C is finite.
Let us consider the following family of cones. For given real positive numbers a, b, c and integers s, t, $C_{a,b,c}^{s,t}$ is the set of functions f in L which satisfy:

- $\forall P \in \mathcal{P}_{s,t}, 0 < \dfrac{1}{\mu(P)} \displaystyle\int_P f dm \leq a \int f dm,$
- $K(f) \leq b \displaystyle\int f dm,$
- $\displaystyle\sup_{x \in P_2} |f(x)| \leq c \int f dm.$

When there won't be any ambiguity, we will simply note C instead of $C_{a,b,c}^{s,t}$. The following properties follow straightforward from the definition of C :

- $C \cap -C = \emptyset$,

- C is a convex cone,
- C is closed for the uniform topology, in particular, it is integrally closed.

Moreover, the following result is easily verified.

Lemma 2.1 *Any $\varphi \in L$ satisfies:*

$$\forall P \in \mathcal{P}_1,\ \forall x \in P,\ \varphi(x) = \frac{1}{m(P)} \int_P \varphi dm \pm K(\varphi) D_1, \tag{2.3}$$

and for $x \in P_2$,

$$\varphi(x) = \frac{1}{m(P_2)} \int_{P_2} \varphi dm \pm 2 \sup_{P_2} |\varphi|. \tag{2.4}$$

So that if φ belongs to C, for any $x \in \Sigma$,

$$\min[-c, Ma - bD_1] \int \varphi dm \leq \varphi(x) \leq \max[c, Ma + bD_1] \int \varphi dm. \tag{2.5}$$

In order to use the cone $C_{a,b,c}^{s,t}$ and its projective metric, we shall need an adapted norm and an adapted homogeneous form. Of course $\rho(f) = \int f dm$ is an adapted homogeneous form. For any $d > 0$, let us consider the norm

$$\|f\|_d = \max\left(d \mid \int f dm|, 2 \left| \frac{\int_P f dm}{m(P)} \right| \ P \in \mathcal{P}_{s,t}, \|f\|_\infty, K(f) \right),$$

the norm $\| \ \|_d$ is equivalent to the norm $\| \ \|$ of L.

Lemma 2.2 *If $d \geq \max(b, c, 2D_1 b)$ then the norm $\| \ \|_d$ is adapted to C in the sense of section 1.2.*

Proof: Let f and g be such that $f+g$ and $f-g$ belong to C, (2.3) gives for $x \in P$, $P \in \mathcal{P}_1$,

$$\begin{aligned} f(x) - g(x) &= \frac{1}{m(P)} \int (f-g) dm \pm bD_1 \int_P (f-g) dm, \\ f(x) + g(x) &= \frac{1}{m(P)} \int_P (f+g) dm \pm bD_1 \int (f+g) dm. \end{aligned}$$

By substracting these inequations, we obtain,

$$|g(x)| \leq 2 \max(1/m(P) | \int_P g dm|, bD_1 \int f dm).$$

Since $f - g \in C$ and $f + g \in C$, we have $|\int_P g dm| \leq \int_P f dm$ for any $P \in \mathcal{P}_{s,t}$, so that for any $x \in \mathcal{P}_1$, $|g(x)| \leq \|f\|_d$ if $d \geq 2bD_1$. Moreover, if $x \in P_2$, the inequality $|g(x)| \leq c \int f dm$ follows from (2.4) in the same way. We also have $|\int g dm| \leq \int f dm$. It remains to take care of the part of the norm which is given by the Lipschitz constant. Since $f-g \in C$ and $f+g \in C$, we have for x and y in the same 1-cylinder,

$$\begin{aligned} f(x)-g(x)-(f(y)-g(y)) &= \pm b\ d(x,y) \int (f-g)dm, \\ f(x)+g(x)-(f(y)+g(y)) &= \pm b\ d(x,y) \int (f+g)dm \end{aligned}$$

By substracting these two inequalities, we get

$$K(g) \leq b \int f dm.$$

Finally, if $d \geq \max(b, c, 2bD_1)$, we have $\|g\|_d \leq \|f\|_d$. $\triangle$

2.2 Contraction of the cone.

We are going to prove that the cone $C^{\prime s,t}_{a,b,c}$ is strictly contracted by $\mathcal{L}^k_\Phi$ provided k, a, b, c, s and t are large enough.

Lemma 2.3 *There exist $k \in \mathbb{N}^*$, $a > 0$, $b > 0$, $c > 0$, $s \in \mathbb{N}^*$, $t \in \mathbb{N}^*$ and $0 < \gamma < 1$ such that*

$$\mathcal{L}^k_\Phi C^{\prime s,t}_{a,b,c} \subseteq C^{\prime s,t}_{\gamma a,\gamma b,\gamma c}.$$

Proof: Let us fix $\alpha < 1 < \alpha'$ and $\beta' < \beta < 1$. The parameters a, b, c, s and t are chosen to verify:

1. $a > \alpha'(1+\frac{\alpha}{2\alpha'})\beta^{-1}$,
2. $c \geq Ma + 1$,
3. $b > \frac{Rc}{\beta-\beta'}$,
4. $D_1 < 1/b\frac{\alpha}{4\alpha'}$,
5. t is such that $t \geq n_1$ and $D_2 < \frac{\alpha}{8\alpha' c}$.

Moreover, let k_0 satisfies (**Exp1**), be such that (2.2) is verified for α and α' and $\forall k > k_0$, $Mr^k < \beta'$. Let us fix $k > k_0$, $\gamma = \max(\beta, \rho_k) < 1$ and f in $C^{\prime s,t}_{a,b,c}$. Let $P \in \mathcal{P}$,

$$\begin{aligned} \frac{1}{\mu(P)} \int_P \mathcal{L}^k_\Phi f dm &= \frac{1}{\mu(P)} \int_{\sigma^{-k}P} f dm \\ &= \frac{1}{\mu(P)} \sum_{P' \in \mathcal{P}_{s,t}} \int_{\sigma^{-k}P \cap P'} f dm, \end{aligned}$$

using (2.3) and (2.4),
$$\frac{1}{\mu(P)} \int_P \mathcal{L}^k_\Phi f dm = \sum_{P' \in \mathcal{P}_{s,t}} \frac{m(\sigma^{-k}P \cap P')}{m(P')\mu(P)} \int_{P'} f dm \pm$$

$$\left(D_1 K(f) \sum_{P' \in \mathcal{P}_1} \frac{m(\sigma^{-k} P \cap P')}{m(P')\mu(P)} m(P') \; + \; 2 \frac{m(\sigma^{-k} P \cap P_2)}{m(P_2)\mu(P)} m(P_2) \sup_{P_2} |f| \right).$$

Since f belongs to $C'^{s,t}_{a,b,c}$, this leads to (using (2.2)):

$$\begin{aligned} \frac{1}{\mu(P)} \int_P \mathcal{L}_\Phi^k f dm &\leq \alpha' \int f dm \, [1 + D_1 \, b + 2D_2 \, c] \text{ and} \\ \frac{1}{\mu(P)} \int_P \mathcal{L}_\Phi^k f dm &\geq [\alpha - \alpha'(D_1 b + 2D_2 c)] \int f dm, \end{aligned}$$

by the choices (4. and 5.) of a, b, c and $\mathcal{P}_{s,t}$ we obtain,

$$\alpha/2 \int f dm \leq \frac{1}{\mu(P)} \int_P \mathcal{L}_\Phi^k f dm \leq \alpha'(1 + \alpha/(2\alpha')) \int f dm. \tag{2.6}$$

For any $\varphi \in L$ and $k \in \mathbb{N}$, the fact that Φ verifies (**BD**) and (**K**) leads to the following standard inequality

$$K(\mathcal{L}_\Phi^k \varphi) \leq M r^k K(\varphi) + R \sup |\varphi|, \tag{2.7}$$

Combined with (2.5) and the definition of the cone, this gives for $f \in C'^{s,t}_{a,b,c}$

$$K(\mathcal{L}_\Phi^k f) \leq \int f dm \left[b M r^k \; + \; R \max(c, Ma + bD_1) \right]$$

and with the choices 2. and 4. above:

$$K(\mathcal{L}_\Phi^k f) \leq \beta' b \int f dm + cR \int f dm, \tag{2.8}$$

so, $K(\mathcal{L}_\Phi^k f) \leq \beta b \int f dm$ if $b > \frac{cR}{\beta - \beta'}$ (which is 3.).
Finally, for $x \in P_2$,

$$\begin{aligned} |\mathcal{L}_\Phi^k f(x)| &= \left| \sum_{x' \in P_2} g_k(x') f(x') + \sum_{x' \notin P_2} g_k(x') f(x') \right| \\ &\leq c \int f dm \sum_{x' \in P_2} g_k(x') + (Ma + bD_1) \int f dm \sum_{x' \notin P_2} g_k(x'), \text{ using (2.3)} \\ &\leq \max(c, Ma + bD_1) \int f dm \, [\sup_{x \in P_2} \mathcal{L}_\Phi^k \mathbf{1}(x)], \\ &\leq \rho_k c \int f, \text{ by the choices 2., 4. and 5. above and hypothesis } (\mathbf{Exp1}), \end{aligned}$$

this conclude the proof of the lemma. △

2.3 Computation of the projective diameter and conclusion.

In order to obtain the speed of convergence of the iterates of the transfer operator to the spectral projection, it remains to estimate the projective diameter of $\mathcal{L}_\Phi^k(C)$ in C.

