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Trotter-Kato product formula and symmetrically-normed ideals

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Abstract

It is proven that the Trotter product formula converges in the norm of a symmetrically-normed ideal of compact operators away from $t_0 > 0$ if the Kac operator (the transfer matrix) $F(t) = e^{-tB/2}e^{-tA}e^{-tB/2}$ belongs to this ideal for $t = t_0$. The result is generalized to the Trotter-Kato product formula. Moreover, if the perturbation B is small relative to A, then error bounds for convergence are obtained. The results apply to the Dixmier trace.

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Contents

1 Introduction

Let A and B be two non-negative self-adjoint operators on the separable Hilbert space $\mathcal{H}$, i.e.

$$A \geq 0 \qquad \text{and} \qquad B \geq 0. \tag{1.1}$$

By $\mathcal{H}_0$ we denote the subspace

$$\mathcal{H}_0 := \overline{\mathrm{dom}(A^{1/2}) \cap \mathrm{dom}(B^{1/2})}. \tag{1.2}$$

It may happen that $\mathcal{H}_0 = \{0\}$. The form sum $H = A \dot{+} B$ is well-defined on the subspace $\mathcal{H}_0$. Kato has shown in [10] that under these conditions the Trotter product formula strongly converges to e^{-tH} uniformly in $t \in [0,T]$, $0 < T < +\infty$, i.e., one has

$$s - \lim_{n\to\infty} \left(e^{-tA/n}e^{-tB/n}\right)^n = e^{-tH}P_0, \qquad t \geq 0, \tag{1.3}$$

where P_0 denotes the orthogonal projection from $\mathcal{H}$ onto $\mathcal{H}_0$. Furthermore, it was found by Kato in [9, 10] that the Trotter product formula is true not only for the exponential function e^{-x}, $x \geq 0$, but for a whole class of Borel measurable functions $f(\cdot)$ and $g(\cdot)$ which are defined on $[0,\infty)$ and characterized by

$$0 \leq f(x) \leq 1, \qquad f(0) = 1, \qquad f'(0) = -1 \tag{1.4}$$

$$0 \leq g(x) \leq 1, \qquad g(0) = 1, \qquad g'(0) = -1. \tag{1.5}$$

It was proved that besides (1.3) one always has

$$s - \lim_{n\to\infty} \left(f(tA/n)g(tB/n)\right)^n = e^{-tH}P_0, \qquad t \geq 0, \tag{1.6}$$

uniformly in $t \in [0,T]$, $0 < T < +\infty$. Product formulas of this type we call *Trotter-Kato product formulas* in the following. In a series of papers [11]-[14], [16] we were interested in the following problems:

(a) to indicate a natural topology γ in which the Trotter-Kato product formula (1.6) converges,

(b) to give convergence error bounds in this topology,

(c) to find a suitable class of Borel measurable functions $f(\cdot)$ and $g(\cdot)$ which allows to solve the previous problems in an optimal manner.

We define

$$0 \leq \varphi_0(x) := \inf_{0<s\leq x} s^{-1}\left(\frac{1}{f(s)} - 1\right), \tag{1.7}$$

where $0^{-1} := +\infty$, and set

$$f_0(x) := \begin{cases} 1 & x = 0 \\ (1 + x\varphi_0(x))^{-1} & x > 0. \end{cases} \tag{1.8}$$

It was shown in [11] that under the condition

$$f_0(tA) \in \mathcal{L}_p(\mathcal{H}), \qquad t > 0, \qquad 1 \le p < +\infty, \tag{1.9}$$

the natural topology γ is that one which is generated by the trace norm $\|\cdot\|_1$, i.e, one has $e^{-tH} \in \mathcal{L}_1(\mathcal{H}_0)$ for $t > 0$ and

$$\lim_{n\to\infty} \sup_{t\in[\tau_0,\tau]} \left\|(f(tA/n)g(tB/n))^n - e^{-tH}P_0\right\|_1 = 0 \tag{1.10}$$

for any bounded interval $[\tau_0, \tau] \subseteq (0, \infty)$. Here by $\mathcal{L}_p(\mathcal{H})$ and $\|\cdot\|_p$, $1 \le p < +\infty$, we denote the so-called von Neumann-Schatten ideals of compact operators and their norms ($p \ge 1$). The ideal of all compact operators is denoted by $\mathcal{L}_\infty(\mathcal{H})$. Its norm $\|\cdot\|_\infty$ is identical with the operator-norm $\|\cdot\|$ on $\mathcal{H}$.

If $f(x) = e^{-x}$, $x \ge 0$, then $\varphi_0(x) = 1$ which yields $f_0(x) = (1 + x)^{-1}$. Therefore, condition (1.9) reduces to

$$(I + tA)^{-1} \in \mathcal{L}_p(\mathcal{H}), \qquad t > 0, \qquad 1 \le p < +\infty, \tag{1.11}$$

and our result (1.10) transforms into *exponential* Trotter formula for Gibbs semigroups [16]:

$$\lim_{n\to\infty} \left\|\left(e^{-tA/n}e^{-tB/n}\right)^n - e^{-tH}P_0\right\|_1 = 0. \tag{1.12}$$

This result for the *exponential* product formula has been recently improved by Hiai [7]. Assuming instead of (1.11) the condition

$$e^{-A} \in \mathcal{L}_1(\mathcal{H}) \tag{1.13}$$

and using essentially Araki's results [1, 2], he found that $e^{-H} \in \mathcal{L}_1(\mathcal{H}_0)$ and

$$\lim_{r\downarrow 0} \left\|(e^{-rB/2}e^{-rA}e^{-rB/2})^{1/r} - e^{-H}P_0\right\|_1 = 0, \tag{1.14}$$

$$\lim_{r\downarrow 0} \left\|(e^{-rA/2}e^{-rB}e^{-rA/2})^{1/r} - e^{-H}P_0\right\|_1 = 0 \tag{1.15}$$

and

$$\lim_{n\to\infty} \left\|\left(e^{-A/n}e^{-B/n}\right)^{n+1} - e^{-H}\right\|_1 = 0. \tag{1.16}$$

In the following we set

$$F(t) := g(tB)^{1/2}f(tA)g(tB)^{1/2}, \qquad t \ge 0, \tag{1.17}$$

and

$$T(t) := f(tA)^{1/2}g(tB)f(tA)^{1/2}, \qquad t \ge 0. \tag{1.18}$$

The question, which we initiated in [11], is obviously a part of the following general problem: Let ϕ be any *symmetric norming function* (see Section 2). Let $\mathcal{L}_\phi(\mathcal{H})$ be the corresponding *symmetrically-normed ideal* of compact operators and $\|\cdot\|_\phi$ its norm (see Section 2). Which conditions guarantee that the Trotter-Kato product formula converges *locally uniformly away from* $t_0 > 0$ *in* $\mathcal{L}_\phi(\mathcal{H})$ for, e.g., the family $\{F(t)\}_{t\ge 0}$? This means that for any bounded interval $[\tau_0, \tau] \subseteq (t_0, +\infty)$ there is a $r_0 \ge 1$ such that

$$e^{-tH} \in \mathcal{L}_\phi(\mathcal{H}_0) \qquad \text{and} \qquad F(t/r)^r \in \mathcal{L}_\phi(\mathcal{H}) \tag{1.19}$$

for $t \in [\tau_0, \tau]$ and $r \geq r_0$ as well as

$$\lim_{r\to\infty} \sup_{t\in[\tau_0,\tau]} \|F(t/r)^r - e^{-tH}P_0\|_\phi = 0. \tag{1.20}$$

Similarly, one can extend this notion of convergence of the Trotter-Kato product formula to the families $\{T(t)\}_{t\geq0}$, $\{f(tA)g(tB)\}_{t\geq0}$ and $\{g(tB)f(tA)\}_{t\geq0}$ though in the last two cases one has to use the *discrete* parameter n instead of the *continuous* parameter r. If the convergence takes place for all these families, then we will say that the Trotter-Kato product formula converges *locally uniformly away from* t_0 *in* $\mathcal{L}_\phi(\mathcal{H})$ *for all families generated by* $f(\cdot)$ *and* $g(\cdot)$. It turns out (see Proposition 3.1) that if the Trotter-Kato product formula converges locally uniformly away from $t_0 > 0$ in $\mathcal{L}_\phi(\mathcal{H})$ for *one* of the families $\{F(t)\}_{t\geq0}$, $\{T(t)\}_{t\geq0}$, $\{f(tA)g(tB)\}_{t\geq0}$ and $\{g(tB)f(tA)\}_{t\geq0}$, then this formula converges locally uniformly away from t_0 in $\mathcal{L}_\phi(\mathcal{H})$ for *all* these families.

To solve the above problem we first attack three subproblems: Firstly, one has to find conditions that the Trotter-Kato product formula converges *locally uniformly away from zero in the operator-norm*, i.e., that for example

$$\lim_{r\to\infty} \sup_{t\in[a,b]} \|F(t/r)^r - e^{-tH}P_0\| = 0 \tag{1.21}$$

for any bounded interval $[a, b] \subset (0, \infty)$. Secondly, one has to guarantee that the sequence $\{F(t/r)^r\}_{r\geq1}$ is *locally uniformly bounded away from* $t_0 > 0$ *in* $\mathcal{L}_\phi(\mathcal{H})$, i.e., for any bounded interval $[\tau_0, \tau] \subseteq (t_0, +\infty)$ there is a $r_0 \geq 1$ such that for $r \geq r_0$ and $t \in [\tau_0, \tau]$ one has $F(t/r)^r \in \mathcal{L}_\phi(\mathcal{H})$ and

$$M([\tau_0, \tau]) := \sup_{r\geq r_0} \sup_{t\in[\tau_0,\tau]} \|F(t/r)^r\|_\phi < +\infty. \tag{1.22}$$

Thirdly, one has to verify that the limit e^{-tH} belongs to the ideal $\mathcal{L}_\phi(\mathcal{H}_0)$ for $t > t_0$.

To solve the first subproblem we essentially use our results of [12]-[14]. The solution of the second subproblem is based on an estimate due to Araki [2] (see also Proposition 2.1 of [7]) and the notion of *dominated* Kato-functions. We say that Kato-functions $f(\cdot)$ and $g(\cdot)$ are *dominated by Borel measurable functions* $f^D(\cdot) : \mathbf{R}^1_+ \longrightarrow \mathbf{R}^1_+$ *and* $g^D(\cdot) : \mathbf{R}^1_+ \longrightarrow \mathbf{R}^1_+$, respectively, if

$$f(qx)^{1/q} \leq f^D(x) \qquad \text{and} \qquad g(qx)^{1/q} \leq g^D(x), \qquad 0 < q \leq 1, \tag{1.23}$$

for $x \geq 0$. Now if the condition

$$F^D(t_0) := g^D(t_0B)^{1/2} f^D(t_0A) g^D(t_0B)^{1/2} \in \mathcal{L}_\phi(\mathcal{H}). \tag{1.24}$$

is satisfied for some $t_0 > 0$, then we can prove that away from t_0 the uniform boundedness of $\{F(t/r)\}_{r\geq1}$ in $\mathcal{L}_\phi(\mathcal{H})$ takes place and, moreover, that $e^{-tH} \in \mathcal{L}_\phi(\mathcal{H}_0)$ for $t > t_0$. Then, combining those results with the operator-norm convergence of the Trotter-Kato product formula established in [14], we obtain conditions for the locally uniform convergence of the Trotter-Kato product formula away from t_0 in $\mathcal{L}_\phi(\mathcal{H})$. Moreover, restricting the class of admissible Kato-functions to *self-dominated* ones, i.e., to Kato-functions $f(\cdot)$ and $g(\cdot)$ obeying

$$f(qx)^{1/q} \leq f(x) \qquad \text{and} \qquad g(qx)^{1/q} \leq g(x), \qquad 0 < q \leq 1, \tag{1.25}$$

for $x \geq 0$, which in addition satisfy conditions

$$C_0 := \sup_{x>0} \frac{xf(x)}{1-f(x)} < +\infty \qquad \text{and} \qquad S_0 := \sup_{x>0} \frac{xg(x)}{1-g(x)} < +\infty, \tag{1.26}$$

we obtain that there is a $t_0 > 0$ such that the Trotter-Kato product formula converges locally uniformly away from $t_0 > 0$ in $\mathcal{L}_\phi(\mathcal{H})$ if and only if for some $s_0 > 0$ and some integer p one has

$$F(s_0)^p \in \mathcal{L}_\phi(\mathcal{H}). \tag{1.27}$$

As an extension of our results of [14] we get that the Trotter-Kato product formula converges uniformly away from zero in the *operator-norm* topology if for some $t_0 > 0$ the condition $F(t_0) \in \mathcal{L}_\infty(\mathcal{H})$ is satisfied.

