SUB-STABLE AND WEAK-STABLE MANIFOLDS ASSOCIATED WITH FINITELY NON-RESONANT SPECTRAL SUBSPACES

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Abstract. In this work we study $C^{k,\delta}, 2 \leq k \leq \infty, 0 \leq \delta \leq 1$, maps of a Banach space near a fixed point. We show the existence and uniqueness of a class of $C^{k,\delta}$ local invariant sub-manifolds of the stable manifold which correspond to a spectral subspace satisfying a finite non-resonance condition of order $L \leq k$ and an overriding condition of order $L \leq k$ (condition (3) of Theorem 1). We study the dependence of these invariant manifolds on a parameter that lies in a Banach space. We also show that a $C^{k,\delta}, 2 \leq k \leq \infty, 0 \leq \delta \leq 1$, local weak-stable manifold that satisfies these two conditions is unique.

1. Introduction

Invariant manifolds at a fixed point of a map have been studied extensively since the works of Perron and Hadamard. See the partial list of references we give and references therein. Let Φ be a map of a Banach space Z to itself with a fixed point, say 0. Let $\sigma(D\Phi(0))$ be the spectrum of $D\Phi(0)$. Assume that $\sigma(D\Phi(0)) = \sigma_1 \cup \sigma_2$. Also assume that Z has a direct sum decomposition $Z = X \times Y$ which is invariant under $D\Phi(0)$. Let $\sigma_1 = \sigma(D\Phi(0) \mid X)$ and $\sigma_2 = \sigma(D\Phi(0) \mid Y)$. An invariant manifold associated with σ_1 (respectively σ_2) is a sub-manifold through the fixed point 0, which is invariant under Φ and tangent to X (respectively Y) at the fixed point. The traditional invariant manifolds (stable, pseudo-stable, weak-stable, centre-stable, and strong-stable manifolds and their unstable cousins) correspond to an annular gap between σ_1 and σ_2. The smoothness of any invariant manifold from this traditional group depends on the so called gap condition. More precisely, assume that $|\sigma_1| < a < b < |\sigma_2|$ where $|\sigma_i| = \{|\lambda| \mid \lambda \in \sigma_i\}$. In this case the spectral decomposition theorem guarantees the existence of a decomposition $Z = X \times Y$ which is invariant under $D\Phi(0)$. The gap condition can be stated as follows: If $a^r < b$, then there is a C^r invariant manifold associated with σ_1. On the other hand if $a < b^r$, then there is a C^r invariant manifold associated with σ_2. In either case, if the map is $C^k, k > r$, one cannot guarantee that an invariant manifold will be C^k. The reason is that all the proofs reduce the problem to a transformation on the appropriate Banach space, which will be a contraction if $a^r < b$ ($a < b^r$), a condition that does not imply that $a^{r+1} < b$ ($a < b^{r+1}$). It is obvious that the gap condition is trivially satisfied in the case of the stable, unstable, centre-stable, centre-unstable, strong-stable, and strong-unstable manifolds, but not in the other cases.

In this work we study a new class of invariant manifolds, which was studied previously only in [14]. To illustrate the object of our study we give an example in a finite dimensional space.

Date: November 11, 1998.
Key words and phrases. Invariant manifolds, weak-stable manifold, linearization, resonance.

Although our example is finite dimensional, our theorem is proved for infinite dimensional Banach spaces. This example will illustrate most of the characteristics of the new object which we call a *sub-stable manifold* since it is a sub-manifold of the stable manifold. The *local weak-stable manifold* is a special case of this new class of invariant manifolds. A *local weak-stable manifold* that satisfies the condition of our theorem is unique. Now for the example: Suppose the eigenvalues of $D\Phi(0)$ are

$$0 = \alpha_1 < \beta_1 = \beta_2 < \alpha_2 < \alpha_3 = \beta_3 < \alpha_4 = \alpha_5 < \beta_4 < \beta_5 = 1 < \beta_6$$

Let the α-generalized eigenvectors be $< u_1, \cdots, u_5 >$ and the β-ones be $< v_1, \cdots, v_6 >$ with matching subscripts. We assume that the eigenvalue $\alpha_3 = \beta_3$ is a multiple eigenvalue with two linearly independent eigenvectors u_3 and v_3. Notice also that $|\alpha_i| < 1, i = 1, \cdots, 5$. Let $Z = \mathbb{R}^{11} = X \times Y$ where $X = span\{u_1, \cdots, u_5\}$ and $Y = span\{v_1, \cdots, v_6\}$. Write $z = (x, y), D\Phi(0) = blockdiag(A, B)$, and $\Phi = (A + f, B + g)$.

In terms of this example our two main Theorems 1 and 2 can be stated as follows. Assume the following:

1. There is a constant $0 < a < 1$ such that $0 \le |\alpha| < a < 1, \alpha \in \sigma(A)$.
2. There is a constant $0 < b$ such that $|\beta| > b > 0, \beta \in \sigma(B)$.
3. $a^L < b$ for some $L \ge 2$.
4. $\beta_j \ne \alpha_1^{m_1} \cdots \alpha_5^{m_5}, 2 \le m_1 + \cdots + m_5 \le L, j = 1, \cdots, 6$.
5. The map Φ is $C^{k,\delta}, k = L, L+1, L+2, \cdots, \infty, 0 \le \delta \le 1$.

Then the following are true:

a: In a sufficiently small neighbourhood of the fixed point $(0, 0)$, there is a unique $C^{k,\delta}$ invariant manifold of the form $y = w(x)$ whose graph is tangent to $X \times \{0\}$ at the fixed point.

b: If Φ varies continuously with a parameter λ in the $C^{k,\delta}$ topology, then w varies continuously with λ in the $C^{k,\gamma}$ topology for all $0 \le \gamma < \delta \le 1$. If $\delta = 0$, then w varies continuously with λ in the $C^{k-1,s}$ topology for all $0 \le s < 1$. The parameter λ is allowed to be in a Banach space.

A family $\{h_\lambda\}$ varies continuously with λ in the $C^{k,\beta}$ topology iff $\|h_\lambda - h_\mu\|_{C^{k,\beta}} \to 0$ as $\lambda \to \mu$.

1.1. Remarks. 1. If we apply our theorem to the *local weak-stable manifold*, in both finite and infinite dimensional spaces, we obtain the following new results:

(a) A *local weak-stable manifold* which satisfies the conditions of our theorem is unique in the class of $C^{k,\delta}$ functions. Thus, any other *local weak-stable manifold* will not be as smooth as ours.

(b) If it is $C^{L,\delta}$, then it is $C^{k,\delta}$ for all $L \le k \le \infty$ as long as the map is $C^{k,\delta}$.

The *local weak-stable manifold* obtained by the gap-condition method is not unique because one has to use a cut-off function. For each cut-off function one obtains a different local weak-stable manifold. Our method does not require a cut-off function.

The situation with the gap-condition method is worse in infinite dimensional Banach spaces because an infinite dimensional Banach space does not necessarily admit a cut-off function. In [13], it is shown that the space of continuous functions on an interval does not admit a C^1 cut-off function. In [17], it is shown that a separable Banach space admits a Fréchet differentiable cut-off function iff its dual space is separable.

2. In general, the spectrum of $D\Phi(0)$ does not have to consist of only real numbers as in the example.
3. Hypothesis (3) is not a gap condition. In fact the two parts of the spectrum do not have to be disjoint. In this case, if Z is infinite dimensional, the spectral decomposition theorem does not guarantee a decomposition of the Banach space Z. In Theorem 1, it will be assumed that such a decomposition exists. In fact we will use adapted norms on X and Y in which $\|A\| < a$ and $\|B\| > b$. Thus $\|A\|^L < \|B\|$. When hypothesis (3) is satisfied we will say that *A overrides B in L steps* or *A L-overrides B*.
4. Hypothesis (4) is a finite nonresonance condition of order L similar to the one we need in studying finitely smooth linearization near a fixed point. In fact this problem is closer to the linearization problem than to the invariant manifold problem. The nonresonance condition (4) allows us to eliminate the j-monomials in x from $g(x, y)$ for $j = 2, 3, \cdots, L$.
5. However, the degree of smoothness of our invariant manifold does not stop at L in the nonresonance condition (L in (4)) as it does in linearization theorems. This is because the L-overriding condition (3) allows us to eliminate the j-terms of degree $j = L + 1, \cdots, k$. In this way we can show that if the invariant manifold is $C^{L,\delta}$, then it is $C^{k,\delta}$ for any $L \le k \le \infty$ as long as the map is $C^{k,\delta}$.
6. Article [14] is the only place known to the author where this class of invariant manifolds was studied. However, in [14] there is a loss of differentiability when $\delta = 0$; that is, if the map is C^k then the invariant manifold obtained in [14] is only $C^{k-1,1}$. In this work we show how to overcome this problem. There is no loss of differentiability in the invariant manifolds that we obtain.
7. The only places in the literature, known to the author, where the possible loss of differentiability of an invariant manifold when the map is $C^k, k \in \mathbb{N}$ was studied are [2], [7], and [16].
8. In [14] it is shown how the class of invariant manifolds studied in this work and in [14] is relevant to some finite dimensional applications of renormalization groups. Also in [14] you will find counter examples when the nonresonance condition fails as well as some examples of the lack of smoothness of invariant circles with periodic orbits which appear after a Hopf bifurcation.

Acknowledgment. The author would like to thank Rafael de la Llave for suggesting this problem and for fruitful conversations. Most of this work was done while the author was on sabbatical at The Department of Mathematics, University of Texas at Austin. The author would like to thank its members for their great hospitality.

2. Main Results

2.1. General setup.

1. Let Z be a Banach space which has a direct sum decomposition $Z = X \times Y$ where X and Y are closed subspaces of Z and the projections $\pi^X : Z \to X$ and $\pi^Y : Z \to Y$ are bounded.
2. Accordingly, write $z = (x, y), z \in Z, x \in X, y \in Y$. Use the box norm $|z| = \max\{|x|, |y|\}$.

3. Let $\Phi : Z \to Z$ be a $C^{0,1}$ map which takes the form $\Phi(x, y) = (x^*, y^*)$ where

$$x^* = Ax + f(x, y),$$
$$y^* = By + g(x, y), \tag{2.1}$$

and $F = (f, g), F(x, y) = o(|x| + |y|)$.

2.2. Definition. 1. Let $\sigma(A)$ and $\sigma(B)$ be the spectra of A and B respectively. We will say that A is L-nonresonant relative to B and denote this by $NR(A^L, B)$ iff

$$\beta \neq \alpha_1\alpha_2 \cdots \alpha_j, \qquad \beta \in \sigma(B), \alpha_i \in \sigma(A), \qquad 1 \leq i \leq j, \qquad 2 \leq j \leq L. \tag{2.2}$$

In other words,

$$\sigma(A)^j \cap \sigma(B) = \emptyset, \qquad 2 \leq j \leq L,$$

where

$$\sigma(A)^j = \{\alpha_1\alpha_2 \cdots \alpha_j \mid \alpha_i \in \sigma(A), 1 \leq i \leq j\}.$$

2. If A and B satisfy $NR(A^j, B)$ for all $j \geq 2$ we will say that A is infinitely nonresonant relative to B.
3. A $C^{k,\delta}$ function $h : U \to Y$ is a C^k function whose k^{th} derivative is uniformly δ-Hölder continuous. In this case we define

$$H_\delta(D^k h) = \sup \left\{ \frac{\|D^k h(x) - D^k h(x')\|}{|x - x'|^\delta} \mid x \neq x', x, x' \in U \right\}.$$

Notice that when $\delta = 1, H_1 = Lip$.

Now we state our three main results.

Theorem 1 (Existence and uniqueness). *Consider the map Φ given by(2.1) in the setup of 2.1. Let $2 \leq L \leq k$. Assume the following in a neighbourhood of the origin:*

1. *There is $0 < a < 1$ such that $|\alpha| < a$ for all $\alpha \in \sigma(A)$.*
2. *There is $b > 0$ such that $|\beta| > b$ for all $\beta \in \sigma(B)$.*
3. *$a^L < b$.*
4. *A is L-nonresonant relative to B. That is A and B satisfy the condition $NR(A^L, B)$.*
5. *$F(x, y)$ is $C^{k,\delta}$, $L \leq k \leq \infty$, $0 \leq \delta \leq 1$.*

Then the following are true:

1. *For sufficiently small $\gamma > 0$, the map (2.1) has an invariant manifold of the form $y = w(x), w \in C^{k,\delta}(B_\gamma(0), Y)$, where $B_\gamma(0) \subset X$ is the closed ball of radius γ.*
2. *Moreover, $w = w^{(L)} - u_2 - \cdots - u_L$, where u_j is unique in the space $\mathcal{M}_j(X, Y)$ of symmetric j-multilinear functions from X to Y, $2 \leq j \leq L$.*
3. *Let $s_o = \cdots = s_{L-1} = 1$ and $s_L = \delta$. Then $w^{(L)}$ is unique in the class of functions that satisfy the following for $0 \leq j \leq L$ and $|x| \leq \gamma$:*

$$\|D^j v(x)\| \leq \frac{1}{\gamma^{L+\delta-1}} |x|^{L+\delta-j}, \tag{2.3}$$

$$\|D^j v(x) - D^j v(x')\| \leq \frac{1}{\gamma^{L+\delta-1}} |x-x'|^{s_j} (|x|+|x'|)^{L+\delta-j-s_j}. \tag{2.4}$$

When $\delta = 0$ omit (2.4) for $j = L$.

4. *Moreover, w is unique in $C^{k,\delta}(B_\gamma(0), Y)$.*
5. *The tangent space of the graph of w at $(0,0)$ is $X \oplus \{0\}$.*

Theorem 2 (Dependence on parameters). *Let Λ be an open subset of a Banach space. Let $\mu \in \Lambda$ be fixed but arbitrary. Let $\mathcal{U} = B_c^o(\mu) \subset \Lambda$ be the open ball of radius c centered at μ. Let $\{\Phi_\lambda : \lambda \in \Lambda\}$ be a family of maps each of which satisfies the hypotheses of Theorem 1.*

Assume that $\{\Phi_\lambda : \lambda \in \Lambda\}$ depends continuously on λ in the $C^{k,\delta}$ topology. That is, for all $\lambda \in \Lambda$

$$\|\Phi_\lambda - \Phi_{\lambda'}\|_{C^{k,\delta}} \to 0,$$

as $\lambda' \to \lambda$. Let $y = w_\lambda(x), w_\lambda \in C^{k,\delta}(B_{\gamma_\lambda}(0), Y)$, be the unique invariant manifold of Φ_λ guaranteed by Theorem 1. Then the following are true:

1. *For sufficiently small $c > 0$, the neighbourhood $B_{\gamma_\lambda}(0)$ can be chosen uniformly in $\lambda \in \mathcal{U}$. That is, we can assume that $\gamma_\lambda = \gamma$ uniformly in $\mathcal{U}$.*
2. *For sufficiently small $\gamma > 0, w_\lambda$ depends continuously on λ in the $C^{k,\beta}(B_\gamma(0), Y)$ topology, for all $0 \leq \beta < \delta$. That is,*

$$\|w_\lambda - w_\mu\|_{C^{k,\beta}} \to 0,$$

as $\lambda \to \mu$.

3. *If $\delta = 0$, w_λ depends continuously on λ in the $C^{k-1,s}(B_\gamma(0), Y)$ for all $0 \leq s < 1$.*

If we apply our theorems to the *local weak-stable manifold* we obtain some new results. The following theorem follows immediately from Theorem 1 and Theorem 2.

Theorem 3 (The local weak-stable manifold). *Consider the map (2.1). In addition to the assumptions of Theorem 1, assume that there are two additional numbers $0 < b_1 < a_1 < 1$ such that*

1. *$0 < b < b_1 < a_1 < a$.*
2. *$|\beta| < b_1, \beta \in \sigma(B)$.*
3. *$|\alpha| > a_1, \alpha \in \sigma(A)$.*

Then,

1. *The invariant manifold guaranteed by Theorem 1is a local weak-stable manifold.*
2. *The local weak-stable manifold which Theorem 1 guarantees is unique in $C^{k,\delta}(B_\gamma(0), Y)$.*
3. *Moreover, if such a local weak-stable manifold is $C^{L,\delta}$, then it is $C^{k,\delta}$ for all $L \leq k \leq \infty$ as long as the map is $C^{k,\delta}$.*

4. *Each member of the family of local weak-stable manifolds guaranteed by Theorem 2 is unique. Moreover, the family depends continuously on the parameter in the $C^{k,\beta}$ topology for all $0 \le \beta < \delta$. If $\delta = 0$, then it depends continuously on the parameter in the $C^{k-1,s}$ topology for all $0 \le s < 1$.*

2.3. Remarks. 1. Notice that we do not require that $\sigma(A) \cap \sigma(B) = \emptyset$. All we require is the existence of a direct-sum decomposition. This might occur in a finite dimensional space if a multiple eigenvalue has, say, two linearly independent eigenvectors one of them lying in X and the other in Y.

2. The first step in our proof is to take advantage of hypotheses (3) and (4) of Theorem 1 and eliminate nonresonant terms of order 2 to k. In fact it is essential to do so since we only assume that$a^L < b$. It is possible that $a \not< b$. In fact if $a < b$ we have the previously known strong-stable manifold. We will accomplish this elimination in Lemma 2.4, Lemma 2.6 and Lemma 2.7.

2.4. Lemma. *The linear operator A is infinitely nonresonant relative to B, i.e. they satisfy $NR(A^j, B)$ for $j = 2, 3, \cdots$.*

Proof. By assumption 4, A and B satisfy $NR(A^L, B)$. For $j > L$, let $\beta \in \sigma(B)$, and $\alpha_i \in \sigma(A), 1 \le i \le j$. Since $a < 1$ and $a^j < a^L < b$, it follows that

$$|\beta| > b > a^L > a^j \ge |\alpha_1| \cdots |\alpha_j|$$

Thus $\beta \ne \alpha_1 \cdots \alpha_j$ for $j \ge L$. □

2.5. Definition. For $j \ge 1$, let $\mathcal{S}_j(X, Y)$ be the space of symmetric j-multilinear functions from X to Y. Define $I_j : X \to X^j$ by $I_j(x) = (x, \cdots, x)$. Let

$$\mathcal{M}_j(X, Y) = \{s \circ I_j \ \mid \ s \in \mathcal{S}_j(X, Y)\}$$

Although X and Y are infinite dimensional Banach spaces, we will continue to call elements in $\mathcal{M}_j(X, Y)$ j-monomials or monomials of degree j. Notice that $\mathcal{M}_1(X, Y) = \mathcal{S}_1(X, Y) = L(X, Y)$.