Lemma 2.4 *The projective diameter of $\mathcal{L}_\Phi^k C$ in C is bounded by $2\log\max\left(\frac{1+\gamma}{1-\gamma}, \frac{2\alpha'+\alpha}{\alpha}\right)$.*

Proof: Let f and g be in $\mathcal{L}_\Phi^k C$ and $\eta > 0$ such that $\eta f - g$ belongs to C, η should verify:

1. $\forall P \in \mathcal{P}_{s,t}$, $0 < \dfrac{\eta}{\mu(P)} \displaystyle\int_P f dm - \dfrac{1}{\mu(P)} \int_P g dm \leq a\eta \int f dm - a \int g dm$,

2. for any x and y in the same 1-cylinder,
$$\begin{aligned} -b\eta \int f dm + b \int g dm &\leq \frac{\eta(f(x) - f(y)) - (g(x) - g(y))}{d(x,y)} \\ &\leq b\eta \int f dm - b \int g dm, \end{aligned}$$

3. for any $x \in P_2$,
$$-c(\eta \int f dm - \int g dm) \leq \eta f(x) - g(x) \leq c(\eta \int f dm - \int g dm).$$

To have 1., η should verify:

$$\eta \geq \sup_{P \in \mathcal{P}_{s,t}} \frac{a \displaystyle\int g dm - 1/\mu(P) \int_P g dm}{a \displaystyle\int f dm - 1/\mu(P) \int_P f dm}, \text{ and } \eta \geq \sup_{P \in \mathcal{P}_{s,t}} \frac{\displaystyle\int_P g dm}{\displaystyle\int_P f dm},$$

By (2.6), it is sufficient to have:

$$\eta \geq \frac{\displaystyle\int g dm}{\displaystyle\int f dm} \max\left(\frac{1}{1-\gamma}, \frac{2\alpha' + \alpha}{\alpha}\right).$$

To have 2. and 3., by lemma 2.3, it suffices that η satisfies:

$$\eta \geq \frac{\displaystyle\int g dm}{\displaystyle\int f dm} \frac{1+\gamma}{1-\gamma}.$$

Similarly, let $\zeta > 0$ be such that $g - \zeta f \in C$. It suffices that ζ verifies:

$$\zeta \leq \frac{\displaystyle\int g dm}{\displaystyle\int f dm} \min\left[1-\gamma, \frac{1-\gamma}{1+\gamma}, \frac{\alpha}{2\alpha' + \alpha}\right].$$

So, the diameter Δ of $\mathcal{L}_\Phi^k C$ in C is bounded by $2\log\max\left(\frac{1+\gamma}{1-\gamma}, \frac{2\alpha'+\alpha}{\alpha}\right)$. $\triangle$

The following lemma shows that any function in L can be pushed into the cone C.

Lemma 2.5 *For any $f \in L$, if $a > 1$, $b > K(h)$ and $c > \sup_{x \in P_2} |h(x)|$ then there exists $C(f) \geq 0$ such that $f + C(f)h$ belong to C, moreover $C(f) \leq Ct\, \|f\|$. In particular, h belongs to C*

Proof: Let $f \in L$, $C(f)$ should satisfies

- $C(f) \geq \max\left(-\frac{\int_P f dm}{\mu(P)}, \frac{1/\mu(P)\int_P f d\mu - a\int f d\mu}{a-1} \; P \in P_1\right),$
- $C(f) \geq \frac{K(f) - b\int f}{b - K(h)},$
- $C(f) \geq \sup_{x \in P_2} \frac{|f(x)| - c\int f dm}{c - |h(x)|}.$

$\triangle$

We will always assume that a, b and c satisfy the hypothesis of lemma 2.5.
Let $\kappa = (\tanh(\Delta/4))^{1/k}$. Using the fact that h belongs to C, the results of section 1.2 give for $d \geq \max(b, 2bD_1, c)$:

$$\forall f \in C, \; \|\mathcal{L}_\Phi^n f - h\int f dm\|_d \leq \text{Ct } \kappa^n \int f dm$$

since the norms $\| \; \|$ and $\| \; \|_d$ are equivalent,

$$\forall f \in C, \; \|\mathcal{L}_\Phi^n f - h\int f dm\| \leq \text{Ct } \kappa^n \int f dm$$

Let $f \in L$ and $f_C = f + C(f)h$.

$$\begin{aligned} \|\mathcal{L}_\Phi^n f - h\int f dm\| &\leq \|\mathcal{L}_\Phi^n f_C - h\int f_C dm\| + C(f)\|\mathcal{L}_\Phi^n h - h\int h dm\| \\ &= \|\mathcal{L}_\Phi^n f_C - h\int f_C dm\| \\ &\leq \text{Ct } \kappa^n \int f_C dm \text{ since } f_C \in C \\ &\leq \text{Ct } \kappa^n \|f\|. \end{aligned} \tag{2.9}$$

(2.9) implies exponential mixing for $g \in L^1(m)$ and f such that fh belong to L (recall (1.1)):

$$|\mu(g \circ \sigma^n f) - \mu(g)\mu(f)| \leq \text{Ct } \kappa^n \|fh\| \|g\|_{L^1(m)},$$

it also implies that $\mathcal{L}_\Phi$ is quasi-compact on L. This concludes the proof of theorem 1.3. $\triangle$

3 Dynamics without big branches at infinity

In this section, we prove theorem 1.4. So we don't assume (**Exp1**) anymore but we assume that Σ has no big branches at infinity and that Φ satisfies (**SA**) and (**S-Exp1**) (definition page 5). Let us recall that (**S-Exp1**) implies (**K**). So, let us note $\mu = hm$ where h is the normalized fixed point for $\mathcal{L}_\Phi$ given by theorem 1.1.

Remark 3.1 Moreover, we will assume that $\lim_{j\to\infty} \delta'_{k,j} = 1$. Indeed, if $\limsup_{j\to\infty} \delta'_{k,j} < 1$, then Φ satisfies (**Exp1**) and the results of the preceding section show that the convergence of the iterates of $\mathcal{L}_\Phi$ to the spectral projection is uniform on Σ and exponential.

For $N \in \mathbb{N}$, $\| \, \|_N$ is the uniform norm on $[\![0, N]\!]$ (notation page 9). The cones of the preceding section may be adapted to give the following result.

Proposition 3.1 *If Σ has no big branches at infinity and Φ verifies (**S-Exp1**) then, $\forall N \geq n_1, \forall f \in L$, $\forall k \geq k_1$, there exists a sequence $(\alpha_j(N))_{j\in\mathbb{N}}$, $\alpha_j(N) \to 0$ such that*

$$\|\mathcal{L}_\Phi^{kj} f - h \int f\|_N \leq \alpha_j \; \|f\| + m([\![0, N]\!]^c) \sup f, \tag{3.1}$$

Moreover, α_j can be expressed in terms of $\prod_{\ell=0}^{j} \delta'_{k,\ell}$. A suitable choice of N with respect to j gives theorem 1.4.

The proof of this proposition follows theorem 1.3's proof. The point is that for k and n large enough, $\mathcal{L}_\Phi^k \mathbf{1}(x)$, $x \in [n]$, is strictly smaller than 1 but this bound is not uniform in n. This is why we shall use a family of cones specified only on the set $[\![0, N{+}kKj]\!]$, that is away from infinity.

For fixed $N \geq n_1$ and $k \geq k_1$, let $D(j)$ denotes the set $[\![0, N + kKj]\!]$ and for $f \in L$, $K_j(f)$ denotes the Lipschitz constant of the function $f : D(j) \to \mathbb{R}$.
Let us consider the following family of cones. Let a, b, c be positive real numbers, $j, s, t \in \mathbb{N}$ and $\mathcal{P}_{s,t}$ the finite partition of Σ defined page 9, $C_N^j(a, b, c)$ is the set of functions f of L such that:

1. $\forall P \in \mathcal{P}_{s,t}, P \subset D(j), 0 < \dfrac{1}{\mu(P)} \displaystyle\int_P f dm \leq a \int_{D(j)} f dm,$
2. $K_j(f) \leq b \displaystyle\int_{D(j)} f dm,$
3. $\displaystyle\sup_{x\in P_2\cap D(j)} |f(x)| \leq c \int_{D(j)} f dm.$

The arguments of the proof of lemma 2.2 prove that the norm

$$\|f\|_{d,j} = \max(d \, |\int_{D(j)} f dm|, 2 \left| \frac{\int_P f dm}{m(P)} \right| \; P \in \mathcal{P}_{s,t} \cap D(j), \|f\|_{N+kKj}) \tag{3.2}$$

is adapted to the cone $C_N^j(a, b, c)$ provided $d \geq \max(2bD_1, c)$. Let us remark that for any $d > 0$, the norm $\| \, \|_{d,j}$ is equivalent to the uniform norm on $D(j)$. Moreover, $C_N^j(a, b, c)$ is a convex cone which is closed for the norm $\| \, \|_{D(j)}$ and $C_N^j(a, b, c) \cap -C_N^j(a, b, c) = \emptyset$. Of course, $f \to \displaystyle\int_{D(j)} f dm$ is also adapted to $C_N^j(a, b, c)$. When there won't be any ambiguity, we will simply write C_N^j instead of $C_N^j(a, b, c)$.

Outline of the proof of proposition 3.1 *We will prove that provided a, b, c, s, t, N and k are large enough and well chosen, for all $j \in \mathbb{N}$,*

$$\mathcal{L}_\Phi^k C_N^j(a,b,c) \subset C_N^{j-1}(\gamma a, \gamma b, \delta_j c) \subset C_N^{j-1}(a,b,c)$$

where $0 < \gamma < 1$, δ_j satisfies for any $\ell \in \mathbb{N}$, $\prod_j^\ell \delta_j \le Ct \prod_j^\ell \delta'_{k,j}$ and $\tanh \frac{\Delta_j}{4} \le \delta_j$ where Δ_j is the diameter of $\mathcal{L}_\Phi^k C_N^j(a,b,c)$ in $C_N^{j-1}(a,b,c)$.
Now, let $\mathcal{C} = \bigcap_j C_N^j(a,b,c)$. Provided a, b and c are large enough, h belong to $\mathcal{C}$. Then, the results of section 1.2 give for any $f \in \mathcal{C}$:

$$\theta_{C_N^0(a,b,c)}(\mathcal{L}_\Phi^{kj} f, h) \le \left(\prod_{\ell=1}^{j-1} \delta_\ell\right) \Delta_j,$$

*(**S-Exp1**) gives that $(\prod_{\ell=1}^{j-1} \delta_\ell)\ \Delta_j$ goes to 0 when j goes to infinity. Using $\|\ \|_{d,0}$ and $\int_{D(0)}$ as adapted norm and homogeneous form, we obtain proposition 3.1 and theorem 1.4.*

Let us begin the proof of proposition 3.1 with the following simple property of the sets $D(j)$.

