COEXISTENCE OF DIFFERENT SPECTRAL TYPES FOR ALMOST PERIODIC SCHRÖDINGER EQUATIONS IN DIMENSION ONE

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Abstract. This paper is an enhanced version of a talk given at the conference QMath 7 that was held in Prague (Czech Republic), 22-26 June 1998. It is devoted to the study of almost periodic Schrödinger equations in dimension 1. The main result is the proof of the coexistence of absolutely continuous and singular spectrum for such equations. We also give a criterion for the existence of absolutely continuous and singular spectrum.

The results presented here are taken from [7] (see also [8]). In [7], we deal with a more general class of equations, but to simplify and shorten the assumptions and statements we will restrict ourself to the following family of almost periodic Schrödinger equations

$$H_{\phi,\varepsilon,\alpha}\psi = -\frac{d^2}{dx^2}\psi(x) + (V(x-\phi) + \alpha\cos(\varepsilon x))\psi(x) = E\psi(x), \quad x \in \mathbb{R} \tag{1}$$

where

- V is L^2_{loc}, real valued and 1-periodic i.e. $V(x+1) = V(x)$ for $x \in \mathbb{R}$,
- ε is a positive number,
- $\alpha > 0$ is a coupling constant,
- ϕ is a real parameter indexing the family of equations.

To be able to control the spectral behavior of $H_{\phi,\varepsilon,\alpha}$, we will need ε to be in a Diophantine class that we define now: for $\aleph > 0$, we introduce

$$\mathcal{D}_\aleph := \left\{\varepsilon \in \mathbb{R};\ \forall q \in \mathbb{Z}\setminus\{0\},\ \inf_{p\in\mathbb{N}}\left|\frac{2\pi}{\varepsilon} - \frac{p}{q}\right| \geq \frac{\aleph}{|q|^3}\right\}.$$

It is well known that $\cup_{\aleph>0}\mathcal{D}_\aleph$ is a set of full measure in $\mathbb{R}$. Moreover, for any $0 < \aleph < 4/9$, 0 is an accumulation point of $\mathcal{D}_\aleph$. We will study the family (1) in the adiabatic limit i.e. when $\varepsilon \to 0$ within a fixed $\mathcal{D}_\aleph$.

1. The periodic equation and the complex momentum

Consider the periodic Schrödinger equation

$$-\frac{d^2}{dx^2}\psi(x) + V(x) = E\psi(x), \quad x \in \mathbb{R} \tag{2}$$

The spectrum of equation (2) on $L^2(\mathbb{R})$ consists of intervals $[E_1, E_2]$, $[E_3, E_4]$, $\ldots$, $[E_{2n+1}, E_{2n+2}]$, $\ldots$, of the real axis such that

$$E_1 < E_2 \leq E_3 < E_4 \ldots E_{2n} \leq E_{2n+1} < E_{2n+2} \leq \ldots$$
$$E_n \to +\infty, \quad n \to +\infty.$$

In the sequel we will assume that $E_2 < E_3$ i.e. that the first gap of (2) is open. Denote the quasi-momentum associated to (2) by $k(E)$, and its dispersion law by $\mathcal{E}(\kappa)$. Then the points $(E_n)_{n\in\mathbb{N}}$ are branching points of k; they are of square root type (see [14, 8] section

2 and references therein).
The *complex momentum* associated to the family of equations (1) is defined by

$$\kappa_\alpha(\varphi) = k(E - \alpha\cos(\varphi)).$$

The function κ_α is multi-valued and its branching points are given by the equation

$$E - \alpha\cos(\varphi) = E_n, \;\; n \in \mathbb{N}.$$

These branching points form a 2π-periodic lattice. In Fig. 1, we show how these points move when E is increasing. We represented 3 different cases:

1. when $E < E_1 - \alpha$.
2. when $E_1 - \alpha < E < \inf(E_2 - \alpha, E_1 + \alpha)$.
3. when $\sup(E_1 + \alpha, E_2 - \alpha) < E < \inf(E_3 - \alpha, E_2 + \alpha)$.

