CRITICAL BEHAVIOR FOR 2D UNIFORM AND DISORDERED FERROMAGNETS AT SELF–DUAL POINTS

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Abstract. We consider certain two-dimensional systems with self–dual points including uniform and disordered q–state Potts models. For systems with continuous energy density (such as the disordered versions) it is established that the self–dual point exhibits critical behavior: Infinite susceptibility, vanishing magnetization and power law bounds for the decay of correlations.

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Introduction

In this note we will consider the Potts ferromagnets and related systems on the square lattice. The Potts models are defined by the Hamiltonian

$$\mathcal{H} = -\sum_{\langle x,y \rangle} J_{x,y} \delta_{\sigma_x,\sigma_y} \tag{1}$$

with $\sigma_x = 1, 2, \ldots, q$ and $\delta_{\sigma_x,\sigma_y}$ the usual Kronecker delta. Here $J_{x,y}$ is non–zero only when x and y are nearest neighbors and it is assumed that these couplings cannot be negative.

Interest in the disordered version of these systems has recently been revived, in particular by J. Cardy and coworkers [Ca] who have discovered an apparent close connection between these problems and systems with random fields. For the disordered Potts models (among other systems) it was established in [AW_1], [AW_2] that for all temperatures, the energy is continuous. Thus, by conventional definitions, the magnetic ordering transition is continuous. However, as pointed out by Cardy in [Ca] – as well as in a number of public forums – for these systems it has not been established that all aspects of the transition meet the conventional criteria of a continuous transition: Vanishing of the order parameter, power law decay for correlations and infinite susceptibility. Here we establish that, at least at the self–dual points, these systems behave critically in the sense of all the above mentioned (with a power law *bound* for decay of correlations). Our results apply to a variety of systems under the hypothesis of a continuous energy density.

The method is to employ graphical representations (e.g. the random cluster representation for the Potts models) and in fact applies to the non–integer cases provided the model is attractive ($q \geq 1$). In essence, the results here are complimentary to one recently proved in a paper coauthored by one of us [BC]. There it was shown that if (at some point) the energy density is *not* continuous then the discontinuity (a) is unique, (b) occurs precisely at the self–dual point and (c) coincides with the magnetic ordering transition. For these cases the picture is, by and large, complete. Unfortunately, for the continuous cases, our methods do not rule out the possibility that the self–dual point is simply a critical point in the interior of a critical phase. (Nor does it rule out the possibility that at the low–temperature edge of this purported phase, the magnetization exhibits a jump akin to the Thouless effect in one–dimensional long–range systems [T], [AY], [ACCN].) Nevertheless, taken together, the two sets of results imply that the self–dual point is always a point of non–analyticity.

This work will be organized along the following lines: We will start with the uniform $q \geq 1$ random cluster cases which are the simplest illustration of the basic method. Next we will treat certain straightforward generalizations, e.g. the Ashkin–Teller model and in the second section we treat the disordered Potts model.

Uniform Systems

The random cluster model. We shall begin by setting notation. Consider the random cluster model on some finite connected $\Lambda \subset \mathbb{Z}^2$. If ω is a configuration of bonds, the probability of ω, in the setup with "free" boundary conditions is given by

$$\mu_{\Lambda^f}^{q,R}(\omega) \propto R^{N(\omega)} q^{C^f(\omega)} \tag{2}$$

Where $N(\omega)$ is the number of "occupied" bonds of the configuration and $C^f(\omega)$ is the number of connected components. If q is an integer, this is the representation of the of the model described by Equ.(1) – for free boundary conditions – with $J_{i,j} \equiv 1$ and $R = e^{\beta} - 1$.

For other boundary conditions, the formula for the weights must be modified. Of primary interest are the *wired* boundary conditions which, back in the spin–system correspond to setting each spin on the boundary to the same value. Then the formula for the weights of configurations is the same as in Equ.(2) with $C^f(\omega)$ replaced by $C^w(\omega)$ where the latter counts all sites connected to the boundary as part of the same cluster.

In this note we will restrict attention to the random cluster measures that are (weak, possibly subsequential) limits of random cluster measures defined in finite volume. Here the boundary conditions that we will consider – essentially those that are handed down from the spin–systems – are defined as follows: In volume Λ, the boundary, $\partial\Lambda$ is divided into k disjoint sets. All sites of the individual sets are identified as the same site. Thus the boundary consists of k effective sites (components) $v_1, \dots v_k$. No interior connections between $v_1, \dots v_k$ are permitted. (I.e. any configuration with such a connection is assigned zero weight.) All interior sites connected to the same boundary component are considered as part of the same connected component. Finally, we will allow couplings between boundary sites and their neighboring interior sites to take arbitrary values in $[0, \infty)$. And, of course, we will also consider arbitrary superpositions of all of the above.

For the case of free and wired boundary conditions, infinite volume limits, ergodicity etc. follow in a straightforward fashion from the monotonicity (FKG) properties of the $q \geq 1$ random cluster measures. (In the disordered cases, some of these points must be rediscussed and we will do so at the appropriate time.) We will assume general familiarity on the part of the reader concerning these properties. Most of the relevant material can be found in [ACCN] or [BC]. However, if available at the time of reading, the authors highly recommend the forthcoming article [GHM].

The dual model, defined on the lattice Λ^* that is dual to Λ, has weights of the same form as those in Equ.(2) with R replaced by $R^* = qR^{-1}$. The general problem of boundary conditions for the dual model are a little intricate but for the purposes of this work, it is sufficient to note that the free and wired boundary conditions are exchanged under duality.