The paper is organized as follows. In Section 2 we recall some basic facts about symmetric norming functions and the corresponding symmetrically-normed ideals, see e.g. [6], [15]. Section 3 concerns the convergence problem of the Trotter-Kato product formula in $\mathcal{L}_\phi(\mathcal{H})$. In Section 4 we indicate *compactness conditions* implying the convergence of the Trotter-Kato product formula in $\mathcal{L}_\phi(\mathcal{H})$. In Section 5 we show that under suitable compactness assumptions the operator-norm error bounds lead to error bounds in symmetrically normed ideals. In particular, the operator-norm error bounds obtained in [12] and [13] for smallness conditions ensure the same bounds in symmetrically normed ideals. Finally, in Section 6 we apply the previous results to traces, in particular, to the Dixmier trace. Section 7 is devoted to remarks and discussions of the above results.

2 Symmetrically-normed ideals

Let c_0 be the space of all sequences $\xi = \{\xi_j\}_{j=1}^\infty$ of real numbers which tend to zero. We denote by c_f the subspace of c_0 consisting of all sequences with finite number of non-zero terms (finite sequences). A real-valued function $\phi(\xi) = \phi(\xi_1, \xi_2, \ldots)$ defined on c_f is called a *norming function* if it has the following properties:

$$\phi(\xi) > 0, \qquad \forall \xi \in c_f, \quad \xi \neq 0, \tag{2.1}$$

$$\phi(\alpha\xi) = |\alpha|\phi(\xi), \qquad \forall \xi \in c_f, \quad \forall \alpha \in \mathbf{R}^1, \tag{2.2}$$

$$\phi(\xi + \eta) \leq \phi(\xi) + \phi(\eta), \qquad \forall \xi, \eta \in c_f, \tag{2.3}$$

$$\phi(1, 0, \ldots) = 1. \tag{2.4}$$

A norming function ϕ is called to be *symmetric* if it has the additional property

$$\phi(\xi_1, \xi_2, ..., \xi_n, 0, 0, \ldots) = \phi(|\xi_{j_1}|, |\xi_{j_2}|, ..., |\xi_{j_n}|, 0, 0, \ldots) \tag{2.5}$$

for any $\xi \in c_f$ and any permutation $j_1, j_2, \ldots, j_n$ of integers $1, 2, \ldots, n$. It turns out (see, e.g., [6], Ch. III.3) that for any *symmetric norming function* and any sequences $\xi, \eta \in c_f$ obeying $\xi_1 \geq \xi_2 \geq \ldots \geq 0$, $\eta_1 \geq \eta_2 \geq \ldots \geq 0$ and

$$\sum_{j=1}^n \xi_j \leq \sum_{j=1}^n \eta_j, \qquad n = 1, 2, \ldots, \tag{2.6}$$

one has

$$\phi(\xi) \leq \phi(\eta) \tag{2.7}$$

and

$$\xi_1 \leq \phi(\xi) \leq \sum_{j=1}^{\infty} \xi_j, \qquad \xi \in c_f. \tag{2.8}$$

Let $\xi = \{\xi_1, \xi_2, \ldots\} \in c_0$ and let

$$\xi^{(n)} := \{\xi_1, \xi_2, \ldots, \xi_n, 0, 0, \ldots\}. \tag{2.9}$$

If ϕ is a symmetric norming function we set

$$c_\phi := \{\xi \in c_0 : \sup_n \phi(\xi^{(n)}) < +\infty\}. \tag{2.10}$$

Therefore, one gets

$$c_f \subseteq c_\phi \subseteq c_0. \tag{2.11}$$

By Lemma III.3.1 of [6] it follows that the limit

$$\phi(\xi) := \lim_{n\to\infty} \phi(\xi^{(n)}), \qquad \xi \in c_\phi, \tag{2.12}$$

exists. The space c_ϕ is linear over real numbers. Denoting by $\xi^* = \{\xi_1^*, \xi_2^*, \ldots\}$ the sequence defined by $\xi_1^* = \sup_j |\xi_j|$, $\xi_1^* + \xi_2^* = \sup_{i\neq j}\{|\xi_i| + |\xi_j|\}$ etc., one obviously has $\xi_1^* \geq \xi_2^* \geq \ldots$. Moreover, $\xi \in c_\phi$ implies $\xi^* \in c_\phi$ and

$$\phi(\xi) = \phi(\xi^*), \qquad \xi \in c_\phi. \tag{2.13}$$

If X is a compact operator, i.e. $X \in \mathcal{L}_\infty(\mathcal{H})$, then by

$$s(X) := \{s_1(X), s_2(X), \ldots\} \tag{2.14}$$

we denote the sequence of singular numbers of X counting multiplicities. We always assume that

$$s_1(X) \geq s_2(X) \geq \ldots \geq s_n(X) \geq \ldots . \tag{2.15}$$

Let ϕ be a symmetric norming function. We set

$$\mathcal{L}_\phi(\mathcal{H}) := \{X \in \mathcal{L}_\infty(\mathcal{H}) : s(X) \in c_\phi\}. \tag{2.16}$$

The set $\mathcal{L}_\phi(\mathcal{H})$ performs a two-sided ideal in the algebra $\mathcal{L}(\mathcal{H})$ of all bounded operators on $\mathcal{H}$. Setting

$$\|X\|_\phi := \phi(s(X)), \qquad X \in \mathcal{L}_\phi(\mathcal{H}), \tag{2.17}$$

one obtains a norm on the two-sided ideal $\mathcal{L}_\phi(\mathcal{H})$ such that $\mathcal{L}_\phi(\mathcal{H})$ becomes a Banach space. By (2.7) and (2.8) one always has that

$$\mathcal{L}_1(\mathcal{H}) \subseteq \mathcal{L}_\phi(\mathcal{H}) \subseteq \mathcal{L}_\infty(\mathcal{H}) \tag{2.18}$$

and

$$\|X\|_\phi \leq \|X\|_1, \qquad X \in \mathcal{L}_1(\mathcal{H}). \tag{2.19}$$

Moreover, by (2.7) the so-called *dominance property* (see [6], Ch. III.3) takes place: If $X \in \mathcal{L}_\phi(\mathcal{H})$, $Y \in \mathcal{L}_\infty(\mathcal{H})$ and

$$\sum_{j=1}^{n} s_j(Y) \le \sum_{j=1}^{n} s_j(X), \qquad n = 1, 2, \ldots, \tag{2.20}$$

then $Y \in \mathcal{L}_\phi(\mathcal{H})$ and $\|Y\|_\phi \le \|X\|_\phi$.

Example 2.1 The widely known von Neumann-Schatten ideals $\mathcal{L}_p(\mathcal{H})$ are generated by the symmetric norming functions l_p,

$$l_p(\xi) = \left(\sum_{j=1}^{\infty} |\xi_j|^p\right)^{1/p}, \qquad \xi \in c_f, \tag{2.21}$$

for $1 \le p < +\infty$ and

$$l_\infty(\xi) = \sup_j |\xi_j|, \qquad \xi \in c_f, \tag{2.22}$$

for $p = +\infty$. We set $\mathcal{L}_p(\mathcal{H}) := \mathcal{L}_{l_p}(\mathcal{H})$. □

Example 2.2 Let $\Pi = \{\pi_j\}_{j=1}^\infty$ be a non-increasing sequence of positive numbers with $\pi_1 = 1$. We associate with Π the function

$$\phi_\Pi(\xi) = \sup_n \left\{ \frac{1}{\sum_{j=1}^n \pi_j} \sum_{j=1}^{n} \xi_j^* \right\}, \qquad \xi \in c_f. \tag{2.23}$$

It turns out that ϕ_Π is a symmetric norming function. The set c_ϕ is characterized by

$$c_\phi := \left\{ \xi \in c_0 : \sup_n \frac{1}{\sum_{j=1}^n \pi_j} \sum_{j=1}^{n} \xi_j^* < +\infty \right\}. \tag{2.24}$$

Hence the two-sided ideal $\mathcal{L}_\Pi(\mathcal{H}) := \mathcal{L}_{\phi_\Pi}(\mathcal{H})$ consists of all those compact operators X for which

$$\sup_n \frac{1}{\sum_{j=1}^n \pi_j} \sum_{j=1}^{n} s_j(X) < +\infty. \tag{2.25}$$

If $\Pi = \{\pi_j\}_{j=1}^\infty$ satisfies

$$\sum_{j=1}^{\infty} \pi_j = +\infty \qquad \text{and} \qquad \lim_{j\to\infty} \pi_j = 0, \tag{2.26}$$

then $\mathcal{L}_\Pi(\mathcal{H}) \ne \mathcal{L}_\infty(\mathcal{H})$ and $\mathcal{L}_\Pi(\mathcal{H}) \ne \mathcal{L}_1(\mathcal{H})$. If in addition to (2.26) the sequence $\Pi = \{\pi_j\}_{j=1}^\infty$ obeys

$$\sum_{j=1}^{n} \pi_j = O(n\pi_n), \qquad n \longrightarrow \infty, \tag{2.27}$$

then $X \in \mathcal{L}_\Pi(\mathcal{H})$ if and only if

$$s_n(X) = O(\pi_n), \qquad n \longrightarrow \infty. \tag{2.28}$$

A natural choice of $\{\pi_j\}_{j=1}^\infty$ is $\pi_j = j^{-\alpha}$, $j = 1, 2, \ldots$, $0 < \alpha \le 1$. If $0 < \alpha < 1$, then $\Pi = \{\pi_j\}_{j=1}^\infty$ satisfies (2.26) and (2.27). Hence, the two-sided ideal $\mathcal{L}_\Pi(\mathcal{H})$ consists of all those compact operators X which singular numbers obey

$$s_n(X) = O(n^{-\alpha}), \quad 0 < \alpha < 1, \quad n \longrightarrow \infty. \tag{2.29}$$

In accordance with [3] the ideals $\mathcal{L}_\Pi(\mathcal{H})$, $0 < \alpha \le 1$, $\Pi = \{j^{-\alpha}\}_{j=1}^\infty$, are denoted by $\mathcal{L}_{p,\infty}(\mathcal{H})$ where $p = 1/\alpha$. A norm which is equivalent to $\|\cdot\|_\Pi := \|\cdot\|_{\phi_\Pi}$ for $0 < \alpha < 1$ is given by

$$\|X\|_{p,\infty} = \sup_n \frac{1}{n^{1-\alpha}} \sum_{j=1}^n s_j(X). \tag{2.30}$$

These ideals correspond to the notion of *weak* $\mathcal{L}_p$-spaces in classical analysis. The ideal $\mathcal{L}_{1,\infty}(\mathcal{H})$ consists of all those compact operators which singular numbers obey

$$\mathcal{L}_{1,\infty}(\mathcal{H}) := \{X \in \mathcal{L}_\infty(\mathcal{H}) : \sum_{j=1}^n s_j(X) = O(\ln(n)), \quad n \longrightarrow \infty\}. \tag{2.31}$$

Since $\Pi = \{j^{-1}\}_{j=1}^\infty$ does *not* satisfy (2.27), the characterization $s_n(X) = O(n^{-1})$, $n \longrightarrow \infty$, cf. (2.29), is not true. An equivalent norm can be given by

$$\|X\|_{1,\infty} := \sup_{n \ge 2} \frac{1}{\ln(n)} \sum_{j=1}^n s_j(X). \tag{2.32}$$

The ideal $\mathcal{L}_{1,\infty}(\mathcal{H})$ is basic for the *Dixmier trace*, see e.g. [3]. □

Example 2.3 With the sequence $\Pi = \{\pi_j\}_{j=1}^\infty$ one can associate another symmetric norming function ϕ_π given by

$$\phi_\pi(\xi) := \sum_{j=1}^\infty \pi_j \xi_j^*, \qquad \xi \in c_f. \tag{2.33}$$

As above, we set $\mathcal{L}_\pi(\mathcal{H}) := \mathcal{L}_{\phi_\pi}(\mathcal{H})$. If Π satisfies (2.26), then the corresponding ideal $\mathcal{L}_\phi(\mathcal{H})$ does not coincide neither with $\mathcal{L}_\infty(\mathcal{H})$ nor with $\mathcal{L}_1(\mathcal{H})$. If $\pi_j = j^{-\alpha}$, $j = 1, 2, \ldots$, $0 < \alpha \le 1$, then the corresponding ideal is denoted by $\mathcal{L}_{\infty,p}(\mathcal{H})$, $p = 1/\alpha$, cf. [3]. In this case the norm is given by

$$\|X\|_{\infty,p} := \sum_{j=1}^\infty j^{-\alpha} s_j(X). \tag{2.34}$$

The ideal $\mathcal{L}_{\infty,1}(\mathcal{H})$ is called the *Macaev ideal* (see [6], Ch. III.15). It turns out that $\mathcal{L}_{\infty,1}(\mathcal{H})^* = \mathcal{L}_{1,\infty}(\mathcal{H})$. □

Let $X \ge 0$ be a compact operator. Let $\lambda(X) = \{\lambda_j(X)\}_{j=1}^\infty$ be its sequence of eigenvalues counting multiplicities. We always assume that

$$\lambda_1(X) \ge \lambda_2(X) \ge \ldots \ge 0. \tag{2.35}$$

Obviously one has that $s(X) = \lambda(X)$. If $X \ge 0$ and $Y \ge 0$ are two compact operators, then we write

$$Y \prec_{\log} X \tag{2.36}$$

if
$$\prod_{j=1}^{n} \lambda_j(Y) \le \prod_{j=1}^{n} \lambda_j(X), \qquad n = 1, 2, \ldots. \tag{2.37}$$
The following result was announced in [7].