Lemma 3.2 *Since Σ has no big branches at infinity, for any $\ell \in \mathbb{N}$ we have,*

$$\sigma^{-k} D(\ell - 1) \subset D(\ell).$$

Moreover, the sequence $m(D(\ell) \setminus \sigma^{-k} D(\ell-1))$ is summable.

Proof: The fact that Σ is without big branches at infinity directly leads to $\sigma^{-k} D(\ell-1) \subset D(\ell)$.
Let us prove that the sequence $u_j = m(D(j) \setminus \sigma^{-k} D(j-1))$ is summable. We have
$u_j = m(D(j)) - m(\sigma^{-k} D(j-1)) = m(D(j)) - (1 - m(\sigma^{-k} D(j-1)^c))$ and

$$\begin{aligned} m(\sigma^{-k} D(j-1)^c)) &= \int \mathbf{1}_{D(j-1)^c} \circ \sigma^k dm \\ &= \int_{D(j-1)^c} \mathcal{L}_\Phi^k \mathbf{1} dm. \end{aligned}$$

(**S-Exp1**) implies $\mathcal{L}_\Phi^k \mathbf{1}(x) \le 1$ on $D(j-1)^c \subset [\![0, N]\!]^c$, so that

$$m(\sigma^{-k} D(j-1)^c)) \le m(D(j-1)^c).$$

This leads to $u_j \le m(D(j)) - m(D(j-1))$ and u_j is summable. $\triangle$

3.1 Contraction of the cone.

We are going to prove that for any $\gamma < 1$ and N, k, a, b, c, s, t well chosen,
$\mathcal{L}_\Phi^k C_N^j(a,b,c) \subset C_N^{j-1}(\gamma a, \gamma b, \delta_j c)$ and the diameter of $\mathcal{L}_\Phi^k C_N^j(a,b,c)$ in $C_N^{j-1}(a,b,c)$ is bounded by $2 \log \frac{1+\delta_j}{1-\delta_j}$ where $\delta_j = \delta'_{j,k}(1 - m(j))^{-1}$ and $m(j) = c\ m(D(j) \setminus \sigma^{-k} D(j-1))$.

Remark 3.2 For any fixed c, it exists j_0 such that for any $j \ge j_0$, $m(j) < 1$ and $\delta_j < 1$. In order to make the reading easier, we will always assume that $j_0 = 1$.

Let us note $m = \sup_j m(j)$, since $m(j) < 1$ for all $j \geq 1$ and $m(j) \to 0$, we have $m < 1$.

Let us fix $\alpha < 1 < \alpha'$, $0 < \beta < (1-m)$, $\gamma = \frac{\beta}{1-m}$ and $\beta' < \beta$. The parameters a, b, c, s, t and k are chosen in the following way:

1. $a > \alpha'(1 + \frac{\alpha}{2\alpha'})\beta^{-1}$,
2. k_0 is such that $\forall k > k_0$, $Mr^k < \beta'$,
3. $c \geq Ma + 1$,
4. $b > \frac{Rc}{\beta - \beta'}$,
5. $D_1 < 1/b\frac{\alpha}{4\alpha'}$,
6. let t_0 be such that for all $t > t_0$ we have $D_2 < \frac{\alpha}{16\alpha' c}$. Let N and $t \leq N$ be such that $t > t_0$ and
$$D_2 \leq 2 \sum_{n=t}^{N} m([n]), \tag{3.3}$$
7. k_1 is such that (2.2) is satisfied for $\mathcal{P}_{s,t}$ and k_1 verifies (**S-Exp1**),
8. $k \geq \max(k_0, k_1)$.

Let $f \in C_N^j(a,b,c)$ and $P \subset D(j-1)$, lemma 3.2 implies that $\sigma^{-k}P \subset D(j)$. Let us remark that, by the choice 6. above,

$$\frac{m(\sigma^{-k}P \cap P_2 \cap D(j))}{\mu(P)m(P_2 \cap D(j))} = \frac{m(\sigma^{-k}P \cap P_2)}{\mu(P)m(P_2)} \frac{m(P_2)}{m(P_2 \cap D(j))} \leq 2\frac{m(\sigma^{-k}P \cap P_2)}{\mu(P)m(P_2)}.$$

Using this and following the proof of lemma 2.3, we obtain (using (2.2) and the definition of the cone):

$$\begin{aligned}
\frac{1}{\mu(P)} \int_P \mathcal{L}_\Phi^k f dm &\leq \alpha' \int_{D(j)} f dm \, [1 + D_1 \, b + 4D_2 \, c] \\
\frac{1}{\mu(P)} \int_P \mathcal{L}_\Phi^k f dm &\geq \left[\alpha - \alpha'(D_1 b + 4D_2 c)\right] \int_{D(j)} f dm.
\end{aligned}$$

which leads, by the choices (5. and 6.) above, to

$$\alpha/2 \int_{D(j)} f dm \leq \frac{1}{\mu(P)} \int_P \mathcal{L}_\Phi^k f dm \leq \alpha'(1 + \alpha/(2\alpha')) \int_{D(j)} f dm.$$

Moreover, since Σ is without big branches at infinity, (2.7) becomes

$$K_{j-1}(\mathcal{L}_\Phi^k f) \leq Mr^k K_j(f) + R \sup_{D(j)} |f|,$$

and, for any $f \in C_N^j(a,b,c)$ and $x \in D(j)$ we have, (as in lemma 2.1),

$$\min[-c, Ma - bD_1] \int_{D(j)} f dm \leq f(x) \leq \max[c, Ma + bD_1] \int_{D(j)} f dm$$

This gives $K_{j-1}(\mathcal{L}_\Phi^k f) \leq \int\limits_{D(j)} fdm \left[bMr^k + R\max(c, Ma + bD_1)\right]$ so, by the choices (3. and 5.) above,

$$K_{j-1}(\mathcal{L}_\Phi^k f) \leq \beta b \int\limits_{D(j)} fdm \text{ if } b > \frac{cR}{\beta - \beta'}.$$

Finally, for $x \in P_2 \cap D(j-1)$, (**S-Exp1**) gives:

$$|\mathcal{L}_\Phi^k f(x)| \leq \delta'_{j,k} c \int\limits_{D(j)} fdm.$$

To prove that $\mathcal{L}_\Phi^k C_N^j(a,b,c) \subset C_N^{j-1}(\gamma a, \gamma b, \delta_j c)$, it remains to compare $\int\limits_{D(j)} fdm$ and $\int\limits_{D(j-1)} \mathcal{L}_\Phi^k fdm$.

Lemma 3.3 *For any $f \in C_N^j(a,b,c)$, $\int\limits_{D(j-1)} \mathcal{L}_\Phi^k fdm = \int\limits_{D(j)} fdm[1 \pm m(j)]$.*

Proof: Let $f \in C_N^j(a,b,c)$, following lemma 3.2, we have $\sigma^{-k}D(j-1) \subset D(j)$,

$$\int\limits_{D(j-1)} \mathcal{L}_\Phi^k fdm = \int\limits_{\sigma^{-k}D(j-1)} fdm = \int\limits_{D(j)} fdm - \int\limits_{D(j)\setminus\sigma^{-k}D(j-1)} fdm.$$

If x belongs to $D(j) \setminus \sigma^{-k}D(j-1)$ then

$$\min[-c, Ma - bD_1] \int\limits_{D(j)} fdm \leq f(x) \leq \int\limits_{D(j)} fdm \ \max[c, Ma + bD_1],$$

this gives $\int\limits_{D(j-1)} \mathcal{L}_\Phi^k fdm = \int\limits_{D(j)} fdm[1 \pm c\ m(D(j) \setminus \sigma^{-k}D(j-1))]$ (we have chosen $bD_1 < 1$ and $c \geq aM + 1$) and the lemma is proven. △

So we have $\mathcal{L}_\Phi^k C_N^j(a,b,c) \subset C_N^{j-1}(\gamma a, \gamma b, \delta_j c)$. It remains to estimate the projective diameter.

Lemma 3.4 *The projective diameter Δ_j of $\mathcal{L}_\Phi^k C_N^j(a,b,c)$ in $C_N^{j-1}(a,b,c)$ is bounded by $2\log\frac{1+\delta_j}{1-\delta_j}$ and for f and g in $C_N^j(a,b,c)$,*

$$\theta_{j-1}(\mathcal{L}_\Phi^k f, \mathcal{L}_\Phi^k g) \leq \delta_j \theta_j(f,g)$$

where θ_j denote the projective metric of the cone $C_N^j(a,b,c)$.

Proof: Following the proof of lemma 2.4, we obtain $\Delta_j \leq 2\log\max(\frac{1+\delta_j}{1-\delta_j}, \frac{2\alpha'+\alpha}{\alpha}, \frac{1+\gamma}{1-\gamma})$ since $\lim\delta_j = 1$ (see remark 3.1), we may assume that $\frac{1+\delta_j}{1-\delta_j} \geq \frac{2\alpha'+\alpha}{\alpha}$ and $\frac{1+\delta_j}{1-\delta_j} \geq \frac{1+\gamma}{1-\gamma}$ so we have $\Delta_j \leq 2\log\frac{1+\delta_j}{1-\delta_j}$ and $\tanh\frac{\Delta_j}{4} \leq \tanh\left[2\log\frac{1+\delta_j}{1-\delta_j}\right] = \delta_j$.

Let f and g belong to $C_N^j(a,b,c)$, proposition 1.5 and the estimate of Δ_j imply

$$\theta_{C_N^{j-1}(a,b,c)}(\mathcal{L}_\Phi^k f, \mathcal{L}_\Phi^k g) \leq \delta_j \theta_{C_N^j(a,b,c)}(f,g).$$

△

Finally, let us remark that since $\sum_j m(j) < \infty$, the product of $(1-m(j))^{-1}$ goes to some positive limit so that , $\prod\limits_{\ell=0,\ldots,j} \delta_\ell \leq \mathrm{Ct} \prod\limits_{\ell=0,\ldots,j} \delta'_{k,\ell}$.

3.2 Estimate of the decay of correlations.

In this section, we conclude the proof of proposition 3.1 and we show how (3.1) can be used to estimate the speed of convergence of $\mathcal{L}_\Phi^n$ to the spectral projection, on compact sets and the decay of correlations. Let $\mathcal{C}=\bigcap_{j>0}C_N^j(a,b,c)$. The cone $\mathcal{C}$ is non empty indeed, we have the following result.