In these models (with integer q) the relationship between the bond density in the random cluster model and energy density in the spin–system is straightforward. In particular, let $\mathfrak{e}_{x,y}$

denote the event of an occupied bond that connects the neighboring pair $\langle x, y\rangle$. Then, as shown in [CM_I],

$$\langle \delta_{\sigma_x,\sigma_y}\rangle^{q,R}_{\Lambda^\#} = \frac{1+R}{R}\mu^{q,R}_{\Lambda^\#}(\mathfrak{e}_{x,y}) \tag{3}$$

where $\langle - \rangle^{q,R}_{\Lambda^\#}$ denotes thermal expectation in the spin–system in boundary condition $\# = f, w$ (or, for that matter any other boundary condition). Thus, in these cases, continuity in the energy density is manifested as continuity in the bond density. Hereafter, we will focus on $q \geq 1$ random cluster models and use continuous *bond* density for our working hypotheses.

Let us finally remark that for almost every R, the bond density is, in fact, a well defined concept. This point (which is fairly standard) has recently been detailed in [BC] so here we will be succinct. Consider the free energy, $\Phi(R)$, defined here by

$$\Phi(R) = \lim_{\Lambda\nearrow\mathbb{Z}^2}\frac{1}{|\Lambda|}\log \mathcal{Z}_{\Lambda^\#} \tag{4}$$

where $\mathcal{Z}_{\Lambda^\#}$ is the sum of the weights in Equ.(2) and $\Lambda \nearrow \mathbb{Z}^2$ means a *thermodynamic* limit – a regular sequence of boxes. The function $\Phi(R)$ is convex (as a function of $\log R$) and hence has a left and right derivative for every R which agree for almost every R. At points of continuity of the derivative, there is uniqueness among the translation invariant states and the bond density in this state is given by the derivative of Φ (with respect to $\log R$). At points of discontinuity, the upper value for the bond density is achieved in the wired state and the lower value in the free state.

We are ready for our first result:

Theorem 1. *Consider the 2d random cluster models with parameter $q \geq 1$. Then if the self–dual point, $R = R^* = \sqrt{q}$ is a point of continuity of the bond density, the percolation density vanishes.*

Proof. This result can in fact be obtained as a consequence of Theorem 2.1 in [BC]. For completeness we will provide a direct proof. Here we will establish the contrapositive statement; i.e. assume that the percolation probability *is* positive at the self–dual point and show that this implies a discontinuity in the bond density.

Percolation is defined in reference to the wired measures (and limits thereof). These measures are ergodic under $\mathbb{Z}^2$ translations, respect the x, y–axis symmetry and have the FKG property. In short, these measures satisfy all the conditions of the theorem in [GKR] which forbids coexisting infinite clusters of the opposite type. Thus, with probability one, whenever there is percolation, all dual bonds reside in finite clusters. However, if there is percolation (in the wired state) at the self–dual point, the same cannot be said in the limiting free boundary condition measure. Indeed, from the perspective of the *dual* bonds, this is a wired state. Hence, in this state, the dual bonds percolate and the regular bonds do not. It is thus evident that the limiting free and wired measures are distinct; $\mu_{\Lambda^f}^{q,R}(-)$ is strictly below $\mu_{\Lambda^w}^{q,R}(-)$. By the (corollary to) Strassen's theorem (see [L] p. 75) this implies that the bond density in the wired state is strictly larger than that of the free state; the self–dual point is thus not a point of continuity for the derivative of Φ $\square$

Corollary. *Under the hypotheses of Theorem 1, there is a unique limiting Gibbs state/ random cluster measure at the self–dual point.*

Proof. It was proved in [ACCN] (Theorem A.2) that the absence of percolation (for $q \geq 1$) implies that a unique limiting random cluster measure and, for (integer $q \geq 2$) a unique Gibbs state in the corresponding spin–system. So this follows immediately. $\square$

Let A and B denote disjoint sets in $\mathbb{Z}^2$. We will denote by $\{A \longleftrightarrow B\}$ the event that some site in A is connected to some site in B by occupied bonds. Further, if D contains both A and B, we let $\{A \longleftrightarrow B\}_D$ denote the event that such a connection occurs by a path that lies entirely in D. For x and y (distinct) points in $\mathbb{Z}^2$, let $g_{x,y} = g^{\#}_{x,y}(R, q) = \mu^{q,R}_{\#}(\{x \longleftrightarrow y\})$ be the probability that x and y belong to the same cluster. We will call this object the *connectivity function.* For integer $q = 2, 3, \dots$ the connectivity function is equal to (or proportional to) the spin–spin correlation function. In these cases, the susceptibility and the average cluster size are also identified. For non–integer q, the geometric quantities are *defined* to be the objects of interest. These quantities are the subject of our next theorem which is a direct consequence of Theorem 1. The proof below borrows heavily from the argument in [A].