Lemma 2.4 *Let X and Y be non-negative self-adjoint operators and let ϕ be a symmetric norming function. If $X \in \mathcal{L}_\phi(\mathcal{H})$ and $Y \prec_{\log} X$, then $Y \in \mathcal{L}_\phi(\mathcal{H})$ and*
$$\|Y\|_\phi \le \|X\|_\phi. \tag{2.38}$$

Proof: From (2.36) and (2.37) we get that
$$\sum_{j=1}^{n} \ln(\lambda_j(Y)) \le \sum_{j=1}^{n} \ln(\lambda_j(X)), \qquad n = 1, 2, \ldots, \tag{2.39}$$
By Lemma II.3.4 of [6] we obtain
$$\sum_{j=1}^{n} e^{\ln(\lambda_j(Y))} \le \sum_{j=1}^{n} e^{\ln(\lambda_j(X))}, \qquad n = 1, 2, \ldots, \tag{2.40}$$
which yields that
$$\sum_{j=1}^{n} \lambda_j(Y) \le \sum_{j=1}^{n} \lambda_j(X), \qquad n = 1, 2, \ldots. \tag{2.41}$$
By the dominance property of $\mathcal{L}_\phi(\mathcal{H})$ we find $Y \in \mathcal{L}_\phi(\mathcal{H})$ and $\|Y\|_\phi \le \|X\|_\phi$. □

In the following we apply Lemma 2.4 in a slightly modified form.

Lemma 2.5 *Let X, X_0, Y and Y_0 be non-negative self-adjoint operators and let ϕ be a symmetric norming function. If for some $r \ge 1$ one has $X^r \le X_0$ and $Y^r \le Y_0$ and $Y_0^{1/2} X_0 Y_0^{1/2} \in \mathcal{L}_\phi(\mathcal{H})$, then one has*
$$\left(Y^{1/2} X Y^{1/2}\right)^r \prec_{\log} Y_0^{1/2} X_0 Y_0^{1/2}, \tag{2.42}$$
$\left(Y^{1/2} X Y^{1/2}\right)^r \in \mathcal{L}_\phi(\mathcal{H})$ *and*
$$\|(Y^{1/2} X Y^{1/2})^r\|_\phi \le \|Y_0^{1/2} X_0 Y_0^{1/2}\|_\phi. \tag{2.43}$$

Proof: By the Araki inequality [2] (see Proposition 2.1 in [7]) we have
$$(Y^{1/2} X Y^{1/2})^r \prec_{\log} Y^{r/2} X^r Y^{r/2}. \tag{2.44}$$
Since $X^r \le X_0$ one gets
$$Y^{r/2} X^r Y^{r/2} \le Y^{r/2} X_0 Y^{r/2}. \tag{2.45}$$
Moreover, since $Y^r \le Y_0$, there is a contraction $\Gamma_0 : \overline{\mathrm{ran}(Y_0)} \longrightarrow \mathcal{H}$ such that
$$Y^{r/2} = \Gamma_0 Y_0^{1/2} \tag{2.46}$$
which leads to
$$Y^{r/2} X^r Y^{r/2} \le \Gamma_0 Y_0^{1/2} X_0 Y_0^{1/2} \Gamma_0^*. \tag{2.47}$$

By

$$\lambda_j(Y^{r/2}X^rY^{r/2}) \le \lambda_j(\Gamma_0 Y_0^{1/2}X_0Y_0^{1/2}\Gamma_0^*) \le \lambda_j(Y_0^{1/2}X_0Y_0^{1/2}) \tag{2.48}$$

for $j = 1, 2, \ldots$ we finally obtain

$$(Y^{1/2}XY^{r/2})^r \prec_{\log} Y^{r/2}X^rY^{r/2} \prec_{\log} Y_0^{1/2}X_0Y_0^{1/2}. \tag{2.49}$$

Now applying Lemma 2.4 we complete the proof. □

Lemma 2.6 *Let $X = X^* \in \mathcal{L}_\phi(\mathcal{H})$, $Y = Y^* \in \mathcal{L}_\infty(\mathcal{H})$ and $Z = Z^* \in \mathcal{L}(\mathcal{H})$. If $\{Z(t)\}_{t\ge0}$, $Z(t) = Z(t)^*$ is a family of self-adjoint bounded operators such that*

$$s - \lim_{t\to+0} Z(t) = Z, \tag{2.50}$$

then

$$\lim_{r\to\infty}\sup_{t\in[0,\tau]} \|(Z(t/r) - Z)YX\|_\phi = \lim_{r\to\infty}\sup_{t\in[0,\tau]} \|XY(Z(t/r) - Z)\|_\phi = 0 \tag{2.51}$$

for any $\tau \in (0, \infty)$.

Proof: Notice that

$$s - \lim_{r\to\infty} Z(t/r) = Z \tag{2.52}$$

uniformly in $t \in [0, \tau]$. Since $Y \in \mathcal{L}_\infty(\mathcal{H})$, this implies $\lim_{r\to\infty}\sup_{t\in[0,\tau]} \|(Z(t/r) - Z)Y\| = 0$ (see e.g. [15], Ch. 2). Using the estimate

$$\|(Z(t/r) - Z)YX\|_\phi \le \|(Z(t/r) - Z)Y\|\|X\|_\phi \tag{2.53}$$

we prove (2.51). □

3 Convergence in ideals: General results

We start with a demonstration that it is enough to consider the Trotter-Kato product formula only for one of the families $\{F(t)\}_{t\ge0}$, $\{T(t)\}_{t\ge0}$, $\{f(tA)g(tB)\}_{t\ge0}$ and $\{g(tB)f(tA)\}_{t\ge0}$.

Proposition 3.1 *Let A and B be non-negative self-adjoint operators on the separable Hilbert space $\mathcal{H}$ and let $f(\cdot)$ and $g(\cdot)$ be Kato-functions. If the Trotter-Kato product formula for one of the families $\{F(t)\}_{t\ge0}$, $\{T(t)\}_{t\ge0}$, $\{g(tB)f(tA)\}_{t\ge0}$ and $\{f(tA)g(tB)\}_{t\ge0}$ converges locally uniformly away from $t_0 > 0$ in $\mathcal{L}_\phi(\mathcal{H})$, then the Trotter-Kato product formula converges locally uniformly away from t_0 in $\mathcal{L}_\phi(\mathcal{H})$ for all families generated by $f(\cdot)$ and $g(\cdot)$.*

Proof: We prove that $\{F(t)\}_{t\ge0} \Longrightarrow \{T(t)\}_{t\ge0} \Longrightarrow \{f(tA)g(tB)\}_{t\ge0} \Longrightarrow \{g(tB)f(tA)\}_{t\ge0} \Longrightarrow \{F(t)\}_{t\ge0}$.

$\{F(t)\}_{t\ge0} \Longrightarrow \{T(t)\}_{t\ge0}$. We use the representation

$$T(t/r)^r = T(t/r)^p f(tA/r)^{1/2} g(tB/r)^{1/2} F(t/r)^{n-1} g(tB/r)^{1/2} f(tA/r)^{1/2} \tag{3.1}$$

for $t \ge 0$ and $r \ge 1$, where $r := n + p$, $n = 1, 2 \ldots$, $0 \le p < 1$. Since

$$F(t/r)^{n-1} = F(t_n/(n-1))^{n-1}, \tag{3.2}$$

where $t_n := \frac{n-1}{n+p}t$, we obtain from (3.1) that $T(t/r)^r \in \mathcal{L}_\phi(\mathcal{H})$ if $t_n \in [\tau_0', \tau] \subseteq (t_0, \infty)$. Since $t_n \in [\tau_0', \tau] \subseteq (t_0, \infty)$ is equivalent to $t \in [(1+\frac{p+1}{n-1})\tau_0', (1+\frac{p+1}{n-1})\tau]$, we find that for $[\tau_0, \tau] \subseteq (\tau_0', \tau]$ and for sufficiently large r one has $[\tau_0, \tau] \subseteq [(1+\frac{p+1}{n-1})\tau_0', (1+\frac{p+1}{n-1})\tau]$ and $T(t/r)^r \in \mathcal{L}_\phi(\mathcal{H})$ for $t \in [\tau_0, \tau]$. To prove that the Trotter-Kato product formula for $\{T(t)\}_{t\geq 0}$ converges locally uniformly away from t_0 in $\mathcal{L}_\phi(\mathcal{H})$ we note that this convergence for $\{F(t)\}_{t\geq 0}$ implies

$$\lim_{n\to\infty} \sup_{t\in[\tau_0',\tau]} \|F(t/(n-1))^{n-1} - e^{-tH}P_0\|_\phi = 0. \tag{3.3}$$

Replacing t by t_n we get

$$\lim_{n\to\infty} \sup_{t_n\in[\tau_0',\tau]} \|F(t_n/(n-1))^{n-1} - e^{-tH}P_0\|_\phi = 0 \tag{3.4}$$

Since $t_n \in [\tau_0', \tau]$ for $t \in [\tau_0, \tau]$, we obtain

$$\lim_{n\to\infty} \sup_{t\in[\tau_0,\tau]} \|F(t/r)^{n-1} - e^{-tH}P_0\|_\phi = 0. \tag{3.5}$$

Taking into account the representation

$$\begin{aligned}
&T(t/r)^r - e^{-tH}P_0 = \\
&\quad T(t/r)^p f(tA/r)^{1/2} g(tB/r)^{1/2}\{F(t/r)^{n-1} - e^{-tH}P_0\} g(tB/r)^{1/2} f(tA/r)^{1/2} + \\
&\quad T(t/r)^p f(tA/r)^{1/2} g(tB/r)^{1/2} e^{-tH}P_0\{g(tB/r)^{1/2} - I\} f(tA/r)^{1/2} + \\
&\quad T(t/r)^p f(tA/r)^{1/2} g(tB/r)^{1/2} e^{-tH}P_0\{f(tA/r)^{1/2} - I\} + \\
&\quad T(t/r)^p f(tA/r)^{1/2}\{g(tB/r)^{1/2} - I\} e^{-tH}P_0 + T(t/r)^p\{f(tA/r)^{1/2} - I\} e^{-tH}P_0 + \\
&\quad T(t/r)^p\{f(tA/r)^{1/2} - I\} e^{-tH}P_0 + \{T(t/r)^p - I\} e^{-tH}P_0
\end{aligned} \tag{3.6}$$

we get the estimate

$$\begin{aligned}
&\|T(t/r)^r - e^{-tH}P_0\|_\phi \leq \\
&\qquad \|F(t/r)^{n-1} - e^{-tH}P_0\|_\phi + 2\|\{I - g(tB/r)^{1/2}\} e^{-tH}P_0\|_\phi + \\
&\qquad 2\|\{I - f(tA/r)^{1/2}\} e^{-tH}P_0\|_\phi + \|\{I - T(t/r)^p\} e^{-tH}P_0\|_\phi.
\end{aligned} \tag{3.7}$$

Since $s-\lim_{t\to+0} f(tA)^{1/2} = I$, $s-\lim_{t\to+0} g(tB)^{1/2} = I$ and $s-\lim_{t\to+0} T(t) = I$, by Lemma 2.6 we get

$$\lim_{r\to\infty} \sup_{t\in[\tau_0,\tau]} \|\{I - g(tB/r)^{1/2}\} e^{-tH}P_0\|_\phi = \lim_{r\to\infty} \sup_{t\in[\tau_0,\tau]} \|\{I - f(tA/r)^{1/2}\} e^{-tH}P_0\|_\phi = 0 \tag{3.8}$$

and

$$\lim_{r\to+\infty} \sup_{t\in[\tau_0,\tau]} \|\{I - T(t/r)^p\} e^{-tH}P_0\|_\phi \leq \lim_{r\to+\infty} \sup_{t\in[\tau_0,\tau]} \|\{I - T(t/r)\} e^{-tH}P_0\|_\phi = 0. \tag{3.9}$$

Taking into account (3.5), (3.7), (3.8) and (3.9), we immediately prove the locally uniform convergence away from t_0 of the Trotter-Kato product formula for $\{T(t)\}_{t\geq 0}$.