Lemma 3.5 *If a, b and c are large enough, then for any $f\in L$, it exists $R(f)\geq 0$ such that $f+R(f)h\in\mathcal{C}$, moreover $R(f)\leq Ct\,\|f\|$*

Proof: It suffices that $R(f)$ verify (recall that $\forall j\in\mathbb{N}$, $[\![0,n_1]\!]\subset D(0)\subset D(j)$)

- $R(f)\geq \dfrac{M\sup f-a\displaystyle\int_{[\![0,n_1]\!]}f}{a\displaystyle\int_{[\![0,n_1]\!]}h-M\sup h}$,
- $R(f)\geq \dfrac{K(f)-b\displaystyle\int_{[\![0,n_1]\!]}f}{b\displaystyle\int_{[\![0,n_1]\!]}h-K(h)}$,
- $R(f)\geq \dfrac{\sup_{P_2}f-c\displaystyle\int_{[\![0,n_1]\!]}f}{c\displaystyle\int_{[\![0,n_1]\!]}h-\sup_{P_2}h}$.

The parameters a, b and c are chosen in order to ensure that the three denominators are strictly positive. △

In what follows, a,b,c are large enough to guaranty that lemma 3.5 is verified. In particular, h belongs to $\mathcal{C}$.

Let $f\in\mathcal{C}$ and $j\in\mathbb{N}$, f belongs to $C_N^j(a,b,c)$ and $\mathcal{L}_\Phi^{k\ell}f$ belongs to $C_N^{j-\ell}(a,b,c,)$ for $\ell=0,\dots,j$.

$$\begin{aligned}\theta_{C_N^0}(\mathcal{L}_\Phi^{kj}f,h) &\leq \delta_1\theta_{C_N^1}(\mathcal{L}_\Phi^{k(j-1)}f,h)\\ \dots &\leq \Delta_j\prod_{\ell=1}^{j-1}\delta_\ell.\end{aligned}$$

The norm $\|\ \|_{d,0}$ defined by (3.2) and $\rho(f)=\displaystyle\int_{D(0)}fdm=\int_{[\![0,N]\!]}fdm$ are adapted to C_N^0, moreover the norms $\|\ \|_{\alpha,0}$ and $\|\ \|_N$ are equivalent, so for $f\in\mathcal{C}$,

$$\|\mathcal{L}_\Phi^{kj}f-\frac{h}{\int_{[\![0,N]\!]}h}\int_{[\![0,N]\!]}\mathcal{L}_\Phi^{kj}f\|_N\leq\Delta_j\prod_{\ell=1}^{j-1}\delta_\ell\ \exp\left(\Delta_j\prod_{\ell=1}^{j-1}\delta_\ell\right)\int_{D(0)}fdm.$$

If $p=kj+r$,

$$\begin{aligned}\|\mathcal{L}_\Phi^{p}f-\frac{h}{\int_{[\![0,N]\!]}h}\int_{[\![0,N]\!]}\mathcal{L}_\Phi^{p}f\|_N &\leq \|\mathcal{L}_\Phi^r\|\ \|\mathcal{L}_\Phi^{kj}f-\frac{h}{\int_{[\![0,N]\!]}h}\int_{[\![0,N]\!]}\mathcal{L}_\Phi^{kj}f\|_N\\ &\leq M\ \|\mathcal{L}_\Phi^{kj}f-\frac{h}{\int_{[\![0,N]\!]}h}\int_{[\![0,N]\!]}\mathcal{L}_\Phi^{kj}f\|_N.\end{aligned}$$

It is easy to prove that (recall that $\int hdm = 1$),

$$\|\frac{h}{\int\limits_{[\![0,N]\!]} hdm} \int\limits_{[\![0,N]\!]} \mathcal{L}_\Phi^p f dm - h \int f dm\|_N \leq \text{Ct } \sup|f| m([\![0,N]\!]^c)$$

Finally, for $f \in \mathcal{C}$ and $p = kj + r$, $r < j$ we have

$$\|\mathcal{L}_\Phi^p f - h \int f\|_N \leq \text{Ct } \Delta_j \prod_{\ell=1}^{j-1} \delta_\ell \ \exp\left(\Delta_j \prod_{\ell=1}^{j-1} \delta_\ell\right) \ |\int f dm| + \text{Ct } m([\![0,N]\!]^c) \sup|f|. \quad (3.4)$$

Remark 3.3 $\Delta_j \prod_{\ell=1}^{j-1} \delta_\ell \ \exp\left(\Delta_j \prod_{\ell=1}^{j-1} \delta_\ell\right)$ depends on N and k, (**S-Exp1**) implies that for fixed N and k, this expression goes to zero when j goes to infinity.

Lemma 3.5 and (3.4) imply for $f \in L$,

$$\|\mathcal{L}_\Phi^{kj} f - h \int f\|_N \leq \text{Ct } \alpha_j \ \|f\| + m([\![0,N]\!]^c) \sup f,$$

with $\alpha_j(N) = \Delta_j \ \prod_{\ell=1}^{j-1} \delta_\ell \ \exp\left(\Delta_j \prod_{\ell=1}^{j-1} \delta_\ell\right)$. This conclude the proof of proposition 3.1. $\triangle$

Now, we are going to show how (3.4) leads to the estimate of the speed of convergence on compact sets of Σ and to the decay of correlations. Let $q \in \mathbb{N}$ and f belongs to $\mathcal{C}$, so that f belongs to $C_N^{k(j+2q)}$ and $\mathcal{L}_\Phi^{k(j+q)} f \in C_N^{kq}$. This leads to the following estimate which will be used to bound the speed of convergence on compacts.

$$\begin{aligned} &\|\mathcal{L}_\Phi^{k(j+q)} f - h \int f\|_{N+Kkq} \leq \\ &\qquad \exp[\Delta_{j+q} \prod_{\ell=1}^{j-1} \delta_{\ell+q}] \Delta_{j+q} \int\limits_{D(0)} f dm \prod_{\ell=1}^{j-1} \delta_{\ell+q} + m([\![0,N+Kkq]\!]^c) \sup|f|. \end{aligned} \quad (3.5)$$

We choose a sequence $q(j)$ such that $q(j) \overset{j\to\infty}{\longrightarrow} \infty$ and

$$\Delta_{j+q(j)} \prod_{\ell=1}^{j-1} \delta_{\ell+q(j)} \ \exp\left(\Delta_{j+q(j)} \prod_{\ell=1}^{j-1} \delta_{\ell+q(j)}\right) := \widetilde{\alpha_j} \longrightarrow 0.$$

For example, if $\delta'_{k,j} \leq (1 - \frac{1}{N+Kkj})^\alpha$, since $\Delta_j \leq 2\log\frac{1+\delta_j}{1-\delta_j}$, for any $0 < \varepsilon < \varepsilon' < 1$, we can choose $q(j) = j^\varepsilon$, then we have, $\widetilde{\alpha_j} \leq C(N,\varepsilon,\varepsilon')\frac{1}{j^{\alpha-\varepsilon'}}$.

Remark 3.4 The condition (**S-Exp1**) may be replaced by: it exists a sequence $q(j)$ which goes to infinity with j and such that

$$\Delta_{j+q(j)} \prod_{\ell=1}^{j-1} \delta_{\ell+q(j)} \overset{j\to\infty}{\longrightarrow} 0.$$

Now, let x belongs to some compact Q, it exists j_0 such that $Q \subset [0, N+Kkq(j_0)] \subset [0, N+Kkq(j)]$ $\forall j \geq j_0$, so, for any $f \in L$, $j \geq j_0$ and $p = k(j+q(j))+r$, lemma 3.5 and (3.5) give,

$$\left|\mathcal{L}_\Phi^p f(x) - h(x)\int f dm\right| \leq \text{Ct } [\widetilde{\alpha_j} + m([0, N+Kkq(j)]^c)]\|f\|.$$

If $u_j(N) = \widetilde{\alpha_j} + Rm[0, N+kKq(j)]^c]$, this can be written as

$$\|\mathcal{L}_\Phi^p f - h\int f dm\|_Q \leq C(j_0)u_j\|f\| \tag{3.6}$$

where $C(j_0)$ goes to infinity with j_0:

$$C(j_0) = \sup_{\substack{k(j+q(j))\leq p<k(j+1+q(j+1))\\ j\leq j_0}} \frac{\|\mathcal{L}_\Phi^p f - h\int f dm\|_Q}{u_j\ \|f\|} \leq \text{Ct } \sup_{j\leq j_0} \frac{1}{u_j}.$$

Let us also remark that if $q(j) = o(j)$ then, for $p \in \mathbb{N}$, the integer j which verify $k(j+q(j)) \leq p < k(j+1+q(j+1))$ has the same order as $\frac{p}{k}$.