Theorem 2. *For self–dual $q \geq 1$ random cluster models with vanishing percolation probability, the function $g_{x,y}$ has a power law lower bound. Explicitly, if 0 is the origin and $\mathbf{L}$ is the point $(L, 0)$ on the x axis then $g_{0,\mathbf{L}} \geq \frac{1}{8}L^{-2}$. Finally, the average cluster size is infinite.*

Proof. Since there is a unique limiting state then, in particular, the limiting free and wired measures coincide. Consider the $L \times L$ square centered at the origin which we denote by S_L. In every configuration, there is either a left–right crossing by regular bonds or a top–bottom crossing by dual bonds. By duality, in the limiting measure, these probabilities are both one half.[1] Letting $\mathcal{L}_L$ denote the left edge and $\mathcal{R}_L$ the right edge of the square, this implies that

$$\sum_{x \in \mathcal{L}_L, y \in \mathcal{R}_L} g_{x,y} \geq \frac{1}{2}. \tag{5}$$

Hence, for some (deterministic) $x^* \in \mathcal{L}_L$ and $y^* \in \mathcal{R}_L$ we have

$$g_{x^*,y^*} \geq \frac{1}{2L^2}. \tag{6}$$

[1] To ensure that this is strictly true, one must carefully construct the square so that it is *exactly* self–dual. However, for the arguments here, it is actually sufficient to observe that one of the probabilities must be greater or equal to one half.

This is, in essence the bound on the correlation function. For æsthetic purposes we will show that a similar bound holds for $g_{0,\mathbf{L}}$ but let us first attend to the susceptibility. Following the logic of Eqs.(5) and (6), there is an x^{**} in $\mathcal{L}_L$ that is connected to $\mathcal{R}_L$ by a path inside S_L with probability of order L^{-1}:

$$\mu^{q,R=\sqrt{q}}(\{x^{**} \longleftrightarrow \mathcal{R}_L\}_{S_L}) \geq \frac{1}{2L}. \tag{7}$$

Regarding the point x^{**} as being at the center of a square of side $2L$ and using translation invariance, we find

$$\mu^{q,R=\sqrt{q}}(\{0 \longleftrightarrow \partial S_{2L}\}) \geq \frac{1}{2L} \tag{8}$$

i.e. $X_L \equiv \mu^{q,R=\sqrt{q}}(\{0 \longleftrightarrow \partial S_L\}) \geq L^{-1}$. This immediately implies a divergent susceptibility/cluster size. Indeed writing

$$\mathcal{X} = \sum_x g_{0,x} = \sum_L \sum_{x \in \partial S_L} g_{0,x} \geq \sum_L X_L \tag{9}$$

this result follows. Finally, let us obtain our bound for the correlation function along the coordinate axes. Consider the event $(\{x^{**} \longleftrightarrow \mathcal{R}_L\}_{S_L}$ along with along with its mirror image reflected along the midline of the square. I.e. a connection between $\mathcal{L}_L$ and y^{**} by a path inside S_L where $y^{**} = x^{**} + \mathbf{L}$. If these two connections occur in tandem with a top bottom crossing of S_L, we achieve a connection between y^{**} and x^{**}. These events are all positively correlated hence

$$g_{0,\mathbf{L}} = g_{y^{**},x^{**}} \geq (\frac{1}{2})(\frac{1}{2L})(\frac{1}{2L}). \tag{10}$$

□

It is clear that the above generalizes to other random cluster systems. However since at present there are not too many examples of physically relevant models that satisfy all of the required conditions, we will be content with a small selection.

The [r,s]-cubic (generalized Ashkin–Teller) model. Consider two copies of $\mathbb{Z}^2$ with two sets of Potts spins: $\tau_i \in \{1, \dots r\}$ and $\kappa_i \in \{1, \dots s\}$. It is convenient to envision the model as two layers of $\mathbb{Z}^2$, the τ–layer and the κ–layer with the τ–layer just above the k–layer. In any case, the Hamiltonian is given by

$$H = -\sum_{\langle x,y \rangle} [a\delta_{\tau_i,\tau_j}\delta_{\kappa_i,\kappa_j} + b\delta_{\kappa_i,\kappa_j} + c\delta_{\tau_i,\tau_j}] \tag{11}$$

where, as it turns out, we will be interested in the cases $a, b, c \geq 0$.

The dual relations for this model (at least for integer r and s) were derived some time ago in [dN], [DR] by algebraic methods. (Of course the special case $r = s = 2$ was derived much earlier starting, in fact, with [AT].) More recently, graphical representations for this model have been discovered [$\mathrm{CM_I}$] [PfV], (and see also [SS]) in which the duality is manifest. Consider bond configurations $\omega = (\omega_\tau, \omega_\kappa)$ i.e. separate bond configurations in the τ– and κ–layers. As usual, we will start in finite volume. Let $N(\omega_\tau)$ and $N(\omega_\kappa)$ denote the number of occupied bonds in the τ– and κ– layers respectively. Let $N(\omega_\tau \vee \omega_\kappa)$ denote the number of edges where at least one of the τ– or κ– layers have occupied bonds and finally let $N(\omega_\tau \wedge \omega_\kappa)$ denote the number of edges where both the τ– and κ– layers have occupied bonds. The graphical representation is defined by the weights

$$W(\omega) = \mathbf{A}^{N(\omega_\tau \vee \omega_\kappa)} \mathbf{B}^{N(\omega_\tau \wedge \omega_\kappa)} \mathbf{C}^{[N(\omega_\kappa) - N(\omega_\tau)]} r^{C(\omega_\tau)} s^{C(\omega_\kappa)} \tag{12}$$

where $C(\omega_\tau)$ and $C(\omega_\kappa)$ are the number of connected components as in the usual random cluster problems. (And typically must be augmented with some boundary conditions.) The relationship between $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ and a, b and c is as follows:

$$\mathbf{A} = [(e^{\beta b} - 1)(e^{\beta c} - 1)]^{\frac{1}{2}} \tag{13a}$$

$$\mathbf{B} = \frac{e^{\beta(a+b+c)} - e^{\beta b} - e^{\beta c} + 1}{[(e^{\beta b} - 1)(e^{\beta c} - 1)]^{\frac{1}{2}}} \tag{13b}$$

$$\mathbf{C} = \left[\frac{(e^{\beta b} - 1)}{(e^{\beta c} - 1)}\right]^{\frac{1}{2}}. \tag{13c}$$

In order for the graphical representation to make sense, we require $b, c \geq 0$. However, this is not the case with a but it turns out that the FKG property – which we will need – only holds if $\mathbf{B} \geq \mathbf{A}$ [BC] thus we actually require all couplings in Equ.(13) to be ferromagnetic. Under these conditions for the case $r = s$, $b = c$ it was shown in [$\mathrm{CM_I}$] that there is a single ordering transition as the temperature is varied.