$\{T(t)\}_{t\geq 0} \Longrightarrow \{g(tB)f(tA)\}_{t\geq 0}$. Using the representation

$$(f(tA/n)g(tB/n))^n = f(tA/n)^{1/2} T(t/n)^{n-1} f(tA/n)^{1/2} \tag{3.10}$$

and follow the line of reasoning of the part $\{F(t)\}_{t\geq 0} \Longrightarrow \{T(t)\}_{t\geq 0}$ one can verify that the Trotter-Kato product formula converges locally uniformly away from $t_0 > 0$ in $\mathcal{L}_\phi(\mathcal{H})$.

$\{g(tB)f(tA)\}_{t\geq 0} \Longrightarrow \{f(tA)g(tB)\}_{t\geq 0}$. This proof is based on the representation

$$(g(tB/n)f(tA/n))^n = g(tB/n)(f(tA/n)g(tB/n))^{n-1}f(tA/n) \tag{3.11}$$

and considerations made above.

$\{f(tA)g(tB)\}_{t\geq 0} \Longrightarrow \{F(t)\}_{t\geq 0}$. The assertion is a consequence of the the representation

$$F(t/r)^r = F(t/r)^p g(tA/r)^{1/2} f(tA/r)(g(tB/r)f(tA/r))^{n-1} g(tB/r)^{1/2}, \tag{3.12}$$

where $r = n + p$, $n = 1, 2, \ldots$, $0 \leq p < 1$, and the reasoning above. □

In virtue of Proposition 3.1 it is enough to consider below only the family $\{F(t)\}_{t\geq 0}$.

Proposition 3.2 *Let A and B be non-negative self-adjoint operators on the separable Hilbert space $\mathcal{H}$ and let $f(\cdot)$ and $g(\cdot)$ be Kato-functions. If the Trotter-Kato product formula for $\{F(t)\}_{t\geq 0}$ converges locally uniformly away from zero in the operator-norm, further, the sequence $\{F(t/r)\}^r$ is locally uniformly bounded away from $t_0 > 0$ in $\mathcal{L}_\phi(\mathcal{H})$ and $e^{-tH} \in \mathcal{L}_\phi(\mathcal{H}_0)$ for $t > t_0$, then the Trotter-Kato product formula converges locally uniformly away from t_0 in $\mathcal{L}_\phi(\mathcal{H})$ for all families generated by $f(\cdot)$ and $g(\cdot)$.*

Proof: If $t \in [\tau_0, \tau] \subset (t_0, \infty)$, then there is an $\alpha \in (0,1)$ such that $\tau_0' := \alpha\tau_0 > t_0$ and $\alpha t \in [\tau_0', \tau] \subset (t_0, \infty)$. Since

$$F(t/r)^{\alpha r} = F(\alpha t/\alpha r)^{\alpha r}, \qquad t \geq 0, \tag{3.13}$$

there is a $r_0 \geq 1$ such that $F(t/r)^{\alpha r} \in \mathcal{L}_\phi(\mathcal{H})$ and

$$\|F(t/r)^{\alpha r}\|_\phi \leq M([\tau_0', \tau]) \tag{3.14}$$

for $t \in [\tau_0, \tau]$ and $r \geq r_0$. Since $e^{-tH} \in \mathcal{L}_\phi(\mathcal{H})$ for $t > t_0$, the operator e^{-tH} is compact for $t > 0$ which yields that the spectrum of H is discrete with the only accumulation point at infinity. Denoting by $E_H(\cdot)$ the spectral measure of H we consider decomposition

$$e^{-tH} = e^{-tH}E_H([0, N)) + e^{-tH}E_H([N, \infty)), \qquad N = 1, 2, \ldots, \quad t \geq 0. \tag{3.15}$$

Since the spectrum of H is discrete, the projection $E_H([0, N))$ is a finite-dimensional operator for each $N = 1, 2, \ldots$. Hence for any $N = 1, 2, \ldots$ one gets

$$\lim_{r\to\infty} \sup_{t\in[0,\tau]} \|E_H([0, N))P_0(F(t/r)^{\alpha r} - e^{-\alpha tH})\|_\phi = 0. \tag{3.16}$$

Since $HE_H([N, \infty)) \geq NE_H([N, \infty))$ we find

$$\|e^{-(1-\alpha)tH}E_H([N, \infty))\| \leq e^{-(1-\alpha)\tau_0 N}, \qquad N = 1, 2, \ldots, \tag{3.17}$$

for $t \in [\tau_0, \tau]$. Taking into account the estimate

$$\begin{aligned} &\|e^{-(1-\alpha)tH}E_H([0, N))(F(t/r)^{\alpha r} - e^{-\alpha tH}P_0)\|_\phi \leq \\ &\qquad \|e^{-(1-\alpha)tH}E_H([N, \infty))\| \left\{\|F(t/r)^{\alpha r}\|_\phi + \|e^{-\alpha tH}P_0\|_\phi\right\} \end{aligned} \tag{3.18}$$

as well as (3.14) and (3.17) we finally get

$$\|e^{-(1-\alpha)tH}E_H([N,\infty))(F(t/r)^{\alpha r} - e^{-\alpha tH}P_0)\|_\phi \le e^{-(1-\alpha)\tau_0 N}\left\{M([\tau_0',\tau]) + \|e^{-\alpha\tau_0 H}\|_\phi\right\} \quad (3.19)$$

for $N = 1, 2, \ldots$, $t \in [\tau_0, \tau] \subseteq (\tau_0', \tau]$ and $r \ge r_0$.

Using the representation

$$\begin{aligned} F(t/r)^r - e^{-tH}P_0 = \\ & \left(F(t/r)^{(1-\alpha)r} - e^{-(1-\alpha)tH}P_0\right) F(t/r)^{\alpha r} + \\ & e^{-(1-\alpha)tH}E_H([0,N))P_0\left(F(t/r)^{\alpha r} - e^{-\alpha tH}P_0\right) + \\ & e^{-(1-\alpha)tH}E_H([N,\infty))P_0\left(F(t/r)^{\alpha r} - e^{-\alpha tH}P_0\right), \end{aligned} \quad (3.20)$$

from (3.14) and (3.19) we obtain the estimate

$$\begin{aligned} \sup_{t\in[\tau_0,\tau]} \|F(t/r)^r - e^{-tH}P_0\|_\phi \le \\ & M([\tau_0',\tau]) \sup_{t\in[\tau_0,\tau]} \|F(t/r)^{(1-\alpha)r} - e^{-(1-\alpha)tH}P_0\| + \\ & \sup_{t\in[\tau_0,\tau]} \|E_H([0,N))P_0(F(t/r)^{\alpha r} - e^{-\alpha tH}P_0)\|_\phi + e^{-(1-\alpha)\tau_0 N}\left\{M([\tau_0',\tau]) + \|e^{-\alpha\tau_0 H}\|_\phi\right\}. \end{aligned} \quad (3.21)$$

for $N = 1, 2, \ldots$, $t \in [\tau_0, \tau]$ and $r \ge r_0$. For any $\epsilon > 0$ we find an integer $N = 1, 2, \ldots$ such that

$$e^{-(1-\alpha)\tau_0 N}\left\{M([\tau_0',\tau]) + \|e^{-\alpha\tau_0 H}\|_\phi\right\} \le \epsilon. \quad (3.22)$$

Then we fix this integer N. Since the Trotter-Kato product formula converges locally uniformly away from zero in the operator-norm, we obtain for sufficiently large r that

$$M([\tau_0',\tau]) \sup_{t\in[\tau_0,\tau]} \|F(t/r)^{(1-\alpha)r} - e^{-(1-\alpha)tH}P_0\| \le \epsilon \quad (3.23)$$

and

$$\sup_{t\in[\tau_0,\tau]} \|E_H([0,N))P_0(F(t/r)^{\alpha r} - e^{-\alpha tH}P_0)\|_\phi \le \epsilon. \quad (3.24)$$

Hence for any $\epsilon > 0$ we get

$$\sup_{t\in[\tau_0,\tau]} \|F(t/r)^r - e^{-tH}P_0\|_\phi \le 3\epsilon \quad (3.25)$$

for an appropriate integer $N = 1, 2, \ldots$ and sufficiently large r. Therefore, for $\{F(t)\}_{t\ge 0}$ the Trotter-Kato product formula converges locally uniformly away from $t_0 > 0$ in $\mathcal{L}_\phi(\mathcal{H})$. Applying Proposition 3.1 we complete the proof. □

The following lemma we need to verify that the sequence $\{F(t/r)^r\}_{r\ge 1}$ is locally uniform bounded away from $t_0 > 0$ in $\mathcal{L}_\infty(\mathcal{H})$.

Lemma 3.3 *Let A and B be non-negative self-adjoint operators on the separable Hilbert space $\mathcal{H}$ and let $f^D(\cdot) : \mathbf{R}^1_+ \longrightarrow \mathbf{R}^1_+$ and $g^D(\cdot) : \mathbf{R}^1_+ \longrightarrow \mathbf{R}^1_+$ be bounded Borel measurable functions* *such that $F^D(t_0)^p \in \mathcal{L}_\phi(\mathcal{H})$ for some $t_0 > 0$ and some integer $p \geq 1$. If the Kato-functions $f(\cdot)$ and $g(\cdot)$ are dominated by $f^D(\cdot)$ and $g^D(\cdot)$, respectively, then*

$$F(t/r)^r \in \mathcal{L}_\phi(\mathcal{H}) \tag{3.26}$$

and

$$\|F(t/r)^r\|_\phi \leq \|F^D(t_0)^p\|_\phi \tag{3.27}$$

for $pt_0 \leq t \leq prt_0$ and $r \geq p$.

Proof: Since $f(\cdot)$ and $g(\cdot)$ are dominated by $f^D(\cdot)$ and $g^D(\cdot)$, by (1.23) we have

$$f(tA/r)^r \leq f(tA/r)^{rt_0/t} \leq f^D(t_0 A) \tag{3.28}$$

and

$$g(tB/r)^r \leq g(tB/r)^{rt_0/t} \leq g^D(t_0 B) \tag{3.29}$$

for $t_0 \leq t \leq rt_0$ and $r \geq 1$. Setting $X_0 := f^D(t_0 A)$ and $Y_0 := g^D(t_0 B)$ we obtain from Lemma 2.5 that

$$F(t/r)^r \prec_{\log} F^D(t_0) \tag{3.30}$$

for $t_0 \leq t \leq rt_0$ and $r \geq 1$. Thus

$$F(t/r)^{rp} \prec_{\log} F^D(t_0)^p, \tag{3.31}$$

which yields $F(t/r)^{rp} \in \mathcal{L}_\phi(\mathcal{H})$ and

$$\|F(t/r)^{rp}\|_\phi \leq \|F(t_0)^p\|_\phi \tag{3.32}$$

for $t_0 \leq t \leq rt_0$ and $r \geq 1$. Since $F(t/r)^{rp} = F(pt/rp)^{rp}$, we get (3.26) and (3.27) for $pt_0 \leq t \leq prt_0$ and $r \geq p$. □

¿From Lemma 3.3 we immediately see that for any interval $[\tau_0, \tau] \subseteq (pt_0, +\infty)$ there is a $r_0 \geq 1$ such that for $r \geq r_0$ and $t \in [\tau_0, \tau]$ conditions $F(t/r)^r \in \mathcal{L}_\phi(\mathcal{H})$ and (1.22) are satisfied. This means that the sequence $\{F(t/r)^r\}_{r \geq 1}$ is locally uniformly bounded away from pt_0 in $\mathcal{L}_\phi(\mathcal{H})$. Moreover, one gets that $M([\tau_0, \tau]) \leq \|F^D(t_0)^p\|_\phi$ for sufficiently large $r \geq 1$ and any interval $[\tau_0, \tau] \subseteq (pt_0, +\infty)$.

