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Invariant sets of degenerate Hamiltonian systems near equilibria

For any collection of $n \geq 2$ numbers $\omega_1, \ldots, \omega_n$, we prove the existence of an infinitely differentiable Hamiltonian system of differential equations X with n degrees of freedom that possesses the following properties: 1) 0 is an elliptic (provided that all the ω_i are different from zero) equilibrium of system X with eigenfrequencies $\omega_1, \ldots, \omega_n$; 2) system X is linear up to a remainder flat at 0; 3) the measure of the union of the invariant n-tori of system X that lie in the ε-neighborhood of 0 tends to zero as $\varepsilon \to 0$ faster than any prescribed function. Analogous statements hold for symplectic diffeomorphisms, reversible flows, and reversible diffeomorphisms. The results obtained are discussed in the context of the standard theorems in the KAM theory, the well-known Rüssmann and Anosov–Katok theorems, and a recent theorem by Herman.

1. Formulation of the results and a discussion

According to the KAM (Kolmogorov–Arnol′d–Moser) theory, any neighborhood of an elliptic equilibrium 0 of a sufficiently smooth Hamiltonian system of differential equations in $\mathbf{R}^{2n}$ and any neighborhood of an elliptic fixed point 0 of a sufficiently smooth symplectic diffeomorphism $(\mathbf{R}^{2n}, 0) \to (\mathbf{R}^{2n}, 0)$ contain invariant Lagrangian n-tori carrying quasiperiodic motions, provided that some nonresonance and nondegeneracy conditions are met [1]–[4] (see also a detailed discussion in [5]–[12]). One usually formulates the nondegeneracy condition in terms of the coefficients of the Birkhoff normal form of degree 4. For instance, in the case of differential equations, the Birkhoff normal form of the Hamilton function of degree 4 is (up to an additive constant)

$$H = \sum_{i=1}^{n} \omega_i \tau_i + \frac{1}{2} \sum_{i,j=1}^{n} \omega_{ij} \tau_i \tau_j + O\left(|\tau|^{5/2}\right), \qquad \omega_{ij} = \omega_{ji}, \tag{1}$$

where $\tau_i = \frac{1}{2}(p_i^2 + q_i^2)$ are the local action variables ($1 \leq i \leq n$), $|\tau| = \tau_1 + \cdots + \tau_n$, and (p, q) are suitable Darboux coordinates with the origin at 0. The nondegeneracy condition in this case is that at least one of the two inequalities

$$\det |\omega_{ij}| \neq 0, \qquad \det \begin{vmatrix} \omega_{ij} & \omega_i \\ \omega_j & 0 \end{vmatrix} \neq 0 \tag{2}$$

holds. The nonresonance condition consists in the possibility itself of the reduction of the Hamilton function to normal form (1), i.e., in the absence of resonances of order 4 and lower among the eigenfrequencies ω_i. The invariant KAM tori are close to the tori $\tau = \text{const}$, and the *relative* Lebesgue measure μ_ε of their union in the polydisc $B_\varepsilon = \left\{\max_{i=1}^n \tau_i \leq \frac{1}{2}\varepsilon^2\right\}$ of radius $\varepsilon > 0$ centered at 0 tends to 1 as $\varepsilon \to 0$.

Analogous results hold for reversible differential equations and mappings (see [9, 12, 13] and references therein).

Various estimates of the decay rate for $1 - \mu_\varepsilon$ as $\varepsilon \to 0$ are known. If the first of the inequalities (2) is valid and there are no resonances of order $l \geq 4$ and lower among the eigenfrequencies ω_i then [14]

$$1 - \mu_\varepsilon = O\left(\varepsilon^{(l-3)/2}\right).$$

For Diophantine collections of eigenfrequencies:

$$|\langle k, \omega\rangle| \geq \gamma |k|^{-\nu} \quad \text{for all} \quad k \in \mathbf{Z}^n \setminus \{0\}$$

(here $\langle k, \omega\rangle = k_1\omega_1 + \cdots + k_n\omega_n$, $|k| = |k_1| + \cdots + |k_n|$, while ν and γ are positive constants independent of k), an exponential estimate of the difference $1 - \mu_\varepsilon$:

$$1 - \mu_\varepsilon = O\left(\exp\left\{-\left[\frac{c}{\varepsilon}\right]^{1/(\nu+1)}\right\}\right), \qquad c = \text{const} > 0 \text{ is independent of } \varepsilon,$$

is obtained in paper [15] under the conditions that the Hamilton function is analytic and the second of the inequalities (2) holds. In paper [16], a similar estimate with exponent $2/(\nu+1)$ in place of $1/(\nu+1)$ is proven under the conditions that the Hamilton function is analytic and the first of the inequalities (2) is valid.