The dual model is defined straightforwardly: edges of the dual lattice in, say, the τ–layer that are traversal to occupied bonds are considered vacant dual bonds, those edges traversal to vacant bonds are the occupied dual bonds and similarly in the κ–layer. The duality conditions are easily obtained from the weights in Equ.(12) (for the simple reason that $\vee \leftrightarrow \wedge$ under duality) and the result is:

$$\mathbf{A}^* = \frac{\sqrt{rs}}{\mathbf{B}} \tag{14a}$$

$$\mathbf{B}^* = \frac{\sqrt{rs}}{\mathbf{A}} \tag{14b}$$

$$\mathbf{C}^* = \sqrt{\frac{r}{s}}\,\frac{1}{\mathbf{C}} \tag{14c}$$

The analog of Theorems 1 and 2 for this system are readily established:

Theorem 3. *Consider the (r,s)–cubic model as described with $\mathbf{A} \geq \mathbf{B}$. Let $\Phi(\mathbf{A},\mathbf{B},\mathbf{C})$ denote the free energy similar to that defined in Equ.(4). Suppose that a self dual point: $(\mathbf{A},\mathbf{B},\mathbf{C}) = (\mathbf{A}^*,\mathbf{B}^*,\mathbf{C}^*)$, is a point of continuity for any first derivative of Φ. Then the percolation probability in either layer vanishes. Let $g^{\tau}_{x,y}$ and $\mathcal{X}^{\tau}$ denote the connectivity function and average size of clusters in the τ–layer and similarly for $g^{\kappa}_{x,y}$ and $\mathcal{X}^{\kappa}$. Then $g^{\tau}_{0,\mathbf{L}} \geq \frac{1}{8}L^{-2}$, $\mathcal{X}^{\tau}$ is infinite and similarly for $g^{\kappa}_{x,y}$ and $\mathcal{X}^{\kappa}$.*

Proof. It is convenient, but not essential, to restrict attention to the "plane" $\mathbf{C} = \mathbf{C}^*$. Indeed, following the argument below, it can be shown that continuity with respect to $\mathbf{A}$ and $\mathbf{B}$ actually implies continuity with respect to $\mathbf{C}$ thus, for all intents and purposes, the $\mathbf{C}$–variable is out of the play. Continuity of the derivative with respect to $\mathbf{B}$ implies that at the point $(\mathbf{A},\mathbf{B},\mathbf{C})$, the density of "doubly occupied" bonds is independent of state for any translation invariant state. Add to this continuity of the derivative with respect to $\mathbf{A}$ and (since $N(\omega_\tau \vee \omega_\kappa) + N(\omega_\tau \wedge \omega_\kappa) = N(\omega_\tau) + N(\omega_\kappa)$) we may conclude that the total bond density is the same in every translation invariant state. However, we claim that this implies the same result for the separate densities. Indeed if the τ–density were discontinuous, to keep the total density continuous would require a compensating discontinuity in the κ–density. But these densities are positively correlated; i.e. the discontinuities must go in the same direction. In particular, we could find a state (at the point $(\mathbf{A},\mathbf{B},\mathbf{C})$) where both densities achieved their lower value and another where they both obtain the upper value.

Constancy of the bond density implies no percolation in either layer at a self–dual point which in turn implies unicity of the state. The rest of the argument follows mutatis mutandis the proof of Theorem 2. □

A loop related model. Our final example appeared in the context of loop models in [CPS]. Let ω denote a bond configuration on $\mathbb{Z}^2$ and let $\tilde{\omega}$ denote the *complimentary* configuration on the dual lattice: If a bond of ω is occupied than so is the traversal bond and similarly for vacancies. (In other words, the vacant bonds of the dual configuration are the occupied bonds of the complimentary configuration.) The weights, in finite volume are given by

$$V(\omega) = L^{N(\omega)} s^{C(\omega)} s^{C(\tilde{\omega})}. \tag{15}$$

We remark that from a technical perspective, unrelated boundary conditions for ω and $\tilde{\omega}$ may be implemented. However it is natural to assert that if one is fully wired so is the other and similarly with free boundary conditions. A derivation identical to the one for the usual random cluster model shows that the dual model is the same model with the parameter

$$L^* = \frac{s^2}{L} \tag{16}$$

along with the usual exchange of boundary conditions. For double–free or double–wired (as well as other) boundary conditions, the FKG property follows easily:

Proposition 4. *For free or wired boundary conditions, the random cluster models defined defined by the weights in Equ.(15) with $s \geq 1$ have the FKG property. Further, for these boundary conditions, if $R_1 \geq R_2$ and $s_1 \leq s_2$ the measure with parameters (R_1, s_1) FKG dominates the one with parameters (R_2, s_2).*

Proof. For the usual random cluster model, the FKG property follows from the FKG lattice condition [FKG]: Let ω_1 and ω_2 denote two bond configurations, $\omega_1 \wedge \omega_2$ the configuration of bonds occupied in both ω_1 and ω_2 and $\omega_1 \vee \omega_2$ the configuration of bonds occupied in either. The lattice condition reads: $\mu(\omega_1 \wedge \omega_2)\mu(\omega_1 \vee \omega_2) \geq \mu(\omega_1)\mu(\omega_2)$ which follows because $C(\omega_1 \wedge \omega_2) + C(\omega_1 \vee \omega_2) \geq C(\omega_1) + C(\omega_2)$ [ACCN]. In the present case, we need only apply this argument twice, once to $C(\omega)$ and once to $C(\tilde{\omega})$. The FKG dominance follows by writing the one set of weights as an increasing function times the other.