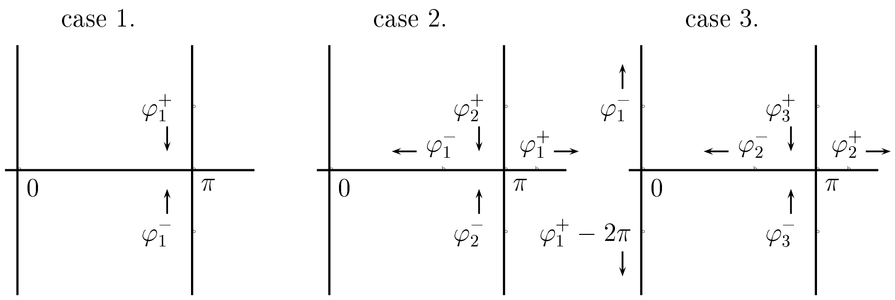

FIGURE 1. The branching points

The arrows indicate the direction in which the branching points are moving as E is increasing.

2. The actions and the phase

Obviously, the spectrum of $H_{\phi,\varepsilon,\alpha}$ is contained in $[E_1 - \alpha, +\infty)$. We will study it near its infimum. More precisely, pick $\delta > 0$ and define the interval J_δ to be

$$J_\delta = [E_1 - \alpha, \inf(E_1 + \alpha, E_2 - \alpha) - \delta].$$

We will study the spectrum of the family of equations (1) in J_δ. For $E \in J_\delta$, the branching points of κ_α are located like in case 2 of Fig. 1.

The main objects needed to describe the spectrum of a typical equation of the family (1) in J_δ will be the actions and phase that we define now. We define the *action integrals* S and S_1 and the *phase* Φ as

$$S(\alpha,E) = -i\int_{\varphi_1^+ - 2\pi}^{\varphi_1^-} \kappa_\alpha d\varphi, \quad S_1(\alpha,E) = i\int_{\varphi_2^-}^{\varphi_2^+} (\kappa_\alpha - \pi) d\varphi, \quad \Phi(\alpha,E) = \int_{\varphi_1^-}^{\varphi_1^+} \kappa_\alpha d\varphi.$$

These integrals depend on the choice of the branch of the multi-valued function κ_α. We choose this branch (see [8]) so that, for $E \in J_\delta$, one has

$$S(\alpha,E) > 0, \quad S_1(\alpha,E) > 0, \quad \Phi(\alpha,E) \geq 0 \text{ and } \Phi'(\alpha,E) > 0.$$

Here Φ' denotes the derivative of Φ with respect to E. The actions S and S_1 and the phase Φ admit a natural geometric interpretation (cf [7, 8]).

3. The location of the spectrum

We first state a result on the location of the spectrum in J_δ. Therefore, we notice that the function $\Phi(\alpha, E)$ is monotonically increasing on J_δ and that its derivative does not vanish on J_δ. Moreover $\Phi(\alpha, E_1 - \alpha) = 0$. In J_δ, consider the points $E^{(l)}$, $l \in \mathbb{N}$, defined by

$$\frac{1}{\varepsilon}\Phi(\alpha, E^{(l)}) = \pi/2 + \pi l, l \in \mathbb{N}.$$

The number of these points is finite; we denote it by L. For sufficiently small ε, the distances between these points satisfy the inequalities

$$c_1\varepsilon \le E^{(l)} - E^{(l-1)} \le c_2\varepsilon, \ \ l = 1, \dots, L$$

where c_1 and c_2 are two positive constant independent of ε. We then prove

Theorem 1. *There exists a collection of intervals $(I_l)_{1\le l\le L}$ such that $I_l \subset J_\delta$ and such that the spectrum of equation (1) in J_δ is contained in $\cup_{1\le l\le L} I_l$ where*

- *the interval I_l lies in an $o(\varepsilon)$-vicinity of $E^{(l)}$,*
- *the measure of I_l is given by*

$$|I_l| = 2\frac{\varepsilon\left(e^{-\frac{1}{\varepsilon}S(\alpha,E)} + e^{-\frac{1}{\varepsilon}S_1(\alpha,E)}\right)}{F(1/2+l)\Phi'(\alpha, E^{(l)})}(1+o(1))$$
$$\textit{where } F(\zeta) = \frac{1}{\sqrt{\pi}}\Gamma(\zeta + 1/2)e^{-\zeta\ln\zeta+\zeta}, \ \zeta > 0.$$

Moreover, if $dN_\varepsilon(E)$ denotes the density of states measure of $H_{\phi,\varepsilon,\alpha}$ at energy E, then, one has

$$\int_{I_l} dN_\varepsilon(E) = \frac{1}{2\pi}\varepsilon. \tag{3}$$

Note that the intervals I_l are exponentially small and separated by a distance of order $O(\varepsilon)$. Theorem 1 can be proved without number theoretical assumptions on ε.