Finally, the decay of correlations is obtained in the same way. Let f be such that $fh \in L$ and $g \in L^\infty(m)$, using (3.5) we obtain:

$$\begin{aligned}
&|\mu(g\circ\sigma^p f) - \mu(g)\mu(f)| \\
&\leq \Big|\int_{[0,N+Kkq(j)]} [\mathcal{L}_\Phi^p(fh) - h\int fhdm]gdm\Big| + \Big|\int_{[0,N+Kkq(j)]^c)} \mathcal{L}_\Phi^p(fh)gdm\Big| \\
&+ \Big|\int_{[0,N+Kkq(j)]^c)} (gh)dm\Big|\ \Big|\int fhdm\Big| \\
&\leq \|\mathcal{L}_\Phi^p(fh) - h\int fhdm\|_{N+Kkq(j)}\|g\|_\infty + \text{ Ct } \|f\|_\infty\|g\|_\infty\mu([0, N+Kkq(j)]^c) \\
&\leq [\widetilde{\alpha_j} + \text{Ct } \mu([0, N+Kkq(j)]^c)]\|f\|\|g\|_\infty \leq \text{Ct } u_j\|f\|\ \|g\|_\infty.
\end{aligned} \tag{3.7}$$

This conclude the proof of theorem 1.4. $\triangle$

4 Some examples.

We will now give some examples of applications of theorems 1.3 and 1.4. We first give large classes of dynamics satisfying (**Exp1**) and (**S-Exp1**). For this dynamics, it will be sufficient to control $\mathcal{L}_\Phi\mathbf{1}(x)$ for $x \in [n]$ for large n. Let us begin with a sufficient condition for φ to satisfy (**K**). We will say that Φ has small contribution at infinity if

$$\exists n_0 \text{ such that } \forall n > n_0, (\mathcal{L}_\Phi\mathbf{1})_\mathbf{n} = \sup_{x\in[n]} \mathcal{L}_\Phi\mathbf{1}(x) \leq 1, \tag{\textbf{H}}$$

Lemma 4.1 *If Φ satisfies (**SA**) and (**H**) then it satisfies (**K**).*

Proof: Recall that we have the bounded distortion for $\mathcal{L}_\Phi$: for x and y in the same 1-cylinder and $k \in \mathbb{N}$, we have $\mathcal{L}_\Phi^k\mathbf{1}(x) \leq e^{Cd(x,y)}\mathcal{L}_\Phi^k\mathbf{1}(y)$. If x belongs to $[n]$, by integrating on the cylinder

$[n]$, we get:

$$\begin{aligned}\mathcal{L}_\Phi^k \mathbf{1}(x) &\leq R\,\frac{1}{m([n])}\int_{[n]}\mathcal{L}_\Phi^k\mathbf{1}dm\\ &\leq \frac{R}{m([n])}\int_\Sigma \mathcal{L}_\Phi^k\mathbf{1}dm = \frac{R}{m([n])} \text{ (because } m \text{ is conformal).}\end{aligned}$$

for some constant $R>0$. Let n_0 be given by $(\mathbf{H})$, $n\leq n_0$ and $k\in\mathbb{N}$.

$$\sup_{x\in[n]}\mathcal{L}_\Phi^k\mathbf{1}(x)\leq R\max_{p\leq n_0}\frac{1}{m([p])} := M'.$$

Let us note $M_k=\sup_\Sigma \mathcal{L}_\Phi^k\mathbf{1}$. If $n>n_0$, we have, for any $x\in[n]$, $\mathcal{L}_\Phi\mathbf{1}(x)\leq 1$ and

$$\mathcal{L}_\Phi^{k+1}\mathbf{1}(x)\leq \mathcal{L}_\Phi\mathbf{1}(x)\sup_\Sigma\mathcal{L}_\Phi^k\mathbf{1}\leq\sup_\Sigma\mathcal{L}_\Phi^k\mathbf{1}\leq \max M_k,$$

so we get, $M_{k+1}\leq\max(M',M_k)$ and by induction $M_k\leq\max(M',1)$. This proves that Φ satisfies $(\mathbf{K})$. $\triangle$

We will say that Σ has *no jumps to infinity* if it exists an integer K such that, for all $n\in\mathbb{N}$ and for all $x\in[n]$, $\sigma x\in[p]$ with $p\leq n+k$. In other words, the matrix which defines Σ has the following form:

$$\begin{pmatrix} * & \cdots & * & 0 & \cdots & \cdots & \cdots \\ \vdots & & & \ddots & \ddots & & \\ * & & & & * & 0 & \\ \vdots & & & & & \ddots & \ddots \end{pmatrix}$$

with $*\in\{0,1\}$.

Example 4.1 [Dynamics without jumps to infinity satisfying (Exp1).]
If Σ has no jumps to infinity and if Φ verifies$(\mathbf{SA})$ and:

$$\exists n_0 \text{ such that } \forall n>n_0, (\mathcal{L}_\Phi\mathbf{1})_\mathbf{n}\leq\rho<1, \tag{\textbf{Exp2}}$$

then Φ verifies $(\mathbf{Exp1})$.

Indeed, $(\mathbf{Exp2})$ implies $(\mathbf{H})$ which implies $(\mathbf{K})$ by lemma 4.1. So, it exists $M>0$ such that for any $n\in\mathbb{N}$, $\|\mathcal{L}_\Phi^n\mathbf{1}\|_\infty\leq M$. If Σ has no jumps to infinity and Φ satisfies $(\mathbf{Exp2})$ then, by induction, we may show,

$$\forall p\geq 1,\ \forall n>n_0+(p-1)K, \sup_{x\in[n]}\mathcal{L}_\Phi^p\mathbf{1}(x)\leq\rho^p.$$

Let us fix $k_1\in\mathbb{N}$ and $n_1=n_0+(k-1)K$, if $k\geq k_1$ and $n\geq n_1$ then:

$$\sup_{x\in[n]}\mathcal{L}_\Phi^k\mathbf{1}(x)\leq\sup(\mathcal{L}_\Phi^{k-k_1}\mathbf{1})\sup_{x\in[n]}\mathcal{L}_\Phi^{k_1}\mathbf{1}(x)\leq M\rho^{k_1}.$$

Then, it suffices to chose k_1 such that $M\rho^{k_1}\leq\beta<1$. $\triangle$

We will say that Σ has *bounded jumps* if it exists an integer K such that for all $n\in\mathbb{N}$, for $x\in[n]$,

$\sigma x \in [p]$ with $n - K \leq p \leq n + K$. In other words, the matrix which defines Σ has the following form:

$$\begin{pmatrix} * & \cdots & \cdots & * & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & & & & \ddots & \ddots & & & & & \\ * & & & & & * & 0 & & & & \\ 0 & * & & & & & * & 0 & & & \\ \vdots & \ddots & \ddots & & & & & \ddots & \ddots & & \\ 0 & & 0 & * & & & & & * & 0 & \\ \vdots & & & \ddots & \ddots & & & & & \ddots & \ddots \end{pmatrix}$$

with $* \in \{0,1\}$.

Example 4.2 [Dynamics satisfying (S-exp1).]
If Σ is aperiodic and has bounded jumps, if Φ verifies (**SA**) and the following two properties:

1. $\exists n_0$ such that $\forall n > n_0$, $\forall x \in [n]$, $\mathcal{L}_\Phi \mathbf{1}(x) \leq (1 - \dfrac{1}{n})^\alpha \ \alpha > 0$, $\qquad$ $(\mathbf{S-Exp2})$
2. the invariant density h goes to zero at infinity (that is, if $\sup\limits_{x\in[n]} h(x) := h_n$ then h_n goes to zero when n goes to infinity),

then Φ verifies (**S-Exp1**).

First of all, let us remark that (**S-Exp2**) implies (**H**) which implies (**K**). Now, if Σ has bounded jumps and Φ verifies (**S-Exp2**) then, by induction, we have: for any $k \geq 1$ and $n > n_0 + (k-1)K$,

$$\sup_{x\in[n]} \mathcal{L}_\Phi^k \mathbf{1}(x) \leq \prod_{j=0}^{k-1} (1 - \frac{1}{n + Kj})^\alpha. \tag{4.1}$$

The following lemma gives an estimate of $\mathcal{L}_\Phi^k \mathbf{1}(x)$ for $x \in [n]$ and $N \leq n \leq N + (k-1)K$ provided N and k are large enough.

Lemma 4.2 *For any $(\frac{1}{2})^\alpha < \eta < 1$, it exists $n_1 \geq n_0$ such that for any $N \geq n_1$ it exists k_1 such that for $k \geq k_1$,*

$$\sup_{N \leq n < N+(k-1)K} \ \sup_{x\in[n]} \mathcal{L}_\Phi^k \mathbf{1}(x) < \eta.$$

Proof: Let $\varepsilon < 1$ such that $2\varepsilon < \eta$, let

1. $N_0 \geq n_0$ such that $n \geq N_0$, $h_n \leq \varepsilon$ and $n_1 = \max(N_0 + K, n_0)$, let us fix $N \geq n_1$.
2. k_0 such that $\forall k \geq k_0$, $\mathcal{L}_\Phi^k \mathbf{1}(x) \leq h(x) + \varepsilon$, for $x \in [n]$, $n \leq N$,[1]
3. $k' > k_0$ such that for $k > k'$, $\left(\dfrac{N + K(k+1)}{N + 2Kk}\right)^\alpha < \eta$ and $k_1 > k'$ such that $k > k_1$, $u_{k'} \left(\dfrac{N + (k'+1)K}{N + (k-2)K}\right)^\alpha < \eta$ where u_k is defined by

$$u_k = \sup\{\mathcal{L}_\Phi^k \mathbf{1}(x) \ / \ x \in [n], N \leq n < N + K(k-1)\}.$$

[1] Since Σ has bounded jumps, the set $[\![0, N]\!]$ is compact. So theorem 1.1 imply that such an integer k_0 exists.

Let $k > k_1$, $N \geq n_1$ and $x \in [n]$ with $N \leq n < N + kK$, if x' is the preimage of x by σ, then $x' \in [n']$ with $N_0 < N - K \leq n' < N + (k+1)K$,

$$\begin{aligned}
\mathcal{L}_\Phi^{k+1}\mathbf{1}(x) &= \sum_{\sigma x' = x} g_0(x')\mathcal{L}_\Phi^k \mathbf{1}(x') \\
&= \underbrace{\sum_{N_0 < n' \leq N} g_0(x')\mathcal{L}_\Phi^k\mathbf{1}(x')}_{[1]} + \underbrace{\sum_{N<n<N+(k-1)K} g_0(x')\mathcal{L}_\Phi^k\mathbf{1}(x')}_{[2]} + \underbrace{\sum_{N+(k-1)K \leq n' < N+(k+1)K} g_0(x')\mathcal{L}_\Phi^k\mathbf{1}(x')}_{[3]}
\end{aligned}$$

The choices of N_0 and k give

$$\begin{aligned}
[1] &\leq \sum_{N_0<n'\leq N} g_0(x')[\varepsilon + h(x')] \text{ (by the choice 2. of } k \geq k_0\text{)}, \\
[2] &\leq u_k \sum_{N<n'<N+(k-1)K} g_0(x'), \\
[3] &\leq \sum_{N+(k-1)K\leq n'<N+(k+1)K} g_0(x') \prod_{j=0}^{k-1}(1 - \frac{1}{n'+Kj})^\alpha \text{ (by (4.1))} \\
&\leq \prod_{j=0}^{k-1}(1 - \frac{1}{N+K(j+k+1)})^\alpha \sum_{N+(k-1)K\leq n'<N+(k+1)K} g_0(x').
\end{aligned}$$