Remark. It would appear that the model under discussion is very close to the $q = s^2$–state random cluster model. This follows by noting that for the former, $C(\omega)$ and $C(\tilde{\omega})$ are identically distributed. Then, we may write $s^{C(\omega)+C(\tilde{\omega})} = s^{2C(\omega)} s^{C(\tilde{\omega})-C(\omega)}$ and suppose that the "fluctuations" $C(\tilde{\omega}) - C(\omega)$ are (thermodynamically) small. However, at present there is no hard evidence of such an equivalence. On the other hand, the self–dual point can in fact be realized as the endpoint of the self–dual line of the symmetric (r = s, C = 1) cubic model corresponding to $A \to \infty$ and $B \to 0$. Here the τ–bonds may be taken to be the occupied bonds and the κ's to be the vacants. The condition $B = 0$ ensures that they cannot coincide while $A = \infty$ implies one bond or the other actually is occupied.

Theorem 5. *For the model defined by the weights in Equ.(15), the results of Theorem 3 apply; If $R = s$ is a point of continuity for the bond density then the connectivity function has a power law lower bound and the average cluster size is infinite.*

Proof. Follows from the same arguments as the proofs of Theorems 1 and 2 $\square$

Remark. In this model, the results of [BC] also apply: If there is any point of discontinuity for the bond density, that point must be the self dual point. (The cases of the Potts model and the cubic model for $r = s$ were the explicit subject of [BC]; the identical arguments apply to the current case.) Thus, one way or another in all these systems the self–dual points are points of "phase transitions". For large q, r, and/or s – at least in the integer cases – it is straightforward to show that discontinuities *do* occur. (Theorem IV.2 in [CM_I] covers all of these cases.) The difficulty is the opposite cases: establishing continuity. Indeed, to the authors' knowledge, the only non–trivial case where this has been done with complete rigor is the Ising magnet. However, the next section features systems where the required continuity has been guaranteed.

Quenched Potts Models

For the remainder of this paper, we will deal exclusively with the q–state Potts model as defined by the Hamiltonian in Equ.(1); we will treat the case where the $J_{x,y}$ are (non–negative) independent random variables. (And ultimately to prove theorems along the lines of Theorems 1 and 2, we will need to focus on distributions that are self–dual.) We strongly suspect that with only minimal labor the forthcoming results could be extended to disordered versions of the various other models discussed in the previous section. But here we will focus on the minimal case.

The approach in this work will be somewhat different from the usual mathematical studies of disordered systems: rather than looking at properties that are "typical" of configurations of couplings, we will construct, from the outset, the *quenched* measure – more precisely, the graphical representation thereof. When all the preliminaries are in place, this has the advantage of allowing a derivation that is essentially indistinguishable from the uniform cases. The disadvantage is that many of the "basic preliminaries" will require some attention.

The quenched measure. Let $\Lambda \subset \mathbb{Z}^2$ (or $\mathbb{Z}^d$ for the duration of the preliminaries) denote a finite volume. In what follows, the inverse temperature β as well as the value of q (≥ 1) will be regarded as fixed and hence will be suppressed notationally. Let $\eta = \{J_{x,y} \mid \langle x, y\rangle \in \Lambda\}$ denote a set of couplings. Let $\#$ denote a boundary condition on $\partial\Lambda$. In general we will allow the boundary condition to depend on the realization of couplings so we will write $\#(\eta)$. (For the case of continuous variables $J_{x,y}$ we must also stipulate that $\#(\eta)$ is a *measurable* function.) We let $\langle - \rangle^{\eta}_{\Lambda;\#(\eta)}$ denote the finite volume Gibbs state (for *this* realization of couplings and *this* boundary condition.) Similarly, we may consider the random cluster measures $\mu^{\eta}_{\Lambda;\#(\eta)}(-)$. Our assumption about the $J_{x,y}$ is that they are i.i.d. non–negative variables. Let $\mathfrak{b}(-)$ denote the product measure for configurations of couplings and $\mathbb{E}_{\mathfrak{b}}(-)$ the expectation with respect to this measure. Then the quenched measures are defined as the $\mathfrak{b}$–averages of the "thermal" averages according to $\langle - \rangle^{\eta}_{\Lambda;\#(\eta)}$ and $\mu^{\eta}_{\Lambda;\#(\eta)}(-)$. Explicitly, if $F(\sigma_{x_1}, \dots \sigma_{x_k})$ is a function of spins (with $x_1, \dots x_k \in \Lambda$) then the quenched average of F is given by

$$\overline{\langle F\rangle}_{\Lambda;\#} = \mathbb{E}_{\mathfrak{b}}(\langle F\rangle^{\eta}_{\Lambda;\#(\eta)}). \tag{17a}$$