The nondegeneracy requirement for the Birkhoff normal form of degree 4 can be relaxed. For example, consider an area preserving diffeomorphism of the plane with elliptic fixed point 0. If the eigenvalues of its linearization at 0 are not roots of unity of degree $l \geq 4$ and lower, then this diffeomorphism is reduced in some Darboux coordinates (p, q) to the Birkhoff normal form of degree l

$$z \mapsto z \exp\left[i\left(\omega_{(0)} + \omega_{(1)}\tau + \cdots + \omega_{(s)}\tau^s\right)\right] + O\left(|z|^l\right),$$

where $z = p + iq$, $\tau = \frac{1}{2}|z|^2 = \frac{1}{2}(p^2 + q^2)$, and $s = [l/2] - 1$. For the existence of invariant KAM curves in any neighborhood of 0, it suffices that at least one of the real coefficients $\omega_{(1)}, \ldots, \omega_{(s)}$ does not vanish [1]–[4].

Nevertheless, one cannot abandon completely the nondegeneracy condition in the local KAM theory, whatever the eigenvalues of the linearization at 0 of the Hamiltonian vector field or symplectic mapping are. This statement follows from theorems 1 and 2 below. It is discussing and proving those theorems that the present note is devoted to.

Before formulating theorems 1 and 2, introduce some notation (which in fact was already used partially). By $|a| = |a_1| + \cdots + |a_N|$, we will always denote the l_1-norm of a vector $a \in \mathbf{R}^N$, and by $\langle a, b\rangle = a_1 b_1 + \cdots + a_N b_N$, the scalar product of two vectors $a, b \in \mathbf{R}^N$. Consider the Euclidean space $\mathbf{R}^{2n}$ ($n \geq 1$) endowed with coordinates $(p_1, \ldots, p_n, q_1, \ldots, q_n)$ and symplectic structure $dp \wedge dq$. In the sequel, we will always have in view this structure while speaking of Hamiltonian flows and symplectic mappings. Set

$$p_i = \sqrt{2\tau_i}\cos\varphi_i, \quad q_i = \sqrt{2\tau_i}\sin\varphi_i, \qquad 1 \leq i \leq n,$$

then $dp \wedge dq = d\tau \wedge d\varphi$. By $P : \mathbf{R}^{2n} \to \mathbf{R}^n_+$, denote the "projection" $P : (p, q) \mapsto \tau$. Fix a closed n-dimensional cube

$$T_{\rho^2/2} = \left\{\tau \in \mathbf{R}^n \;\middle|\; 0 \leq \tau_i \leq \tfrac{1}{2}\rho^2, \;\; 1 \leq i \leq n\right\}$$

with edge length $\rho^2/2$ ($\rho > 0$). Then

$$B_\rho = P^{-1}(T_{\rho^2/2})$$

is a $2n$-dimensional polydisc of radius ρ centered at 0.

Let $\{K_m\}_{m\geq 1}$ be an arbitrary sequence of pairwise disjoint closed subsets in $T_{\rho^2/2}$ possessing the following properties:

$$K_m \subset \operatorname{Int} T_{\rho^2/2} = \left\{\tau \in \mathbf{R}^n \;\middle|\; 0 < \tau_i < \tfrac{1}{2}\rho^2, \;\; 1 \leq i \leq n\right\} \quad \text{for all} \quad m$$

and

$$\max_{\tau \in K_m} |\tau| \to 0 \quad (m \to +\infty). \tag{3}$$

A subset of the space $\mathbf{R}^{2n}$ will be said to be *nonexceptional*, if for any $k \in \mathbf{Z}^n \setminus \{0\}$, this subset contains a point (τ, φ) such that $\tau_1 > 0, \ldots, \tau_n > 0$, $\sin\langle k, \varphi\rangle \neq 0$. Nonexceptional sets are exemplified by n-tori $\tau = f(\varphi)$ for which $f_i(\varphi) > 0$ for all i and φ.