Next we show that under the assumptions of Lemma 3.3 one has $e^{-tH} \in \mathcal{L}_\phi(\mathcal{H}_0)$ for $t > pt_0$. To this end we note that if the Kato-function $f(\cdot)$ is dominated by $f^D(\cdot)$, then by (1.23)

$$f(x/r)^r \leq f^D(x), \qquad r \geq 1, \tag{3.33}$$

for $x \geq 0$. Since $\lim_{r \to \infty} f(x/r)^r = e^{-x}$ for each $x \geq 0$, we find

$$e^{-x} \leq f^D(x) \tag{3.34}$$

for $x \geq 0$, i.e., if $f^D(\cdot)$ dominates the Kato-function $f(\cdot)$, then it dominates the Kato-function $\hat{f}(x) = e^{-x}$, $x \geq 0$, too.

Lemma 3.4 *Let A and B be non-negative self-adjoint operators on the separable Hilbert space $\mathcal{H}$ and let $f^D(\cdot) : \mathbf{R}^1_+ \longrightarrow \mathbf{R}^1_+$ and $g^D(\cdot) : \mathbf{R}^1_+ \longrightarrow \mathbf{R}^1_+$ be bounded Borel measurable functions such that $F^D(t_0)^p \in \mathcal{L}_\phi(\mathcal{H})$ for some $t_0 > 0$ and some integer $p \geq 1$. If the Kato-functions $f(\cdot)$ and $g(\cdot)$ are dominated by $f^D(\cdot)$ and $g^D(\cdot)$, respectively, then*

$$e^{-tH} \in \mathcal{L}_\phi(\mathcal{H}_0) \tag{3.35}$$

and

$$\|e^{-tH}\|_\phi \leq \|F^D(t_0)^p\|_\phi \tag{3.36}$$

for $t \geq pt_0$.

Proof: Since $\hat{f}(x) = \hat{g}(x) = e^{-x}$, $x \geq 0$, are dominated by $f^D(\cdot)$ and $g^D(\cdot)$, respectively, we find by Lemma 3.3 that

$$\hat{F}(t/r)^r \in \mathcal{L}_\phi(\mathcal{H}) \tag{3.37}$$

and

$$\|\hat{F}(t/r)^r\|_\phi \leq \|F^D(t_0)^p\|_\phi, \qquad r \geq 1, \tag{3.38}$$

for $pt_0 \leq t \leq prt_0$ and $r \geq p$, where

$$\hat{F}(t) := e^{-tB/2}e^{-tA}e^{-tB/2}, \qquad t \geq 0. \tag{3.39}$$

By Theorem 2.3 of [7] we have

$$e^{-tH}P_0 \prec_{\log} \hat{F}(t/n)^n, \qquad n = 1, 2, \ldots, \tag{3.40}$$

for $t > 0$. Then by (3.37), (3.38) and Lemma 2.4 we get the proof of (3.35) and (3.36). □

In virtue of Propositions 3.1, 3.2 and Lemmas 3.3, 3.4 the following theorem easily follows.

Theorem 3.5 *Let A and B be non-negative self-adjoint operators on the separable Hilbert space $\mathcal{H}$ and let $f^D(\cdot) : \mathbf{R}^1_+ \longrightarrow \mathbf{R}^1_+$ and $g^D(\cdot) : \mathbf{R}^1_+ \longrightarrow \mathbf{R}^1_+$ be bounded Borel measurable functions such that $F^D(t_0)^p \in \mathcal{L}_\phi(\mathcal{H})$ for some $t_0 > 0$ and some integer $p \geq 1$. If the Kato-functions $f(\cdot)$ and $g(\cdot)$ are dominated by $f^D(\cdot)$ and $g^D(\cdot)$, respectively, and the Trotter-Kato product formula converges locally uniformly away from zero in the operator-norm for the family $\{F(t)\}_{t\geq 0}$, then the Trotter-Kato product formula converges locally uniformly away from pt_0 in $\mathcal{L}_\phi(\mathcal{H})$ for all families generated by $f(\cdot)$ and $g(\cdot)$.*

Proof: By Lemma 3.4 we obtain that $e^{-tH} \in \mathcal{L}_\phi(\mathcal{H}_0)$ for $t > pt_0$. Lemma 3.3 implies that for each bounded interval $[\tau_0, \tau] \subseteq (pt_0, \infty)$ there is a $r_0 \geq 1$ such that $F(t/r)^r \in \mathcal{L}_\phi(\mathcal{H})$ for $r \geq r_0$ and $t \in [\tau_0, \tau]$. By (3.27) one has $\sup_{r\geq r_0} \sup_{t\in[\tau_0,\tau]} \|F(t/r)^r\|_\phi \leq \|F^D(t_0)\|_\phi < \infty$ which verifies that the sequence $\{F(t/r)^r\}_{r\geq 1}$ is locally uniformly bounded away from pt_0 in $\mathcal{L}_\phi(\mathcal{H})$. Applying Proposition 3.2 we obtain that the Trotter-Kato product formula for $\{F(t)\}_{t\geq 0}$ converges locally uniformly away from pt_0 in $\mathcal{L}_\phi(\mathcal{H})$. Finally, taking into account Proposition 3.1 we obtain that this holds for all families generated by $f(\cdot)$ and $g(\cdot)$. □

4 Convergence in ideals: Special results

Let $f(\cdot)$ be any Kato-function. We set

$$b(x) := \sup_{0 \le s \le x} sf(s), \qquad x \ge 0, \tag{4.1}$$

and

$$r(x) := \sup_{s \in [x,+\infty)} f(s), \qquad x \ge 0. \tag{4.2}$$

A Kato-function $f(\cdot)$ is called *regular* if

$$\lim_{x \to +\infty} \frac{b(x)}{x} = 0 \tag{4.3}$$

and

$$0 \le r(x) < 1 \tag{4.4}$$

for $x > 0$ (see [14]). For instance, the standard Kato-functions $f(x) = e^{-x}$ and $f(x) = (1 + x/m)^m$, $m = 1, 2, \ldots$, are *regular*, while $f(x) = 1 - |\sin(x)|$ is *not*.

Theorem 4.1 *Let A and B be non-negative self-adjoint operators on the separable Hilbert space $\mathcal{H}$ and let $f^D(\cdot) : \mathbf{R}^1_+ \longrightarrow \mathbf{R}^1_+$ be a Borel measurable function such that $f^D(t_0 A) \in \mathcal{L}_\phi(\mathcal{H})$ for some $t_0 > 0$. If the Kato-function $f(\cdot)$ is regular and dominated by $f^D(\cdot)$ and $g(\cdot)$ is any Kato-function, then the Trotter-Kato product formula converges locally uniformly away from t_0 in $\mathcal{L}_\phi(\mathcal{H})$ for all families generated by $f(\cdot)$ and $g(\cdot)$.*

Proof: Since $e^{-x} \le f^D(x)$, $x \ge 0$, one gets $e^{-t_0 A} \in \mathcal{L}_\phi(\mathcal{H})$ which implies $(I + t_0 A)^{-1} \in \mathcal{L}_\infty(\mathcal{H})$. Since $f(\cdot)$ is regular, by Theorem 3.6 of [14] one gets that the Trotter-Kato product formula converges locally uniformly away from zero in the operator-norm. Setting $g^D(x) \equiv 1$, $x \ge 0$, one gets $F^D(t_0) \in \mathcal{L}_\phi(\mathcal{H})$. Since any Kato-function is dominated by $g^D(\cdot) \equiv 1$, Theorem 3.5 implies that the Trotter-Kato product formula converges locally uniformly away from t_0 in $\mathcal{L}_\phi(\mathcal{H})$ for all families generated by $f(\cdot)$ and $g(\cdot)$. $\square$

Next we characterize *self-dominated* Kato-functions, see (1.25).

Lemma 4.2 *A Borel function $f(\cdot)$ is a self-dominated Kato-function if and only if there is a non-increasing function $h(\cdot) : \mathbf{R}^1_+ \longrightarrow \mathbf{R}^1_+$ which obeys*

$$\lim_{x \to +0} h(x) = 1 \tag{4.5}$$

such that $f(\cdot)$ admits the representation

$$f(x) = e^{-xh(x)}, \qquad x \ge 0. \tag{4.6}$$

The function $h(\cdot)$ is unique.

Proof: Let $f(\cdot)$ be a self-dominated Kato-function. Let $t := qx$, $x \ge 0$, which yields $0 \le t \le x$. Then from (1.25) we obtain

$$f(t) \le f(x)^{t/x}, \qquad 0 \le t \le x, \quad x > 0, \tag{4.7}$$

which implies

$$f(t)^{1/t} \leq f(x)^{1/x}, \qquad 0 < t \leq x. \tag{4.8}$$

If $f(x) = 0$ for some $x > 0$ one gets $f(t) = 0$ for $0 < t \leq x$ which is impossible by (1.4). Therefore, $f(x) \neq 0$ and it makes sense to define

$$h(x) := -\frac{1}{x}\ln(f(x)) \geq 0, \qquad x > 0. \tag{4.9}$$

Taking into account (4.8) one gets

$$h(t) \geq h(x) \tag{4.10}$$

for $0 < t \leq x$. Thus $h(\cdot)$ is a non-negative and non-increasing function. We note that $f(x) \neq 1$ for sufficiently small $x > 0$. Otherwise, there would exist a sequence $\{x_n\}_{n=1}^{\infty}$ of positive real numbers tending to zero for $n \longrightarrow \infty$ such that $f(x_n) = 1$ and by (1.4)

$$1 = \lim_{n\to\infty} \frac{1 - f(x_n)}{x_n} = 0 \tag{4.11}$$

which is impossible. Therefore, the representation

$$h(x) = -\frac{1 - f(x)}{x} \ln(f(x)^{1/(1-f(x))}) \tag{4.12}$$

makes sense for sufficiently small $x > 0$. Since

$$\lim_{y\to 1-0} y^{1/(1-y)} = e^{-1}, \tag{4.13}$$

one gets (4.5). The representation (4.6) follows from definition (4.9).

Conversely, let $f(\cdot)$ be given by (4.6) where $h(\cdot)$ is a non-increasing function which satisfies (4.5). Since $h(\cdot)$ is not increasing, we find

$$f(qx) = e^{-qxh(qx)} \leq e^{-qxh(x)} = f(x)^q \tag{4.14}$$

for $x \geq 0$ which verifies (1.25). Furthermore, by (4.5) one gets $f(0) = 1$ and

$$\frac{f(x) - 1}{x} = \sum_{n=1}^{\infty}(-1)^n x^{n-1} h(x)^n \frac{1}{n!}, \qquad x > 0, \tag{4.15}$$

which leads to the estimate

$$\left|\frac{1 - f(x)}{x} + h(x)\right| \leq \sum_{n=2}^{\infty} x^{n-1}\frac{1}{n!}, \qquad x > 0. \tag{4.16}$$

Then (4.5) we immediately get $f'(+0) = -1$. Thus (4.6) defines a self-dominated Kato-function. The uniqueness of $h(\cdot)$ is obvious. □

Corollary 4.3 *If $f(\cdot)$ is a self-dominated Kato-function, then*

$$f_0(x) \leq \frac{1}{1 - \ln(f(x))} \tag{4.17}$$

for $x \geq 0$, where $f_0(\cdot)$ is defined by (1.7) and (1.8).

Proof: By definition (1.7) and (4.6) one has

$$\varphi_0(x) = \inf_{0<t\le x} \frac{1}{t}\left(\frac{1}{f(t)} - 1\right) = \inf_{0<t\le x} \frac{1}{t}\left(e^{th(t)} - 1\right) \tag{4.18}$$

which yields

$$f_0(x) = \frac{1}{1+x\varphi_0(x)} \le \frac{1}{1+xh(x)} = \frac{1}{1-\ln(f(x))} \tag{4.19}$$

for $x \ge 0$. □

Lemma 4.4 *Let A be a non-negative self-adjoint operator on the separable Hilbert space $\mathcal{H}$ and let $f^D(\cdot)$ be a self-dominated Kato-function. If $f^D(t_0A) \in \mathcal{L}_\phi(\mathcal{H})$ for some $t_0 > 0$, then*

$$f_0^D(t_0A) \in \mathcal{L}_\infty(\mathcal{H}), \tag{4.20}$$

where $f_0^D(\cdot)$ is defined by (1.7) and (1.8).