Similarly, for a bond event $\mathcal{A}$,

$$\overline{\mu}_{\Lambda;\#}(\mathcal{A}) = \mathbb{E}_{\mathfrak{b}}(\mu^{\eta}_{\Lambda;\#(\eta)}(\mathcal{A})). \tag{17b}$$

Most of our attention will be focused on the quenched random cluster measures as defined in Equ.(17b) – or the infinite volume limits thereof. Our first proposition establishes some FKG properties of these quenched measures:

Proposition 6. *On finite Λ, let w and f denote the boundary conditions that are, respectively wired and free for all η. Then the measures $\overline{\mu}_{\Lambda,w}(-)$ and $\overline{\mu}_{\Lambda,f}(-)$ have the FKG property (in the sense of positive correlations). Furthermore, for any $\# = \#(\eta)$, the wired measure $\overline{\mu}_{\Lambda,w}(-)$ FKG–dominates the measure $\overline{\mu}_{\Lambda,\#}(-)$*

Proof. For the FKG property, we will just do the wired case; the free case is identical. Let $\mathcal{A}$ and $\mathcal{B}$ denote increasing events (defined on the bonds of Λ). By the FKG properties of the measures $\mu^{\eta}_{\Lambda;w}(-)$,

$$\mu^{\eta}_{\Lambda;w}(\mathcal{A} \cap \mathcal{B}) \geq \mu^{\eta}_{\Lambda;w}(\mathcal{A})\mu^{\eta}_{\Lambda;w}(\mathcal{B}). \tag{18}$$

Now we observe that for any increasing event $\mathcal{C}$, the quantity $\mu^{\eta}_{\Lambda;w}(\mathcal{C})$ is an increasing function of η. Indeed, if $\eta_1 \succ \eta_2$ (meaning $J^1_{x,y} \geq J^2_{x,y}$ for each bond in Λ) then the random cluster measure with couplings η_1 FKG dominates the one with couplings η_2. But then

$$\begin{aligned}\overline{\mu}_{\Lambda,w}(\mathcal{A} \cap \mathcal{B}) = \mathbb{E}_{\mathfrak{b}}\left(\mu^{\eta}_{\Lambda;w}(\mathcal{A} \cap \mathcal{B})\right) \geq \mathbb{E}_{\mathfrak{b}}\left(\mu^{\eta}_{\Lambda;w}(\mathcal{A})\mu^{\eta}_{\Lambda;w}(\mathcal{B})\right) \geq \\ \mathbb{E}_{\mathfrak{b}}\left(\mu^{\eta}_{\Lambda;w}(\mathcal{A})\right)\mathbb{E}_{\mathfrak{b}}\left(\mu^{\eta}_{\Lambda;w}(\mathcal{B})\right) = \overline{\mu}_{\Lambda,w}(\mathcal{A})\overline{\mu}_{\Lambda,w}(\mathcal{B}).\end{aligned} \tag{19}$$

The same works for the free case (and various other boundary conditions that are independent of η). Finally, the stated FKG–domination is an obvious consequence of the fact that for each η, the measure $\mu^{\eta}_{\Lambda,w}(-)$ FKG–dominates the measure $\mu^{\eta}_{\Lambda,\#(\eta)}(-)$ □

As a corollary, we obtain the existence of infinite volume limits as in the usual random cluster cases:

Corollary. *The infinite volume limits $\overline{\mu}_f(-)$ and $\overline{\mu}_w(-)$ exist in the sense that if $\Lambda_k \nearrow \mathbb{Z}^d$ is any sequence of boxes with $\Lambda_{k+1} \supset \Lambda_k$ then the limits of $\overline{\mu}_{\Lambda_k,f}(-)$ and $\overline{\mu}_{\Lambda_k,w}(-)$ exist and are independent of the sequence. These limiting measures are translation invariant and invariant under exchange of coordinate axes.*

Proof. The argument is exactly as in the standard proofs and is a consequence of the following observation: If $\Lambda_1 \subset \Lambda_2$ then for any fixed η, the restriction of $\mu^{\eta}_{\Lambda_2,w}(-)$ to Λ_1 is FKG dominated by the wired measure in Λ_1. Thus the same statement holds for the quenched average of these measures. A similar sort of domination, but in the opposite direction is established for the free measures. The remainder of the proof is now identical to the derivations for uniform random cluster models (with occasional use of translation invariance and coordinate symmetry of $\mathfrak{b}(-)$). Such proofs have been written in many places (see e.g. [$\mathrm{CM_I}$] Theorem 3.3) and need not be repeated here. □

We now demonstrate that absence of percolation is the correct criterion for uniqueness. Our working definition of percolation is fairly standard:

Definition. Let $\Lambda \subset \mathbb{Z}^d$ be a finite set that contains the origin. We define

$$P_\infty = \lim_{\Lambda \nearrow \mathbb{Z}^d} \overline{\mu}_{\Lambda,w}(0 \longleftrightarrow \partial\Lambda). \tag{20}$$

We say there is percolation if P_∞ is not zero.

Remark. It is obvious, by the considerations of the corollary to Proposition 6 that this limit exists. Further, if P_∞ vanishes, there is no percolation by any other criterion. Finally it is not difficult to show that P_∞ is exactly the spontaneous magnetization in the spin–system.

Next we establish the quenched analog of Theorem A.2 in [ACCN].