4. The nature of the spectrum

We fix α. The function that will tell us the nature of the spectrum of (1) is

$$\Delta S(\alpha, E) := S_1(\alpha, E) - S(\alpha, E).$$

For $\delta \ge 0$, define the intervals

$$J_\delta^- = \{E \in J_\delta;\ \Delta S(\alpha, E) < -\delta\} \text{ and } J_\delta^+ = \{E \in J_\delta;\ \Delta S(\alpha, E) > \delta\}.$$

If $J_\delta^+ \ne \emptyset$ (resp. $J_\delta^- \ne \emptyset$), then, for sufficiently small ε, the number of intervals I_l lying in J_δ^+ (resp. in J_δ^-) is of order $O(1/\varepsilon)$. We prove

Theorem 2. *Fix $0 < \aleph < 4/9$ and $\delta > 0$. Then, there exists $\varepsilon_0 = \varepsilon_0(\aleph, \delta)$ such that, for $\varepsilon \in (0, \varepsilon_0) \cap \mathcal{D}_\aleph$, we have*

- *each of the intervals I_l contained in J_δ^- contains only singular spectrum (i.e. singular continuous or pure point spectrum),*
- *each of the intervals I_l contained in J_δ^+ contains absolutely continuous spectrum. Moreover, for $I_l \subset J_\delta^+$, one has*

$$|I_l \cap \sigma_{ac}(H_{\phi,\varepsilon,\alpha})|/|I_l| = 1 + o(1) \ \textit{as } \varepsilon \to 0, \ \varepsilon \in \mathcal{D}_\aleph.$$

For fixed α, it is natural to define the *asymptotic mobility edges* as the solutions of the equations $\Delta S(\alpha, E) = 0$. According to Theorem 2, asymptotically, these are the points where a transition from absolutely continuous to singular spectrum (or vice versa) occurs. We also note that our result does not tell us what happens at the transitions.

5. The phase diagram

To complete our understanding of the spectrum, we now need to determine whether transitions actually occur i.e. to study the locus of the asymptotic mobility edges. Define

$$\mathcal{S} = \{(\alpha, E); \quad E_1 - \alpha \leq E \leq \inf(E_2 - \alpha, E_1 + \alpha)\}.$$

and

$$\mathcal{S}_\pm = \{(\alpha, E) \in \mathcal{S}; \ \Delta S(\alpha, E) \gtrless 0\} \text{ and } \mathcal{S}_0 = \{(\alpha, E) \in \mathcal{S}; \ \Delta S(\alpha, E) = 0\}.$$

$\mathcal{S}_+$, $\mathcal{S}_0$ and $\mathcal{S}_-$ form a partition of $\mathcal{S}$. Roughly, Theorem 2 tells us that $\mathcal{S}_+$ (resp. $\mathcal{S}_-$) is the set of (α, E) where we have absolutely continuous (resp. singular) spectrum. We prove

Theorem 3. *Under the assumptions made above, we know that*

- $\mathcal{S}_0$ *is a non-empty piecewise analytic curve.*
- $\mathcal{S}_+$ *is connected and there exists* $\alpha_0 > 0$ *such that* $\mathcal{S} \cap \{\alpha < \alpha_0\} \subset \mathcal{S}_+$.
- $\mathcal{S}_-$ *is connected and there exists* $\alpha_1 > \alpha^* > 0$ *such that* $\mathcal{S} \cap \{\alpha > \alpha_1\} \subset \mathcal{S}_-$.

To illustrate this, we compute (numerically) the action integrals S and S_1 in the case when V is a 1-periodic one-gap potential associated to the periodic spectrum $[E_1, E_2] \cup [E_3, +\infty)$ where $E_1 = 0$, $E_2 = 8$. The value of E_3 is computed so that the potential do have period 1. Such a potential is unique and the quasi-momentum k associated to it can be explicitly expressed in terms of elliptic integrals (see e.g. [14] and the references therein).