2.6. Lemma. *Let (A, B) satisfy the assumptions of Theorem 1. For $j \ge 2$, define the two linear operators*

$$\begin{aligned} K_{A,B} &: \mathcal{M}_j(X, Y) \to \mathcal{M}_j(X, Y), & K_{A,B} u_j &= B^{-1} u_j \circ A\,, \\ L_{A,B} &: \mathcal{M}_j(X, Y) \to \mathcal{M}_j(X, Y)\,, & L_{A,B} &= I - K_{A,B}\,. \end{aligned}$$

Then the following are true for each fixed but arbitrary $j \ge 2$:

1. *The spectrum of $K_{A,B}$ is given by*

$$\sigma(K_{A,B}) \subseteq \{\beta^{-1}\alpha_1 \cdots \alpha_j \mid \beta \in \sigma(B), \alpha_i \in \sigma(A), 1 \le i \le j\}. \tag{2.5}$$

2. $1 \notin \sigma(K_{A,B})$.
3. *$L_{A,B}$ is a bijection.*
4. *$L_{A,B}^{-1}$ is continuous (equivalently bounded).*

5. *Consider a family* $\{(A_\lambda, B_\lambda) \mid \lambda \in \Lambda\}$ *satisfying the assumptions of Theorem 2. Let* $\mu \in \Lambda$ *be fixed but arbitrary. To simplify notation let* $L_\lambda = L_{A_\lambda,B_\lambda}, K_\lambda = K_{A_\lambda,B_\lambda}, \Delta L = L_\lambda - L_\mu$, *and* $\Delta B = B_\lambda - B_\mu, \dots$. *Then for* λ *sufficiently close to* μ

$$B_\lambda^{-1} = [I + B_\mu^{-1}\Delta B]^{-1} B_\mu^{-1}, \tag{2.6}$$

$$L_\lambda^{-1} = [I + L_\mu^{-1}\Delta L]^{-1} L_\mu^{-1}. \tag{2.7}$$

Thus B_λ^{-1} *and* L_λ^{-1} *are continuous in* λ. *In fact,*

$$\|L_\lambda^{-1}\| \le \frac{\|L_\mu^{-1}\|}{1 - \|L_\mu^{-1}\|\|\Delta L\|}, \tag{2.8}$$

$$\|B_\lambda^{-1}\| \le \frac{\|B_\mu^{-1}\|}{1 - \|B_\mu^{-1}\|\|\Delta B\|}.$$

Thus L_λ^{-1} *and* B_λ^{-1} *are uniformly bounded for* λ *sufficiently close to* μ. *(We know from the assumptions of Theorem 2 that* $\|B_\lambda^{-1}\| < b^{-1}, \lambda \in \Lambda$*).*

Proof. Assertion (1) is shown in [15]. Assertion (2) follows from Lemma 2.4. Assertion (3) follows from assertion (2) and the concept of a spectrum. Assertion (4) follows from the Inverse Mapping Theorem (which is an immediate corollary to the Open Mapping Theorem in the present case).

To prove assertion (5) we write

$$B_\lambda = B_\mu[I + B_\mu^{-1}\Delta B], \tag{2.9}$$

$$L_\lambda = L_\mu[I + L_\mu^{-1}\Delta L] = L_\mu[I - L_\mu^{-1}\Delta K]. \tag{2.10}$$

By the assumptions of Theorem 2, B_λ is continuous in λ and $\|B_\mu^{-1}\| < b^{-1}$. Thus we obtain (2.6) for λ sufficiently close to μ. It follows from (2.6) that B_λ^{-1} is continuous in λ.

Moreover, K_λ, and hence L_λ, are continuous in λ since B_λ^{-1} is continuous in λ and since for all $u \in \mathcal{M}_j(X, Y)$

$$\|u \circ A_\lambda - u \circ A_\mu\| \le Lip(u)\|\Delta A\| \le \|u\|_{C^{0,1}}\|\Delta A\|.$$

Also notice that L_μ^{-1} is bounded by assertion (4) above. Thus, for λ sufficiently close to μ we can invert (2.10) and obtain (2.7). It follows also that L_λ^{-1} is continuous in λ. $\square$

Now we are in position to eliminate all j-monomials in x of degree $2 \le j \le k$ from $g(x, y)$.

2.7. Lemma. *The following are true under the assumptions of Theorem 1:*

1. *There is a change of variables of the form*

$$(x, y) \mapsto (x, \hat{y}), \qquad \hat{y} = y + u(x),$$

which eliminates all j*-monomials in* $x, 2 \le j \le k$, *from* $g(x, y)$.

2. *Moreover,* $u(x)$ *is a sum*

$$u(x) = u_2(x) + \cdots + u_k(x),$$

where $u_j \in \mathcal{M}_j(X, Y), 2 \le j \le k$. *Each* $u_j(x)$ *is unique in* $\mathcal{M}_j(X, Y)$. *Thus* $u(x)$ *is unique in* $\bigcup_{j=2}^k \mathcal{M}_j(X, Y)$.

3. *The new map takes the form* $(x, \hat{y}) \mapsto (x^*, \hat{y}^*)$*, where*

$$\begin{aligned} x^* &= Ax + f(x, \hat{y} - u(x)), \\ \hat{y}^* &= B\hat{y} + h_{k+1}(x, \hat{y}). \end{aligned} \tag{2.11}$$

and

$$|h_{k+1}(x, \hat{y})| = O(|x|^{k+\delta}) + O(|\hat{y}|[|x| + |\hat{y}|]). \tag{2.12}$$

If $\delta = 0$*, replace* $O(|x|^{k+\delta})$ *by* $o(|x|^j)$*.*

4. *Let*

$$y = v(x) = v_1(x) + v_2(x) + \cdots + v_k(x) + v^{(k)}(x),$$

where $v_1 \in L(X, Y)$*,* $v_j \in \mathcal{M}_j(X, Y)$ *for* $2 \le j \le k$*,* $|v^{(k)}(x)| = O(|x|^{k+\delta})$ *when* $0 < \delta \le 1$*, and* $|v^{(k)}(x)| = o(|x|^k)$ *when* $\delta = 0$*. Then* $y = v(x)$ *is invariant under the original map iff* $v_1(x) \equiv 0$*,* $v_j = -u_j, 2 \le j \le k$*, and* $\hat{y} = v^{(k)}(x)$ *is invariant under the new map (2.11). Thus,*

$$v = v^{(k)} - u_2 - \cdots - u_k. \tag{2.13}$$

5. *Consider the family of maps in Theorem 2. Let* $\mu \in \Lambda$ *be fixed but arbitrary. Then, as* $\lambda \to \mu$*,*

$$\|u_{\lambda,j} - u_{\mu,j}\|_{C^{k,\delta}} \to 0. \tag{2.14}$$

Proof. Assume that we have already eliminated symmetric i-monomials in x for $2 \le i \le j-1$. In order not to complicate the notation we will continue to denote the new nonlinear y-component by $g(x, y)$.

Write the new $g(x, y)$ as $g(x, y) = g_j(x) + \bar{h}_{j+1}(x, y)$, where $g_j(x) \in \mathcal{M}_j(X, Y)$ and $|\bar{h}_{j+1}(x, y)| = O(|x|^{j+s}) + O(|y|[|x| + |y|])$, where $s = 1$ for $2 \le j \le k-1$ and $s = \delta$ for $j = k$. If $\delta = 0$, replace $O(|x|^{k+\delta})$ by $o(|x|^k)$. If we can find $u_j \in \mathcal{M}_j(X, Y)$ such that

$$u_j(x) = B^{-1}u_j(Ax) + B^{-1}g_j(x), \tag{2.15}$$

we set $\bar{y} = y + u_j(x)$. Since $f(0, 0) = 0$, it follows that for some $0 \le c \le 1$

$$\begin{aligned} |u_j(x^*) - u_j(Ax)| &\le \|Du_j(Ax + cf(x, y))\| |f(x, y)| \\ &\le O([|x|^{j-1} + |y|][|x|^2 + |y|^2]) \le O(|x|^{j+1} + |y|[|x| + |y|]). \end{aligned}$$

It follows that after eliminating $g_j(x)$ the map takes the following form in the $(x, \bar{y})$ variables:

$$\begin{aligned} x^* &= Ax + f(x, \bar{y} - u_j(x)), \\ \bar{y}^* &= y^* + u_j(x^*) = B\bar{y} + h_{j+1}(x, \bar{y}), \\ h_{j+1}(x, \bar{y}) &= \bar{h}_{j+1}(x, \bar{y} - u_j(x)) + u_j(x^*) - u_j(Ax). \end{aligned} \tag{2.16}$$

But $|u_j(x)| = O(|x|^j)$. Thus,

$$\begin{aligned} |\bar{h}_{j+1}(x, \bar{y} - u_j(x))| &= O(|x|^{j+s}) + O(|\bar{y}|(|x| + |\bar{y}|)), \\ \|h_{j+1}(x, \bar{y})\| &= O(|x|^{j+s}) + O(|\bar{y}|(|x| + |\bar{y}|)). \end{aligned}$$

When $\delta = 0$, replace $O(|x|^{j+s})$ by $o(|x|^j)$.

It remains to solve the homological equation (2.15) which can be rewritten as

$$L_{A,B}u_j = B^{-1}g_j. \tag{2.17}$$

By (3) and (4) of Lemma 2.6, $L_{A,B} : \mathcal{M}_j(X,Y) \to \mathcal{M}_j(X,Y)$ is a bijection and $L_{A,B}^{-1}$ is continuous (equivalently bounded). It follows that we can solve (2.17) uniquely and find the desired

$$u_j = L_{A,B}^{-1}B^{-1}g_j.$$

Applying this procedure successively for $j = 2,3,\cdots,k$, we obtain the desired $u(x) = u_2(x) + \cdots + u_k(x)$. Since each $u_j(x)$ is unique in $\mathcal{M}_j(X,Y)$, it follows that $u(x)$ is unique in $\bigcup_{j=2}^k \mathcal{M}_j(X,Y)$.

Assertion (3) follows by straightforward calculations. Assertion (4) follows from the same calculations that led to (2.15).

To prove assertion (5) notice that

$$\begin{aligned} u_{\lambda,j} &= L_\lambda^{-1}B_\lambda^{-1}g_{\lambda,j}, \\ \|u_{\lambda,j} - u_{\mu,j}\|_{C^{k,\delta}} &\le b^{-1}\|g_{\lambda,j}\|_{C^{k,\delta}}\|L_\lambda^{-1} - L_\mu^{-1}\| \\ &\quad + \|L_\mu^{-1}\|\|g_{\lambda,j}\|_{C^{k,\delta}}\|B_\lambda^{-1} - B_\mu^{-1}\| \\ &\quad + b^{-1}\|L_\mu^{-1}\|\|g_{\lambda,j} - g_{\mu,j}\|_{C^{k,\delta}}. \end{aligned} \tag{2.18}$$

By Lemma 2.6, L_λ^{-1} and B_λ^{-1} are continuous near $\lambda = \mu$ and the family $\{L_\lambda^{-1}\}$ is uniformly bounded near $\lambda = \mu$. It follows that $\|u_{\lambda,j} - u_{\mu,j}\|_{C^{k,\delta}} \to 0$ as $\lambda \to \mu$. □

The following lemma is obvious.

2.8. Lemma. *Let $0 < \gamma < 1$ and define*

$$\Phi^{(\gamma)}(z) := \frac{1}{\gamma}\Phi(\gamma z) = D\Phi(0) + F^{(\gamma)}(z) = D\Phi(0) + \frac{1}{\gamma}F(\gamma z), \quad |z| \le 1. \tag{2.19}$$

Then the graph of

$$y = w_{(\gamma)}(x), \qquad |x| \le 1,$$

is invariant under $\Phi^{(\gamma)}$ iff the graph of

$$w(x) = \gamma w_{(\gamma)}(x/\gamma), \qquad |x| \le \gamma, \tag{2.20}$$

is invariant under Φ.

2.9. Corollary. *Let $k \ge 1, r = k + \delta$ and $0 \le \delta \le 1$. In view of Lemma 2.7 and Lemma 2.8, for all $\epsilon > 0$, for sufficiently small $\gamma > 0$, $F^{(\gamma)}(x,y)$ satisfies the following estimates for $(x,y) \in B_1(0,0)$:*
For $0 \le j \le k$,

$$\|D^jF^{(\gamma)}\| \le O(\epsilon), \tag{1}$$

$$H_\delta(D^kF^{(\gamma)}) \le O(\epsilon), \tag{2}$$

$$|g^{(\gamma)}(x,y)| \le O(\epsilon)(|x|^{k+\delta} + |y|), \tag{3}$$

$$\|D_x^j g^{(\gamma)}(x,y)\| \le O(\epsilon)(|x|^{r-j} + |y|), \tag{4}$$

$$\|D_x^k g^{(\gamma)}(x,y) - D_x^k g^{(\gamma)}(x',y')\| \le O(\epsilon)[|x-x'|^\delta + |y-y'|^\delta]. \tag{5}$$

For $0 \le j \le k-1$,

$$\|D_x^j g^{(\gamma)}(x,y) - D_x^j g^{(\gamma)}(x',y)\| \le O(\epsilon)|x-x'|[(|x|+|x'|)^{r-j-1} + |y|], \tag{6}$$

$$\|D_x^j g^{(\gamma)}(x,y) - D_x^j g^{(\gamma)}(x,y')\| \le O(\epsilon)|y-y'|, \tag{7}$$

$$\|D_x^j g^{(\gamma)}(x,y) - D_x^j g^{(\gamma)}(x,y')\| \le O(\epsilon)|y-y'|^\delta[(|y|+|y'|)^{1-\delta} + |x|^{r-j-\delta}]. \tag{8}$$

Proof. The $O(\epsilon)$ factors in all these estimates follow from Lemma 2.8.

Estimates (3) and (4) follow from Lemma 2.7.

Estimate (5) follows from the fact that $g^{(\gamma)}(x,y)$ is $C^{k,\delta}$.

When $0 \le j \le k-1$, $D_x^j g^{(\gamma)}(x,y)$ is $C^{k-j,\delta}$. Thus, for some $0 \le c \le 1$

$$\|D_x^j g^{(\gamma)}(x,y) - D_x^j g^{(\gamma)}(x',y)\| \le \|D_x^{j+1} g^{(\gamma)}(x + c(x'-x), y)\| |x-x'|. \tag{*}$$

Now, (6) follows from (*) and (4).

Estimate (7) follows from the fact that $D^j g^{(\gamma)}$ is uniformly Lipschitz continuous for $0 \le j \le k-1$.

Estimate (8) follows from (7) if we notice that $|y-y'| \le |y-y'|^\delta(|y|+|y'|)^{1-\delta}$. □

2.10. Remarks.

1. Notice that small ϵ for $F^{(\gamma)}|_{B_1(0,0)}$ in Lemma 2.8 corresponds to small γ for $F|_{B_{(\gamma)}(0,0)}$.
2. Corollary 2.9 and Lemma 2.8 will allow us to reduce Theorem 1 to Theorem 4 below where we study $\Phi^{(\gamma)}$ on $B_1(0,0)$ under the additional assumptions (H5)-(H9).
3. To simplify notation we will drop the superscript γ. However, we will bring it back when we Theorem 1 in Section 7.
4. From now on we will use adapted norms on X and Y such that $\|A\| < a$ and $\|B^{-1}\| < b^{-1}$.

Theorem 4. *Consider the map (2.1) in the setup of section 2.1. Let* $0 \le n \le k$. *When* $n = 0$, *let* $\delta = 1$. *When* $1 \le n \lneqq k$, *let* $0 \lneqq \delta \le 1$. *When* $n = k$, *let* $0 \le \delta \le 1$. *Let* $r = k + \delta$. *Assume the following in* $B_1(0,0)$*:*

- **(H1):** *There is* $0 < a < 1$ *such that* $\|A\| < a$.
- **(H2):** *There is* $b > 0$ *such that* $\|B^{-1}\| < b^{-1}$.
- **(H3):** $a^r < b$.
- **(H4):** $F(x,y)$ *is* $C^{n,\delta}$.
- **(H5):** $|F(x,y)| \le \epsilon o(|x|+|y|)$ *and* $\|D^j F\| < \epsilon, 0 \le j \le n$.
- **(H6):** $H_\delta(D^n F) < \epsilon$. *(Notice that when* $n = 0, \delta = 1$, *and hence* $H_\delta = Lip$*).*
- **(H7):** $\|D_x^j g(x,y)\| \le \epsilon(|x|^{k-j+\delta} + |y|), 0 \le j \le n$.
- **(H8):** *Let* $s_1 = \cdots = s_{n-1} = 1$ *and* $s_n = \delta$. *Assume that for* $0 \le j \le n$

$$\|D_x^j g(x,y) - D_x^j g(x',y)\| \le \epsilon |x-x'|^{s_j}[(|x|+|x'|)^{r-j-s_j} + |y|]$$

(H9): *Let* $l_j = 1 - s_j, 0 \le j \le n-1$. *If* $n < k$, *let* $l_n = 1 - s_n$. *If* $n = k$, *let* $l_n = 0$. *Assume*

$$\|D_x^j g(x,y) - D_x^j g(x,y')\| \le \epsilon |y-y'|^{s_j}[(|y|+|y'|)^{l_j} + |x|^{r-j-s_j}] \qquad 0 \le j \le n$$

Then for sufficiently small $\epsilon > 0$ *the map (2.1) has an invariant manifold of the form* $y = w(x), x \in B_1(0) \subset X$ *and* $w \in C^{n,\delta}(B_1(0), Y)$. *Moreover,* $w(x)$ *is unique in the class of functions that satisfy the following for* $0 \le j \le n$ *and* $|x| \le 1$*:*

$$\|D^j w(x)\| \le |x|^{r-j} \tag{2.21}$$

$$\|D^j w(x) - D^j w(x')\| \le |x-x'|^{s_j}(|x|^{r-j-s_j} + |x'|^{r-j-s_j}) \tag{2.22}$$

If $\delta = 0$ *omit (2.22) for* $j = n$.

2.11. Remarks. 1. Hypothesis (H9) needs some explanation. The map of Theorem 1 is $C^{k,\delta}$. Using Lemma 2.7 and Lemma 2.8 we showed in (5)and (7) of Corollary 2.9 that $g(x,y)$ satisfies

$$\|D_x^j g(x,y) - D_x^j g(x,y')\| \le \epsilon |y-y'|^{s_j}, \qquad 0 \le j \le k. \tag{2.23}$$

We also showed that $g(x,y)$ can be assumed to satisfy (8), and hence, (H9).

But why (H9) and not (2.23)? We will prove Theorem 4 by induction on n. In estimate (4.9) in the proof of part (6) of Lemma 4.3 we will need $D_x g$ to satisfy an estimate of the form (H9) and not (2.23) to cover the case $n = 1$ and $s_1 = \delta \lneqq 1$. If $s_1 = 1$, we can use (2.23). Also, in (4) of Lemma 5.4 we will need $D_x^m g$ to satisfy an estimate of the form (H9) to cover the case $n = m$ and $s_m = \delta \lneqq 1$.

2. Theorem 1 will follow from Theorem 4. Briefly, let k and L be as in Theorem 1 and let $n = k$ and $r = n + \delta = k + \delta$. Thus $a^r < b$ which is hypothesis (H3). By Lemma 2.7, Corollary 2.9, and Lemma 2.8 we can assume that F and g satisfy assumptions (H5)-(H9).
3. Notice that in Theorem 4 we did not need the non-resonance condition nor the high degree of smoothness. These two requirements are needed in Theorem 1 in order to reduce $g(x,y)$ to a form that satisfies (H7)-(H9). What is really needed in the proof of Theorem 4 is to have the same exponent r in both (H3) and (H7)-(H9).
4. We will prove Theorem 4 by induction on $n \ge 1$. However, we will find it essential and possible to use the inequality $a^{k+\delta} < b$. If we used induction on k we would need to use the inequalities $a^{j+\delta} < b$ for $0 \le j \le L-1$. But these inequalities are not part of our assumptions.
5. In Section 3 we show the existence of a $C^{0,1}$ invariant manifold for the map in Theorem 4. In this case, $\delta = 1$. We will need to use norms which are appropriate for the class of functions satisfying (2.21) and (2.22). In particular we cannot use the usual Lipschitz constant $Lip(u)$ since $Lip(B^{-1}u \circ A - B^{-1}v \circ A) \le b^{-1}aLip(u-v)$. But $b^{-1}a \not< 1$. However, $b^{-1}a^{k+\delta} < 1$. Instead of $Lip(u)$ we use $L_{r,o}(u)$ which is given in Definition 3.1 below.
6. In section 4 we prove Theorem 4 when $n = 1$ and $0 \lneqq \delta \le 1$ using the Fiber Contraction Theorem [10, page 136]. It is here that the problem of a possible loss of differentiability arises. It is straight forward to show that we have a fiber contraction. The difficulty is to

show that *the fiber contraction is continuous in the base points*. (The base points are the functions $u(x)$). When $0 \lneqq \delta \leq 1$ we have some form of uniform Hölder continuity which allows us to show that the fiber contraction is continuous in the base points and conclude that the manifold is $C^{1,\delta}$ in Section 4. If $\delta = 0$ and we are in a finite dimensional space (it is enough that X is finite dimensional) then we can use the fact that a continuous map on a closed bounded ball is uniformly continuous. This statement is not true in an infinite dimensional Banach space.