Proposition 9. *If $P_\infty = 0$ there is a unique limiting quenched random cluster measure and a unique limiting quenched Gibbs measure.*

Proof. Our proof will in essence be to show that any sequence of finite volume measures converges to the free measure. Let $\mathcal{A}$ denote any local increasing event. Let Λ denote a large (finite) box – the bonds of which determine the event $\mathcal{A}$. Now consider a much larger box Ξ along with some boundary condition $\#(\eta)$; the measure $\overline{\mu}_{\Xi,\#}(-)$ may be thought of as "well along the way" towards the construction of some infinite volume measure.

Since the percolation probability is assumed to vanish, it is clear that if Ξ is sufficiently large, then, for $\epsilon > 0$

$$\overline{\mu}_{\Xi,\#}(\partial\Lambda \longleftrightarrow \partial\Xi) \leq \epsilon^2. \tag{21}$$

Thus if $\mathcal{D}_{\Lambda,\Xi} = \{\eta \mid \mu^\eta_{\Xi,\#(\eta)}(\partial\Lambda \longleftrightarrow \partial\Xi) > \epsilon\}$ then $\mathfrak{b}(\mathcal{D}_{\Lambda,\Xi}) < \epsilon$.

Now for any $\eta \in \mathcal{D}_{\Lambda,\Xi}$, with $\mu^\eta_{\Xi,\#(\eta)}$–probability greater than $1-\epsilon$, there is a "ring" (separating surface) of vacant bonds in the region between $\partial\Lambda$ and $\partial\Xi$. Conditioning to the "outermost" such ring gives us a measure which, in the interior of the ring, is equivalent to free boundary conditions on the ring. For any η, this in turn is dominated by the measure with free boundary conditions on $\partial\Xi$ and dominates (in Λ) the measure with free boundary conditions on $\partial\Lambda$. Thus, for $\eta \in \mathcal{D}_{\Lambda,\Xi}$,

$$(1-\epsilon)[\mu^\eta_{\Lambda,f}(\mathcal{A})] \leq \mu^\eta_{\Xi,\#(\eta)}(\mathcal{A}) \leq (1-\epsilon)[\mu^\eta_{\Xi,f}(\mathcal{A})] + \epsilon \tag{22}$$

and hence

$$(1-\epsilon)[\overline{\mu}_{\Lambda,f}(\mathcal{A})] \leq \overline{\mu}_{\Xi,\#}(\mathcal{A}) \leq (1-\epsilon)[\overline{\mu}_{\Xi,f}(\mathcal{A})] + 2\epsilon \tag{23}$$

where the extra ϵ comes from the $\epsilon \notin \mathcal{D}_{\Lambda,\Xi}$. ¿From Equ.(23) it is easy to see that all sequences of finite volume quenched measures converge to the limiting free measure.

The argument for the uniqueness of the quenched Gibbs state follows from the above by noting that the thermal average of any local spin–function can be expressed as expectations of random cluster functions (which themselves are finite combinations of increasing events.)

This proves (a) the existence of a limiting $\overline{\langle - \rangle}_f$ (and for that matter a limiting $\overline{\langle - \rangle}_w$) and (b) that if the magnetization vanishes that this is the unique limiting state. $\square$

The final result we will need is the ergodic property for the free and wired quenched random cluster measures.

Theorem 10. *The measures $\overline{\mu}_w(-)$ and $\overline{\mu}_f(-)$ are ergodic under $\mathbb{Z}^d$ translations.*

Proof. We will do the wired case, the free case is nearly identical. Let $\mathcal{A}$ and $\mathcal{B}$ denote local events assumed, without loss of generality, to be increasing. Let $\vec{r} \in \mathbb{Z}^d$ and let $T_{\vec{r}}(\mathcal{B})$ denote the event $\mathcal{B}$ translated by $\vec{r}$. We will show that $\lim_{\vec{r}\to\infty} \overline{\mu}_w(\mathcal{A} \cap T_{\vec{r}}(\mathcal{B})) = \overline{\mu}_w(\mathcal{A})\overline{\mu}_w(\mathcal{B})$.

By FKG and translation invariance we have, for any $\vec{r}$,

$$\overline{\mu}_w(\mathcal{A} \cap T_{\vec{r}}(\mathcal{B})) \geq \overline{\mu}_w(\mathcal{A})\overline{\mu}_w(\mathcal{B}). \tag{24}$$

Now consider $|\vec{r}|$ large – far larger than the scale of the regions that determine the events $\mathcal{A}$ and $\mathcal{B}$. Let $s \leq |\vec{r}|$ be chosen so that Λ_s, the box of side s centered at the origin and its translate by $\vec{r}$, which we denote by $T_{\vec{r}}(\Lambda_s)$, are disjoint but within a few lattice spacings of each other. Finally, let us consider an L that is very large compared with $\vec{r}$; we will approximate $\overline{\mu}_w(\mathcal{A}\cap T_{\vec{r}}(\mathcal{B}))$ by $\overline{\mu}_{\Lambda_L,w}(\mathcal{A}\cap T_{\vec{r}}(\mathcal{B}))$. By the FKG property, $\overline{\mu}_{\Lambda_L,w}(\mathcal{A}\cap T_{\vec{r}}(\mathcal{B}))$ is less than the corresponding probability given that all bonds on the outside of Λ_s and $T_{\vec{r}}(\Lambda_s)$ are occupied. But given these occupations, the measure inside Λ_s is equivalent to wired boundary conditions on Λ_s and similarly for $T_{\vec{r}}(\Lambda_s)$. Now for each η, the wirings make these interior measures independent. Thus we have

$$\mu^{\eta}_{\Lambda_L,w}(\mathcal{A} \cap T_{\vec{r}}(\mathcal{B})) \leq \mu^{\eta}_{\Lambda_s,w}(\mathcal{A})\mu^{\eta}_{T_{\vec{r}}(\Lambda_s),w}(T_{\vec{r}}(\mathcal{B})). \tag{25}$$