THEOREM 1. *Let $n \geq 2$ and let a Hamilton function $H_0 : B_\rho \to \mathbf{R}$ depend on the local action variables τ only:*

$$H_0(p, q) = F(\tau),$$

where $F \in C^\infty(T_{\rho^2/2}, \mathbf{R})$. If the image of the frequency mapping

$$\Omega : T_{\rho^2/2} \to \mathbf{R}^n, \qquad \Omega(\tau) = \partial F(\tau)/\partial\tau \tag{4}$$

lies in some hyperplane in $\mathbf{R}^n$ *passing through* 0*, then there exists a* C^∞*-function* $H : B_\rho \to \mathbf{R}$ *possessing the following properties:*

i) *the difference* $H(p,q) - H_0(p,q)$ *is flat at* 0*;*

ii) *the Hamiltonian system* X_H *with Hamilton function* H *does not admit nonexceptional invariant sets lying wholly in* $\bigcup_{m=1}^{\infty} P^{-1}(K_m)$.

If function F *is linear* (*i.e.,* $\Omega = \mathrm{const}$ *and the system* X_{H_0} *with Hamilton function* H_0 *is linear*)*, then property* ii) *can be replaced with the following one:*

ii′) *system* X_H *does not admit invariant* n*-tori of the form* $\tau = f(\varphi)$ *that intersect* $\mathrm{Int}\,[\,\bigcup_{m=1}^{\infty} P^{-1}(K_m)\,]$.

THEOREM 2. *Let* $A_0 : B_\rho \to B_\rho$ *be a symplectic diffeomorphism of the form*

$$A_0 : (\tau,\varphi) \mapsto (\tau, \varphi + \Omega(\tau)), \qquad \Omega(\tau) = \partial F(\tau)/\partial\tau,$$

where $F \in C^\infty(T_{\rho^2/2}, \mathbf{R})$*. If the image of the frequency mapping* $\Omega : T_{\rho^2/2} \to \mathbf{R}^n$ *lies in some affine hyperplane*[1] *in* $\mathbf{R}^n$*, then there exists a symplectic* C^∞*-diffeomorphism* $A : B_\rho \to B_\rho$ *possessing the following properties:*

i) *the difference* $A(p,q) - A_0(p,q)$ *is flat at* 0*;*

ii) *diffeomorphism* A *does not admit nonexceptional invariant sets lying wholly in* $\bigcup_{m=1}^{\infty} P^{-1}(K_m)$.

If function F *is linear* (*i.e.,* $\Omega = \mathrm{const}$ *and diffeomorphism* A_0 *is linear*)*, then property* ii) *can be replaced with the following one:*

ii′) *diffeomorphism* A *does not admit invariant* n*-tori of the form* $\tau = f(\varphi)$ *that intersect* $\mathrm{Int}\,[\,\bigcup_{m=1}^{\infty} P^{-1}(K_m)\,]$.

These theorems are proven in section 2.

REMARK 1. Recall that a C^∞-function defined in a neighborhood of a point $a \in \mathbf{R}^N$ is said to be *flat* at a, if all the derivatives of this function of all the orders $l \geq 0$ vanish at a.

REMARK 2. The statement of theorem 1 holds for $n = 1$ as well, but we have excluded the case $n = 1$ because of its triviality. Indeed, for $n = 1$ the hypothesis of the theorem means that $\Omega \equiv 0$, i.e., $H_0 = \mathrm{const}$ (one can set $H_0 \equiv 0$). Let $\chi : \mathbf{R} \to \mathbf{R}$ be an arbitrary C^∞-function flat at 0 and such that $d\chi/dx \neq 0$ for $x \neq 0$. Then the only bounded solutions of the system governed by the Hamilton function $H(p,q) = \chi(pq)$ are equilibria filling up the coordinate axes $\{p = 0\}$ and $\{q = 0\}$.