7. In Section 5 we use the Fiber Contraction Theorem again to prove Theorem 4 for $2 \leq n \leq k$ and $0 \lneqq \delta \leq 1$.
8. When $\delta = 0$ we have, from Section 5, a $C^{k-1,1}$ invariant manifold. We devote Section 6 to showing that this manifold is in fact C^k. We do this as follows: Let the $C^{k-1,1}$ invariant manifold be given by $y = w(x)$. Although the fiber contraction we have is not necessarily continuous in the base points, it still has an invariant fiber $\mathcal{F}_{\Delta^{k-1}w}$ above $\Delta^{k-1}w$ since $\Delta^{k-1}w$ is a unique attractive fixed point in the base. Thus, the fiber $\mathcal{F}_{\Delta^{k-1}w}$ has a unique attractive fixed point $\tau \in \mathcal{F}_{\Delta^{k-1}w}$. Although τ is attractive only in the invariant fiber $\mathcal{F}_{\Delta^{k-1}w}$, it is our candidate for $D^k w$. We show that $D^k w = \tau$ by showing that given any fixed but arbitrary x and h

$$D^{k-1}w(x+h) - D^{k-1}w(x) = \int_0^1 \tau(x+th)hdt.$$

In order to do that we pick any C^k function u_o and let $u_n = \Gamma u_{n-1}, n \geq 1$, where Γ is the transformation (3.2) given below. Thus, u_n is C^k and $\Gamma u_n \to w$ as $n \to \infty$. Then, we restrict our fiber contraction to the sequence $\{\Delta^{k-1}u_n|_J\}_{n=0}^{n=\infty}$, where $J\{\xi = x + th \mid 0 \leq t \leq 1\}$ and $u_\infty := w$. This way we obtain a fiber contraction on the compact line segment $J = \xi([0,1])$ instead of the whole unit ball. We will need to show that the three sets

$$\begin{aligned}
\mathcal{U} &= \bigcup_{n=0}^{n=\infty} \{u_n(\xi) \mid \xi \in J\}, \\
\mathcal{V} &= \bigcup_{n=0}^{n=\infty} \{u_n(z_n) \mid \xi \in J\}, \\
\mathcal{Z} &= \bigcup_{n=0}^{n=\infty} \{H_{u_n}(\xi) \mid \xi \in J\}, \\
z_n &= H_{u_n}(\xi) := A\xi + f(\xi, u_n(\xi)),
\end{aligned}$$

are compact. This is not particulary hard. Now, a continuous function on a compact set is uniformly continuous. This allows us to show that the restricted fiber contraction is continuous in the base points.
9. For the rest of this article the main statements will be given as propositions. Each proposition will be preceded by one or more lemmas in which we prove the estimates needed for the proposition.

3. The $\mathbf{C^{0,1}}$ Manifold

In this section we prove Theorem 4 for $n = 0$. In this case $\delta = 1$. So, we show the existence of a unique $C^{0,1}$ invariant manifold. Also, when $n \geq 1$, the map is automatically $C^{0,1}$ regardless of the value of δ. So, the conclusions of this section apply to the following sections.

3.1. Definition. Let $C^0(B_1(0), Y)$ be the space of continuous functions from $B_1(0)$ to Y. Let $s_o = 1, q_o = r - 1$, and $p_o = r$. Also define the following:

$$\begin{aligned} N_{r,o}(u) &= \sup \left\{ \frac{|u(x)|}{|x|^{p_o}} \mid 0 < |x| \leq 1 \right\}, \\ L_{r,o}(u) &= \sup \left\{ \frac{|u(x) - u(x')|}{|x - x'|^{s_o}(|x| + |x'|)^{q_o}} \mid x \neq x'; x, x' \in B_1(0) \right\}, \\ \mathfrak{Z}_{r,o} &= \{u \in C^0(B_1(0), Y) \mid N_{r,o}(u) < \infty\}, \\ \mathfrak{X}_{r,o} &= \{u \in \mathfrak{Z}_{r,o} \mid u(0) = 0, L_{r,o}(u) \leq 1, N_{r,o}(u) \leq 1\}. \end{aligned}$$

3.2. Remarks. 1. Since convergence in the $N_{r,o}$ norm implies uniform convergence on bounded sets, it follows that $(\mathfrak{Z}_{r,o}, N_{r,o})$ is a Banach space and that $\mathfrak{X}_{r,o}$ is a closed subset.

2. The graph of a function $y = u(x)$ is invariant under the map (2.1) if and only if $u(x)$ satisfies

$$\begin{aligned} u(x) &= B^{-1}u(z) - B^{-1}g(x, u(x)), \\ z &= Ax + f(x, u(x)). \end{aligned} \tag{3.1}$$

Therefore we define a transformation $\Gamma_F : \mathfrak{X}_{r,o} \to \mathfrak{X}_{r,o}$ by

$$\begin{aligned} \Gamma_F u(x) &= B^{-1}u(z) - B^{-1}g(x, u(x)), \\ z &= H_u(x) := Ax + f(x, u(x)). \end{aligned} \tag{3.2}$$

3. We will be looking for a fixed point for Γ_F. The transformation (3.2) is not the usual graph transform. The usual graph transformation is

$$T_F u(x) = Bu(z) + g(z, u(z)), \qquad x = Az + f(z, u(z)). \tag{3.3}$$

To use (3.3) we need to invert A which is not necessarily invertible. Even if A were invertible, we would need to replace the condition $a^r < b$ by the gap condition $\|B\|\|A^{-1}\|^r < 1$. But this condition would give us just the usual pseudo-unstable manifold theorem and not the new theorem we are trying to prove.

4. Now, u is a fixed point for Γ_F if and only if it satisfies (3.1). But u satisfies (3.1) if and only if it is a fixed point of T_F.

5. For the rest of this work we will use the convention that $t = t(\epsilon) \to 0$ as $\epsilon \to 0$, where $t(\epsilon)$ is continuous and may vary from one estimate to another but depends always on F and not on $u \in \mathfrak{X}_{r,o}$.

6. Since $b^{-1}a^r < 1$ we have the following: For sufficiently small $\epsilon > 0$, $\theta = b^{-1}a^r(1 + t) < 1$. So, for the rest of this work θ will stand for such a quantity that can be made less than one by taking $\epsilon > 0$ small.

The following lemma is obvious.

3.3. Lemma. *Let $u \in \mathfrak{X}_{r,o}$. Let $x \neq x'$. Then, for $1 \leq j \leq n$, the following are true:*

$$\begin{aligned} |z| = |H_u(x)| \leq\ & \|A\||x| + |f(x, u(x)| \\ \leq\ & a(1+t)|x| \leq 1, \end{aligned} \tag{3.4}$$

$$H_u(0) = 0, \tag{3.5}$$

$$|H_u(x) - H_u(x')| \leq a(1+t)|x - x'|. \tag{3.6}$$

Proof. The assertions follow immediately from the facts that $f(0,0) = 0, Lip(f) < \epsilon, N_{r,o}(u) \leq 1$ and $|x| \leq 1$. □

3.4. Proposition. *For sufficiently small $\epsilon > 0$, there is $\theta = b^{-1}a^r(1+t) \in (0,1)$ such that the following are true:*

1. *For all $u \in \mathfrak{X}_{r,o}$, $N_{r,o}(\Gamma_F u) < \theta N_{r,o}(u) + t < 1$.*
2. *For all $u \in \mathfrak{X}_{r,o}$, $L_{r,o}(\Gamma_F u) < \theta L_{r,o}(u) + t < 1$.*
3. *Thus Γ_F indeed maps $\mathfrak{X}_{r,o}$ to itself.*

Proof. (1) Let $u \in \mathfrak{X}_{r,o}$. By (H7) with $j = 0$ and (3.4)we obtain

$$\begin{aligned} |\Gamma_F u(x)| &\leq b^{-1} N_{r,o}(u)|z|^r + t|x|^r \\ &\leq b^{-1}(a+t)^r N_{r,o}(u)|x|^r + t|x|^r \\ &\leq (\theta N_{r,o}(u) + t)|x|^{p_o} \leq (\theta + t)|x|^{p_o}, \\ \theta &= b^{-1}a^r(1+t), \end{aligned}$$

for sufficiently small $\epsilon > 0, 0 < \theta + t < 1$. It follows that $N_{r,o}(\Gamma_F u) < \theta N_{r,o}(u) + t < 1$.
(2) Let $x \neq x'$, $z = H_u(x)$, and $z' = H_u(x')$. Recall that $s_o + q_o = p_o = r$. By Lemma 3.3 and (H8) we obtain

$$\begin{aligned} |\Gamma_F u(x) - \Gamma_F u(x')| \leq\ & b^{-1} L_{r,o}(u)|z - z'|^{s_o}(|z| + |z'|)^{q_o} \\ & + \epsilon|x - x'|^{s_o}(|x| + |x'|)^{q_o} \\ \leq\ & [\theta L_{r,o}(u) + t]|x - x'|^{s_o}(|x| + |x'|)^{q_o}. \end{aligned} \tag{3.7}$$

Thus, for sufficiently small ϵ, $L_{r,o}(\Gamma_F u) \leq \theta L_{r,o}(u) + t < 1$.
(3) Also, $u(0) = 0$ implies that $\Gamma_F u(0) = 0$. Then by (1) and (2) $\Gamma_F u \in \mathfrak{X}_{r,o}$. □

3.5. Lemma. *Let u and v be in $\mathfrak{X}_{r,o}$. Let $z = H_u(x)$ and $z^* = H_v(x)$. Then*

$$|z - z^*| \leq \epsilon N_{r,o}(u - v)|x|^{p_o}, \tag{3.8}$$

$$|u(z) - v(z^*)| \leq a^r[1+t]N_{r,o}(u - v)|x|^{p_o}. \tag{3.9}$$

Proof. Since $F \in C^{0,1}(B_1(0), Y)$, we have

$$\begin{aligned} |z - z^*| &\leq |f(x, u(x)) - f(x, v(x))| \\ &\leq \epsilon|u(x) - v(x)| \leq \epsilon N_{r,o}(u - v)|x|^{p_o}. \end{aligned}$$

By (3.4) and(3.8),

$$\begin{aligned}|u(z) - v(z^*)| &\le |u(z) - v(z)| + |v(z) - v(z^*)| \\ &\le N_{r,o}(u - v)(a + t)^{p_o}|x|^{p_o} + L_{r,o}(v)|z - z^*|^{s_o}(|z| + |z^*|)^{q_o} \\ &\le N_{r,o}(u - v)a^r[1 + t]|x|^{p_o}.\end{aligned}$$

□

3.6. Proposition. *For sufficiently small $\epsilon > 0$, $\Gamma_F u$ is uniformly Lipschitz continuous in u in the norm $N_{r,o}$. More precisely, there is $\theta \in (0, 1)$ such that for all $u, v \in \mathfrak{X}_{r,o}$*

$$N_{r,o}(\Gamma_F u - \Gamma_F v) < \theta N_{r,o}(u - v).$$

Thus $\Gamma_F : \mathfrak{X}_{r,o} \to \mathfrak{X}_{r,o}$ is a contraction, and hence, has a unique attractive fixed point $w \in \mathfrak{X}_{r,o}$.

Proof. Let u and v be in $\mathfrak{X}_{r,o}$. Let $z = H_u(x)$ and $z^* = H_v(x)$. By Lemma 3.3 and Lemma 3.5 we have

$$\begin{aligned}|\Gamma_F u(x) - \Gamma_F v(x)| &\le b^{-1}[|u(z) - u(z^*)| + |g(x, u(x)) - g(x, v(x))|] \\ &\le b^{-1}(a^r + t)N_{r,o}(u - v)|x|^r + b^{-1}\epsilon|u(x) - v(x)| \\ &\le b^{-1}a^r(1 + t)N_{r,o}(u - v)|x|^r \\ &= \theta N_{r,o}(u - v)|x|^r,\end{aligned}$$

and $\theta \in (0, 1)$, for sufficiently small $\epsilon > 0$. Thus $N_{r,o}(\Gamma_F u - \Gamma_F v) < \theta N_{r,o}(u - v)$. □

4. The $\mathbf{C^{1,\delta}}$ Manifold, $\mathbf{0} \lneqq \delta \le \mathbf{1}$

Let $w \in \mathfrak{X}_{r,o}$ be the unique fixed point obtained in Section 3. If w is C^1 it satisfies

$$\begin{aligned}Dw(x) = B^{-1}Dw(z)[A + Df(x, w(x))(I, Dw(x))] \\ - B^{-1}Dg(x, w(x))(I, Dw(x))\end{aligned} \tag{4.1}$$

It is obvious that Dw will be a fixed point for a transformation over an appropriately constructed function-space which we define presently.

4.1. Definition. When $1 \le n + \delta \le r = k + \delta$ define the following:

1. Extend (s_o, q_o, p_o) as defined in Definition 3.1 to

$$s_j = 1, \qquad 0 \le j \le n - 1, \tag{4.2}$$

$$s_j = \delta, \qquad j = n, \tag{4.3}$$

$$q_j = r - j - s_j, \qquad 0 \le j \le n, \tag{4.4}$$

$$p_j = r - j, \qquad 0 \le j \le n. \tag{4.5}$$

2. Let $X^j = X \times \cdots \times X$, j times. Let $L(X^j, Y)$ be the space of symmetric continuous multilinear maps from X^j to Y. Define the following for $\sigma : B^1(0) \to L(X^j, Y), j \geq 2$:
$$\begin{aligned}
\|\sigma\| &= \sup_{|x|\leq 1} \|\sigma(x)\|,\\
N_{r,j}(\sigma) &= \sup_{0<|x|\leq 1} \frac{\|\sigma(x)\|}{|x|^{p_j}},\\
L_{r,j}(\sigma) &= \sup\left\{ \frac{\|\sigma(x)-\sigma(x')\|}{|x-x'|^{s_j}(|x|+|x'|)^{q_j}} \mid x \neq x', x, x' \in B_1(0)\right\}.
\end{aligned}$$
3. Also define the following spaces for $1 \leq j \leq k$:
$$\begin{aligned}
\mathfrak{Z}_{r,j} &= \{\sigma \in C^0(B_1(0), L(X^j, Y)) \mid N_{r,j}(\sigma) < \infty\},\\
\mathfrak{X}_{r,j} &= \{\sigma \in \mathfrak{Z}_{r,j} \mid N_{r,j}(\sigma) \leq 1, L_{r,j}(\sigma) \leq 1, \sigma(0) = 0\}.
\end{aligned}$$
4. If h is C^1 define
$$\Delta h(x,\xi) = (h(x), Dh(x)\xi), \qquad x \in B_1(0), \xi \in X.$$
5. Define the following transformation on $\mathfrak{X}_{r,o} \times \mathfrak{X}_{r,1}$:
$$\varGamma_{\Delta F}(u,\sigma) = (\varGamma_F u, \varGamma^u\sigma), \tag{4.6}$$
$$\begin{aligned}
\varGamma^u\sigma(x) &= B^{-1}\sigma(z)\mathcal{H}^u\sigma(x) + \mathcal{G}^u\sigma(x), \qquad (4.7)\\
\mathcal{H}^u\sigma(x) &= A + Df(x,u(x))(I,\sigma(x)),\\
\mathcal{G}^u\sigma(x) &= -B^{-1}Dg(x,u(x))(I,\sigma(x)),\\
z &= H_u(x) = Ax + f(x,u(x)),
\end{aligned}$$
where $\varGamma_F(u)$ is given by (3.2).

Since convergence in the $N_{r,j}$ norm implies uniform convergence on bounded sets, it follows that $(\mathfrak{Z}_{r,j}, N_{r,j})$ is a Banach space and that $\mathfrak{X}_{r,j}$ is a closed subset.

If we added $L_{r,j}(u) < \infty$ to the definition of $\mathfrak{Z}_{r,j}$ and used the norm $\max\{N_{r,j}, L_{r,j}\}$, it would be a Banach space and $\mathfrak{X}_{r,j}$ would be a closed subset. However, we would be able to prove that $\varGamma^u\sigma$ is continuous in u only if we replace δ by $0 \leq \gamma < \delta$ in the definition of $N_{r,1}$ and $L_{r,1}$.

In order to study $\varGamma_{\Delta F}$ we need the fiber contraction theorem [10, page 136].

4.2. The Fiber Contraction Theorem. *Let $\mathfrak{B}$ be a metric space and $\phi : \mathfrak{B} \to \mathfrak{B}$ a map having an attractive fixed point p (i.e. $\forall u \in \mathfrak{B} : \phi^n(u) \to p$ as $n \to \infty$.) Let $\mathfrak{Y}$ be a metric space and $\{\psi^u \mid u \in \mathfrak{B}\}$ a family of maps $\psi^u : \mathfrak{Y} \to \mathfrak{Y}$ such that the formula $\varGamma(u,\sigma) = (\phi(u), \psi^u(\sigma))$ defines a continuous map $\varGamma : \mathfrak{B} \times \mathfrak{Y} \to \mathfrak{B} \times \mathfrak{Y}$. Let $q \in \mathfrak{Y}$ be a fixed point for ψ^p. Then, (p,q) is an attractive fixed point for $\varGamma$ provided that*
$$\limsup_{n\to\infty} Lip(\psi^{\phi^n(u)}) < 1, \qquad \forall u \in \mathfrak{B}. \tag{*}$$

We will apply the Fiber Contraction Theorem to $\varGamma_{\Delta F}(u,\sigma)$. We will show that there is $\theta \in (0,1)$ such that $Lip\varGamma^u < \theta < 1, \forall u \in \mathfrak{X}_{r,o}$, which is stronger than (*). In order to apply the Fiber Contraction Theorem we need a few estimates that are given in the following few lemmas.