However as functions of η, the two objects on the right of Equ.(25) are independent – they take place on disjoint sets. It is clear that $\mu^{\eta}_{T_{\vec{r}}(\Lambda_s),w}(T_{\vec{r}}(\mathcal{B}))$ averages to $\overline{\mu}_{\Lambda_s,w}(\mathcal{B})$ and thus

$$\overline{\mu}_{\Lambda_L,w}(\mathcal{A} \cap T_{\vec{r}}(\mathcal{B})) \leq \overline{\mu}_{\Lambda_s,w}(\mathcal{A})\overline{\mu}_{\Lambda_s,w}(\mathcal{B}). \tag{26}$$

Letting $L \to \infty$ we get

$$\overline{\mu}_w(\mathcal{A} \cap T_{\vec{r}}(\mathcal{B})) \leq \overline{\mu}_{\Lambda_s,w}(\mathcal{A})\overline{\mu}_{\Lambda_s,w}(\mathcal{B}). \tag{27}$$

and hence

$$\lim_{\vec{r}\to\infty} \overline{\mu}_w(\mathcal{A} \cap T_{\vec{r}}(\mathcal{B})) \leq \lim_{s\to\infty} \overline{\mu}_{\Lambda_s,w}(\mathcal{A})\overline{\mu}_{\Lambda_s,w}(\mathcal{B}) = \overline{\mu}_w(\mathcal{A})\overline{\mu}_w(\mathcal{B}). \tag{28}$$

This completes the proof for the wired case, the free case works the same way, here we use decreasing events for $\mathcal{A}$ and $\mathcal{B}$. $\square$

Main results. We are ready for the disordered analogs of Theorems 1 and 2. However, in this case, we will not need to hypothesize the required continuity: this is the central subject of [AW_1], [AW_2]. Let us first briefly discuss duality in the disordered case. In the general setup, let

$$J^*(J) = \log\left[1 + \frac{q}{e^J - 1}\right]. \tag{29}$$

(I.e. $e^J - 1 = q/[e^{J^*} - 1]$). Then what is needed, in the discrete case, to have $\beta = 1$ a point of self–duality is that $\mathfrak{b}(J_{x,y} = J) = \mathfrak{b}(J_{x,y} = J^*(J))$. (A similar statement holds for continuous or other distributions) Indeed, if this is the case we see that the probability of bonds and dual bonds of equivalent *strength* are the same. Then, in finite volume, the probabilities of two coupling configurations that are equivalent under duality (including the usual exchange of boundary conditions) are equal.

Sometimes it is convenient to parameterise the distribution and allow β to vary. For example, suppose there are two bond values J_1 and J_2 with $\mathfrak{b}(J_{x,y} = J_1) = \mathfrak{b}(J_{x,y} = J_2) = 1/2$. Since temperature is back in the problem, we may assume, without loss of generality that $J_1 = 1$ and write $J_2 = \lambda$ with $0 \leq \lambda \leq 1$. Then, in the λ, β plane it is not hard to see that the model with parameters λ, β is equivalent under duality to the one with parameters λ^*, β^* where

$$\beta^* = \log\left[1 + \frac{q}{e^{\lambda\beta} - 1}\right] \tag{30a}$$

and

$$\lambda^* = \frac{\log\left[1 + \frac{q}{e^{\beta}-1}\right]}{\log\left[1 + \frac{q}{e^{\lambda\beta}-1}\right]}. \tag{30b}$$

The system is self–dual ($\lambda = \lambda^*, \beta = \beta^*$) when $(e^\beta - 1)(e^{\lambda\beta} - 1) = q$.

Theorem 1′. *Consider a disordered Potts model of the type described at a self–dual point. Then there is a unique limiting state with zero magnetization.*

Proof. The proof is the same as the proof of Theorem 1 which we will recapitulate for continuity. If the magnetization (percolation probability) were non–zero then in the free boundary state, the percolation probability for dual bonds would be non–vanishing. Proposition 6 and Theorem 10 allow us to use the result in [GKR]. Thus the free and wired states would be distinguished. But these are FKG ordered states so it would follows that the bond (and hence energy) density would differ in the two states implying a discontinuity in the energy density (or bond density). This however, is contradicted by the results of [AW_1], [AW_2]. Proposition 9 connects the absence of magnetization to uniqueness. □

Theorem 2′. *Under the hypotheses of Theorem 1′, in the limiting state the quenched correlation function satisfies*

$$\overline{\langle \delta_{\sigma_0,\sigma_L} - \frac{1}{q}\rangle} \geq \frac{1}{8}\frac{1}{L^2}.$$

Further the quenched susceptibility, defined as

$$\overline{\mathcal{X}} = \sum_y \overline{\langle \delta_{\sigma_0,\sigma_y} - \frac{1}{q}\rangle}$$

is infinite.

Proof. Again this follows exactly the proof of Theorem 2 once we can conclude that the probability of a square crossing is one half. Although this follows from self–duality on "general principles" it is comforting to consider the square $L \times L$ square, S_L in the middle of a finite (but much larger) square with wired boundary conditions on the top and right and with free boundary conditions on the left and bottom. Then the quenched crossing probability is manifestly exactly one half and the thermodynamic limit can be taken, which gets us to our unique state, and we conclude that the probability in the limiting state is one half. □

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