So, for $k > k'$,

$$\begin{aligned}
\mathcal{L}_\Phi^{k+1}\mathbf{1}(x) &\leq \max[\varepsilon + \sup_{N_0<n'\leq N} h_{\mathbf{n}'}, u_k, \prod_{j=0}^{k-1}(1 - \frac{1}{N+K(j+k+1)})^\alpha]\mathcal{L}_\Phi\mathbf{1}(x) \\
&\leq \max[2\varepsilon, u_k, (\frac{N+K(k+1)}{N+2Kk})^\alpha](1 - \frac{1}{N+kK})^\alpha \\
&\leq \max[u_k, \eta](1 - \frac{1}{N+kK})^\alpha,
\end{aligned}$$

and finally,

$$u_k \leq \max[u_{k'} \prod_{\ell=k'+1}^{k-2}(1 - \frac{1}{N+\ell K})^\alpha, \eta](1 - \frac{1}{N+(k-1)K})^\alpha \leq \eta$$

△

So, if $\frac{1}{2^\alpha} < \eta < 1$ is fixed, we have: for all $N \geq n_1$, it exists k_1 such that for $k \geq k_1$,

$$\begin{aligned}
\delta'_{k,j} &= \sup\{Lo^k\mathbf{1}(x) \,/\, x \in [n],\ N \leq n \leq N + kKj\} \\
&\leq \max\left[\sup_{N\leq n\leq N+kK} \mathcal{L}_\Phi^k\mathbf{1}(x)\ x \in [n], \sup_{N+kK\leq n\leq N+kKj} \mathcal{L}_\Phi^k\mathbf{1}(x)\ x \in [n]\right] \\
&\leq \max\left[\eta, \prod_{i=0}^{k-1}\left(1 - \frac{1}{N+K(kj+i)}\right)^\alpha\right] \text{ (using 4.1)}
\end{aligned}$$

this is sufficient to get (**S-Exp1**). △

To finish, we give some explicit estimates on the decay of correlations for some maps of the interval and estimates on the speed of convergence of some positively recurrent birth-and-death process.

4.1 Uniformly expanding maps of the interval and birth-and-death process.

We are going to adapt our methods to some uniformly expanding maps of the interval which do not satisfy the "big branches property" ([Sa]) nor the "covering" property of [L, S, V1] and some non uniformly expanding maps.

In what follows, I is the interval $[0,1]$, λ is the Lebesgue measure on I. For a given partition $\mathcal{I} = (I_n)_{n\in\mathbb{N}}$ (mod 0) of I in open subintervals, B is the Banach space of functions on I which are Lipschitz on each I_n with uniformly bounded Lipschitz constant. Let $K(f)$ be the sup of the Lipschitz constants for $f \in B$. For $f \in B$, let $\|f\| = \max(\|f\|_\infty, K(f))$, $\|\ \|$ is a norm on B which turns it into a Banach space. If $T : I \to I$ is C^1 and injective on each I_n, then the Lebesgue measure is conformal for the transfer operator associated to the potential $-\log T'$. We will denote by $\mathcal{L}$ this transfer operator. We assume that T verifies the Markov property: for any $n \in \mathbb{N}$, TI_n is a union (mod 0) of elements of $\mathcal{I}$ and that the partition $\mathcal{I}$ generates the borelian sigma-field under T. Such systems are always conjugated to some sub-shift of finite type. We will say that T is expanding if $T'(x) > 1$ for any $x \in \bigcup_n I_n$ and that T is uniformly expanding if it exists $D > 1$ such that $T'(x) \geq D$ for any $x \in \bigcup_n I_n$. If T is uniformly expanding then B injects naturally in L and we may work on the symbolic dynamic as well as on the interval. But, since we also wish to consider non uniformly expanding maps, it is preferable to work directly on I. The techniques that we have developed in sections 2 and 3 are directly applicable to the uniformly expanding case and are applicable with some modifications to the non uniformly expanding case. In the interval maps setting, the k-cylinders correspond to subintervals of I of the form $\bigcap_{i=0}^{k-1} T^{-i} I_{n_i}$. We will treat the following example.

Example 4.3
Let $T : I \to I$ be defined in the following way:
T is C^2, monotone and increasing on each I_n, it may be continued to a continuous function on the closure of I_n and there exists some integer K such that

$$TI_n = \bigcup_{n-K\leq p\leq n+K} I_p$$

(with the convention that $I_n = \emptyset$ if $n < 0$). Moreover, T is uniformly expanding and there exists $R > 0$ such that $T''(x) \leq R$ for any $x \in \bigcup I_n$.

These dynamics are aperiodic and with bounded jumps. We will denote by

$$\rho_n = \frac{\lambda_n}{\displaystyle\sum_{p=n-K}^{n+K} \lambda_p} \text{ with } \lambda_n = \lambda(I_n).$$

Let us remark that

$$\rho_n^{-1} = \lambda_n^{-1} \int_{I_n} T' d\lambda.$$

Lemma 4.3 *If the sequence $(\lambda_n)_{n\in\mathbb{N}}$ satisfies one of the two following properties:*

1. $\displaystyle\lim_{n\to\infty} \frac{\lambda_{n+1}}{\lambda_n} < 1$, *in this case, we would say that the sequence λ_n is of exponential type,*

2. $\displaystyle\frac{\lambda_{n+1}}{\lambda_n}$ *is increasing to 1 for $n \geq n_0$, and $\lambda_n = o(n^{-2})$.*

then $\mathcal{L}$ satisfies ($\mathbf{H}$).

Proof: We have for any x and y in I_n,

$$|T'(x) - T'(y)| \leq \sup_{z \in I_n} T''(z)|x-y|.$$

By integrating on I_n, we get:

$$-R\lambda(I_n) + \frac{1}{\rho_n} \leq T'(x) \leq R\lambda(I_n) + \frac{1}{\rho_n}. \tag{4.2}$$

Let us assume that $\lim_{n\to\infty} \frac{\lambda_{n+1}}{\lambda_n} = \theta < 1$, this implies that ρ_n goes to $\frac{\theta^K}{1+\cdots+\theta^{2K}} < (2K+1)^{-1}$ and $\sup_{x\in[n]} \mathcal{L}\mathbf{1}(x)$ to $(2K+1)\frac{\theta^K}{1+\cdots+\theta^{2K}} < 1$ (it uses (4.2)) so that ($\mathbf{H}$) is satisfied.
Let us assume that it exists n_0 such that from n_0, the sequence $\frac{\lambda_{n+1}}{\lambda_n}$ increases to 1, let $0 < u_n < 1$ be such that for $n \geq n_0$, $u_{n+1} \leq u_n$, the sequence u_n goes to zero and

$$\frac{\lambda_{n+1}}{\lambda_n} = (1-u_n),$$

for any $j = 1, \cdots, K$ and $n \geq n_0 + K$, it exists a sequence $u_{n,j}$ such that $u_{n,j}$ goes to zero when n goes to infinity, $0 < u_{n+1,j} \leq u_{n,j}$ and:

$$\frac{\lambda_{n+j}}{\lambda_n} = (1-u_{n,j}).$$

We get for some $n_0' \geq n_0$ and for any $n \geq n_0' + K$:

$$\rho_n^{-1} = (2K+1) - \sum_{j=1}^{K} u_{n,j} + \sum_{j=1}^{K} [u_{n-j,j} + \sum_{i=2}^{\infty} (u_{n-j,j})^i] \geq (2K+1).$$

Remark that $\sum_n u_{n,j} = \infty$ for all j, since $\lambda_n = o(n^{-2})$, we have that $\frac{\lambda_n}{u_{n,j}^2}$ goes to zero for all j. So, using (4.2), we get $\mathcal{L}\mathbf{1_n} \leq 1$ if $n \geq n_0 + K$ and ($\mathbf{H}$) is verified. $\triangle$

As we already noticed, ($\mathbf{H}$) and the fact that λ is a conformal measure for $\mathcal{L}$ imply ($\mathbf{K}$), let h be the invariant density and $\mu = h\lambda$. The proof of lemma 4.3 shows that if $(\lambda_n)_{n\in\mathbb{N}}$ is of exponential type, then $\mathcal{L}$ satisfies ($\mathbf{Exp2}$). Since T has bounded jumps, example 4.1 proves that $\mathcal{L}$ satisfies ($\mathbf{Exp1}$), so that the decay of correlations is exponential on the space B.
Moreover, if T is affine on each I_n then it is easy to see that φ defined by $\varphi(x) = \lambda_{n+K} + \cdots + \lambda_{n-K}$ if $x \in I_n$ is a fixed point for $\mathcal{L}$ (so it is the only one up to a normalization by theorem 1.1). Let us consider $J_N = \bigcup_{n>N+K} I_n$. We have

$$|\mu(T^{-N}J_N \cap I_0) - \mu(J_N)\mu(I_0)| = \mu(J_N)\mu(I_0) \text{ because } T^{-n}J_N \cap I_0 = \emptyset.$$

In particular, it is not possible to have an exponential decay of correlations of type (1.3) if the sequence λ_n is not majored by an exponential sequence. The decay of correlations may, nevertheless, be estimated in some cases. For $P \in \mathbb{N}$, let $\|\ \|_P$ denotes the uniform norm on $\bigcup_{j\leq P} I_j$.