4.3. Lemma. *Let $u \in \mathfrak{X}_{r,o}$, $v \in \mathfrak{X}_{r,o}$ and $\sigma \in \mathfrak{X}_{r,1}$. Then,*

1. $\|\mathcal{H}^u\sigma(x)\| \leq \|A\| + \epsilon \leq a + \epsilon$.
2. $\|\mathcal{G}^u\sigma(x)\| \leq t|x|^{p_1}$.
3. $\|\mathcal{H}^u\sigma(x) - \mathcal{H}^v\sigma(x)\| \leq \epsilon[N_{r,o}(u-v)]^{s_1}|x|^{p_1+rs_1}$.
4. $\|\mathcal{G}^u\sigma(x) - \mathcal{G}^v\sigma(x)\| \leq t[N_{r,o}(u-v)]^{s_1}|x|^{p_1+rs_1}$.
5. $\|\mathcal{H}^u\sigma(x) - \mathcal{H}^u\sigma(x')\| \leq t|x-x'|^{s_1}$.
6. $\|D_xg(x,u(x)) - D_xg(x',u(x'))\| \leq t|x-x'|^{s_1}(|x|+|x'|)^{q_1}$
7. $\|\mathcal{G}^u\sigma(x) - \mathcal{G}^u\sigma(x')\| \leq t|x-x'|^{s_1}(|x|+|x'|)^{q_1}$.
8. *Let $z = H_u(x)$ and $z^* = H_v(x)$. Then,*
$$\begin{aligned}\|\sigma(z) - \sigma(z^*)\| &\leq t[N_{r,o}(u-v)]^{s_1}\|x\|^{q_1+rs_1}\\ &\leq t[N_{r,o}(u-v)]^{s_1}\|x\|^{p_1}.\end{aligned}$$

Proof. Since $N_{r,1}(\sigma) \leq 1$ and $\|x\| \leq 1$, we have $\|\sigma(x)\| \leq 1$.
(1) Assertion (1) follows from (H5) and the fact that $\|\sigma(x)\| \leq 1$.
(2) Using (H5) and (H7)we obtain
$$\|\mathcal{G}^u\sigma(x)\| \leq b^{-1}\|D_xg(x,u(x))\| + b^{-1}\|D_yg(x,u(x))\|\|\sigma(x)\| \leq t|x|^{p_1}.$$
(3)&(4) We prove (3). The proof of (4) is similar. By hypothesis (H6) on DF we have
$$\begin{aligned}\|\mathcal{H}^u\sigma(x) - \mathcal{H}^v\sigma(x)\| &\leq \|Df_y(x,u(x)) - Df_y(x,v(x))\|\|\sigma(x)\|\\ &\leq \epsilon|u(x)-v(x)|^{s_1}|x|^{p_1} \leq \epsilon(N_{r,o}(u-v))^{s_1}|x|^{p_1+rs_1}.\end{aligned}$$
(5) As for the fifth inequality, we have
$$\begin{aligned}\|\mathcal{H}^u\sigma(x) - \mathcal{H}^u\sigma(x')\| &\leq \|D_yf(x',u(x'))\|\|\sigma(x)-\sigma(x')\|\\ &\quad + \|D_yf(x,u(x)) - D_yf(x',u(x'))\|\|\sigma(x)\|\\ &\quad + \|D_xf(x,u(x)) - D_xf(x',u(x'))\|\\ &\leq \epsilon L_{r,1}(\sigma)|x-x'|^{s_1}(|x|+|x'|)^{q_1}\\ &\quad + t|x-x'|^{s_1}N_{r,1}(\sigma)|x|^{p_1} + t|x-x'|^{s_1}\\ &\leq t|x-x'|^{s_1}.\end{aligned}$$
(6) Notice that $p_1 \geq q_1$ and hence, $|x|^{p_1} \leq |x|^{q_1} \leq (|x|+|x'|)^{q_1}$. Also, $q_o = q_1 + 1$. Using (H8) we obtain
$$\begin{aligned}\|D_xg(x,u(x)) - D_xg(x',u(x))\| &\leq \epsilon|x-x'|^{s_1}[(|x|+|x'|)^{q_1} + |u(x)|]\\ &\leq \epsilon|x-x'|^{s_1}[(|x|+|x'|)^{q_1} + |x|^{p_1}]\\ &\leq 2\epsilon|x-x'|^{s_1}(|x|+|x'|)^{q_1}.\end{aligned} \tag{4.8}$$

Notice that $(s_1+l_1)q_o = q_o = 1 + q_1$ and that $s_1r \geq s_1q_o$. Using (H9) we obtain

$$\begin{aligned}\|D_xg(x',u(x)) - D_xg(x',u(x'))\| &\leq \epsilon|u(x)-u(x')|^{s_1}[(|u(x)|+|u(x')|)^{l_1} + |x'|^{q_1}]\\ &\leq t|x-x'|^{s_1}(|x|+|x'|)^{\delta q_o}[(|x|+|x'|)^{l_1q_o} + |x'|^{q_1}]\\ &\leq t|x-x'|^{\delta}(|x|+|x'|)^{q_1}.\end{aligned} \tag{4.9}$$

Estimate (6) follows from estimates (4.8) and (4.9).
(7) Using estimate (6) of this Lemma, we obtain

$$\begin{aligned}\|\mathcal{G}^u\sigma(x)-\mathcal{G}^u\sigma(x')\| &\le \|D_yg(x',u(x'))\|\|\sigma(x)-\sigma(x')\|\\ &\quad+\|D_yg(x,u(x))-D_yg(x',u(x'))\|\|\sigma(x)\|\\ &\quad+\|D_xg(x,u(x))-D_xg(x',u(x'))\|\\ &\le \epsilon L_{r,1}(\sigma)|x-x'|^{s_1}(|x|+|x'|)^{q_1}\\ &\quad+t|x-x'|^{s_1}N_{r,1}(\sigma)|x|^{p_1}\\ &\quad+t|x-x'|^{s_1}(|x|+|x'|)^{q_1}\\ &\le t|x-x'|^{s_1}(|x|+|x'|)^{q_1}.\end{aligned}$$

(8) Recall that we are considering the case $s_1=\delta\in(0,1]$. By (3.8) and (3.4), we have

$$\begin{aligned}\|\sigma(z)-\sigma(z^*)\| &\le L_{r,1}(\sigma)|z-z^*|^{s_1}(|z|+|z^*|)^{q_1}\\ &\le t[N_{r,o}(u-v)]^{s_1}|x|^{q_1+rs_1}.\end{aligned}$$

The estimate follows from the fact that $q_1+rs_1\ge q_1+s_1=p_1$. □

4.4. Proposition. *For sufficiently small $\epsilon>0$, the transformation $\Gamma_{\Delta F}$ maps $\mathfrak{X}_{r,o}\times\mathfrak{X}_{r,1}$ to itself. More precisely, for sufficiently small $\epsilon>0$ there is $\theta=b^{-1}a^r(1+t)\in(0,1)$ such that for all $(u,\sigma)\in\mathfrak{X}_{r,o}\times\mathfrak{X}_{r,1}$, $\Gamma^u\sigma(0)=0$ and*

$$\begin{aligned}N_{r,1}(\Gamma^u\sigma)&\le\theta N_{r,1}(\sigma)+t<1,\\ L_{r,1}(\Gamma^u\sigma)&\le\theta L_{r,1}(\sigma)+t<1.\end{aligned}$$

Proof. We already know that Γ_F maps $\mathfrak{X}_{r,o}$ to itself. Let $(u,\sigma)\in\mathfrak{X}_{r,o}\times\mathfrak{X}_{r,1}$. Since $u(0)=0$ and $Df(0,u(0))=0$, it follows that if $\sigma(0)=0$ then $\Gamma^u\sigma(0)=0$. It remains to show that $N_{r,1}(\Gamma^u\sigma)\le1$ and $L_{r,1}(\Gamma^u\sigma)\le1$. By (1) and (2) of Lemma 4.3 we have

$$\begin{aligned}\|\Gamma^u\sigma(x)\| &\le b^{-1}N_{r,1}(\sigma)|z|^{p_1}\|\mathcal{H}^u\sigma(x)\|+\|\mathcal{G}^u\sigma(x)\|\\ &\le b^{-1}a^r(1+t)[N_{r,1}(\sigma)+t]|x|^{p_1}.\end{aligned}$$

Since $b^{-1}a^r<1$ and $N_{r,1}(\sigma)\le1$, we have $N_{r,1}(\Gamma^u\sigma)\le1$ for sufficiently small ϵ. It remains to show that $L_{r,1}(\Gamma^u\sigma)\le1$. Let $x\ne x', z=H_u(x)$, and $z'=H_u(x')$. By assertions (5) and (7) of Lemma 4.3 and (3.6); and since $q_1+s_1=p_1$, we have

$$\begin{aligned}\|\Gamma^u\sigma(x)-\Gamma^u\sigma(x')\| &\le b^{-1}\|\sigma(z)-\sigma(z')\|\|\mathcal{H}^u(x)\|\\ &\quad+b^{-1}\|\sigma(z')\|\|\mathcal{H}^u\sigma(x)-\mathcal{H}^u\sigma(x')\|\\ &\quad+\|\mathcal{G}^u(x)-\mathcal{G}^u(x')\|\\ &\le b^{-1}(a+t)L_{r,1}(\sigma)|z-z'|^{s_1}(|z|+|z'|)^{q_1}\\ &\quad+tN_{r,1}(\sigma)|x-x'|^{s_1}|x'|^{p_1}\\ &\quad+t|x-x'|^{s_1}(|x|+|x'|)^{q_1}\\ &\le[b^{-1}a^r(1+t)L_{r,1}(\sigma)+t]|x-x'|^{s_1}(|x|+|x'|)^{q_1}.\end{aligned}$$

Since $b^{-1}a^r < 1$, it follows that for sufficiently small ϵ

$$L_{r,1}(\Gamma^u\sigma) \leq [b^{-1}a^r(1+t)+t] < 1.$$

□

4.5. Proposition. *Let $u \in \mathfrak{X}_{r,o}$, $v \in \mathfrak{X}_{r,o}$ and $\sigma \in \mathfrak{X}_1$. Then*

$$N_{r,1}(\Gamma^u\sigma - \Gamma^v\sigma) \leq t[N_{r,o}(u-v)]^{s_1}. \tag{4.10}$$

Thus $\Gamma_{\Delta F}(u,\sigma)$ is uniformly s_1-Hölder continuous in u.

Proof. Let $z = H_u(x)$ and $z^* = H_v(x)$. By (3), (4), and (8) of Lemma 4.3 we have

$$\begin{aligned}\|\Gamma^u\sigma(x) - \Gamma^v\sigma(x)\| &\leq b^{-1}\|\sigma(z) - \sigma(z^*)\|\|\mathcal{H}^u\sigma(x)\| \\ &\quad + b^{-1}\|\sigma(z^*)\|\|\mathcal{H}^u\sigma(x) - \mathcal{H}^v\sigma(x)\| \\ &\quad + \|\mathcal{G}^u\sigma(x) - \mathcal{G}^v\sigma(x)\| \\ &\leq t[N_{r,o}(u-v)]^{s_1}|x|^{p_1},\end{aligned}$$

which implies (4.10). We also know from Proposition 3.6 that $\Gamma_F u$ is uniformly Lipschitz continuous in u. Thus $\Gamma_{\Delta F}(u,\sigma)$ is uniformly s_1-Hölder continuous in u. □

It remains to show that $\Gamma_{\Delta F}(u,\sigma)$ is continuous in σ and that $\Gamma^u\sigma$ is uniformly Lipschitz continuous in σ with Lipschitz constant less than one. This will be done simultaneously, but first we need the following lemma.

4.6. Lemma. *Let $u \in \mathfrak{X}_{r,o}$ and $\sigma, \rho \in \mathfrak{X}_1$. Then,*

1. $\|\mathcal{H}^u\sigma(x) - \mathcal{H}^u\rho(x)\| \leq \epsilon N_{r,1}(\sigma-\rho)|x|^{p_1}$.
2. $\|\mathcal{G}^u\sigma(x) - \mathcal{G}^u\rho(x)\| \leq \epsilon N_{r,1}(\sigma-\rho)|x|^{p_1}$.

Proof. The proof of (1) and (2) are identical and simple, so we prove (1).

$$\begin{aligned}\|\mathcal{H}^u\sigma(x) - \mathcal{H}^u\rho(x)\| &\leq \|D_y f(x,u(x))\|\|\sigma(x) - \rho(x)\| \\ &\leq \epsilon N_{r,1}(\sigma-\rho)|x|^{p_1}.\end{aligned}$$

□

4.7. Proposition. *For sufficiently small $\epsilon > 0$, there is $0 < \theta < 1$ such that for all u in $\mathfrak{X}_{r,o}$, and for all σ and ρ in $\mathfrak{X}_{r,1}$*

$$N_{r,1}(\Gamma^u\sigma - \Gamma^u\rho) \leq \theta N_{r,1}(\sigma-\rho).$$

In fact $\theta = b^{-1}a^r(1+t) < 1$ for sufficiently small $\epsilon > 0$. It follows that $\Gamma_{\Delta F}(u,\sigma)$ is uniformly Lipschitz continuous in σ.

Proof. Recall that $|z| = |H_u(x)| \leq (a+\epsilon)|x|$ and that $1 + p_1 = r = p_o$. By Lemma 4.6 we have

$$\begin{aligned}\|\Gamma^u\sigma(x) - \Gamma^u\rho(x)\| &\leq b^{-1}\|\sigma(z) - \rho(z)\|\|\mathcal{H}^u\sigma(x)\| \\ &\quad + b^{-1}\|\rho(z)\|\|\mathcal{H}^u\sigma(x) - \mathcal{H}^u\rho(x)\| \\ &\quad + \|\mathcal{G}^u\sigma(x) - \mathcal{G}^u\rho(x)\| \\ &\leq b^{-1}N_{r,1}(\sigma-\rho)|z|^{p_1}(a+\epsilon) + 2\epsilon N_{r,1}(\sigma-\rho)|x|^{p_1} \\ &\leq b^{-1}a^r(1+t)N_{r,1}(\sigma-\rho)|x|^{p_1}.\end{aligned}$$

Thus $N_{r,1}(\Gamma^u\sigma - \Gamma^u\rho) \leq \theta N_{r,1}(\sigma-\rho)$ with $\theta = b^{-1}a^r(1+t) < 1$ for sufficiently small $\epsilon > 0$. $\square$

4.8. Proposition. *For sufficiently small $\epsilon > 0$, $\Gamma_{\Delta F} : \mathfrak{X}_{r,o} \times \mathfrak{X}_{r,1} \to \mathfrak{X}_{r,o} \times \mathfrak{X}_{r,1}$ has a unique attractive fixed point (w, τ), and $Dw = \tau$. Let $G(w)$ is the graph of w. Then, $T_{(0,0)}G(w) = X \times \{0\}$.*

Proof. By Proposition 4.5 and Proposition 4.7, $\Gamma_{\Delta F}$ is a continuous fiber contraction. We know from Lemma 3.6 that Γ_F has a unique attractive fixed point $w \in \mathfrak{X}_{r,o}$. The fiber $\{w\} \times \mathfrak{X}_{r,1}$ is invariant under $\Gamma_{\Delta F}$. But Γ^w is a contraction. Thus Γ^w has a unique attractive fixed point $\tau \in \mathfrak{X}_{r,1}$. By the Fiber Contraction Theorem 4.2 (w, τ) is a unique attractive fixed point for $\Gamma_{\Delta F}$.

Now we would like to show that $Dw = \tau$. Recall (3) of Definition 4.1 and notice that

$$\Gamma_{\Delta F}(u, Du) = \Delta(\Gamma_F u),$$

$$(*) \qquad (\Gamma_{\Delta F})^n(u, Du) = \Delta(\Gamma_F^n u) \qquad n = 1, 2, 3, \cdots .$$

Let $u_0 = 0$ and $u_n = \Gamma_F^n u$. Then, $\lim_{n\to\infty} u_n = w$ and u_n is C^1 for $n \geq 1$. Thus , by (*), we have

$$\begin{aligned}\lim_{n\to\infty}(u_n, Du_n) &= \lim_{n\to\infty} \Delta u_n = \lim_{n\to\infty} \Delta(\Gamma_F^n u_0)\\ &= \lim_{n\to\infty}(\Gamma_{\Delta F})^n(u_0, Du_0) = (w, \tau),\end{aligned}$$

where all the limits are taken in the norms we have been using. Thus $\lim_{n\to\infty} N_{r,1}(Du_n - \tau) = 0$. Since convergence in the norm $N_{r,1}$ implies uniform convergence on bounded sets, it follows that $Dw = \tau$.

Since $Dw(0) = \tau(0) = 0$, it follows that $T_{(0,0)}G(w) = X$. Finally,

$$N_{r,1}(Dw) = N_{r,1}(\tau) \leq 1,$$
$$L_{r,1}(Dw) = L_{r,1}(\tau) \leq 1.$$

$\square$

So far we have proven the first induction step in the proof of Theorem 4 for $0 < \delta \leq 1$.

5. The $\mathbf{C^{n,\delta}}$ Manifold, $\mathbf{0 \lneqq \delta \leq 1}$

In Section 4 we proved Theorem 4 for $n = 1$ and $0 \lneqq \delta \leq 1$. The reason we did not include $\delta = 0$ is that we could not use the standard Fiber Contraction Theorem 4.2 because $\Gamma_{\Delta F}(u, \sigma)$ is not necessarily continuous in u when $\delta = 0$. That is, Proposition 4.5 is not necessarily true. The points where the argument fails are assertions (3), (4), and (8) of Lemma 4.3. These are the places where $s_1 = \delta$ appears as an exponent of $N_{r,o}(u - v)$. In assertion (8), for example, the exponent appears because $\sigma(x)$ is C^{0,s_1}, $s_1 = \delta \neq 0$.

In this section we prove Theorem 4 for $2 \leq n \leq k$ and $0 \lneqq \delta \leq 1$. In Section 6 we will prove Theorem 4 for $n = k$ and $\delta = 0$. We will explain in Remark 6.9 why we have to take this path.

5.1. Remarks. 1. Let $D^{[j]}w = (D^1w, \cdots , D^jw), j = 1, \cdots , k.$

2. Assume we have proved Theorem 4 for $n = m-1 < k$ and $0 \lneqq \delta \leq 1$ and hence have a $C^{m-1,\delta}$ invariant manifold $y = w(x), |x| \leq 1$, unique in the class of functions satisfying (2.21) and (2.22). If w is C^m, then $D^m w(x)$ satisfies

$$
\begin{aligned}
D^m w(x) = {} & B^{-1} D^m w(z)[A + Df(x, w(x))(I, Dw(x))]^m \\
& + B^{-1} Dw(z) D^m f(x, w(x))[(I, Dw(x))]^m \\
& + B^{-1} Dw(z) D_y f(x, w(x)) D^m w(x) \\
& - B^{-1} D_y g(x, w(x)) D^m w(x) \\
& + S_m^{\Delta^{m-1} w}(x) \\
& + T_m^{\Delta^{m-1} w}(D^m g(x, w(x)), Dw(x)) \\
& + P_{m-1}^{\Delta^{m-1} w}(D^{[m-1]} w(z), \Delta^{m-1} f(x, w(x)), D^{[m-1]} w(x)) \\
& + Q_{m-1}^{\Delta^{m-1} w}(\Delta^{m-1} g(x, w(x)), D^{[m-1]} w(x)), \\
z = {} & Ax + f(x, w(x)), \\
S_m^{\Delta^{m-1} w}(x) = {} & -B^{-1} D_x^m g(x, w(x)), \\
T_m^{\Delta^{m-1} w}(Dw(x), D^m g(x, u(x)) = {} & -B^{-1} D^m g(x, w(x))[(I, Dw(x))]^m - S_m^{\Delta^{m-1} w}(x),
\end{aligned}
\tag{5.1}
$$

where $P_{m-1}^{\Delta^{m-1} w}(\gamma, \alpha, \beta)$ and $Q_{m-1}^{\Delta^{m-1} w}(\alpha, \beta)$ are multilinear in the components of α, β and γ and satisfy some vanishing properties which will be described in Definition 5.2 below.

3. It is obvious that $D^m w$ should be a fixed point for some transformation on an appropriately constructed function-space which we define presently.