REMARK 3. The requirement that the image of the unperturbed frequency mapping should not lie in any hyperplane (the so called *Rüssmann condition* [17]–[19]) is the optimal

[1] i.e., in a hyperplane not necessarily passing through 0

nondegeneracy condition in the global KAM theory for analytic systems [12, 20]. For the last several years, this condition and other close requirements have become more or less standard, see [12], [17]–[27] (an extensive bibliography is given in [12]).

REMARK 4. Theorems 1 and 2 guarantee the absence of *any* invariant tori of the form $\tau = f(\varphi)$ with the properties indicated, not only that of tori carrying quasiperiodic motions (and even not only that of Lagrangian tori).

REMARK 5. If the linear part $\langle \omega, \tau \rangle$ of function $F(\tau)$ in theorem 1 satisfies some non-resonance conditions, then system X_H possesses n invariant two-dimensional C^∞-surfaces passing through 0 and foliated into closed trajectories with periods close to $2\pi/|\omega_1|, \dots,$ $2\pi/|\omega_n|$ (the Lyapunov center theorem, see, e.g., [4, 8, 28]).

One can choose the sequence of sets $\{K_m\}_{m\geq 1}$ in such a way that the measure of the difference

$$B_\varepsilon \setminus \left[\bigcup_{m=1}^{\infty} P^{-1}(K_m) \right] \tag{5}$$

will decrease as $\varepsilon \to 0$ faster than any prescribed function. Moreover, within this framework, each of the sets K_m can be chosen to be a polyhedron. Indeed, let $\chi : [0; \rho] \to \mathbf{R}_+$ be an arbitrary monotonically increasing function such that $\chi(0) = 0$ and $\chi(\varepsilon) > 0$ for $\varepsilon > 0$. Introduce the notation

$$x_m = \frac{\rho}{m}, \qquad y_m = \chi\left(\frac{\rho}{m+1}\right).$$

Fix a polyhedron

$$K_m \subset \left(\operatorname{Int} T_{x_m^2/2} \setminus T_{x_{m+1}^2/2}\right)$$

such that the measure of the difference

$$B_{x_m} \setminus \left[B_{x_{m+1}} \bigcup P^{-1}(K_m)\right]$$

is less than

$$\min_{r=1}^{m} \frac{y_r}{2^{m-r+1}}.$$

Then for any $0 < \varepsilon \leq \rho$ ($x_{s+1} < \varepsilon \leq x_s$ for some $s \geq 1$) the measure of difference (5) will be less than

$$\sum_{m=s}^{\infty} \min_{r=1}^{m} \frac{y_r}{2^{m-r+1}} \leq y_s \sum_{m=s}^{\infty} \frac{1}{2^{m-s+1}} = y_s \leq \chi(\varepsilon).$$

Taking this into account, we arrive at the following corollaries of theorems 1 and 2.

COROLLARY 1. *Let $n \geq 2$ and let 0 be an elliptic (if all the ω_i are different from zero) equilibrium of a linear Hamiltonian system in $\mathbf{R}^{2n}$ with quadratic Hamilton function $H_0 = \langle \omega, \tau \rangle$. Then there exists a Hamilton C^∞-function H such that*

i) *the difference* $H(p,q) - H_0(p,q)$ *is flat at* 0;

ii) *the measure of the union of the* n*-tori of the form* $\tau = f(\varphi)$ *invariant under the system with Hamilton function* H *and lying in the polydisc* B_ε *of radius* $\varepsilon > 0$ *centered at* 0 *tends to zero as* $\varepsilon \to 0$ *faster than any prescribed function.*

COROLLARY 2. *Let* $A_0 : \mathbf{R}^{2n} \to \mathbf{R}^{2n}$ *be a linear symplectic diffeomorphism* $(\tau, \varphi) \mapsto (\tau, \varphi + \omega)$ *with elliptic (if none of the ratios* ω_i/π *is an integer) fixed point* 0. *There exists a symplectic* C^∞*-diffeomorphism* A *such that*

i) *the difference* $A(p,q) - A_0(p,q)$ *is flat at* 0;

ii) *the measure of the union of the* n*-tori of the form* $\tau = f(\varphi)$ *invariant under diffeomorphism* A *and lying in the polydisc* B_ε *of radius* $\varepsilon > 0$ *centered at* 0 *tends to zero as* $\varepsilon \to 0$ *faster than any prescribed function.*

It is useful to compare these corollaries with the following three statements.