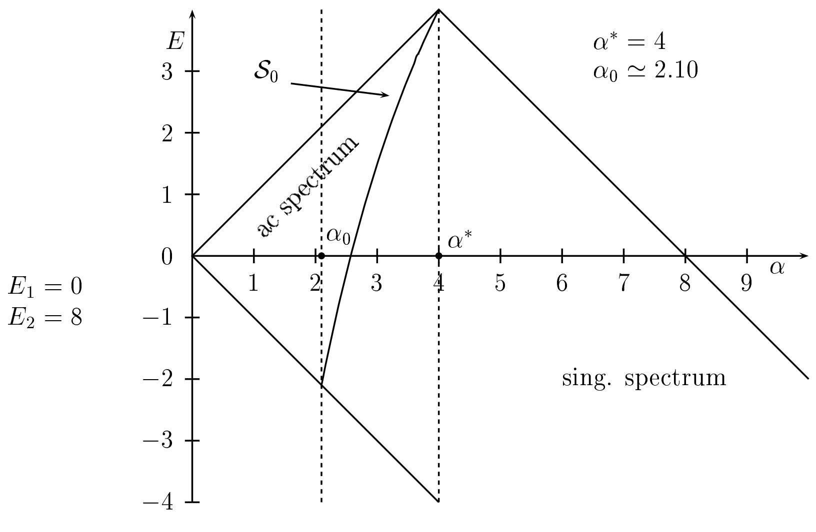

Figure 2. An example of phase diagram

In this case, the *phase diagram* (i.e. the curve $\mathcal{S}_0$ in (α, E)) is given in Fig. 2. We see that, for any $\delta > 0$, for ε small enough,

- for $\alpha < \alpha_0$: we have no transition and most of the spectrum is ac (i.e absolutely continuous). This is the case of "a small perturbation" of a periodic Schrödinger operator.
- for $\alpha_0 < \alpha < \alpha^*$: there exists a single mobility edge $E_{\mathrm{mob}}(\alpha)$ such that

– for $E < E_{\rm mob} - \delta$, the spectrum is singular.
– for $E > E_{\rm mob} + \delta$, most of the spectrum is ac.

- for $\alpha > \alpha^*$: there is no transition and the spectrum is purely singular. We are in a large perturbation regime.

The theory developed in [7] allows more general potentials than $V(x) + \alpha\cos(\varepsilon x)$ i.e. one can replace cos by a more general 2π-periodic function. In this case the phase diagram can be quite different from the one shown in Fig. 2; e.g. for a given α, one may have two or more mobility edges (see [7]).

6. A word about the proofs

The method used to prove the results described above is to reduce the family of equations (1) to an effective Hamiltonian. This is done locally in energy. Therefore we introduce a notion of *monodromy matrix* suitable for (1) (see [8, 7]). The monodromy matrix is a very general idea. For periodic Schrödinger equations, the notion of a monodromy matrix is classical (see, for example, [16]). For difference equations with periodic coefficients, the monodromy matrix was introduced in [2]. For almost periodic Schrödiger equations, it was introduced and its role was understood in [8].

For any fixed ϕ, let $\psi_{1,2}(x,\,\phi)$ be two linearly independent solutions of equation (1). We say that they form a *consistent basis* if their Wronskian is independent of ϕ and these solutions are 1-periodic in ϕ i.e. $\forall x, \phi,\ \psi_{1,2}(x,\,\phi+1) = \psi_{1,2}(x,\,\phi)$.

Let $\psi_{1,2}$ be a consistent basis. As the cosine is 2π-periodic, the functions $\psi_{1,2}(x+2\pi/\varepsilon, \phi+2\pi/\varepsilon)$ are linearly independent solutions of equation (1). Therefore, one can write

$$\Psi(x + 2\pi/\varepsilon, \phi + 2\pi/\varepsilon) = M(\phi)\Psi(x, \phi), \tag{4}$$

where Ψ is the transposed of the vector $(\psi_1(x,\phi), \psi_2(x,\phi))$ and $M(\phi)$ is a matrix with coefficients independent of x. $M(\phi)$ is the *monodromy matrix* associated to the consistent basis $\psi_{1,2}$. As $\psi_{1,2}$ is consistent, M is 1-periodic in ϕ and $\forall\phi$, $\det M(\phi) \equiv 1$.