5.2. Definition. 1. Let $\mathfrak{Z}_{r,o}, \cdots, \mathfrak{Z}_{r,m}, \mathfrak{X}_{r,o}, \cdots, \mathfrak{X}_{r,m}$ be as in Definitions 3.1 and 4.1. Let

$$
\begin{aligned}
\mathfrak{B}_{m-1} &= \mathfrak{X}_{r,o} \times \mathfrak{X}_{r,1} \times \cdots \times \mathfrak{X}_{r,m-1}, \\
U &= (u, \Sigma) = (u, \sigma_1, \cdots, \sigma_{m-1}) \in \mathfrak{B}_{m-1}, \\
\|\Sigma\|_* &= \max\{N_{r,1}(\sigma_1), \cdots, N_{r,m-1}(\sigma_{m-1})\}, \\
\|U\|_* &= \max\{N_{r,o}(u), N_{r,1}(\sigma_1), \cdots, N_{r,m-1}(\sigma_{m-1})\}.
\end{aligned}
$$

2. Inductively, define $\mathit{\Gamma}_{\Delta^m F} : \mathfrak{B}_{m-1} \times \mathfrak{X}_{r,m} \to \mathfrak{B}_{m-1} \times \mathfrak{X}_{r,m}$ by

$$
\mathit{\Gamma}_{\Delta^m F}(U, \rho) = (\mathit{\Gamma}_{\Delta^{m-1} F}(U), \mathit{\Gamma}^U \rho), \tag{5.2}
$$

$$
\begin{aligned}
\mathit{\Gamma}^U \rho(x) = {} & B^{-1} \rho(z)[\mathcal{H}^u \sigma_1(x)]^m \\
& + B^{-1} \sigma_1(z) D_y f(x, u(x)) \rho(x) \\
& - B^{-1} D_y g(x, u(x)) \rho(x) \\
& + B^{-1} \sigma_1(z) D^m f(x, u(x))[(I, \sigma_1(x))]^m \\
& + S_m^U(x) + T_m^U(x) \\
& + P_{m-1}^U(x) + Q_{m-1}^U(x), \\
z = {} & H_u(x) = Ax + f(x, u(x)), \\
\mathcal{H}^u \sigma_1(x) = {} & A + Df(x, u(x))(I, \sigma_1(x)),
\end{aligned}
\tag{5.3}
$$

where

$$S_m^U(x) = -B^{-1} D_x^m g(x, u(x)), \tag{5.4}$$

$$\begin{aligned} T_m^U(x) &= T_m^U(D^m g(x, u(x)), \sigma_1(x)) \\ &= -B^{-1} D^m g(x, u(x))[(I, \sigma_1(x))]^m - S_m^U(x) \\ &= \sum_{\substack{i+j=m \\ j\geq 1}} \theta^{i,j}(x), \end{aligned} \tag{5.5}$$

$$\theta^{i,j}(x) = D_x^i D_y^j g(x, u(x))[\sigma_1(x)]^j,$$

$$T_m^U(\cdot, 0) \equiv 0, \tag{5.6}$$

$$T_m^U(0, \cdot) \equiv 0,$$

$$P_{m-1}^U(x) = P_{m-1}^U(\Sigma(z), \Delta^{m-1} f(x, u(x)), \Sigma(x)), \tag{5.7}$$

$$P_{m-1}^U(0, \cdot, \cdot) \equiv 0, \tag{5.8}$$

$$P_{m-1}^U(\cdot, 0, \cdot) \equiv 0,$$

$$Q_{m-1}^U(x) = Q_{m-1}^U(\Delta^{m-1} g(x, u(x)), \Sigma(x)) = \sum \psi^{i,j}_{r_1 \cdots r_{m-1}}(x), \tag{5.9}$$

$$\psi^{i,j}_{r_1 \cdots r_{m-1}}(x) = D_x^i D_y^j g(x, u(x))[\sigma_1(x)]^{r_1} \cdots [\sigma_{m-1}(x)]^{r_{m-1}},$$

$$Q_{m-1}^U(\cdot, 0) \equiv 0, \tag{5.10}$$

$$Q_{m-1}^U(0, \cdot) \equiv 0,$$

where the sum in (5.9)is taken over the finite set of indices

$$\left\{ (i, j, r_1, \cdots, r_{m-1}) \;\middle|\; i + j \leq m - 1,\ 1 \leq j = r_1 + \cdots + r_{m-1},\ i + \sum_{l=1}^{m-1} l r_l = m \right\},$$

and $P_{m-1}^U(\gamma, \alpha, \beta)$ is multilinear in the components of α, β and γ.

3. To simplify notation we will write

$$\Gamma_j := \Gamma_{\Delta^j F}, \qquad j = 0, 1, 2, \cdots, k.$$

Since convergence in the norm $\|\cdot\|_*$ implies uniform convergence on bounded sets, it follows that $\mathfrak{B}_{m-1}$ is a closed subset of the Banach space $\mathfrak{Z}_{r,o} \times \cdots \times \mathfrak{Z}_{r,m-1}$. We do not need to write $P_{m-1}^U(x)$ in great detail as we did with $Q_{m-1}^U(x)$ and $T_m^U(x)$ because of the presence of a component of $\Sigma(z)$ in each term which causes $P_{m-1}^U(x)$ to be of order $|x|^{m-1}$.

5.3. Induction Hypothesis IH(m). Assume that $\Gamma_{\Delta^{m-1}F}$ maps $\mathfrak{B}_{m-1}$ to itself and has a unique attractive fixed point $W := \Delta^{m-1} w = (w, \Omega) \in \mathfrak{B}_{m-1}$.

5.4. Lemma. *For sufficiently small $\epsilon > 0$, the following are true for $M = P, Q$ and $R = S, T$:*

1. $\|M_{m-1}^U(x)\| \leq t|x|^{p_{m-1}} \leq t|x|^{p_m}$.
2. $\|R_m^U(x)\| \leq t|x|^{p_m}$.
3. $\|M_{m-1}^U(x) - M_{m-1}^U(x')\| \leq t|x - x'|(|x| + |x'|)^{q_{m-1}} \leq t|x - x'|(|x| + |x'|)^{q_m}$.
4. $\|R_m^U(x) - R_m^U(x')\| \leq t|x - x'|^{s_m}(|x| + |x'|)^{q_m}$.

5. $\|M^U_{m-1}(x) - M^V_{m-1}(x)\| \le t\|U - V\|_*|x|^{p_{m-1}} \le t\|U - V\|_*|x|^{p_m}$.
6. $\|R^U_m(x) - R^V_m(x)\| \le t[\|U - V\|_*]^{s_m}|x|^{p_m}$.
7. *Let* $z = H_u(x)$ *and* $z^* = H_v(x)$. *Then*

$$\|\rho(z) - \rho(z^*)\| \le t[N_{r,o}(u - v)]^{s_m}|x|^{p_m} \le t[\|U - V\|_*]^{s_m}|x|^{p_m}.$$

Proof. The Lemma follows by straightforward calculations similar to those of Lemma 4.3 if we recall properties (5.4)-(5.10) and the multilinearity of P^U_{m-1}. Notice also that $\Delta^{m-1}f(x, y)$ and $\Delta^{m-1}g(x, y)$ are $C^{1,\delta}$ and that $\Sigma(x)$ is $C^{0,1}$ since $s_o = \cdots = s_{m-1} = 1$. ☐

5.5. Proposition. *For sufficiently small* $\epsilon > 0$, *the following are true:*

1. Γ_m *maps* $\mathfrak{B}_{m-1} \times \mathfrak{X}_{r,m}$ *to itself.*
2. $\Gamma^U_m\rho$ *is uniformly* s_m*-Hölder continuous in* U. *In fact,*

$$\|\Gamma^U_m\rho(x) - \Gamma^V_m\rho(x)\| \le t[\|U - V\|_*]^{s_m}|x|^{p_m}.$$

3. $\|\Gamma^U\rho(x) - \Gamma^U\tau(x)\| \le \theta N_{r,m}(\rho - \tau)|x|^{p_m}$ *where* $\theta = b^{-1}a^r(1 + t) < 1$.
4. *Thus,* Γ^W *has a unique attractive fixed point* μ *in the invariant fiber* $\{W\} \times \mathfrak{X}_{r,m}$.
5. *By the Fiber Contraction Theorem, it follows that* (W, μ) *is a unique attractive fixed point for* $\Gamma_m : \mathfrak{B}_{m-1} \times \mathfrak{X}_{r,m} \to \mathfrak{B}_{m-1} \times \mathfrak{X}_{r,m}$.
6. $D^m w = \mu$.

Proof. Again assertions (1)-(5) follow by straightforward calculations if we recall the points mentioned in the proof of Lemma 5.4.

Assertion (6) follows from the fact that $\Gamma_{\Delta^m F}(\Delta^m u) = \Delta^m(\Gamma_F u)$ using an argument similar to that in the proof of Proposition 4.8. ☐

6. The $\mathbf{C^k}$ Manifold, $\delta = 0$

In this section we prove Theorem 4 for $n = k = r$ and $\delta = 0$. In Remark 6.9 at the end of this section we will explain why it is necessary to consider the case $\delta = 0$ for $n = k$ and not $n < k$.

When $\delta = 0$, $\Gamma_k(U, \rho)$ of Section 5 is still a fiber contraction but not necessarily continuous in U. More precisely, $\Gamma^U\rho$ is not continuous in the base point U. Even at the C^1 level, $\Gamma^u\rho$ is not necessarily continuous in u. That is, Proposition 4.5 is not true. The points where the argument fails are assertions (3), (4), and (8) of Lemma 4.3. These are the places where $s_1 = \delta = 0$ appears as an exponent of $N_{r,o}(u-v)$. At the C^k level, assertion (2) of Proposition 5.5 is false due to the falsehood of assertions (6) and (7) of Lemma 5.4 when $m = k$, $s_k = \delta = 0$.

In spite of this superficial setback, the argument in Section 5 still shows that Γ_k maps $\mathfrak{B}_{k-1} \times \mathfrak{X}_{k,k}$ to itself and is a fiber contraction which is not necessarily continuous in the base point U. Since W is the unique attractive fixed point for Γ_{k-1}, the fiber $\{W\} \times \mathfrak{X}_{k,k}$ is invariant under Γ_k. Thus $\Gamma^W : \mathfrak{X}_{k,k} \to \mathfrak{X}_{k,k}$ is a contraction and, by the standard contraction mapping theorem, has a unique attractive fixed point μ which is our candidate for $D^k w$.

Theorem 4, with $n = k = r$ and $\delta = 0$, follows from the following Proposition 6.1:

6.1. Proposition. *If* x *and* h *are in* X *such that both* x *and* $x + h$ *are in* $B_1(0)$, *then*

$$D^{k-1}w(x + h) - D^{k-1}w(x) = \int_0^1 \mu(x + th)hdt. \tag{6.1}$$

We prove Proposition 6.1 by taking a fixed but arbitrary sequence $\{u_n \in \mathfrak{X}_{r,o}\}_{n=0}^{\infty}$, such that $\Delta^k u_o \in \mathfrak{B}_{k-1} \times \mathfrak{X}_{k,k}$, and $u_n = (\Gamma_F)^n u_o$. Thus $\|\Delta^{k-1} u_n - \Delta^{k-1} w\|_* \to 0$. Then we reformulate the problem on the line segment $J = \{x + th \mid 0 \le t \le 1\}$ which is compact.

The reason for using this approach is the simple fact that a continuous map on a compact set is obviously uniformly continuous but a continuous map on a closed bounded subset of an infinite dimensional Banach space is not necessarily uniformly continuous. We will need compactness in assertions (4)-(7) of Lemma 6.6.

If X is finite dimensional, then $B_1(0)$ is compact and the argument of this section can be used on all of $B_1(0)$.

If $0 < \delta \le 1$, a $C^{0,\delta}$ function is always uniformly continuous. This is what we used in Lemma 4.3, especially assertion (8).

6.2. Definition. Let x and h be as in Proposition 6.1. Then define the following:

1. Let $I = [0,1]$ and $\xi : [0,1] \to X$ be given by $\xi(t) = x + th$. Let $J = \xi(I)$.
2. Let $u_o \in \mathfrak{X}_{r,o}$ be such that $\Delta^k u_o \in \mathfrak{B}_{k-1} \times \mathfrak{X}_{k,k}$. Let $u_n = \Gamma_o u_{n-1}, n \ge 1$. Thus, for $n \ge 0, \Delta^k u_n \in \mathfrak{B}_{k-1} \times \mathfrak{X}_{k,k}$. Let $u_\infty = w$.
3. Let $U_n = \Delta^{k-1} u_n$ and $U_\infty = W$. Thus, for $n \ge 0, U_n \in \mathfrak{B}_{k-1}, U_n = (\Gamma_{k-1})^n (U_o)$, and $\|U_n - W\|_* \to 0$ as $n \to \infty$.
4. Let $\phi_n = U_n|_J$, for $n \ge 0$, and $\phi_\infty = W|_J = U_\infty|_J$.
5. Let

$$\begin{aligned}
\mathbb{V} &= \{\phi_n \mid n \ge 0\} \cup \{W|_J\} = \{\phi_n \mid 0 \le n \le \infty\}, \\
\mathbb{Y} &= \{\hat\rho = \rho|_J \;\mid\; \rho \in \mathfrak{X}_{k,k}\}, \\
\|\phi_n\|_o &= \|U_n|_J\|_*, \\
\|\hat\rho\|_1 &= \|\rho|_J\| = \sup_{\xi \in J} \|\rho(\xi)\|, \qquad \hat\rho = \rho|_J \in \mathbb{Y}.
\end{aligned}$$

6. Let $\Lambda : \mathbb{V} \times \mathbb{Y} \to \mathbb{V} \times \mathbb{Y}$ be the obvious restriction of Γ_k to $\mathbb{V} \times \mathbb{Y}$ as follows:

$$\Lambda(\phi_j, \hat\rho) := (\Lambda_o \phi_j \;,\; \Lambda^{\phi_j} \hat\rho) := [\Gamma_k(U_j, \rho)]|_J. \tag{6.2}$$

Thus, for $\xi \in J$,

$$\begin{aligned}
\Lambda_o \phi_j(\xi) &:= \Lambda_o(U_j|_J)(\xi) := [\Gamma_{k-1} U_j]|_J(\xi), \\
\Lambda^{\phi_j} \hat\rho(\xi) &:= [\Gamma^{U_j} \rho]|_J(\xi) \\
&= B^{-1} \rho(z_j) [\mathcal{H}^{u_j} \sigma_{j,1}(\xi)]^k \\
&\quad + B^{-1} \sigma_{j,1}(z_j) D_y f(\xi, u_j(\xi)) \rho(\xi) \\
&\quad - B^{-1} D_y g(\xi, u_j(\xi)) \rho(\xi) \\
&\quad + B^{-1} \sigma_{j,1}(z_j) D^k f(\xi, u_j(\xi)) [(I, \sigma_{j,1}(\xi))]^k \\
&\quad + S_k^{U_j}(\xi) + T_k^{U_j}(\xi) \\
&\quad + P_{k-1}^{U_j}(\xi) + Q_{k-1}^{U_j}(\xi), \\
z_j &= H_{u_j}(\xi) = A\xi + f(\xi, u_j(\xi)), \\
\mathcal{H}^{u_j} \sigma_{j,1}(\xi) &= [A + Df(\xi, u_j(\xi))(I, \sigma_{j,1}(\xi))],
\end{aligned} \tag{6.3}$$

$$U_j = (u_j, \Sigma_j) = (u_j, \sigma_{j,1}, \cdots, \sigma_{j,k-1}).$$

6.3. Remarks. 1. It is obvious that $(\mathbb{V}, \|\cdot\|_o)$ is a complete metric space since it consists of one sequence and its limit point. Also $(\mathbb{Y}, \|\cdot\|_1)$ is a complete metric space.

2. Recall that

$$(\varGamma_{k-1}U_j\ ,\ \varGamma^{U_j}D^k u_j) = \varGamma_k(U_j, D^k u_j) = \varGamma_k(\Delta^k u_j) \\ = \Delta^k(\varGamma_o u_j) = \Delta^k u_{j+1} = (U_{j+1}, D^k u_{j+1}). \tag{6.4}$$

Restricting (6.4) to J, we obtain

$$\varLambda_o\phi_j = [\varGamma_{k-1}U_j]|_J = U_{j+1}|_J = \phi_{j+1}, \\ \varLambda^{\phi_j}(D^k u_j|_J) = [\varGamma^{U_j}D^k u_j)]|_J = [D^k u_{j+1}]|_J. \tag{6.5}$$

3. When $\delta = 0, p_k = 0$. Thus, $N_{k,k}(\cdot) = \|\cdot\|$, the standard sup-norm.

6.4. Proposition. *For sufficiently small* $\epsilon > 0$,

1. $\varLambda$ *maps* $\mathbb{V} \times \mathbb{Y}$ *to itself.*
2. *The fiber* $\mathfrak{Y}_\infty := \{\phi_\infty\} \times \mathbb{Y}$ *is invariant under* $\varLambda$.
3. $\varLambda : \mathbb{V} \times \mathbb{Y} \to \mathbb{V} \times \mathbb{Y}$ *is a fiber contraction (not necessarily continuous in* U_j *yet).*
4. $\varLambda^{\phi_\infty} : \mathfrak{Y}_\infty \to \mathfrak{Y}_\infty$ *has a unique attractive fixed point* τ.
5. $\tau = \mu|_J$ *where* μ *is the unique attractive fixed point of* $\varGamma^W : \mathfrak{X}_{k,k} \to \mathfrak{X}_{k,k}$.

Proof. Assertions (1)-(4) follow from Proposition 5.5. As for (5), it is straightforward to see that $\mu|_J$ is a fixed point for $\varLambda^{\phi_\infty}$. But τ is unique fixed point. Thus $\tau = \mu|_J$. □

In order to be able to apply the Fiber Contraction Theorem we need to show that $\varLambda(\phi_j, \hat{\rho})$ is continuous in ϕ_j. We need the following elementary but important lemma.

6.5. Lemma. *The following sets are compact:*

$$\begin{aligned} &\mathcal{V}_j = u_j(H_{u_j}(J)), && \mathcal{U}_j = u_j(J), && \mathcal{Z}_j = H_{u_j}(J), \\ &\mathcal{V}_\infty = w(H_w(J)), && \mathcal{U}_\infty = w(J), && \mathcal{Z}_\infty = H_w(J), \\ &\mathcal{V} = \bigcup_{j\geq 0}^{j=\infty} \mathcal{V}_j, && \mathcal{U} = \bigcup_{j\geq 0}^{j=\infty} \mathcal{U}_j, && \mathcal{Z} = \bigcup_{j\geq 0}^{j=\infty} \mathcal{Z}_j, \\ &\mathcal{W} = \mathcal{U} \cup \mathcal{V} \cup \mathcal{Z}. \end{aligned}$$

Proof. Recall that X and Y are Banach spaces. All the sets defined, except $\mathcal{Z}, \mathcal{U}, \mathcal{V}$ and $\mathcal{W}$, are continuous images of the interval I, and hence, compact. $\mathcal{W}$ is compact, being the union of three compact sets. First we will show that $\mathcal{V}$ is compact.

Since $\mathcal{V}_\infty$ is compact, for all $\eta > 0$ there is a finite set of points $\{x_1, \cdots, x_{n_\eta}\} \subset \mathcal{V}_\infty$ such that

$$\mathcal{V}_\infty \subset \mathcal{O}(\eta) = \bigcup_{i=1}^{n_\eta} \mathcal{O}(x_i, \eta), \qquad \mathcal{O}(x, \eta) = \{x' \in B_1(0) \mid |x' - x| < \eta\}.$$

By (3.9) of Lemma 3.5 we have

$$\|u_j(H_{u_j}(\xi)) - w(H_w(\xi))\| \leq a^r(1+t)N_{r,o}(u_j - w)|\xi|^{p_o} \to 0 \tag{6.6}$$

as $j \to \infty$ uniformly in $|\xi| \leq 1$.

It follows that for all $\eta > 0$ there is j_o such that: If $j \geq j_0$, then $\mathcal{V}_j \subset \mathcal{O}(\eta)$. The finite union of compact sets $\mathcal{V}_0 \cup \mathcal{V}_1 \cup \cdots \cup \mathcal{V}_{j_o-1}$, is compact. Thus, $\mathcal{V}$ has a finite cover of balls of radius η. Since Y is also a metric space, it follows that $\mathcal{V}$ is compact.

We can sho in a similar fashion that $\mathcal{U}$ and $\mathcal{Z}$ are compact. We only need to replace (6.6) by

$$|u_j(\xi) - w(\xi)\| \leq N_{r,o}(u_j - w)|\xi|^{p_o} \to 0,$$
$$|H_{u_j}(\xi) - H_w(\xi)| \leq tN_{r,o}(u_j - w)|\xi|^{p_o} \to 0,$$

respectively. The first inequality is obvious. The second follows from (3.8). ☐

6.6. Lemma. *Let $\phi = U|_J$ and $\psi = V|_J$ be in $\mathbb{V}$ and $\rho|_J$ be in $\mathbb{Y}$. Write $z = H_u(\xi)$ and $z^* = H_v(\xi)$. The following are true:*

1. $|z - z^*| < tN_{r,o}((u-v)\,|_J)|\xi|^r \leq t\|\phi - \psi\|_o|\xi|^r$.
2. $|u(z) - v(z^*)| \leq a^k(1+t)N_{r,o}((u-v)|_J)|\xi|^r \leq a^k(1+t)\|\phi - \psi\|_o|\xi|^r$.
3. $\|M^U_{k-1}(\xi) - M^V_{k-1}(\xi)\| < t\|(U-V)|_J\|_*|\xi| = t\|\phi - \psi\|_o|\xi|, M = P, Q$.
   Moreover, for all $\eta > 0$, there is $c > 0$ such that: if $\|\phi - \psi\|_o < c$, then the following hold for all $\xi \in J$:
4. $\|\rho(z) - \rho(z^*)\| < \eta$.
5. $\|D^k h(\xi, u(\xi)) - D^k h(\xi, v(\xi))\| < \eta, h = f, g$.
6. $\|S^U_k(\xi) - S^V_k(\xi)\| < \eta$.
7. $\|T^U_k(\xi) - T^V_k(\xi)\| < \eta$.