THE RÜSSMANN THEOREM [29]. *Let* $A : (\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$ *be an area preserving analytic mapping of the plane into itself for which* 0 *is an elliptic fixed point. Let the eigenvalues of the linearization of mapping* A *at* 0 *be equal to* $\exp(\pm i\alpha)$, *where angle* α *is strongly incommensurable with* 2π, *i.e., there exist positive constants* ν *and* γ *such that*

$$|s\alpha - 2\pi r| \geq \gamma s^{-\nu} \quad \text{for all} \quad s \in \mathbf{N}, \quad r \in \mathbf{Z}.$$

Suppose that by a formal canonical transformation, one can reduce mapping A *to its linear part, i.e., the rotation by angle* α. *Then this mapping is reducible to its linear part by a convergent canonical transformation as well. An analogous (and even a stronger) statement holds for analytic Hamiltonian systems of differential equations as well (with any number* n *of degrees of freedom). Namely, if the collection of the eigenfrequencies* $\omega_1, \ldots, \omega_n$ *of the system is Diophantine and the formal Birkhoff normal form of the Hamilton function is a function of its linear part* $\langle \omega, \tau \rangle$ *(e.g., coincides with* $\langle \omega, \tau \rangle$*), then one can reduce the Hamilton function to this normal form by a convergent canonical transformation.*

REMARK 6. In paper [30, § 12], various generalizations (pertaining to the case of differential equations) of the Rüssmann theorem are obtained.

THE HERMAN THEOREM [31]. *Let* $A : (\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$ *be an area preserving* C^∞*-mapping of the plane into itself for which* 0 *is an elliptic fixed point. Let the eigenvalues of the linearization of mapping* A *at* 0 *be equal to* $\exp(\pm i\alpha)$, *where angle* α *is strongly incommensurable with* 2π. *Then in any neighborhood of point* 0, *mapping* A *possesses invariant circles*[2] *whose union is of positive measure. An analogous statement holds for*

[2]i.e., invariant closed curves surrounding 0

Hamiltonian C^∞-systems of differential equations with two degrees of freedom as well.

A PARTICULAR CASE OF THE ANOSOV–KATOK THEOREM [32, 33]. *In any neighborhood (in the C^∞-topology) of the rotation of a two-dimensional disc by any angle α, there are area preserving C^∞-diffeomorphisms of this disc that are ergodic.*

We arrive at the following description of invariant curves of symplectic diffeomorphisms of the plane. Let $A : (\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$ be an area preserving C^∞-mapping of the plane into itself, and let the linear part at 0 of this mapping be the rotation by angle α strongly incommensurable with 2π. Consider the formal Birkhoff normal form of mapping A at 0:

$$z \mapsto z \exp\left[i\left(\alpha + \sum_{s=1}^{\infty} \omega_{(s)}\tau^s\right)\right], \qquad z = p + iq, \quad \tau = \tfrac{1}{2}|z|^2. \tag{6}$$

We are interested in the relative measure μ_ε of the union of invariant circles of mapping A in the ε-neighborhood of fixed point 0. If at least one of the coefficients $\omega_{(s)}$ in (6), $s \geq 1$, is different from zero then $\mu_\varepsilon \to 1$ as $\varepsilon \to 0$ according to the standard KAM theory. On the other hand, if $\omega_{(s)} = 0$ for all s then still $\mu_\varepsilon > 0$ for each $\varepsilon > 0$ (by the Herman theorem), but μ_ε can *decrease* as $\varepsilon \to 0$, and faster than any prescribed function (according to corollary 2). In both the cases, fixed point 0 of mapping A is Lyapunov stable. Finally, if $\omega_{(s)} = 0$ for all s but A is analytic then by the Rüssmann theorem, $\mu_\varepsilon \equiv 1$ (to be more precise, some punctured neighborhood of point 0 is wholly foliated into invariant circles of mapping A).

The linear parts of Anosov–Katok ergodic diffeomorphisms are rotations by angles α that are not strongly incommensurable with 2π.