Set $h = \dfrac{2\pi}{\varepsilon}\mathrm{mod}1$ so that $0 < h < 1$. The spectral analysis of (1) can be reduced to the investigation of the solutions of the *monodromy equation*

$$\chi\,(\phi + h) = M\,(\phi)\,\chi\,(\phi) \qquad \forall\phi \in \mathbb{R}. \tag{5}$$

Indeed, it is well known that the spectral properties of one dimensional Schrödinger equations can be completely described in terms of the behavior of its solutions as $x \to \pm\infty$ (see e.g. [9]). It appears that the behavior of solutions of (1) for $x \to \pm\infty$ mimics the behavior of solutions of the monodromy equation for $\phi \to \mp\infty$. More precisely, consider the restriction $\chi_n = \chi(\phi + nh)$ of a solution χ of (5) to the lattice $\phi + h\mathbb{Z}$. It satisfies the equation

$$\chi_{n+1} = M(\phi + nh)\chi_n, \quad \forall n. \tag{6}$$

One proves

Proposition 1. *Let $\psi_{1,2}$ be consistent basis solutions of (1), and let M be the corresponding monodromy matrix. Fix $\phi = \phi_0 \in \mathbb{R}$. Then, there exists $C > 1$ such that, for any vector solution $(F_n)_{n\in\mathbb{Z}}$ of the equation (6), there exists a unique solution f of (1) such that*

$$\frac{1}{C}\|F_{-n}\| \leq \left\| \begin{pmatrix} f(x + 2\pi n/\varepsilon) \\ f'(x + 2\pi n/\varepsilon) \end{pmatrix} \right\| \leq C\|F_{-n}\|, \quad \forall x \in \left[0, \frac{2\pi}{\varepsilon}\right], \ \forall n \in \mathbb{Z}. \tag{7}$$

This proposition reduces the spectral analysis of equations (1) to the analysis of the solutions of the finite difference equations (6). This is the effective Hamiltonian referred to in the beginning of this section.

This now leaves us with 2 problems. First we have to find a consistent basis for which we can compute the monodromy matrix (at least to some degree of precision). And then we have to analyze the thus obtained matrix.

To solve the first problem, we use a WKB method for quasiperiodic ordinary differential equations in the adiabatic regime $\varepsilon \to 0$. This method was developed in [8]. The form of the monodromy matrix (i.e. which of the coefficients are of leading order when $\varepsilon \to 0$) that one obtains from this computation essentially depends on the picture of the branching points obtained in Fig. 1 (as this picture is related to the geometry of the canonical domains used in the WKB method). In the energy region we study (i.e. in case 2 in Fig. 1), the asymptotics of the leading coefficients of the monodromy matrix are given in terms of the actions S, S_1 and the phase Φ.

The first order vector valued finite difference equation (6) can be rewritten as a second order scalar finite difference equation. It turns out that, when the branching points are in the configuration of case 2 in Fig. 1, this equation is nothing but a small perturbation of Harper's equation

$$(u_{n+1} + u_{n-1}) + 2\lambda(\alpha, E)\cos(nh + \phi)u_n = F(\alpha, E)u_n. \tag{8}$$

Harper's equation and its perturbations have been the subject of many studies during the last 20 years (for more references, see e.g. [11, 12, 13, 2, 15, 3, 10, 4]). Roughly, it is known that, if $\lambda(\alpha, E)$ is large then the spectrum of (8) is pure point, and in $\lambda(\alpha, E)$ is small then it is absolutely continuous.

In our case, the parameter $\lambda(\alpha, E)$ is given quite explicitly in terms of the action difference ΔS: one has

$$\lambda(\alpha, E) \sim e^{-\Delta S(\alpha,E)/\varepsilon}.$$

This explains roughly the results of Theorem 2. To get a rigorous proof, we need to take into account the fact that (6) is a small perturbation of (8) and the spectrum is very sensitive to these perturbations. Hence, to conclude our study, we use results on more general quasi-periodic finite difference equations (see [1, 5, 6]). To complete this section, let us add that the fact that both the effective energy F and the coupling constant in (8) depend on the spectral parameter E of the initial equation leads to another technical problem that, in particular, prevents us from showing that the singular spectrum is pure point.

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