Proof: Since

$$f^D(t_0A) \in \mathcal{L}_\infty(\mathcal{H}), \tag{4.21}$$

we have

$$(I + \ln(f^D(t_0A)^{-1}))^{-1} \in \mathcal{L}_\infty(\mathcal{H}). \tag{4.22}$$

By Corollary 4.3 one has that

$$f_0^D(t_0A) \le (I - \ln(f^D(t_0A)))^{-1} \tag{4.23}$$

which together with (4.22) yields (4.20). □

Theorem 4.5 *Let A and B be non-negative self-adjoint operators on the separable Hilbert space $\mathcal{H}$ and let $f^D(\cdot)$ be a self-dominated Kato-function such that $f^D(t_0A) \in \mathcal{L}_\phi(\mathcal{H})$ for some $t_0 > 0$. Then for any Kato-function $f(\cdot)$ obeying $f(x) \le f^D(x)$, $x \ge 0$, and any Kato-function $g(\cdot)$ the Trotter-Kato product formula converges locally uniformly away from t_0 in $\mathcal{L}_\phi(\mathcal{H})$ for all families generated by $f(\cdot)$ and $g(\cdot)$.*

Proof: First we note that each Kato-function $f(\cdot)$ obeying $f(x) \le f^D(x)$, $x \ge 0$, is dominated by $f^D(\cdot)$. This follows from (1.25) and

$$f(qx)^{1/q} \le f^D(qx)^{1/q} \le f^D(x), \qquad x \ge 0. \tag{4.24}$$

Further, by Lemma 4.4 and Theorem 3.2 of [14] one gets that the Trotter-Kato product formula converges locally uniformly away from zero in the operator-norm. Then applying Theorem 3.5 we complete the proof. □

Till now we did not impose any *additional conditions* on the Kato-function $g(\cdot)$. If we do this, we obtain a new sort of conditions implying convergence in $\mathcal{L}_\phi(\mathcal{H})$.

Theorem 4.6 *Let A and B be non-negative self-adjoint operators on the separable Hilbert space $\mathcal{H}$ and let $f^D(\cdot) : \mathbf{R}^1_+ \longrightarrow \mathbf{R}^1_+$ and $g^D(\cdot) : \mathbf{R}^1_+ \longrightarrow \mathbf{R}^1_+$ be Borel measurable functions such that $F^D(t) \in \mathcal{L}_\phi(\mathcal{H})$ for some $t_0 > 0$. If the Kato-functions $f(\cdot)$ and $g(\cdot)$ are dominated by $f^D(\cdot)$ and $g^D(\cdot)$, respectively, and if they satisfy conditions (1.26), then the Trotter-Kato product formula converges locally uniformly away from t_0 in $\mathcal{L}_\phi(\mathcal{H})$ for all families generated by $f(\cdot)$ and $g(\cdot)$.*

Proof: First we check that

$$(I+A)^{-1}(I+B)^{-1} \in \mathcal{L}_\infty(\mathcal{H}). \tag{4.25}$$

Using the reasoning of Lemma 3.4 we find (in accordance with (3.37)) that

$$e^{-t_0 B/2} e^{-t_0 A} e^{-t_0 B/2} \in \mathcal{L}_\infty(\mathcal{H}). \tag{4.26}$$

Setting $C := e^{-t_0 A/2} e^{-t_0 B/2}$ one gets $C^* C \in \mathcal{L}_\infty(\mathcal{H})$ which yields

$$e^{-t_0 A/2} e^{-t_0 B/2} \in \mathcal{L}_\infty(\mathcal{H}). \tag{4.27}$$

Repeating this process we obtain

$$e^{-t_0 A/2^m} e^{-t_0 B/2} \in \mathcal{L}_\infty(\mathcal{H}). \tag{4.28}$$

for $m = 1, 2, \ldots$. However, this implies

$$e^{-tA} e^{-t_0 B/2} \in \mathcal{L}_\infty(\mathcal{H}) \tag{4.29}$$

for $t > 0$. Similarly, we prove

$$e^{-tA} e^{-t_0 B/2^k} \in \mathcal{L}_\infty(\mathcal{H}) \tag{4.30}$$

for $t > 0$ and $k = 0, 1, 2, \ldots$. Therefore, one gets

$$e^{-tA} e^{-sB} \in \mathcal{L}_\infty(\mathcal{H}) \tag{4.31}$$

for $t > 0$ and $s > 0$. Taking into account the representation

$$(I+A)^{-1}(I+B)^{-1} = \int_0^\infty \int_0^\infty dt\, ds\, e^{-(t+s)} e^{-tA} e^{-sB}, \tag{4.32}$$

we obtain from (4.31) and (4.32) that (4.25) holds. Then (4.25), Theorem 5.3 of [14] and Theorem 3.5 give the proof of the assertion. □

Theorem 4.7 *Let A and B be non-negative self-adjoint operators on the separable Hilbert space $\mathcal{H}$ and let $f^D(\cdot) : \mathbf{R}^1_+ \longrightarrow \mathbf{R}^1_+$ and $g^D(\cdot) : \mathbf{R}^1_+ \longrightarrow \mathbf{R}^1_+$ be self-dominated Kato-functions such that conditions (1.26) and $F^D(t_0) \in \mathcal{L}_\phi(\mathcal{H})$ for some $t_0 > 0$ are satisfied. If the Kato-functions $f(\cdot)$ and $g(\cdot)$ obey $f(x) \le f^D(x)$ and $g(x) \le g^D(x)$ for $x \ge 0$, then the Trotter-Kato product formula converges locally uniformly away from t_0 in $\mathcal{L}_\phi(\mathcal{H})$ for all families generated by $f(\cdot)$ and $g(\cdot)$.*

Proof: The statement is a consequence of Theorem 4.6 and the fact that $f(x) \le f^D(x)$ and $g(x) \le g^D(x)$ for $x \ge 0$ imply that $f(\cdot)$ and $g(\cdot)$ are dominated by $f^D(\cdot)$ and $g^D(\cdot)$, respectively, i.e., they satisfy conditions (1.26). □

Corollary 4.8 *Let A and B be non-negative self-adjoint operators on the separable Hilbert space $\mathcal{H}$ such that*

$$(I+B)^{-1/2}(I+A)^{-1}(I+B)^{-1/2} \in \mathcal{L}_\phi(\mathcal{H}). \tag{4.33}$$

If $f(\cdot)$ and $g(\cdot)$ are Kato-functions such that $f(x) \le \frac{1}{1+x}$ and $g(x) \le \frac{1}{1+x}$ for $x \ge 0$, then the Trotter-Kato product formula converges locally uniformly away from zero (sic!) in $\mathcal{L}_\phi(\mathcal{H})$ for all families generated by $f(\cdot)$ and $g(\cdot)$.

Proof: First of all we note that condition (4.33) implies

$$(I + t_0 B)^{-1/2}(I + t_0 A)^{-1}(I + t_0 B)^{-1/2} \in \mathcal{L}_\phi(\mathcal{H}) \tag{4.34}$$

for each $t_0 > 0$. Further, we set $f^D(x) = (1+x)^{-1}$ and $g^D(x) = (1+x)^{-1}$ for $x \geq 0$. The Kato-functions $f^D(\cdot)$ and $g^D(\cdot)$ are obviously self-dominated (1.25) and obey condition (1.26). Therefore, the Kato-functions $f(\cdot)$ and $g(\cdot)$ satisfy condition (1.26), too. Applying Corollary 4.7 one finishes the proof. □

Corollary 4.9 *Let A and B be non-negative self-adjoint operators on the separable Hilbert space $\mathcal{H}$ such that*

$$e^{-t_0 B/2} e^{-t_0 A} e^{-t_0 B/2} \in \mathcal{L}_\phi(\mathcal{H}). \tag{4.35}$$

If $f(\cdot)$ and $g(\cdot)$ are Kato-functions such that $f(x) \leq e^{-x}$ and $g(x) \leq e^{-x}$ for $x \geq 0$, then the Trotter-Kato product formula converges locally uniformly away from t_0 in $\mathcal{L}_\phi(\mathcal{H})$ for all families generated by $f(\cdot)$ and $g(\cdot)$.

Proof: We set $f^D(x) = g^D(x) = e^{-x}$, $x \geq 0$. Obviously, the Kato-functions $f^D(\cdot)$ and $g^D(\cdot)$ are self-dominated (1.25) and satisfy conditions (1.26). Applying Theorem 4.7 we get the proof. □

Theorem 4.7 allows in some sense a converse.

Theorem 4.10 *Let A and B be non-negative self-adjoint operators on the separable Hilbert space $\mathcal{H}$ and let $f(\cdot)$ and $g(\cdot)$ be self-dominated Kato-functions which satisfy conditions (1.26). There is a $t_0 > 0$ such that the Trotter-Kato product formula converges locally uniformly away from $t_0 > 0$ in $\mathcal{L}_\phi(\mathcal{H})$ for all families generated by $f(\cdot)$ and $g(\cdot)$ if and only if there is $s_0 > 0$ and an integer p such that $F(s_0)^p \in \mathcal{L}_\phi(\mathcal{H})$*

Proof: If the Trotter-Kato product formula converges locally uniformly away from t_0 in $\mathcal{L}_\phi(\mathcal{H})$ for all families generated by $f(\cdot)$ and $g(\cdot)$, then it converges in particular for the family $\{F(t)\}_{t\geq 0}$. Hence, there is a $t > t_0$ such that for sufficiently large r one has $F(t/r)^r \in \mathcal{L}_\phi(\mathcal{H})$. Setting $p = r$ and $s_0 = t/r > 0$ we find $F(s_0)^p \in \mathcal{L}_\phi(\mathcal{H})$. Thus the necessity is proven.

To prove the converse we note that $F(s_0)^p \in \mathcal{L}_\phi(\mathcal{H})$ yields $F(s_0) \in \mathcal{L}_\infty(\mathcal{H})$. Following the proof of Theorem 4.6 we get (4.25). Then applying Theorem 5.3 of [14] we find that the Trotter-Kato product formula converges locally uniformly away from zero in the operator-norm. Finally, setting $t_0 = ps_0$ and taking into account Proposition 3.5 we complete the proof. □

Theorem 4.10 allows us to generalize the results of [7] and [11] as follows.

Corollary 4.11 *Let A and B be non-negative self-adjoint operators on the separable Hilbert space $\mathcal{H}$ and let $f(\cdot)$ and $g(\cdot)$ be self-dominated Kato-functions (1.25) which satisfy conditions (1.26). Then there is a $t_0 > 0$ such that the Trotter-Kato product formula converges locally uniformly away from $t_0 > 0$ in $\mathcal{L}_1(\mathcal{H})$ for all families generated by $f(\cdot)$ and $g(\cdot)$ if and only if there is a $s_0 > 0$ and an integer p such that $F(s_0) \in \mathcal{L}_p(\mathcal{H})$.*

Proof: From Theorem 4.10 one gets that the condition $F(s_0)^p \in \mathcal{L}_1(\mathcal{H})$ has to be satisfied for some $s_0 > 0$ and some integer $p \geq 1$. However, this is equivalent to $F(s_0) \in \mathcal{L}_p(\mathcal{H})$. □

In particular, let $f(x) = g(x) = \frac{1}{1+x}$ or $f(x) = g(x) = e^{-x}$, $x \geq 0$. Then Corollary 4.11 yields that the Trotter-Kato product formula converges locally uniformly away from some t_0

in the trace-class norm for all families generated by these $f(\cdot)$ and $g(\cdot)$ if and only if for some $s_0 > 0$ one has

$$(I + s_0 B)^{-1/2}(I + s_0 A)^{-1}(I + s_0 B)^{-1} \in \mathcal{L}_p(\mathcal{H}) \tag{4.36}$$

or

$$e^{-s_0 B/2} e^{-s_0 A} e^{-s_0 B/2} \in \mathcal{L}_p(\mathcal{H}) \tag{4.37}$$

for some $p \geq 1$, respectively. In the case $f(x) = g(x) = \frac{1}{1+x}$ one gets in addition that the convergence takes place away from *zero* (see Corollary 4.8).

Moreover, the notion of *self-dominated* Kato-functions allows to generalize Theorem 5.3 of [14] in the following way.