Proof. Assertions (1) and (2) follow from Lemma 3.5. Assertion (3) follows from assertion (5) of Lemma 5.4 since $p_{k-1} = 1$.

Assertion (3) follows from the fact that $\Delta^{k-1}h$ is C^1, the vanishing properties (5.8) and (5.10), the multilinearity of P^U_{k-1} and Q^U_{k-1} in their arguments and the fact that $\Sigma(x)$ is uniformly Lipschitz continuous in $\mathcal{W}$ since $s_o = \cdots = s_{k-1} = 1$.

To prove assertion (4) notice that ρ is uniformly continuous on $\mathcal{W}$ since $\mathcal{W}$ is compact. Thus for all $\eta > 0$ there is $c > 0$ such that: if $x, x' \in \mathcal{W}$ and $|x - x'| < c$, then $\|\rho(x) - \rho(x')\| < \eta$. Let $\|\phi - \psi\|_o < c$. Since $z, z^* \in \mathcal{W}$, it follows from (1) that $|z - z^*| \leq \|\phi - \psi\|_o < c$. Thus $\|\rho(z) - \rho(z^*)\| < \eta$.

Similar arguments prove assertions (5) and (6) since $D^k h$ is uniformly continuous on $\mathcal{W}$ for $h = f, g$.

Assertion (7) follows from (5) and (6) and the fact that $\sigma_{j,1}$ is uniformly Lipschitz continuous because $s_1 = 1$. ☐

6.7. Proposition. *$\Lambda(\phi, \hat{\rho})$ is continuous in ϕ.*

Proof. We know that $\Lambda_o\phi$ is continuous in ϕ. Therefore we only need to show that $\Lambda^\phi\hat{\rho}$ is continuous in ϕ.

The last four terms in (6.3) are continuous in ϕ in view of (3) and (5)-(7) of Lemma 6.6. In view of (4) of the same Lemma, $\rho(z)$ in the first term of (6.3) is also continuous in ϕ. Also,

$\sigma_{j,1}(z)$, appearing in the second and fourth terms, is continuous in ϕ because $\sigma_{j,1}$ is uniformly Lipschitz continuous (since $s_1 = 1$). Moreover, Df and Dg, in the first, second, and third terms respectively, are C^1. Finally, $D^k f(\xi, u(\xi))$ is continuous in u by (5) of Lemma 6.6. Therefore, for all $\eta > 0$ there is a $c > 0$ such that if $\|\phi - \psi\|_o < c$, then

$$\|\Lambda^\phi \hat{\rho} - \Lambda^\psi \hat{\rho}\| \leq \eta,$$

which proves that $\Lambda(\phi, \hat{\rho})$ is continuous in ϕ. □

6.8. Proposition. *For sufficiently small $\epsilon > 0, \Lambda : \mathbb{V} \times \mathbb{Y} \to \mathbb{V} \times \mathbb{Y}$ has a unique attractive fixed point $(\phi_\infty, \mu|_J)$.*

Proof. We know from Proposition 6.4 that $\Lambda : \mathbb{V} \times \mathbb{Y} \to \mathbb{V} \times \mathbb{Y}$ is a fiber contraction, that the fiber $\mathfrak{Y}_\infty = \{\phi_\infty\} \times \mathbb{Y}$ is invariant under Λ and that Λ^{ϕ_∞} has a unique attractive fixed point $\tau = \mu|_J \in \mathfrak{Y}_\infty$. By Proposition 6.7, Λ is continuous in both arguments. Thus, by the Fiber Contraction Theorem $(\phi_\infty, \mu|_J)$ is a unique attractive fixed point for $\Lambda : \mathbb{V} \times \mathbb{Y} \to \mathbb{V} \times \mathbb{Y}$. □

Proof of Proposition 6.1. Each u_n is C^k. Therefore

$$\mathcal{L}_n(x,h) := D^{k-1}u_n(x+h) - D^{k-1}u_n(x) = \int_0^1 D^k u_n(x+th)h dt =: \mathcal{R}_n(x,h). \tag{6.7}$$

We also know that $N_{k,k-1}(D^{k-1}u_n - D^{k-1}w) \to 0$ as $n \to \infty$, which implies uniform convergence on all of $B_1(0)$. Thus $\mathcal{L}_n(x,h)$ converges uniformly to the left hand side of (6.1). It remains to show that $\mathcal{R}_n(x,h)$ converges to the right hand side of (6.1).

Recall that $(\phi_\infty, \mu|_J)$ is the unique attractive fixed point of Λ. Thus, by (6.5),

$$\|\mu|_J - D^k u_n|_J\| \to 0,$$

as $n \to 0$. It follows that,

$$\|\int_0^1 D^k u_n(x+th)h - \mu(x+th)hdt\| \leq |h|\|D^k u_n|_J - \mu|_J\| \to 0$$

as $n \to 0$. □

6.9. Remark. At this point, we have proven Theorem 4 for $n = k$ and $\delta = 0$. It is not possible to consider the $\delta = 0$ case for $n < k$ because by Lemma 9.10, we can replace (4) of Lemma 6.6 only by

$$\begin{aligned}\|\rho(z) - \rho(z^*)\| &\leq \eta^{\gamma/p_n}[N_{k,n}(\rho)]^{1-(\gamma/p_n)}(|z| + |z^*|)^{p_n - \gamma}, \\ &\leq \eta^{\gamma/p_n}[N_{k,n}(\rho)]^{1-(\gamma/p_n)}a^{p_n-\gamma}(1+t)|x|^{p_n-\gamma},\end{aligned}$$

for any $0 < \gamma \leq 1$. We can replace (5)-(7) of the same lemma by similar estimates. These estimates are not good enough because of the factor $|x|^{p_n-\gamma}$. We would need a factor of $|x|^{p_n}$ to be able to use the norm $N_{k,n}$.

7. **Proof of Theorem 1**

7.1. Remarks. We would like to start this section by emphasizing a few points.

1. The smallness assumptions on F in $B_1(0,0)$ ((H5)-(H9) of Theorem 4) were obtained in Lemma 2.8, where ϵ_k corresponds to a closed ball $B_k := B_{\gamma_k}(0)$.
2. We have proven Theorem 4 for any fixed but arbitrary $k \geq 2$ and shown that for sufficiently small $\epsilon_k > 0$ we have a unique invariant $C^{k,\delta}$ manifold which is the graph of a $C^{k,\delta}$ function $y = w_{(\gamma_k)}(x)$ defined on $B_1(0) \subset X$. Moreover, $D^j w_{(\gamma_k)} \in \mathfrak{X}_{r,j}, 0 \leq j \leq k$.
3. Recall also that Theorem 4 is stated for the map in Theorem 1 after eliminating j-monomials in $x, 2 \leq j \leq k$, from $g(x,y)$. This was accomplished in Lemma 2.7. Notice that when we eliminate $g_2(x)$ we change the remainder of $g(x,y)$ in particular $g_i(x), 3 \leq i \leq k$. The same thing happens at each step. In order not to complicate the notation, at each step we will continue to denote the new y-component of the map by $g(x,y)$ and the new j-monomials in x by $g_j(x)$.
4. It follows that the following is true for Theorem 1 after performing the elimination: For every $k \geq L$, there is an invariant $C^{k,\delta}$ manifold given by $y = w^{(k)}(x), x \in B_k := B_{\gamma_k}(0)$, and possibly $\gamma_k \to 0$ as $k \to \infty$. However, $w^{(k)}|_{B_{k+1}} \neq w^{(k+1)}$ because we obtained $w^{(k+1)}$ after eliminating of $g_{k+1}(x)$. In Proposition 7.2 we will show that $w^{(k+1)} = [w^{(k)}|_{B_{k+1}}] + u_{k+1}$, where u_{k+1} is as in Lemma 2.7 (i.e. the change of variables $(x,y) \mapsto (x,\bar{y}), \bar{y} = y + u_{k+1}(x)$, eliminates $g_{k+1}(x)$).
5. By (2.20) we can see that for $0 \leq j \leq k$

$$N_{r,j}(D^j w_{(\gamma_k)}) \leq 1 \Longleftrightarrow N_{r,j}(D^j w^{(k)}) \leq \frac{1}{\gamma_k^{k+\delta-1}}, \tag{7.1}$$

$$L_{r,j}(D^j w_{(\gamma_k)}) \leq 1 \Longleftrightarrow L_{r,j}(D^j w^{(k)}) \leq \frac{1}{\gamma_k^{k+\delta-1}}.$$

6. Theorem 4 was proved for $w_{(\gamma_k)}$ and F_{γ_k}. The correspondences in (7.1) allow us to translate the uniqueness of Theorem 4 to that of Theorem 1.

7.2. Proposition. *Let $w^{(k)}|_{B_{k+1}}$ be the restriction of $w^{(k)}$ to B_{k+1}. Let $u_{k+1} \in \mathcal{M}_{k+1}(X,Y)$ be as in Lemma 2.7 (i.e. the change of variables $(x,y) \mapsto (x,\bar{y}), \bar{y} = y + u_{k+1}(x)$, eliminates the $(k+1)$-monomials $g_{k+1}(x)$). Then, $w^{(k+1)} = [w^{(k)}|_{B_{k+1}}] + u_{k+1}$. Thus $w^{(k)} \mid B_{k+1}$ is $C^{k+1,\delta}$.*

Proof. To simplify notation we will write $f(x;u) := f(x,u(x)), \cdots$. Thus $w^{(k)}$ and u_{k+1} satisfy the following:

$$w^{(k)}(Ax + f(x;w^{(k)})) = Bw^{(k)}(x) + g_{k+1}(x) + \bar{h}_{k+2}(x;w^{(k)}), \tag{7.2}$$

$$u_{k+1}(Ax) = Bu_{k+1}(x) - g_{k+1}(x). \tag{7.3}$$

It follows from (2.16) that after eliminating $g_{k+1}(x)$ with the change of variables $(x,y) \mapsto (x,\bar{y})$ the map takes the form

$$x^* = Ax + f(x, \bar{y} - u_{k+1}(x)), \tag{7.4}$$

$$\bar{y}^* = B\bar{y} + \bar{h}_{k+2}(x, \bar{y} - u_{k+1}(x)) + u_{k+1}(x^*) - u_{k+1}(Ax), \tag{7.5}$$

with invariant manifold $\bar{y} = w^{(k+1)}(x)$ that satisfies

$$w^{(k+1)}(Ax + f(x; w^{(k+1)} - u_{k+1})) = Bw^{(k+1)}(x) + \bar{h}_{k+2}(x; w^{(k+1)} - u_{k+1}) \\ + u_{k+1}(Ax + f(x; w^{(k+1)} - u_{k+1})) - u_{k+1}(Ax). \tag{7.6}$$

Let $\tilde{v}(x) = w^{(k)}(x) + u_{k+1}(x), x \in B_{k+1}$. Straightforward computations, using (7.2) and (7.3), show that $\tilde{v}$ satisfies the same equation (7.6) that $w^{(k+1)}$ satisfies, namely,

$$\tilde{v}(Ax + f(x; \tilde{v} - u_{k+1})) = B\tilde{v}(x) + \bar{h}_{k+2}(x; \tilde{v} - u_{k+1}) \\ + u_{k+1}(Ax + f(x; \tilde{v} - u_{k+1})) - u_{k+1}(Ax). \tag{7.7}$$

Straightforward calculations, using (2.20), show that (7.7) can be rewritten as

$$\Gamma_F \tilde{v}_{(\gamma_{k+1})} = \tilde{v}_{(\gamma_{k+1})},$$

where $F = F^{(\gamma_{k+1})}$. Notice that

$$|z| = |Ax + f(x; \tilde{v}_{(\gamma_{k+1})} - (u_{k+1})_{(\gamma_{k+1})})| = a + |f(x; w_{(\gamma_{k+1})})| \leq a(1+t)|x|.$$

The same calculations that led to assertions (1) and (2) of Proposition 3.4 lead to the estimates

$$N_{r,o}(\tilde{v}_{(\gamma_{k+1})}) = N_{r,o}(\Gamma_F \tilde{v}_{(\gamma_{k+1})}) \leq \theta N_{r,o}(\tilde{v}_{(\gamma_{k+1})}) + t,$$
$$L_{r,o}(\tilde{v}_{(\gamma_{k+1})}) = L_{r,o}(\Gamma_F \tilde{v}_{(\gamma_{k+1})}) \leq \theta L_{r,o}(\tilde{v}_{(\gamma_{k+1})}) + t.$$

Thus,

$$N_{r,o}(\tilde{v}_{(\gamma_{k+1})}) \leq t/(1-\theta) < 1,$$
$$L_{r,o}(\tilde{v}_{(\gamma_{k+1})}) \leq t/(1-\theta) < 1.$$

It follows that $\tilde{v}_{(\gamma_{k+1})} \in \mathfrak{X}_{r,o}$. But $w_{(\gamma_{k+1})}$ is a unique fixed point for $\Gamma_F : \mathfrak{X}_{r,o} \to \mathfrak{X}_{r,o}$. Hence $\tilde{v}_{(\gamma_{k+1})} = w_{(\gamma_{k+1})}$. By (2.20), $\tilde{v} = w^{(k+1)}$. □

7.3. Proposition. *The following are true:*

1. $w^{(L)} \in C^{k,\delta}(B_L, Y)$ *for all* $k = L, L+1, L+2, \cdots,$ *as long as the map is* $C^{k,\delta}$.
2. *If* $k = \infty$, *then* $w^{(L)} \in C^{\infty}(B_L, Y)$.
3. *Moreover,* $w^{(L)}$ *is unique in the class of functions that satisfy (2.3) and (2.4).*

Proof. (1) We know that $w^{(L)} \in C^{L,\delta}(B_L, Y)$. Assume that $k \geq L+1$. Let B_k^o be the interior of B_k. By Proposition 7.2, for $k > L$

$$u := w^{(L)}|_{B_k^o} = w^{(k)} - u_k - \cdots - u_{L+1}$$

is $C^{k,\delta}$ in B_k^o. Let $v(x) := w^{(L)}(x), x \in B_L \setminus B_k^o$. Let $x_o \in B_L \setminus B_k^o$ be fixed but arbitrary. Since $A + f \circ (id, v)$ is a contraction on the invariant manifold, it follows that after a finite number of iterations x_o will land in B_k^o. Without loss of generality we may assume that this happens after the first iteration. Let $z_o = Ax_o + f(x_o, v(x_o))$. Since $u(z_o) = w^{(L)}(z_o)$ and $w^{(L)}(x_o) = v(x_o)$ we have

$$u(z_o) = Bv(x_o) + g(x_o, v(x_o)). \tag{7.8}$$

Let $\alpha_o = v(x_o)$. Consider the following function in a neighbourhood of (x_o, α_o):

$$\begin{aligned} \mathcal{F}(x,\alpha) &= B\alpha + g(x,\alpha) - u(z), \\ z &= Ax + f(x,\alpha), \\ \mathcal{F}(x_o,\alpha_o) &= 0, \end{aligned} \tag{7.9}$$

$$D_\alpha \mathcal{F}(x,\alpha) = B + D_y g(x,\alpha) - Du(z) D_y f(x,\alpha). \tag{7.10}$$

Recall that B is invertible and Df and Dg are small. Thus $D_\alpha \mathcal{F}(x_o,\alpha_o)$ is nonsingular. By the Implicit Function Theorem we have, in an open neighbourhood of (x_o,α_o), a unique $C^{k,\delta}$ solution and a unique $C^{L,\delta}$ solution of

$$\begin{aligned} \mathcal{F}(x,\alpha(x)) &= 0, \\ \alpha(x_o) &= \alpha_o. \end{aligned} \tag{7.11}$$

The two solutions coincide because $k \geq L$. But $v = w^{(L)}$ is a $C^{L,\delta}$ solution of (7.11) near (x_o,α_o) since

$$\begin{aligned} \mathcal{F}(x,v(x)) &= 0, \\ v(x_o) &= \alpha_o. \end{aligned} \tag{7.12}$$

Thus $w^{(L)}(x) = v(x) = \alpha(x)$ in a small neighbourhood of x_o. Since x_o is fixed but arbitrary in $B_L \setminus B_k^o$, it follows that $w^{(L)}$ is $C^{k,\delta}$ in all of B_L. This finishes the proof of part (1).
(2) In the C^∞ case repeated applications of part (1) show that $w^{(L)}$ is C^k in B_L for all $k \geq L$.
(3) The invariant manifold for the original map of Theorem 1 is

$$y = w(x) = w^{(L)}(x) - u_2(x) - \cdots - u_L(x).$$

In view of (7.1) with $k = L$, we can see that $w^{(L)}$ is unique in the class of functions that satisfy estimates (2.3) and (2.4) with $\gamma = \gamma_L$. □

7.4. Corollary. *Let $v \in C^{k,\delta}(B_L, Y)$. Then the following are true:*

1. *$y = v(x)$ is invariant under the original map of Theorem 1 iff $y = v^{(L)}(x) := v(x) + u_2(x) + \cdots + u_L(x), |x| \leq \gamma_L$, is invariant under the mapping of Theorem 1 after eliminating all j-monomials in x from $g(x,y), 2 \geq j \geq L$.*
2. *Moreover,*
$$y = v_{(\gamma_L)}(x) = \gamma_L^{-1}[v(\gamma_L x) + u_2(\gamma_L x) + \cdots + u_L(\gamma_L x)]$$
is invariant under the map of Theorem 4. That is, $\Gamma_F v_{(\gamma_L)} = v_{(\gamma_L)}, F = F^{(\gamma_L)}$.
3. *$v_{(\gamma_L)} \in \mathfrak{X}_{r,o}$. Moreover, $v_{(\gamma_L)} = w_{(\gamma_L)}$.*
4. *$v^{(L)} = w^{(L)}$.*
5. *$v(x) = w^{(L)}(x) - u_2(x) - \cdots - u_L(x) = w_{(\gamma_L)}, |x| \leq \gamma_L(x)$.*

Proof. Assertion (1) follows from (4) of Lemma 2.7 if we take $k = L$ in the latter. Assertion (2) follows from (2.20) and Lemma 2.8.

Assertion (4) follows from assertion (3) and (2.20). Assertion (5) follows from assertion (4).

It remains to prove assertion (3). Notice that $v_{(\gamma_L)}$ satisfies (5.1) for $0 \le m \le L$. Thus,

$$\|D^L v_{(\gamma_L)}\| \le \frac{t}{1-\theta}$$

with $\theta = b^{-1}a^L(1+t)+t < 1$, and hence, $\|D^L v_{(\gamma_L)}\| < 1$. Since $D^j v_{(\gamma_L)}(0) = 0, 0 \le j \le L$, it follows that

$$\text{(i)} \qquad \|D^j v_{(\gamma_L)}(x)\| \le |x|^{L-j}, \qquad 0 \le j \le L.$$

Moreover, for $0 \le j \le L-1$, for some $0 \le c \le 1$

$$\text{(ii)} \qquad \begin{aligned} \|D^j v_{(\gamma_L)}(x) - D^j v_{(\gamma_L)}(x')\| &\le D^{j+1} v_{(\gamma_L)}(x + c(x'-x)) \\ &\le |x-x'|(|x|+|x'|)^{L-j-1}. \end{aligned}$$

Thus $N_{L,o}(v_{(\gamma_L)}) \le 1$ and $L_{L,o}(v_{(\gamma_L)}) \le 1$. Hence $v_{(\gamma_L)} \in \mathfrak{X}_{L,o}$.