Proposition 4.4 *If $\lambda_n = K\gamma^{n^\alpha} + o(\gamma^{n^\alpha})$, $1/2 \le \alpha < 1$ then, for any $r > 0$ and any $P \in \mathbb{N}$, there exist $C(r) > 0$ and $C(r, P)$ such that: for any f and g such that $fh \in B$ and $g \in L^\infty$,*

$$|\mu(fg \circ T^n) - \mu(f)\mu(g)| \le C(r)\frac{1}{n^r}\|fh\| \, \|g\|_\infty$$

$$\text{and } \|\mathcal{L}^n f - hm(f)\|_P \le C(r, P)\frac{1}{n^r}\|f\| \text{ for any } f \in B.$$

Proof: Let us assume that $\lambda(I_n) = K\gamma^{n^\alpha} + o(\gamma^{n^\alpha})$, $\alpha < 1$. ρ_n satisfies for some positive constant C:

$$\rho_n = \frac{1}{2K+1} - \frac{C}{n^{2(1-\alpha)}} + o(\frac{1}{n^{2(1-\alpha)}}).$$

So, it exists n_0 such that for some positive constant C and $n \ge n_0$,

$$\rho_n \le \frac{1}{2K+1} - \frac{C}{n^{2(1-\alpha)}}.$$

So, using (4.2), if $n \ge n_0$,

$$\sup_{x\in[n]} \mathcal{L}\mathbf{1}(x) \le 1 - \frac{(2K+1)C}{n^{2(1-\alpha)}} := 1 - \frac{c}{n^{2(1-\alpha)}}.$$

Let $\beta = 2(1-\alpha) \le 1$ if $\alpha \ge \frac{1}{2}$, which we will assume. This means that $\mathcal{L}$ verifies (**S-Exp2**). Moreover, for $v(x) = n^{-\alpha}$, $\alpha > 0$ for $x \in I_n$, if $\mathcal{L}_v$ is the transfer operator associated to this change of potential, the arguments of lemma 4.3 prove that $\mathcal{L}_v$ satisfies (**H**); this implies that the invariant density h of $\mathcal{L}$ goes to zero at infinity (indeed, we have that $\frac{h}{v}$ is bounded). So example 4.2, shows the following estimate.

Lemma 4.5 *It exists n_1 such that for $N \ge n_1$, it exists k_0 such that for $k \ge k_0$,*

$$\delta'_{j,k} = \sup_{N \le n \le N+kj} \mathcal{L}^k \mathbf{1_n} \le \prod_{i=0}^{k-1} 1 - \frac{c}{(N+kj+i)^\beta} \le \prod_{i=0}^{k-1} \left(1 - \frac{1}{(N+kj+i)^\beta}\right)^c.$$

So $\mathcal{L}$ satisfies (**S-Exp1**). Moreover, (3.6) and (3.7) give for $f \in L$, $q(j) = j^u$, $0 < u < 1$ and $n = k(j + q(j)) + r$, $j = O(\frac{n}{k})$, if $\beta < 1$,

$$\|\mathcal{L}^n f - hm(f)\|_P \le \mathrm{Ct}(P, N) \, [\exp(-c(N+kj)^{1-\beta}) + \sum_{n \ge N+kq(j)} \gamma^{n^\alpha}]$$

and for f such that $fh \in B$ and $g \in L^\infty$,

$$|\mu(fg \circ T^n) - \mu(f)\mu(g)| \le \mathrm{Ct}(N) \, [\exp(-c(N+kj)^{1-\beta}) + \sum_{n \ge N+kq(j)} \gamma^{n^\alpha}] \tag{4.3}$$

Since $\sum_{p \ge n} \gamma^{p^\alpha} = O(n^{1-\alpha}\gamma^{n^\alpha})$, we have the announced estimate for $\beta < 1$. The same computation leads also to the result for $\beta = 1$. $\triangle$

Remark 4.1 When the convergence to zero of $\lambda(I_n)$ is slower than γ^{n^α}, $\frac{1}{2} \le \alpha < 1$, for example if it is polynomial, we have

$$\sup_{x\in[n]} \mathcal{L}\mathbf{1}(x) \le (1 - \frac{C}{n^\beta}) \text{ with } \beta > 1$$

provided n is large enough, this estimate is not sufficient to use the techniques of section 3. However, it is maybe possible to estimate the decay of correlations by improving the above estimate for iterates of $\mathcal{L}$.

Birth-and-death process. Using the same method, we obtain the following results for birth-and-death process (see [Se] for a review on non negative matrices). We consider a stochastic matrix $P = (p_{i,j})_{i,j\in\mathbb{N}}$ ($\sum_i p_{i,j} = 1$ for all j). We assume that there is an integer K such that $p_{i,j} = 0$ if $|i-j| > K$ (this is why we call these process birth-and-death process) and we assume that the matrix is aperiodic. In this situation, $\mathbf{1}$ is a fixed point for the Markov operator P and we are looking for an stationary measure, i.e. a fixed point for the dual ${}^tP$ of P. Let us denote ${}^tP$ by $\mathcal{L}$. The measure defined by $m_1[i] = 1$ and $m_1[i_1, \cdots, i_n] = p_{i_1 i_2} \cdots p_{i_{n-1} i_n}$ is a conformal measure for $\mathcal{L}$ but it is not finite. To any function v constant on the 1-cylinders and such that $\sum_{n\in\mathbb{N}} v_n < \infty$, we associate a transfer operator $\mathcal{L}_v$ by change of potential (see page 6). The measure $m_v = v m_1$ is finite and conformal for $\mathcal{L}_v$. We make the following assumptions of the matrix.

- For any $n \in \mathbb{N}$, let $Pu(n) = \sum_{i>n} p_{i,n}$ and $Pd(n) = \sum_{i<n} p_{i,n}$.
- We assume that for n large enough, $Pu(n) = a(1-w_n)$ and $Pd(n) = b(1+u_n)$ where a and b are positive numbers and (u_n) and (w_n) are positive sequences that go to zero.

We have the following results:

- If $a + b < 1$ then there exists a change of potential v such that $\mathcal{L}_v$ satisfies (**S-Exp2**). So the matrix is positive recurrent and geometrically ergodic in the sense of D. Vere-Jones ([V-J1], [V-J2]): we have the following exponential convergence

$$\sup_{i,j\in\mathbb{N}} |p_{i,j}^{(n)} - \nu_j| \leq \text{Ct } \gamma^n,$$

where ν is the stationary measure and $0 < \gamma < 1$.

- If $a + b = 1$, $a < b$, for n large enough $w_n < u_n$ and $\sum_n w_n = \infty$ then there exists a change of potential v such that $\mathcal{L}_v$ satisfies (**S-Exp2**). So that the matrix is positive recurrent and we have the following estimate: for any $N \in \mathbb{N}$ and any $r \in \mathbb{N}$ there exists $C(N, r) > 0$ such that if $i, j \leq N$ then

$$|p_{i,j}^{(n)} - \nu_j| \leq \text{C(N,r)} n^{-r}.$$

- If $a + b = 1$, $a = b$, for n large enough, $w_n < u_n$ and if $z_n = u_n - w_n$ then $\sum_n z_n = \infty$ then there exists a change of potential v such that $\mathcal{L}_v$ satisfies (**S-Exp2**). So that the matrix is positive recurrent and we have the following estimate: for any $N \in \mathbb{N}$ and any $r \in \mathbb{N}$ there exists $C(N, r) > 0$ such that if $i, j \leq N$ then

$$|p_{i,j}^{(n)} - \nu_j| \leq \text{C(N,r)} n^{-r}.$$

4.2 Non uniformly expanding maps of the interval.

We conclude this article with the estimation of the decay of correlations for Gaspard-Wang type applications. Let $(I_n)_{n\in\mathbb{N}}$ be a partition (mod 0) of I with $\lambda(I_n) = \frac{K}{(n+1)^\alpha}$, $K > 0$, $\alpha > 1$. Let us consider the following piecewise affine application. T is increasing, affine on each I_n, $TI_n = I_{n-1}$ for $n \geq 1$ and $TI_0 = I$. This is a linearization of smooth non uniformly expanding maps of the interval considered for example by M. Thaler ([T]), C. Liverani, B. Saussol et S. Vaienti ([L, S, V2]) and introduced by P. Gaspard et X.-J. Wang ([G, W], [Wan]) in order to model intermitency phenomenons. It is well known that T admits a unique absolutely continuous invariant measure whose density h verifies $cn \leq h(x) \leq Cn$ if $x \in I_n$ ([La,Si,V]). In particular, $\mu = h\lambda$ is a finite measure if and only if $\alpha > 2$. Moreover, this measure is mixing. Let us notice that the dynamic is without big branches at infinity and aperiodic.
If $d > 0$ we denote by $v_d : I \to \mathbb{R}^+$ the locally constant function:

$$v_d(x) = v_n = n^d \text{ if } x \in I_n,$$

let E be the space of functions f such that

$$\frac{fh}{v_d} \in B \text{ for any } d > 1 \text{ and } \sup_{d>1} \left\| \frac{fh}{v_d} \right\| := \|\|f\|\| < \infty$$

We are going to prove the following result.

Proposition 4.6 *For any $\varepsilon > 0$, there exists $C(\varepsilon)$ such that for all $f \in E$ and $g \in L^\infty$,*

$$|\mu(fg \circ T^n) - \mu(f)\mu(g)| \leq C(\varepsilon)\|\|f\|\| \, \|g\|_\infty \frac{1}{n^{\alpha-2-\varepsilon}}. \qquad (I)$$

Let us remark that since $cn \leq h(x) \leq Cn$ for $x \in I_n$, the space B is included in E and for $f \in B$, we have $\|\|f\|\| \leq \mathrm{Ct}\|f\|$.

Remark 4.2 There exists many results on the decay of correlations for this map (linearized or not). The oldest are from A. Lambert, S. Siboni, S. Vaienti ([La,Si,V]), M. Mori ([Mo]) and N. Chernov ([Ch]). A.M. Fisher and A. Lopes ([F, L]) and S. Isola ([I]) get a speed of convergence in $\frac{1}{n^{\alpha-2}}$ for observables which are finite linear combinations of characteristic functions of cylinders. Concerning the smooth model, using approximation techniques, C. Liverani, B. Saussol et S. Vaienti ([L, S, V2]) obtain a rate of convergence in $\frac{\log n}{n^{\alpha-2}}$ for Lipschitz functions on the interval I, this space is included in E. Using a coupling method, L.-S. Young ([Yo]) get, on the same space, a rate of convergence of order $\frac{1}{n^{\alpha-2}}$; in [H] H. Hu proves the same result and that this result is optimal for Lipschitz functions on I. More recently, M. Pollicott and M. Yuri ([Po,Y]) get an estimation for observables in a space containing $\{\frac{1}{x^\gamma},\ 0 < \gamma < \frac{1}{\alpha+1}\}$.