Theorem 4.12 *Let A and B be non-negative self-adjoint operators on the separable Hilbert space $\mathcal{H}$ and let $f(\cdot)$ and $g(\cdot)$ be self-dominated Kato-functions which satisfy conditions (1.26). If there is a $t_0 > 0$ such that $F(t_0) \in \mathcal{L}_\infty(\mathcal{H})$ for some $t_0 > 0$, then the Trotter-Kato product formula converges locally uniformly away from zero in the operator-norm for all families generated by $f(\cdot)$ and $g(\cdot)$.*

Proof: Following the proof of Theorem 4.6 one obtains (4.25). Applying Theorem 5.3 of [14] we finish the proof. □

5 Error bounds in ideals

Let $\eta(\cdot)$ be a non-negative real function defined on $\mathbf{R}^1_+$ or on $\mathbf{N}_0 = \{0, 1, 2, \ldots\}$ such that

$$\lim_{x \to \infty} \eta(x) = 0. \tag{5.1}$$

The function $\eta(\cdot)$ is called an *operator-norm error bound away from $t_0 > 0$ of the Trotter-Kato product formula for the family $\{F(t)\}_{t \geq 0}$* if for any interval $[a, b] \subseteq (t_0, +\infty)$ there is a $x_0 \geq 1$ and a constant $C > 0$ such that

$$\|F(t/x)^x - e^{-tH} P_0\| \leq C\eta(x) \tag{5.2}$$

for $t \in [a, b]$ and $x \geq x_0$, where x is either the continuous parameter $r \geq 1$ or the discrete parameter $n \in \mathbf{N} := \{1, 2, \ldots\}$. We note that in contrast to the real number x_0 and the constant C in (5.2) the error bound $\eta(\cdot)$ does not depend on the interval $[\tau, \tau_0]$. By the same manner the notion of an operator-norm error bound can be extended to the families $\{T(t)\}_{t \geq 0}$, $\{f(tA)g(tB)\}_{t \geq 0}$ and $\{g(tB)f(tA)\}_{t \geq 0}$, yet in the last two cases only the discrete parameter n makes sense.

The function $\eta_\phi(\cdot)$ is called an *error bound away from $t_0 > 0$ in $\mathcal{L}_\phi(\mathcal{H})$ of the Trotter-Kato product formula for the family $\{F(t)\}_{t \geq 0}$* if for any interval $[\tau_0, \tau] \subseteq (t_0, +\infty)$ there is a $x_0 \geq 1$ and a constant $C > 0$ such that

$$F(t/x)^x - e^{-tH} P_0 \in \mathcal{L}_\phi(\mathcal{H}) \tag{5.3}$$

and

$$\|F(t/x)^x - e^{-tH} P_0\|_\phi \leq C\eta_\phi(x) \tag{5.4}$$

for $t \in [\tau_0, \tau]$ and $x \geq x_0$ where either $x = r$ or $x = n$. As above this notion has an extension to the families $\{T(t)\}_{t\geq 0}$, $\{f(tA)g(tB)\}_{t\geq 0}$ and $\{g(tB)f(tA)\}_{t\geq 0}$.

The following problem naturally arises: Assume that we know the operator-norm error bound $\eta(\cdot)$ away from $t_0 > 0$. Can we find an error bound $\eta_\phi(\cdot)$ away from t_0 in $\mathcal{L}_\phi(\mathcal{H})$?

Theorem 5.1 *Let A and B be non-negative self-adjoint operators on the separable Hilbert space $\mathcal{H}$, and let $f^D(\cdot) : \mathbf{R}^1_+ \longrightarrow \mathbf{R}^1_+$ and $g^D(\cdot) : \mathbf{R}^1_+ \longrightarrow \mathbf{R}^1_+$ be Borel measurable functions such that $F^D(t_0) \in \mathcal{L}_\phi(\mathcal{H})$ for some $t_0 > 0$. Assume that the Kato-functions $f(\cdot)$ and $g(\cdot)$ are dominated by $f^D(\cdot)$ and $g^D(\cdot)$, respectively. Then:*

If $\eta(r)$, $r \in \mathbf{R}^1_+$, (continuous case) is an operator-norm error bound away from $t_0 > 0$ of the Trotter-Kato product formula for the family $\{F(t)\}_{t\geq 0}$, respectively $\{T(t)\}_{t\geq 0}$, then the function $\eta_\phi(r) := \eta(r/2)$, $r \in \mathbf{R}^1_+$, is an error bound away from $2t_0$ in $\mathcal{L}_\phi(\mathcal{H})$ of the Trotter-Kato product formula for the family $\{F(t)\}_{t\geq 0}$, respectively, for the family $\{T(t)\}_{t\geq 0}$.

If $\eta(n)$, $n \in \mathbf{N}_0$, (discrete case) is an operator-norm error bound away from $t_0 > 0$ of the Trotter-Kato product formula for $\{f(tA)g(tB)\}_{t\geq 0}$, respectively $\{g(tB)f(tA)\}_{t\geq 0}$, $\{F(t)\}_{t\geq 0}$ or $\{T(t)\}_{t\geq 0}$, then the function $\eta_\phi(n) := \frac{1}{2}\{\eta([n/2]) + \eta([(n+1)/2])\}$, $n \in \mathbf{N}_0$, where $[x] := \max\{l \in \mathbf{N}_0 : l \leq x\}$, $x \in \mathbf{R}^1_+$, is an error bound away $2t_0$ in $\mathcal{L}_\phi(\mathcal{H})$ of the Trotter-Kato product formula for the family $\{f(tA)g(tB)\}_{t\geq 0}$, respectively, for $\{g(tB)f(tA)\}_{t\geq 0}$, $\{F(t)\}_{t\geq 0}$ or $\{T(t)\}_{t\geq 0}$.

Proof: (*Continuous case*) We use the representation

$$F(t/r)^r - e^{-tH} = \\ (F(t/r)^{r/2} - e^{-tH/2})F(t/r)^{r/2} + e^{-tH/2}(F(t/r)^{r/2} - e^{-tH/2}). \tag{5.5}$$

By Lemma 3.3 we find the estimate

$$\|F(t/r)^{r/2}\|_\phi \leq \|F^D(t_0)\|_\phi \tag{5.6}$$

for $t_0 \leq t/2 \leq rt/2$ and $r/2 \geq 1$. Hence we get

$$\|F(t/r)^r - e^{-tH}P_0\|_\phi \leq 2\|F^D(t_0)\|_\phi \|F(t/r)^{r/2} - e^{-tH/2}P_0\| \tag{5.7}$$

for $2t_0 \leq t \leq rt_0$ and $r \geq 2$. Since $\eta(\cdot)$ is an operator-norm error bound away from t_0 for any interval $[a,b] \subseteq (t_0, +\infty)$, there is a $r_0 \geq 1$ and a constant $C > 0$ (cf. (5.2)) such that

$$\|F(t/r)^{r/2} - e^{-tH/2}P_0\| \leq C\eta(r/2). \tag{5.8}$$

Thus for any interval $[\tau_0, \tau] \subseteq (2t_0, \infty)$ there is a $r_0 \geq 1$ and a constant $C > 0$ such that the estimate (5.8) is valid. Inserting (5.8) into (5.7) we find that $\eta_\phi(r) := \eta(r/2)$, $r \in [0, +\infty)$ is an error bound away from $2t_0$ in $\mathcal{L}_\phi(\mathcal{H})$ for $\{F(t)\}_{t\geq 0}$.

In order to verify the statement for the family $\{T(t)\}_{t\geq 0}$ it is enough to show that $F^D(t_0) \in \mathcal{L}_\phi(\mathcal{H})$ implies $T^D(t_0) \in \mathcal{L}_\phi(\mathcal{H})$, where

$$T^D(t) := f^D(tA)^{1/2} g^D(tB) f^D(tA)^{1/2}, \qquad t \geq 0. \tag{5.9}$$

Let

$$G^D(t) := f^D(tA)^{1/2} g^D(tB)^{1/2}, \qquad t \geq 0. \tag{5.10}$$

Taking into account the polar decomposition

$$G^D(t) = V(t)|G^D(t)| = V(t)F^D(t)^{1/2}, \qquad t \geq 0, \tag{5.11}$$

where $V(t)$ is a partial isometry acting from $\overline{\mathrm{ran}(G^D(t)^*)}$ onto $\overline{\mathrm{ran}(G^D(t))}$, we find that

$$T^D(t) = G^D(t)G^D(t)^* = V(t)F^D(t)V(t)^*, \qquad t \geq 0. \tag{5.12}$$

This yields $T^D(t_0) = V(t)F^D(t_0)V(t)^* \in \mathcal{L}_\phi(\mathcal{H})$.

(*Discrete case*) To prove the assertion for the family $\{f(tA)g(tB)\}_{t\geq 0}$ we use decompositions $n = k+m$, $k \in \mathbf{N} := \{1,2,\ldots\}$, $m = 2,3,\ldots$, $n \geq 3$, and

$$\begin{aligned}(f(tA/n)g(tB/n))^n - e^{-tH}P_0 = \\ \left((f(tA/n)g(tB/n))^k - e^{-ktH/n}P_0\right)(f(tA/n)g(tB/n))^m + \\ e^{-ktH/n}P_0\left((f(tA/n)g(tB/n))^m - e^{-mtH/n}P_0\right).\end{aligned} \tag{5.13}$$

Since

$$(f(tA/n)g(tB/n))^m = f(tA/n)g(tB/n)^{1/2}F(t/n)^{m-1}g(tB)^{1/2}, \tag{5.14}$$

by Lemma 3.3 one gets $(f(tA/n)g(tB/n))^m \in \mathcal{L}_\phi(\mathcal{H})$ and

$$\|(f(tA/n)g(tB/n))^m\|_\phi \leq \|F^D(t_0)\|_\phi \tag{5.15}$$

for $t_0 \leq \frac{m-1}{n}t \leq (m-1)t_0$ and $m-1 \geq 1$. Lemma 3.4 implies $e^{-ktH/n} \in \mathcal{L}_\phi(\mathcal{H}_0)$ and

$$\|e^{-ktH/n}P_0\|_\phi \leq \|F^D(t_0)\|_\phi \tag{5.16}$$

for $kt/n \geq t_0$. Hence from (5.13) we obtain the estimate

$$\begin{aligned}\|(f(tA/n)g(tB/n))^n - e^{-tH}P_0\|_\phi \leq \\ \|F^D(t_0)\|_\phi\left\{\|(f(tA/n)g(tB/n))^k - e^{-ktH/n}P_0\| + \|(f(tA/n)g(tB/n))^m - e^{-mtH/k}P_0\|\right\}\end{aligned} \tag{5.17}$$

for $(1+\frac{k+1}{m-1})t_0 \leq t \leq nt_0$, $m \geq 2$ and $t \geq (1+\frac{m}{k})t_0$. Since $\eta(\cdot)$ is an operator-norm error bound away from t_0 by definition (5.2), for any interval $[a,b] \subseteq (t_0,+\infty)$ there is a $n_0 \geq 1$ and a constant $C > 0$ such that

$$\|(f(tA/n)g(tB/n))^k - e^{-ktH/n}P_0\| \leq C\eta(k) \tag{5.18}$$

for $\frac{k}{n}t \in [a,b] \Leftrightarrow t \in [(1+\frac{m}{k})a, (1+\frac{m}{k})b]$ and

$$\|(f(tA/n)g(tB/n))^m - e^{-mtH/n}P_0\| \leq C\eta(m) \tag{5.19}$$

for $\frac{m}{n}t \in [a,b] \Leftrightarrow t \in [(1+\frac{k}{m})a, (1+\frac{k}{m})b]$. Setting $m := [(n+1)/2]$ and $k = [n/2]$, $n \geq 3$, we satisfy $n = k+m$ and $m \geq 2$ as well as $\lim_{n\to\infty}\frac{k+1}{m-1} = 1$, $\lim_{n\to\infty}\frac{m}{k} = 1$ and $\lim_{n\to\infty}\frac{k}{m} = 1$. Hence for any interval $[\tau_0,\tau] \subseteq (2t_0,+\infty)$ we find that $[\tau_0,\tau] \subseteq [(1+\frac{k+1}{m-1})t_0, nt_0]$ for sufficiently large n. Moreover, choosing $[\tau_0/2,\tau/2] \subseteq (a,b) \subseteq (t_0,+\infty)$ we satisfy $[\tau_0,\tau] \subseteq [(1+\frac{m}{k})a,(1+\frac{m}{k})b]$ and $[\tau_0,\tau] \subseteq [(1+\frac{k}{m})a,(1+\frac{k}{m})b]$ for sufficiently large n too. Thus for any interval