Let $\tilde{w}_{(\gamma_L)} \in \mathfrak{X}_{L,o}$ be the unique fixed point for $\Gamma_F : \mathfrak{X}_{L,o} \to \mathfrak{X}_{L,o}$. Thus $v_{(\gamma_L)} = \tilde{w}_{(\gamma_L)}$. But $w_{(\gamma_L)} \in \mathfrak{X}_{L+\delta,o} \subset \mathfrak{X}_{L,o}$ since all our maps are defined on $|x| \le 1$. Thus $w_{(\gamma_L)} = \tilde{w}_{(\gamma_L)} = v_{(\gamma_L)}$. □

7.5. Finishing the proof of Theorem 1. It remains to prove assertions (4) and (5) of Theorem 1. Assertion (5) is obvious since $w_{(\gamma_L)}(0) = 0$ and $Dw_{(\gamma_L)}(0) = 0$.

To prove assertion (4) let $y = v(x), v \in C^{k,\delta}(B_L, Y)$, be invariant under the original map of Theorem 1. Then, by Corollary 7.4,

$$v = w^{(L)}(x) - u_2(x) - \cdots - u_L(x) = w_{(\gamma_L)}, \qquad |x| \le \gamma_L$$

Therefore $w_{(\gamma_L)}$ is unique in $C^{k,\delta}(B_L, Y)$ which finishes the proof of Theorem 1.

8. **Dependence on Parameters**

In this section we study the dependence of the invariant manifold we obtained on a parameter that lies in a Banach space. We will prove Theorem 2. Notice that we need to consider only the case $0 < \delta \le 1$ since assertion (3) follows from assertion (2).

Let $\mu \in \mathcal{U} = B_c^o(\mu) \subset \Lambda$ be as in Theorem 2. Recall that in order to reduce Theorem 1 to Theorem 4 we used Lemma 2.7 to eliminate j-monomials in x from $g(x,y)$ for $2 \le j \le k$. We know from assertion (5) of Lemma 2.7 that as $\lambda \to \mu$

$$\text{(8.1)} \qquad \|u_{\lambda,j} - u_{\mu,j}\|_{C^{k,\delta}} \to 0.$$

Because of the continuous dependence of Φ_λ on λ we can choose $\epsilon > 0$ in Lemma 2.8 uniformly for $\lambda \in \mathcal{U}$. Thus, terms of the form $b^{-1}a^r(1+t)+t$ can be made less then 1 uniformly for $\lambda \in \mathcal{U}$ This proves assertion (1) of Theorem 2.

It follows that for finite k it is enough to consider the dependence on parameters for maps satisfying the hypotheses of Theorem 4 with $n = k$. The C^∞ case will be studied at the end of this section.

Our first step is to show that g_λ depends continuously on λ in the norm $N_{k+\beta,o}$ for all $0 \le \beta < \delta$. To do that we need Lemma 9.2 and Lemma 9.3 given and proven in the Appendix. Then we will show that as $\lambda \to \mu, w_\lambda \to w_\mu$ in the norm $N_{k+\beta,o}, 0 \le \beta < \delta \le 1$. Recall that the norm $N_{k+\beta,o}$ controls a function at the $C^{0,1}$ level. So, we use some classical results, which we state and prove in the appendix for completeness, to show that $w_\lambda \to w_\mu$ in the $C^{k,\beta}$ topology.

8.1. Lemma. *Let $0 < \beta < \delta \le 1$. The following are true:*

$$|f_\lambda(x,y) - f_\mu(x,y)| \le t\varrho(|\lambda-\mu|)[|x| + |y|], \tag{8.2}$$

$$|g_\lambda(x,y) - g_\mu(x,y)| \le t\varrho(|\lambda-\mu|)[|x|^{k+\beta} + |y|], \tag{8.3}$$

where $\varrho(|\lambda-\mu|) \to 0$ as $\lambda \to \mu$.

Proof. Since the family $\{\Phi_\lambda : \lambda \in \mathcal{U}\}$ is continuous in the $C^{k,\delta}$ norm, the lemma follows from estimate (*) of Lemma 9.3 as follows: Let $\alpha = \lambda - \mu$. For (8.2) take $j = 1$ and $\psi_\alpha = f_\lambda - f_\mu$. For (8.3) take $j = k$ and $\psi_\alpha = g_\lambda - g_\mu$. Also notice that since $|x| \le 1$ and $|y| \le 1$, there is $c > 0$ such that

$$\frac{1}{|x|^{k+\beta} + |y|} \le \frac{1}{|x|^{k+\beta} + |y|^{k+\beta}} \le \frac{c}{(|x| + |y|)^{k+\beta}}.$$

□

8.2. Remarks. 1. Let $y = w(x)$ and $y = u(x)$, $|x| \le 1$, be the unique invariant manifolds in $\mathfrak{X}_{k+\delta,o}$ which Theorem 4 establishes for Φ_λ and Φ_μ respectively. Then,

$$\begin{aligned} w(x) &= B_\lambda^{-1} w(z) - B_\lambda^{-1} g_\lambda(x, w(x)), \\ u(x) &= B_\mu^{-1} u(\zeta) - B_\mu^{-1} g_\mu(x, u(x)), \\ z &= A_\lambda x + f_\lambda(x, w(x)), \\ \zeta &= A_\mu x + f_\mu(x, u(x)). \end{aligned} \tag{8.4}$$

2. Notice that when $0 < \delta \le 1$ all our estimates and conclusions of the previous sections are valid for any $\beta \in (0, \delta]$.
3. Since $N_{k+\beta,o}(\psi) \le N_{k+\delta,o}(\psi)$ and $L_{k+\beta,o}(\psi) \le L_{k+\beta,o}(\psi)$, it follows from the uniqueness assertion of Theorem 4 that $y = w(x)$ and $y = u(x)$, $|x| \le 1$ are also the unique invariant manifolds in $\mathfrak{X}_{k+\beta,o}$ for all $0 \le \beta < \delta$.
4. In what follows, t will stand for a continuous function $t(\epsilon) \to 0$ as $\epsilon \to 0$. Similarly, ϱ will stand for a continuous function $\varrho(|\lambda - \mu|) \to 0$ as $|\lambda - \mu| \to 0$. Both t and ϱ may vary from one line to another.

8.3. Lemma. *Let $0 \le \beta < \delta \le 1$. Then the following estimates are true:*

1. $|f_\lambda(x, w(x)) - f_\mu(x, u(x))| \le t[N_{k+\beta,o}(w-u) + \varrho]|x|$
2. $|g_\lambda(x, w(x)) - g_\mu(x, u(x))| \le t[N_{k+\beta,o}(w-u) + \varrho]|x|^{k+\beta}$
3. $|z - \zeta| \le [tN_{k+\beta,o}(w-u) + (1+t)\varrho]|x|$.
4. $|w(z) - u(\zeta)| \le [a^{k+\beta}(1+t)N_{k+\beta,o}(w-u) + t\varrho]|x|^{k+\beta}$.

Proof. Estimates (1) and (2) follow from (8.2) and (8.3) respectively. Estimate (3) follows from (1). Calculations similar to those in Section 3 lead to the following:

$$\begin{aligned}
|z-\zeta| &\le [\|A_\lambda - A_\mu\| + tN_{k+\beta,o}(w-u) + t\varrho]|x| \\
&\le [tN_{k+\beta,o}(w-u) + (1+t)\varrho]|x|, \\
|w(z)-u(\zeta)| &\le |w(z)-u(z)| + |u(z)-u(\zeta)| \\
&\le a^{k+\beta}(1+t)N_{k+\beta,o}(w-u)|x|^{k+\beta} + L_{k+\beta,o}(u)|z-\zeta|(|z|+|\zeta|)^{k-1+\beta} \\
&\le a^{k+\beta}(1+t)N_{k+\beta,o}(w-u)|x|^{k+\beta} \\
&\quad + [tN_{k+\beta,o}(w-u) + (1+t)\varrho](2a)^{k-1+\beta}(1+t)|x|^{k+\beta} \\
&\le [a^{k+\beta}(1+t)N_{k+\beta,o}(w-u) + \varrho]|x|^{k+\beta}.
\end{aligned}$$

□

8.4. Proposition. $N_{k+\beta,o}(w-u) \to 0$ *as* $\mu \to \lambda$.

Proof. We have the following for sufficiently small $\epsilon > 0$:

$$\begin{aligned}
|w(x)-u(x)| \le \|B_\lambda^{-1} - B_\mu^{-1}\||u(z)| + b^{-1}|w(z)-u(\zeta)| \\
+ \|B_\lambda^{-1} - B_\mu^{-1}\||g_\mu(x,u(x))| + b^{-1}|g_\lambda(x,w(x)) - g_\mu(x,u(x))|.
\end{aligned}$$

By (2) and (4) of Lemma 8.3 we have

$$|w(x)-u(x)| \le [\theta N_{k+\beta,o}(w-u) + \varrho]|x|^{k+\beta},$$

where $\theta = b^{-1}a^r(1+t)+t < 1$ for sufficiently small $\epsilon > 0$. Thus $N_{k+\beta,o}(w-u) \le \varrho/(1-\theta) \to 0$ as $|\lambda - \mu| \to 0$. □

8.5. Corollary. *The following are true as* $\lambda \to \mu$*:*

1. $\|w-u\| \to 0$.
2. *Let* $U_d = \{x \in X \mid |x| \le d < 1\}$ *and* $0 \le \beta < \delta \le 1$. *Then* $\|w-u\|_{k,\beta} \to 0$ *on* U_d.
3. *If* $\delta = 0$, $\|w-u\|_{k-1,s} \to 0$ *for* $0 \le s < 1$ *on* U_d.

Proof. 1. Since $|x| \le 1$, it follows that

$$\|w-u\| = \sup_{0<|x|\le 1} |w(x)-u(x)| \le \sup_{0<|x|\le 1} N_{k+\beta,o}(w-u)|x|^r \le N_{k+\beta,o}(w-u).$$

2. This assertion follows from Lemma 9.9 of the appendix.
3. This assertion follows if we apply the previous assertion with smoothness $C^{k-1,1}$.

□

Proof of Theorem 2. For finite k the theorem follows from (8.1) and Corollary 8.5.

For the C^∞ case we use the metric

$$d(w,u) := \sum_{k=0}^{\infty} 2^{-k} \frac{\|w-u\|_{C^k}}{1+\|w-u\|_{C^k}}.$$

It is obvious that the sum is convergent. Given $\eta > 0$, let N be large enough so that the tail of the series is small, i.e.

$$\sum_{k=N+1}^{\infty} \frac{1}{2^k} \frac{\|w-u\|_{C^k}}{1+\|w-u\|_{C^k}} < \frac{\eta}{2}.$$

Now we can choose $|\lambda - \mu|$ small enough such that the finite sum

$$\sum_{k=0}^{N} \frac{1}{2^k} \frac{\|w-u\|_{C^k}}{1+\|w-u\|_{C^k}} < \frac{\eta}{2}.$$

Thus $d(w,u) \to 0$ as $\lambda \to \mu$. □

9. Appendix: Embedding Lemmas

The following lemmas are from [4]. We include them here, together with their proofs, for completeness.

9.1. Definition. Let X and Y be Banach spaces. Let $U \subset X$ be such that $cl(int(U)) \neq \emptyset$. For $j = 0, 1, 2, \cdots$ and $0 \leq \delta \leq 1$ define the following for $\psi : U \to Y$ and $c > 0$:

$$N_{j+\delta,o}(\psi) = \sup_{|x|>0} \frac{|\psi(x)|}{|x|^{j+\delta}}, \tag{9.1}$$

$$\mathfrak{Z}_{j+\delta,o}(U,Y) = \{\psi \in C^0(U,Y) \mid N_{j+\delta,o}(\psi) < \infty\}, \tag{9.2}$$

$$\mathfrak{Z}^c_{j+\delta,o}(U,Y) := \{\psi \in \mathfrak{Z}_{j+\delta,o}(U,Y) \mid N_{j+\delta,o}(\psi) \leq c\}. \tag{9.3}$$

9.2. Lemma. *The following are true:*

1. *The function space $(\mathfrak{Z}_{j+\delta,o}(U,Y), N_{j+\delta,o})$ is a Banach space.*
2. *$\mathfrak{Z}^c_{j+\delta,o}(U,Y)$ is closed in the $N_{j+\delta,o}$ norm.*
3. *Let $\{\phi_n \mid n = 1,2,3,\cdots\}$ be a sequence in $\mathfrak{Z}^c_{j+\delta,o}(U,Y)$ and $\phi : U \to Y$. If $\|\phi_n - \phi\| \to 0$, then $\phi \in \mathfrak{Z}^c_{j+\delta,o}(U,Y)$. It follows that $\mathfrak{Z}^c_{j+\delta,o}(U,Y)$ is closed in $(C^0(U,Y), \|\cdot\|)$ where $\|\cdot\|$ is the usual sup-norm.*
4. *Let $0 \leq \beta < \delta \leq 1$. Then, for all $\psi \in \mathfrak{Z}_{j+\delta,o}(U,Y)$,*

$$N_{j+\beta,o}(\psi) \leq \|\psi\|^{\frac{\delta-\beta}{j+\delta}} \left[N_{j+\delta,o}(\psi)\right]^{\frac{j+\beta}{j+\delta}}. \tag{9.4}$$

5. *Let $\{\phi_n \mid n = 1,2,3,\cdots\}$ be a sequence in $\mathfrak{Z}^c_{j+\delta,o}(U,Y)$ and $\phi : U \to Y$. Assume that $\|\phi_n - \phi\| \to 0$ as $n \to \infty$. (By (3), $\phi \in \mathfrak{Z}^c_{j+\delta,o}(U,Y)$.) Then, for all $0 \leq \beta < \delta$, $N_{j+\beta,o}(\phi_n - \phi) \to 0$ as $n \to \infty$.*

Proof. The first assertion follows from the fact that convergence in the norm $N_{j+\delta,o}$ implies uniform convergence on bounded sets. Assertion (2) is obvious in view of (1).

As for the third assertion notice that ϕ must be continuous because of the uniform convergence. For any $x \in U$, for all $n > 1$

$$|\phi(x)| \leq N_{j+\delta,o}(\phi_n)|x|^{j+\delta} + \|\phi - \phi_n\| \leq c|x|^{j+\delta} + \|\phi - \phi_n\| \to c|x|^{j+\delta}$$

as $n \to \infty$.

To prove the fourth assertion notice that

$$(*)\qquad \Delta := \frac{|\psi(x)|}{|x|^{j+\beta}} \leq \min\left\{ \frac{\|\psi\|}{|x|^{j+\beta}}\ ,\ |x|^{\delta-\beta} N_{j+\delta,o}(\psi) \right\}.$$

The two quantities on the right hand side of (*) are equal iff

$$|x| = A := \left(\frac{\|\psi\|}{N_{j+\delta,o}(\psi)} \right)^{1/(j+\delta)}.$$

There are two cases. Case 1: If $|x| \geq A$, then

$$\Delta \leq \frac{\|\psi\|}{A^{j+\beta}} = \|\psi\|^{\frac{\delta-\beta}{j+\delta}} \left[N_{j+\delta,o}(\psi)\right]^{\frac{j+\beta}{j+\delta}}.$$

Case 2: If $|x| < A$, then

$$\Delta \leq A^{\delta-\beta} N_{j+\delta,o}(\psi) = \|\psi\|^{\frac{\delta-\beta}{j+\delta}} \left[N_{j+\delta,o}(\psi)\right]^{\frac{j+\beta}{j+\delta}}.$$

Thus, in either case,

$$\frac{|\psi(x)|}{|x|^{j+\beta}} \leq \|\psi\|^{\frac{\delta-\beta}{j+\delta}} \left[N_{j+\delta,o}(\psi)\right]^{\frac{j+\beta}{j+\delta}}.$$

Taking the supremum on the left hand side we obtain the desired estimate.

To prove the first identity notice that by (3) $\phi \in \mathfrak{Z}^c_{j+\delta,o}$. Thus $N_{j+\delta,o}(\phi - \phi_n) \leq 2c$. Now we can apply (9.4) to $\psi_n = \phi - \phi_n$. □

9.3. Lemma. *Let X and Y be Banach spaces. Let $U \subset X$ be such that $cl(int(U)) \neq \emptyset$. Let Λ be an open neighbourhood of 0 in a Banach space (parameter space). Let $j = 0, 1, 2, \cdots$ and $0 < \delta \leq 1$. Let $\mathcal{F} := \{\psi_\alpha \mid \alpha \in \Lambda\} \subset \mathfrak{Z}^c_{j+\delta,o}(U, Y)$ be a family of continuous functions $\psi_\alpha : U \to Y$. Assume that the family $\mathcal{F}$ is continuous in the C^0 norm and that $\|\psi_\alpha\| \to 0$ as $\alpha \to 0$. Then, for all $0 \leq \beta < \delta$, $N_{j+\beta,o}(\psi_\alpha) \to 0$ as $\alpha \to 0$. In fact,*

$$(*)\qquad N_{j+\beta,o}(\psi_\alpha) \leq \|\psi_\alpha\|^{\frac{\delta-\beta}{j+\delta}} \left[N_{j+\delta,o}(\psi_\alpha)\right]^{\frac{j+\beta}{j+\delta}},$$

Proof. The assertion follows from (4) and (5) of Lemma 9.2. □

In the following two lemmas we give a detailed proof of a lemma by D. Henry [9, page 151]. See also [2, page 305, 319].

9.4. Definition. Let X and Y be Banach spaces. Let $U \subset X$ be such that $cl(int(U)) \neq \emptyset$. Let $k = 0, 1, 2, \cdots$ and $0 < \delta \leq 1$. Let $b > 0$ and

$$C^{k,\delta}_b(U, Y) := \{u \in C^{k,\delta}(U, Y) \mid \|u\|_{k,\delta} \leq b\},$$

where

$$\|u\|_{k,\delta} = \max\{\|u\|, \|Du\|, \cdots, \|D^k u\|, H_\delta(D^k u)\},$$

$$H_\delta(D^k u) = \sup\left\{ \frac{\|D^k u(x) - D^k u(x')\|}{|x - x'|^\delta} \mid x \neq x', x \in U, x' \in U \right\}.$$

9.5. Lemma (Estimate H). *Let X and Y be Banach spaces. Let $U \subset X$ be such that $cl(int(U)) \neq \emptyset$. Let $k = 1, 2, 3, \cdots$ and $0 < \delta \leq 1$. Let ∂U be the boundary of U. For $x \in U$, define $d(x) := \min\{1, dist(x, \partial U)\}$. For all $f \in C^{k,\delta}(U, Y)$, for all $x \in U$ the following estimates are true:*

$$d(x)\|Df(x)\| \leq N_1 \|f\|^{\frac{\delta}{1+\delta}} \|f\|_{1,\delta}^{\frac{1}{1+\delta}}, \tag{H1}$$

$$d(x)^{a_k}\|D^k f(x)\| \leq N_k \|f\|^{\beta_k} \|f\|_{k,\delta}^{1-\beta_k}, \tag{Hk}$$

where $N_1 = 4, a_1 = 1, \beta_1 = \delta/(1+\delta)$ and

$$\begin{aligned} N_k &= 2(k!)[2 + \frac{N_1}{1!} + \frac{N_2}{2!} \cdots + \frac{N_{k-1}}{(k-1)!}], \\ a_k &= a_{k-1} + k, \\ \beta_k &= \frac{\delta}{k+\delta}\beta_{k-1} = \frac{\delta^k}{(k+\delta)!}, \\ (k+\delta)! &:= (k+\delta)(k-1+\delta)\cdots(2)(1). \end{aligned}$$

Proof. Let $x \in U$ be fixed but arbitrary. Let $v \in X$ and $|v| = 1$ and let $0 < t < d(x)$. Then

$$\begin{aligned} |Df(x)tv| &\leq |f(x) - f(x+tv)| + |f(x+tv) - f(x) - Df(x)tv| \\ &\leq 2\|f\| + \frac{\|f\|_{1,\delta}}{1+\delta} t^{1+\delta} \leq 2\|f\| + \|f\|_{1,\delta} t^{1+\delta}. \end{aligned}$$

Thus

$$\|Df(x)\| \leq \frac{2\|f\|}{t} + \|f\|_{1,\delta} t^{\delta}, \qquad t < d(x) \leq 1. \tag{*}$$

The two terms on the righthand side of (*) are equal iff

$$\begin{aligned} t &= \left(\frac{R}{S}\right)^{\frac{1}{1+\delta}} := A, \\ R &= 2\|f\|, \\ S &= \|f\|_{1,\delta}. \end{aligned} \tag{**}$$

We have two cases. Case 1: $d(x) \leq A$. Then for all $0 < t < d(x)$

$$t\|Df(x)\| \leq 2R \leq 4\|f\| \leq 4\|f\|^{\frac{\delta}{1+\delta}} \|f\|_{1,\delta}^{\frac{1}{1+\delta}}.$$

Taking the supremum over $0 < t < d(x)$, we obtain

$$d(x)\|Df(x)\| \leq 4\|f\|^{\frac{\delta}{1+\delta}} \|f\|_{1,\delta}^{\frac{1}{1+\delta}}. \tag{a1}$$

Case 2: $d(x) > A$. In this case we evaluate the right hand side of (*) at $t = A$. Thus

$$d(x)\|Df(x)\| \leq \|Df(x)\| \leq 2R^{\frac{\delta}{1+\delta}} S^{\frac{1}{1+\delta}} \leq 4\|f\|^{\frac{\delta}{1+\delta}} \|f\|_{1,\delta}^{\frac{1}{1+\delta}}. \tag{b1}$$

It follows from (a1) and (b1) that estimate (H1) holds.