Proof of of the proposition 4.6: The transfer operator $\mathcal{L}_\Phi$ associated to the potential $\Phi = -\log T'$ satisfies

$$\mathcal{L}_\Phi \mathbf{1_n} = \left(\frac{n+1}{n+2}\right)^\alpha + \lambda(I_0).$$

Since h is not bounded, $\mathcal{L}_\Phi$ cannot verifies $(\mathbf{K})$. This is why we use a cohomologous potential. For $d > 1$, let $\mathcal{L}_d$ be the transfer operator associated to the change of potential v_d, $\mathcal{L}_d f = \frac{1}{v_d}\mathcal{L}_\Phi(fv_d)$.

$$\sup_{x\in[n]} \mathcal{L}_d\mathbf{1}(x) = \left(\frac{n+1}{n+2}\right)^{\alpha-d} + \frac{\lambda(I_0)}{n^d}.$$

So, if n is large enough, $\mathcal{L}_d\mathbf{1_n} \leq 1$, which means that $\mathcal{L}_d$ satisfies $(\mathbf{H})$ so it satisfies $(\mathbf{K})$ provided that the conformal measure $m_d = v_d\lambda$ remains finite: the potential is constant on each I_n so it is uniformly locally Lipschitz on the partition $(I_n)_{n\in\mathbb{N}}$. We have $m_d(I) = \sum v_n\lambda(I_n)$, so $m_d(I)$ is finite if and only if $\alpha - d > 1$, since $d > 1$, we recover the condition $\alpha > 2$ which guaranty the existence of an absolutely continuous invariant measure μ. In what follows, we assume that $\alpha > 2$, $\alpha - d > 1$ and the measure m_d is normalized (i.e. $m_d(I) = 1$). Let h_d be the normalized fixed point of $\mathcal{L}_d$, we have $\mu = h_d m_d = h\lambda$.
Let us prove that $\mathcal{L}_d$ verifies (**S-Exp1**). Let us fix $0 < \eta < \alpha - d$, let $\beta = \alpha - d - \eta$.

Lemma 4.7 *It exists $n_1 = n_1(d,\eta)$, it exists k_0 such that for $N \geq n_1$ and $k \geq k_0$,*

$$\delta'_{k,j} = \sup_{N\leq n\leq N+kj} \sup_{x\in[n]} \mathcal{L}_d^k\mathbf{1}(x) \leq \prod_{\ell=0}^{k-1}\left(1 - \frac{1}{N+kj+\ell+2}\right)^\beta$$

Proof: $\mathcal{L}_d$ satisfies (**K**), so there exists $M > 0$ such that $\|\mathcal{L}_d^n \mathbf{1}\|_\infty \le M$ for all $n \in \mathbb{N}$. Let us fix n_0 such that if $n \ge n_0$ then $(1 - \frac{1}{n+2})^\eta + \frac{2^\beta \lambda(I_0) M}{n^d} \le 1$. This implies that if $n \ge n_0$, $\mathcal{L}_d \mathbf{1_n} \le (1 - \frac{1}{n+2})^\beta$. Now, the following estimate can be proved by induction.

$$\text{For any } n \ge n_0 + k,\ \mathcal{L}_d^k \mathbf{1_n} \le \prod_{\ell=0}^{k-1} \left(1 - \frac{1}{n+\ell+2}\right)^\beta .$$

It remains to estimate $\sup_{x\in[n]} \mathcal{L}_d^k \mathbf{1}(x)$ for $N \le n < n_0 + k$. Let $N \le n < n_0 + k$,

$$\sup_{x\in[n]} \mathcal{L}_d^{k+1}\mathbf{1}(x) \le \left(1 - \frac{1}{n+2}\right)^{\alpha-d} \sup_{x\in[n+1]} \mathcal{L}_d^k \mathbf{1}(x) + \frac{\lambda(I_0)M}{n^d},$$

by induction, we prove

$$\sup_{x\in[n]} \mathcal{L}_d^k \mathbf{1}(x) \le \prod_{\ell=0}^{k-1} (1 - \frac{1}{n+j+2})^{\alpha-d} + \frac{\lambda(I_0)M}{n^d}[1 + \sum_{p=2}^{k-1} \prod_{\ell=1}^{p} (1 - \frac{1}{n+\ell+2})^{\alpha-d}].$$

This leads to

$$\sup_{x\in\mathbb{N}} \mathcal{L}_d^k \mathbf{1}(x) \le \left(\frac{n+1}{n+k}\right)^{\alpha-d} + \frac{K}{n^{d-1}} \text{ where } K \text{ is a constant which depends neither on } n \text{ nor on } k.$$

Let $\left(\frac{1}{2}\right)^{\alpha-d} < \gamma' < \gamma < 1$, and n_1 be such that $n \ge n_1$, $\frac{K}{n^{d-1}} < \gamma - \gamma'$ and $N \ge n_1$, we choose k_0 such that $k \ge k_0$, $(1 - \frac{k}{n_0+2k})^{\alpha-d} < \gamma'$. Since $N \le n < n_0 + k$, $\frac{n+1}{n+k} \le 1 - \frac{k}{n_0+2k}$ we have for $x \in [n]$, $\mathcal{L}_d^k \mathbf{1}(x) \le \gamma$. This is sufficient to get the lemma. △

So, we have,

$$\delta'_{k,j} \le \left(1 - \frac{1}{N + k(j+1) + 2}\right)^{\beta k},$$

and $\mathcal{L}_d$ satisfies (**S-Exp1**). In order to adapt the method of section 3, it suffices to estimate $K_j(\mathcal{L}_d^k f)$ for $f \in C_N^j(a,b,c)$. Let us note $\rho(x) = (T'(x))^{-1}$ and $\rho_k(x) = \prod_{i=0}^{k-1} \rho(T^i x)$. For x and $y \in I_n$, $n \le N + k(j-1)$ and $f \in B$, since T is without big branches at infinity and affine, we get:

$$|\mathcal{L}_d^k f(x) - \mathcal{L}_d^k f(y)| \le K_j(f)\ d(x,y) \sum_{T^k x' = x} g_k(x')\rho_k(x'),$$

- $\rho_k \le 1$, so that for $n \ge N$

$$|\mathcal{L}_d^k f(x) - \mathcal{L}_d^k f(y)| \le K_j(f)\ d(x,y)\ \delta'_{k,j}$$

- Let $0 \le n < N$. For any $p \le k$ and $z \in \Sigma$, we have $\rho_k(z) \le \rho_p(z)$. So,

$$\sum_{T^k x'=x} g_k(x')\rho_k(x') \le \sum_{T^k x'=x} g_k(x')\rho_p(x') = \mathcal{L}_d^k \rho_p(x),$$

if $x \in \bigcup_{n\le N} I_n$, theorem 1.1 applied to $\mathcal{L}_d$ implies that $\mathcal{L}_d^k \rho_p(x)$ goes to $h_d(x)m_d(\rho_p)$ uniformly in x. Birkhoff's ergodic theorem applied to $\log\rho$ and the fact that $\mu(\log\rho) < 0$ imply that $\rho_p(z)$ goes to zero when p goes to infinity for μ-almost all z. We have $\mu = h_d m_d$, so that $\rho_p(z)$ goes to zero when p goes to infinity for m_d-almost all z. Lebesgue's dominated convergence theorem implies that $m_d(\rho_p)$ goes to zero when p goes to infinity. So, there exists $k(N)$ such that for $k > k(N)$,

$$\sum_{T^k x' = x} g_k(x')\rho_k(x') \leq \mathcal{L}_d^k \rho_p(x) \leq \delta'_{k,j}.$$

Finally, for $k \geq k(N)$, we have $K_{j-1}(\mathcal{L}_\Phi^k f) \leq K_j(f)\delta'_{k,j}$. With the notations of section 3, for any $k \geq \max(k(N), k_0)$ and $0 < \gamma < 1$

$$\mathcal{L}_d^k C_N^j(a,b,c) \subset C_N^{j-1}(\gamma a, \delta_j b, \delta_j c)$$

and the hyperbolic diameter of $\mathcal{L}_d^k C_N^j(a,b,c)$ in $C_N^{j-1}(\gamma a, \delta_j b, \delta_j c)$ is majored by $2\log\frac{1+\delta_j}{1-\delta_j}$. The fixed point of $\mathcal{L}_d$ verifies $h_d = \frac{h}{v_d}$. If f is such that $fh_d = \frac{fh}{v_d} \in B$ and $g \in L^\infty$, the estimate (3.7) gives with $q(j) = j^u$, $0 < u < 1$ and $n = k(j + q(j)) + r$, $j = O(\frac{n}{k})$,

$$\begin{aligned}
&|\mu(fg\circ T^n) - \mu(f)\mu(g)| \\
&\leq \ \mathrm{Ct}\left[\log(N + k(j+q(j)))\left(\frac{N+kq(j)}{N+kj}\right)^{\beta k} + \mu([N+kq(j)]^c)\right] \|\frac{fh}{v_d}\|\ \|g\|_\infty \\
&\leq \ \mathrm{Ct}(d,u)\left[\frac{1}{j^{(1-u)\beta k}} + \frac{1}{j^{u(\alpha-2)}}\right] \|\frac{fh}{v_d}\|\ \|g\|_\infty.
\end{aligned}$$

Let f belongs to E. We fix $\varepsilon > 0$ and $1 < d < 1+\varepsilon$, let $u = \frac{\alpha-2-\varepsilon}{\alpha-d-1} < 1$ and $k > \frac{\alpha-2-\varepsilon}{\beta(1-u)}$, then

$$|\mu(fg\circ T^n) - \mu(f)\mu(g)| \leq \mathrm{Ct}(\varepsilon)\frac{1}{j^{\alpha-2-\varepsilon}}\|\!|f|\!\|\ \|g\|_\infty.$$

Since $j = O(\frac{n}{k})$ and $k = k(\varepsilon)$ we deduce:

$$|\mu(fg\circ T^n) - \mu(f)\mu(g)| \leq \mathrm{Ct}(\varepsilon)\frac{1}{n^{\alpha-2-\varepsilon}}\|\!|f|\!\|\ \|g\|_\infty.$$

This conclude the prove of the proposition.

$\triangle$

Remark 4.3 We can also apply the same techniques to affine non uniformly expanding Markovian maps of the interval with bounded jumps provided they satisfy (**S-Exp2**). Moreover, the techniques may be improved to consider dynamics which do not verify the bounded distortion property (but a bounded distortion on each I_n) and then obtain estimates for piecewise smooth non uniformly expanding maps.

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