$[\tau_0, \tau] \subseteq (2t_0, +\infty)$ there is a $n_0 \geq 1$ and a constant $C > 0$ such that (5.17), (5.18) and (5.19) hold for $t \in [\tau_0, \tau]$ and $n \geq n_0$. Therefore, from (5.17) we find the estimate

$$\|(f(tA/n)g(tB/n))^n - e^{-tH}P_0\|_\phi \leq C\|F^D(t_0)\|_\phi\{\eta([n/2]) + \eta([(n+1)/2])\} \tag{5.20}$$

for $t \in [\tau_0, \tau] \subseteq (2t_0, +\infty)$ and $n \geq n_0$. Hence $\eta_\phi(\cdot)$ is an error bound away from $2t_0$ in $\mathcal{L}_\phi(\mathcal{H})$ for the family $\{f(tA)g(tB)\}_{t\geq 0}$. Similarly one can prove this result for the families $\{g(tB)f(tA)\}_{t\geq 0}$, $\{F(t)\}_{t\geq 0}$ and $\{T(t)\}_{t\geq 0}$. □

Now we are going to apply Theorem 5.1 to the results of [12] and [13]. To this end we restrict the class of Kato-functions by conditions (cf. [13]):

$$C_0(\alpha) := \sup_{x>0} \frac{xf(x)^{1/2\alpha}}{1-f(x)} < +\infty, \tag{5.21}$$

$$C_2 := \sup_{x>0} \left| \left(f(x) - \frac{1}{1+x}\right)\frac{1}{x^2} \right| < +\infty, \tag{5.22}$$

$$S_2 := \sup_{x>0} \left| \left(g(x) - \frac{1}{1+x}\right)\frac{1}{x^2} \right| < +\infty, \tag{5.23}$$

for some $\alpha \in (1/2, 1]$. Below, we consider self-adjoint operators $A \geq I$ and $B \geq 0$ on the separable Hilbert space $\mathcal{H}$ such that, as in [12, 13], one has

$$\mathrm{dom}(A^\alpha) \subseteq \mathrm{dom}(B^\alpha) \tag{5.24}$$

and

$$\|B^\alpha f\| \leq a\|A^\alpha f\|, \qquad f \in \mathrm{dom}(A^\alpha) \tag{5.25}$$

for some $\alpha \in (1/2, 1]$. Further, we set ([12],[13])

$$S_1 := \sup_{x>0} \left| (1 - g(x))\frac{1}{x} \right| < +\infty. \tag{5.26}$$

Combining the results of [12] and [13] with Theorem 5.1 we get the following result.

Theorem 5.2 *Let $A \geq I$ and $B \geq 0$ be self-adjoint operators on the separable Hilbert space $\mathcal{H}$ such that conditions (5.24) and (5.25) are satisfied. Further, let $f^D(\cdot) : \mathbf{R}^1_+ \longrightarrow \mathbf{R}^1_+$ and $g^D(\cdot) : \mathbf{R}^1_+ \longrightarrow \mathbf{R}^1_+$ be bounded Borel measurable functions such that $F^D(t_0) \in \mathcal{L}_\phi(\mathcal{H})$ for some $t_0 > 0$. Assume that the Kato-functions $f(\cdot)$ and $g(\cdot)$ are dominated by $f^D(\cdot)$ and $g^D(\cdot)$, respectively. If for some $\alpha \in (1/2, 1]$ the Kato-functions $f(\cdot)$ and $g(\cdot)$ satisfy conditions (5.21) - (5.23) and*

$$a^{1/\alpha}C_0(\alpha)S_1 < 1, \tag{5.27}$$

then the function $\eta_\phi(\cdot)$,

$$\eta_\phi(n) := \begin{cases} 0 & : \quad n = 0 \\ \frac{\ln(n)}{n^{2\alpha-1}} & : \quad n \in \mathbf{N}, \end{cases} \tag{5.28}$$

is an error bound away from $2t_0$ in $\mathcal{L}_\phi(\mathcal{H})$ of the Trotter-Kato product formula for the families $\{T(t)\}_{t\geq 0}$ and $\{f(tA)g(tB)\}_{t\geq 0}$.

If in addition for some $\alpha \in (1/2, 1)$ besides (5.21) - (5.23) and (5.27) the condition $\mathrm{dom}(H^\alpha) \subseteq \mathrm{dom}(A^\alpha)$ *is satisfied, then*

$$\eta_\phi(n) := \begin{cases} 0 & : \quad n = 0 \\ \frac{1}{n^{2\alpha-1}} & : \quad n \in \mathbf{N}, \end{cases} \tag{5.29}$$

is an error bound away from $2t_0$ in $\mathcal{L}_\phi(\mathcal{H})$ of the Trotter-Kato product formula for the familes $\{T(t)\}_{t\geq 0}$ and $\{f(tA)g(tB)\}_{t\geq 0}$.

Proof: Let

$$\eta(n) := \begin{cases} 0 & : \quad n = 0 \\ \frac{\ln(n)}{n^{2\alpha-1}} & : \quad n \in \mathbf{N}. \end{cases} \tag{5.30}$$

Taking into account Theorem 2.7 and Corollary 2.8 of [12] as well as Theorem 5.3 and Corollary 5.4 of [13] we find that $\eta(\cdot)$ is an operator-norm error bound away from $t_0 > 0$ of the Trotter-Kato product formula for the families $\{T(t)\}_{t\geq 0}$ and $\{f(tA)g(tB)\}_{t\geq 0}$. Applying Theorem 5.1 we obtain that $\hat{\eta}_\phi(n) := \frac{1}{2}\{\eta([n/2]) + \eta([(n+1)/2])\}$, $n \in \mathbf{N}_0$, is an error bound away from $2t_0$ in $\mathcal{L}_\phi(\mathcal{H})$ for the families $\{T(t)\}_{t\geq 0}$ and $\{f(tA)g(tB)\}_{t\geq 0}$. By a straightforward computation one gets a constant C_α such that $\hat{\eta}_\phi(n) \leq C_\alpha \eta_\phi(n)$, $n \in \mathbf{N}_0$. Hence $\eta_\phi(\cdot)$ is an error bound away from $2t_0$ in $\mathcal{L}_\phi(\mathcal{H})$ of the Trotter-Kato product formulas for the families $\{T(t)\}_{t\geq 0}$ and $\{f(tA)g(tB)\}_{t\geq 0}$.

Similarly, let

$$\eta(n) := \begin{cases} 0 & : \quad n = 0 \\ \frac{1}{n^{2\alpha-1}} & : \quad n \in \mathbf{N}. \end{cases} \tag{5.31}$$

If in addition conditions $\alpha \in (1/2, 1)$ and $\mathrm{dom}(H^\alpha) \subseteq \mathrm{dom}(A^\alpha)$ are satisfied, then by Theorem 5.5 and Corollary 5.6 of [13] $\eta(\cdot)$ is an operator-norm error bound away from $t_0 > 0$. Applying Theorem 5.1 and proceeding as before we prove the second part of the theorem. □

It is the point to note the following: First, Theorem 2.7 of [12] is proven under the assumption $A \geq I$ and $B \geq I$. But revision of the proof in [12] shows that $B \geq 0$ is sufficient in order to prove the statements of this theorem. Therefore, one can use it in Theorem 5.2.

Second, we note that by Corollary 5.7 of [13] the condition $\mathrm{dom}(H^\alpha) \subseteq \mathrm{dom}(A^\alpha)$ for some $\alpha \in (1/2, 1)$ is satisfied if $\mathrm{dom}(A^\beta) \subseteq \mathrm{dom}(B^\alpha)$ holds for some $\beta \in (0, \alpha)$, in particular, if $\mathrm{dom}(A^{1/2}) \subseteq \mathrm{dom}(B^\alpha)$ is valid.

6 Traces and ideals

If $\mathcal{L}_\phi(\mathcal{H}) = \mathcal{L}_1(\mathcal{H})$, then it turns out that under the assumptions of Theorem 4.5, Theorem 4.6, Corollary 4.7, Corollary 4.8 and Corollary 4.9 the Trotter-Kato product formula converges under the trace, i.e

$$\begin{aligned} &\lim_{r\to+\infty} Tr(F(t/r)^r) = \lim_{r\to+\infty} Tr(T(t/r)^r) = \\ &\qquad \lim_{n\to+\infty} Tr((f(tA/n)g(tB/n))^n) = \lim_{n\to\infty} Tr((g(tB/n)f(tA/n))^n) = Tr(e^{-tH}), \end{aligned} \tag{6.1}$$

for $t > t_0$. Besides this standard trace $Tr(\cdot)$ Dixmier has introduced in [4] a "non-normal" trace which vanishes on the trace class-operators $\mathcal{L}_1(\mathcal{H})$. In accordance with [3] we denote it by $Tr_\omega(\cdot)$ and we call it *Dixmier trace*.

The Dixmier trace is a non-negative linear functional on the ideal $\mathcal{L}_{1,\infty}(\mathcal{H})$ which norming function is given by the sequence $\Pi = \{j^{-1}\}_{j=1}^{\infty}$ (see (2.23)). Since (2.32) the Dixmier trace is continuous with respect to the norm $\|\cdot\|_{1,\infty}$ and since the Trotter-Kato product formula converges in $\|\cdot\|_{1,\infty}$ (provided in Theorem 4.5, Theorem 4.6, Corollary 4.7 and Corollary 4.8 and Corollary 4.9 the ideal $\mathcal{L}_\phi$ is specified to $\mathcal{L}_{1,\infty}$), one gets, instead of (6.1), that

$$\lim_{r\to+\infty} Tr_\omega(F(t/r)^r) = \lim_{r\to+\infty} Tr_\omega(T(t/r)^r) = \tag{6.2}$$
$$\lim_{n\to+\infty} Tr_\omega((f(tA/n)g(tB/n))^n) = \lim_{n\to\infty} Tr_\omega((g(tB/n)f(tA/n))^n) = Tr_\omega(e^{-tH})$$

for $t > t_0$. Moreover, by Theorems 5.1 and 5.2 the error estimates can be obtained for the Dixmier trace $Tr_\omega(\cdot)$ in the same way as for the usual trace $Tr(\cdot)$.

7 Remarks

Let us make some comments concerning the results of this paper.

1. It turns out that the results of [7] and [11] for the von Neumann-Schatten ideals admit an extension to arbitrary symmetrically-normed ideals.

2. For the convergence of Trotter-Kato product formula in the norm of symmetrically-normed ideal away from $t_0 > 0$ it is enough that the generalized Kac operator $F(t)$ (the transfer matrix) belongs to that ideal for $t = t_0$ provided the Kato-functions $f(\cdot)$ and $g(\cdot)$ behaves well. In particular, this is valid for von Neumann-Schatten ideals. This obviously improves [7] and [11] where stronger conditions were used: either $e^{-t_0 A}$ for some $t_0 > 0$ or, respectively, $f_0(tA)$ for $t > 0$, belong to a von Neumann-Schatten ideal. (Of course, these conditions imply that the generalized Kac operator belongs to that ideal.)

3. From the point of view of the class of admissible Kato-functions $f(\cdot)$ and $g(\cdot)$, the present results are more restrictive than in [11] since it is assumed that the Kato-functions are regular, dominated or self-dominated and admit a certain decaying behaviour at infinity. Only Theorems 4.1 and 4.5 allow arbitrary Kato-functions $g(\cdot)$.

4. Notice that the result of [7] establishes convergence in von Neumann-Schatten ideals including $t = t_0$ while we prove it for $t > t_0$.

5. As a by-product we get that the norm-convergence of the Trotter-Kato product formula for the case when the generalized Kac operator $F(t)$ is compact for some $t = t_0 > 0$. This completes the results of paper [14].

6. If the generalized Kac operator belongs to the symmetrically-normed ideal $\mathcal{L}_{1,\infty}(\mathcal{H})$, then the Trotter-Kato product formula converges under the Dixmier trace $Tr_\omega(\cdot)$ which extends the corresponding results of [7] and [11] to non-normal traces.

7. Due to the results of the present paper one easily gets the trace norm-convergence of the Trotter-Kato product formula. Under assumptions of papers [5] and [8], however, error bounds are not available in this abstract setting since the smallness assumptions of Section

5 are not valid for [5] and [8]. Nevertheless, taking into account the operator-norm error bounds found in [5], [8] and applying Theorem 5.1 one immediately gets corresponding error bounds in the trace-norm!

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