Assume that our estimate is true for $i = 1, 2, \cdots, k-1$. Since $\|f\|_{i,\delta} \le \|f\|_{l,\delta}$ and $\beta_l < \beta_i$ for $i < l$, it follows that

$$\|f\|^{\beta_i}\|f\|_{i,\delta}^{1-\beta_i} \le \|f\|^{\beta_l}\|f\|_{l,\delta}^{1-\beta_l}. \tag{9.5}$$

Let $v^{(i)} = (v, \cdots, v)$ with i components. For $i = 1, 2, \cdots, k$, let

$$\begin{aligned}\mathcal{F}_i(t) &= f(x+tv) - f(x) - \frac{t}{1!}Df(x)v - \frac{t^2}{2!}D^2f(x)v^{(2)} - \cdots - \frac{t^i}{i!}D^if(x)v^{(i)}\\ &= \mathcal{F}_{i-1} - \frac{t^i}{i!}D^if(x)v^{(i)},\\ \mathcal{A}_i(t) &= \|\mathcal{F}_i\|.\end{aligned}$$

It follows that for some $0 \le t_k \le t_{k-1} \le \cdots \le t_2 \le t < d(x)$,

$$\mathcal{F}_k(t) = \int_0^t\int_0^{t_2}\cdots\int_0^{t_k}[D^kf(x+sv) - D^kf(x)]v^{(k)}\, d\mu_k d\mu_{k-1}\cdots d\mu_2 d\mu_1. \tag{9.6}$$

Thus

$$\begin{aligned}\mathcal{A}_k(t) &\le H_\delta(D^kf)\int_0^t\int_0^{t_2}\cdots\int_0^{t_k} s^\delta d\mu_k d\mu_{k-1}\cdots d\mu_2 d\mu_1\\ &= \frac{t^{k+\delta}}{(k+\delta)!}H_\delta(D^kf) \le \frac{t^{k+\delta}}{(k+\delta)!}\|f\|_{k,\delta} \le \|f\|_{k,\delta}t^{k+\delta}.\end{aligned} \tag{9.7}$$

Using (9.5) we obtain

$$\begin{aligned}\mathcal{A}_{k-1}(t) &\le 2\|f\| + \frac{tN_1}{1!}\|f\|^{\beta_1}\|f\|_{k,\delta}^{1-\beta_1}d(x)^{-1} + \frac{t^2N_2}{2!}\|f\|^{\beta_2}\|f\|_{k,\delta}^{1-\beta_2}d(x)^{-a_2}\\ &\quad + \cdots + \frac{t^{k-1}N_{k-1}}{(k-1)!}\|f\|^{\beta_{k-1}}\|f\|_{k,\delta}^{1-\beta_{k-1}}d(x)^{-a_{k-1}}\\ &\le \frac{C_k}{d(x)^{a_{k-1}}}\|f\|^{\beta_{k-1}}\|f\|_{k,\delta}^{1-\beta_{k-1}},\\ C_k &= 2 + \frac{N_1}{1!} + \frac{N_2}{2!} + \cdots + \frac{N_{k-1}}{(k-1)!}.\end{aligned} \tag{9.8}$$

Using (9.7) and (9.8) we obtain, for $0 < t < d(x)$,

$$\begin{aligned}\frac{t^k\|D^kf(x)\|}{k!} &\le \mathcal{A}_{k-1}(t) + \mathcal{A}_k(t),\\ \frac{\|D^kf(x)\|}{k!} &\le \frac{R}{t^k} + St^\delta,\\ R &:= \frac{C_k}{d(x)^{a_{k-1}}}\|f\|^{\beta_{k-1}}\|f\|_{k,\delta}^{1-\beta_{k-1}},\\ S &:= \|f\|_{k,\delta},\end{aligned} \tag{9.9}$$

The two quantities on the right hand side of (#) are equal iff

$$t = \left(\frac{R}{S}\right)^{1/(k+\delta)} =: B.$$

As before, there are two cases. Case 1: $d(x) \le B$. Then $0 \le t < d(x) \le B$ and

$$\text{(ak)} \qquad d(x)^k \frac{\|D^k f(x)\|}{k!} \le 2R \le \frac{2C_k}{d(x)^{a_{k-1}}} \|f\|^{\beta_{k-1}} \|f\|_{k,\delta}^{1-\beta_{k-1}} \le \frac{2C_k}{d(x)^{a_{k-1}}} \|f\|^{\beta_k} \|f\|_{k,\delta}^{1-\beta_k}.$$

Case 2: $d(x) > B$. In this case we evaluate (9.9) at $t = B$ and obtain

$$\begin{aligned} \text{(bk)} \qquad \frac{\|D^k f(x)\|}{k!} &\le 2R^{\delta/(k+\delta)} S^{k/(k+\delta)} \\ &\le \frac{2C_k^{\delta/(k+\delta)}}{d(x)^h [(k+\delta)!]^{k/(k+\delta)}} \|f\|^{\delta\beta_{k-1}/(k+\delta)} \|f\|_{k,\delta}^{\delta(1-\beta_{k-1})/(k+\delta)} \|f\|_{k,\delta}^{k/(k+\delta)} \\ &\le \frac{2C_k}{d(x)^h} \|f\|^{\beta_k} \|f\|_{k,\delta}^{1-\beta_k}, \\ h &= \frac{\delta a_{k-1}}{k+\delta} < a_{k-1}. \end{aligned}$$

It follows from (ak) and (bk) that

$$d(x)^{k+a_{k-1}} \|D^k f(x)\| \le 2(k!) C_k \|f\|^{\beta_k} \|f\|_{k,\delta}^{1-\beta_k}$$

which finishes the proof of the lemma. □

9.6. Lemma (Henry's lemma)**.** *The following are true:*

1. *Any closed bounded subset $C_b^{k,\delta}(U,Y)$ is a closed and bounded subset of $C^0(U,Y)$. That is, if $u : U \to Y$, and $\{u_n \mid 0 \le n < \infty\}$ is a sequence in $C_b^{k,\delta}(U,Y)$, and $\|u_n - u\| \to 0$ as $n \to \infty$, then $u \in C_b^{k,\delta}(U,Y)$. Moreover, $D^k u_n(x) \to D^k u(x)$ point-wise as $n \to \infty$.*
2. *Let $U_o \subset U$ be a subset which is uniformly bounded away from the boundary of U and assume $cl(int(U_o)) \ne \emptyset$. Then $D^k u_n(x) \to D^k u(x)$ uniformly on U_o.*

Proof. (1) If $k = 0$, then for any $x \ne x'$,

$$\begin{aligned} |u(x) - u(x')| &\le |u_n(x) - u_n(x')| + 2\|u_n - u\| \\ &\le c|x - x'|^\delta + 2\|u_n - u\| \to c|x - x'|^\delta, \end{aligned}$$

as $n \to \infty$.

When $k \ge 1$, let $f := u_n - u_m$. It follows from estimate (Hk) of Lemma 9.5 that

$$(*) \qquad d(x)^{a_k} \|D^k u_n(x) - D^k u_m(x)\| \le N_k \|u_n - u_m\|^{\beta_k} \|u_n - u_m\|_{k,\delta}^{1-\beta_k}.$$

Thus u is $C^{k,\delta}$and $D^k u_n(x) \to D^k u(x)$ for each $x \in U$. It also follows that $\|u\|_{k,\delta} \le b$.
Part (2) follows immediately from estimate (*) and the fact that $U_o \subset U$ is uniformly bounded away from the boundary of U. □

9.7. Remarks. 1. It is obvious that the proof of Henry's Lemma 9.6 is based on estimates (H1) and (Hk) of Lemma 9.5.

2. Estimate (H1) is given in [9, page 151] and [2, page 319] without the explicit dependence on $\|f\|_{1,\delta}$. It is not clear whether the $k \geq 2$ case of Henry's Lemma follows by induction from the $C^{1,\delta}$ case without estimate (Hk).
3. Moreover, estimates (H1) and (Hk) of Lemma 9.5 are of interest by themselves.
4. Notice that in estimates (H1) and (Hk), as well as all the estimates in this appendix, the sum of the exponents of the different norms of the function that appear on the righthand side is always one. In other words, these inequalities are homogenous so to speak. This is not the case with the estimates in [9, page 151] and [2, page 319].

The following is a classical result. We present a proof here for completeness.

9.8. Lemma. 1. *Let $g \in C^{0,\delta}(U_o, Y)$ and $0 < \beta < \delta \leq 1$. Then*

$$\|g\|_{0,\beta} \leq 2^{(\delta-\beta)/\delta} \|g\|^{(\delta-\beta)/\delta} \|g\|_{0,\delta}^{\beta/\delta}. \tag{9.10}$$

2. *Let f_n and f be $C_b^{0,\delta}(U_o, Y)$ and assume that $\|f_n - f\| \to 0$. Then $\|f_n - f\|_{0,\beta} \to 0$ for all $0 \leq \beta < \delta$.*

Proof. Notice that part (2) follows from part (1) by setting $g = f_n - f$. So, we prove part (1).
Notice that

$$\Delta := \frac{|g(x+h) - g(x)|}{|h|^\beta} \leq \min\left\{ 2\frac{\|g\|}{|h|^\beta} \,,\ |h|^{\delta-\beta}\|g\|_{0,\delta} \right\}. \tag{\#}$$

The two terms on the righthand side of (#) are equal iff

$$|h| = \left(\frac{2\|g\|}{\|g\|_{0,\delta}}\right)^{1/\delta} := A.$$

When $|h| > A$, we use the first term on the righthand side of (#) to estimate Δ. When $|h| \leq A$ we use the second term. In either case we obtain

$$\Delta \leq 2^{(\delta-\beta)/\delta} \|g\|^{(\delta-\beta)/\delta} \|g\|_{0,\delta}^{\beta/\delta},$$

which proves (1). □

9.9. Lemma. *Let u_n be a sequence in $C_b^{k,\delta}(U, Y)$ and $u : U \to Y$. Assume that $\|u_n - u\| \to 0$. Let $U_o \subset U$ be a subset which is uniformly bounded away from the boundary of U and $cl(int(U_o)) \neq \emptyset$. Let $0 < \beta < \delta$. Then $u \in C_b^{k,\delta}(U, Y)$. Moreover, on U_o, $\|u_n - u\|_{k,\beta} \to 0$ as $n \to \infty$.*

Proof. By Henry's Lemma 9.6, $\|D^k u_n - D^k u\| \to 0$ on U_o as $n \to \infty$. Applying (2) of Lemma 9.8 to ,$D^k u_n$ and $D^k u$, we obtain

$$\|D^k u_n - D^k u\|_{0,\beta} \to 0,$$

as $n \to \infty$ on U_o. □

The following lemma shows the relationship between the N-norms we are using and uniform continuity on compact sets. It is stated in more general terms so that we can use it to prove the first part of Lemma 9.11.

9.10. Lemma. *Let X and Y be Banach spaces and $f : K \to Y$ be continuous where $K \subset X$. Assume that K satisfies the following condition : There are $\eta(x, x') > 0$, and $c(x, x') > 0$ such that if $x, x' \in K$ and $|x - x'| < \chi(x, x')$, then $|f(x) - f(x')| \le \eta(x, x')$. Let $j \ge 1$ and $0 < \delta \le 1$.*

Assume that $N_{j+\delta,o}(f) < \infty$. Let $0 < \gamma \le \delta$. The following are true:

1. *if $x, x' \in K$ and $|x - x'| < \chi(x, x')$, then*

$$\text{(a)} \qquad |f(x) - f(x')| \le [\eta(x, x')]^{\gamma/(j+\delta)}[N_{j+\delta,o}(f)]^{1-(\gamma/j+\delta)}(|x| + |x'|)^{j+\delta-\gamma}.$$

2. *Assume K is compact. Then for all $\eta > 0$, there is $c > 0$, such that if $|x - x'| < c$, (a) holds with $\eta(x, x') = \eta$.*

Proof. If $|x - x'| < \chi(x, x')$, then

$$\text{(9.11)} \qquad \Delta := \frac{|f(x) - f(x')|}{(|x| + |x'|)^{j+\delta-\gamma}} \le \min\left\{ \frac{\eta(x, x')}{(|x| + |x'|)^{j+\delta-\gamma}} \ , \ 2(|x| + |x'|)^{\gamma} N_{j+\delta,o}(f) \right\}.$$

The two terms on the right hand side of (9.11) are equal iff

$$|x| + |x'| = \left(\frac{\eta(x, x')}{2N_{j+\delta,o}(f)} \right)^{1/(j+\delta)} := A.$$

There are two cases: either $|x| + |x'| > A$ or $|x| + |x'| \le A$. In the first case we use the first term on the righthand side of (9.11) to estimate Δ; in the second case we use the second term. In either case estimate (a) follows.

If K is compact, the assertion follows from the definition of a compact set. □

The following lemma shows the relationship between the N-norms we are using and uniform Hölder continuity.

9.11. Lemma. *Let X and Y be Banach spaces and $U \subset X$ be such that $cl(int(U)) \ne \emptyset$. Let $f \in C^{0,\delta}(U, Y), j \ge 1$ and $0 < \beta < \delta \le 1$.*

Assume that $N_{j+\delta,o}(f) < \infty$. Then for all $x, x' \in U, x \ne x'$

$$\text{(b)} \quad |f(x) - f(x')| \le (2N_{j+\delta,o}(f))^{1-(\beta/(j+\delta))}(H_\delta(f))^{\beta/(j+\delta)}|x - x'|^{\beta\delta/(j+\delta)}(|x| + |x'|)^{j+\delta-\beta},$$

$$\text{(c)} \quad |f(x) - f(x')| \le (2N_{j+\delta,o}(f))^{1-\beta/\delta}(H_\delta(f))^{\beta/\delta}|x - x'|^{\beta}(|x| + |x'|)^{(j+\delta)(1-\beta/\delta)}.$$

Proof. Let $\gamma = \beta$ and $K = U$ in Lemma 9.10. Since $f \in C^{0,\delta}(U, Y)$, for all $x, x' \in U$,

$$|f(x) - f(x')| \le \eta(x, x') := H_\delta(f)|x - x'|^{\delta}, \qquad x, x' \in U.$$

We can take $\chi(x, x') = |x| + |x'|$. Thus estimate (b) follows from estimate (a) of Lemma 9.10.

We can prove estimate (b) directly as follows: Notice that

$$\text{(9.12)} \quad \Delta := \frac{|f(x) - f(x')|}{|x - x'|^{\beta}(|x| + |x'|)^{j+\delta-\beta}} \le \min\left\{ \frac{|x - x'|^{\delta-\beta} H_\delta(f)}{(|x| + |x'|)^{j+\delta-\beta}} \ , \ \frac{2(|x| + |x'|)^{\beta} N_{j+\delta,o}(f)}{|x - x'|^{\beta}} \right\}.$$

The two terms on the right hand side of (9.12) are equal iff

$$|x| + |x'| = \left(\frac{H_\delta(f)|x - x'|^{\delta}}{2N_{j+\delta,o}(f)} \right)^{1/(j+\delta)} := A.$$

There are two cases: either $|x| + |x'| > A$ or $|x| + |x'| \leq A$. In the first case we use the first term on the righthand side of (9.12) to estimate Δ; in the second case we use the second term. In either case,

$$\Delta \leq (2N_{j+\delta,o}(f))^{1-(\beta/(j+\delta))}(H_\delta(f))^{\beta/(j+\delta)}\frac{|x-x'|^\alpha}{|x-x'|^\beta}, \qquad \alpha = \frac{\beta\delta}{(j+\delta)},$$

which implies estimate (b) follows.

To prove estimate (c) notice that the two terms on the right hand side of (9.12) are equal iff

$$|x-x'| = \left(\frac{2N_{j+\delta,o}(f)(|x|+|x'|)^{j+\delta}}{H_\delta(f)}\right)^{1/\delta} := B.$$

Whether $|x-x'| \leq B$ or $|x-x'| > B$ we have

$$\Delta \leq (2N_{j+\delta,o}(f))^{1-(\beta/\delta)}(H_\delta(f))^{\beta/\delta}\frac{(|x|+|x'|)^\gamma}{(|x|+|x'|)^{j+\delta-\beta}}, \qquad \gamma = (j+\delta)(1-\beta/\delta),$$

which implies estimate (c). □

We can see from Lemma 9.11 that H_δ is not the appropriate norm to measure Hölder continuity for functions that vanish to high order at the origin. The following lemma gives the appropriate way to measure Hölder continuity for such function.

9.12. Lemma. *Let X and Y be Banach spaces and $U \subset X$ be such that $cl(int(U)) \neq \emptyset$. Let $f \in C^{0,\delta}(U,Y), j \geq 1$ and $0 < \beta < \delta \leq 1$. Define*

$$L_{j+\delta,o}(f) = \sup\left\{\frac{|f(x)-f(x')|}{|x-x'|^\delta[|x|+|x'|]^{j+\delta}} \ \mid\ x \neq x', x \in U, x' \in U\right\}$$

1. *Assume that $N_{j+\delta,o}(f) < \infty$ and $L_{j+\delta,o}(f) < \infty$. Then*
$$L_{j+\beta,o}(f) \leq [L_{j+\delta,o}(f)]^{\beta/\delta}[2N_{j+\delta,o}(f)]^{1-\beta/\delta}$$
2. *Let $u_n \in C^{0,\delta}(U,Y), L_{j+\delta,o}(u_n) \leq c$ and $N_{j+\delta,o}(u_n) \leq c$ for $n \geq 1$. Let $u: U \to Y$. Assume that $N_{j+\delta,o}(u_n - u) \to 0$. Then $N_{j+\delta,o}(u) \leq c, L_{j+\delta,o}(u) \leq c$ and $L_{j+\beta,o}(u_n-u) \to 0$ for all $0 < \beta < \delta$.*

The proof uses the same technique we used above and we leave it to the reader.

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