What can one say about area preserving C^∞-mappings $A : (\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$ such that the eigenvalues $\exp(\pm i\alpha)$ of the linearization of A at 0 are roots of unity (the case opposite to the strong incommensurability of α with 2π)? No analogue of the Herman theorem exists for such mappings because for $\alpha = 2\pi r/s$ ($s \in \mathbf{N}$, $r \in \mathbf{Z}$, $2r/s \notin \mathbf{Z}$) fixed point 0 can be unstable even for analytic mappings A [4]. On the other hand, the Rüssmann theorem is carried over to the case $\alpha = 2\pi r/s$. If α is commensurable with 2π then the Birkhoff normal form[3] of mapping A is, generally speaking, more complicated than (6). However, exploiting the methods developed in [30, 34] one can easily show that if an area preserving analytic mapping A with $\alpha = 2\pi r/s$ is formally linearizable then it is linearizable by a convergent canonical transformation as well [35]. In Gong's paper [36] devoted to the normalization of some special reversible diffeomorphisms characterized by a parameter $\lambda \in \mathbf{C}$, $|\lambda| = 1$, the case where $\arg\lambda$ is strongly incommensurable with 2π and the case where $\lambda^s = 1$ are

[3] to be more precise, the Birkhoff–Gustavson normal form, see [8]

treated in parallel.

REMARK 7. Other examples of situations where flat perturbations of Hamiltonian C^∞-flows or symplectic C^∞-diffeomorphisms change the dynamics drastically can be found in paper [37].

REMARK 8. Let A_ε be the mapping of a two-dimensional annulus of the form $A_\varepsilon : (\tau, \varphi) \mapsto (\tau, \varphi + \varepsilon\tau)$, where $0 < \varepsilon \ll 1$. An interesting criterion for the absence of invariant curves of the form $\tau = f(\varphi)$ for perturbations of the mappings A_ε is obtained in recent paper [38].

2. Proofs

In the global KAM theory, the analogue of theorems 1 and 2 is the following proposition: if the image of the unperturbed frequency mapping lies in some hyperplane (passing through 0 in the case of flows) then one can destroy all the invariant tori by an arbitrarily small perturbation of the initial integrable system. This statement is proven in [12, 20] for Hamiltonian flows and in [20] for symplectic diffeomorphisms. By some complication of the constructions of papers [12, 20], one may prove local theorems 1 and 2 of the present note.

We will prove in detail theorem 1 only.

PROOF OF THEOREM 1. Due to condition (3), the sets K_m, $m \geq 1$, admit pairwise disjoint neighborhoods. We can therefore choose two other sequences of closed subsets $\{S_m\}_{m\geq 1}$ and $\{Q_m\}_{m\geq 1}$ in $T_{\rho^2/2}$ possessing the following properties:

a) $K_m \subset \operatorname{Int} S_m$, $S_m \subset \operatorname{Int} Q_m$, $Q_m \subset \operatorname{Int} T_{\rho^2/2}$;
b) the sets Q_m, $m \geq 1$ are pairwise disjoint;
c) $\max_{\tau \in Q_m} |\tau| \to 0$ $(m \to +\infty)$.

The condition that the image of mapping (4) lies in some hyperplane in $\mathbf{R}^n$ passing through 0 means that $\langle c, \Omega(\tau)\rangle \equiv 0$ for some $c \in \mathbf{R}^n \setminus \{0\}$.

A new Hamilton function $H(p, q)$ will be looked for in the form

$$H = H_0 + \sum_{m=1}^{\infty} h_m,$$

where each C^∞-function $h_m : B_\rho \to \mathbf{R}$ vanishes outside $P^{-1}(Q_m)$. Fix $m \in \mathbf{N}$ and a C^∞-function $\psi : B_\rho \to \mathbf{R}$ equal to unit in $P^{-1}(S_m)$ and equal to zero outside $P^{-1}(Q_m)$; it suffices to consider a partition of unity ψ, Ψ of class C^∞ subordinate to the covering $P^{-1}(\operatorname{Int} Q_m)$, $\mathbf{R}^{2n} \setminus P^{-1}(S_m)$ of space $\mathbf{R}^{2n}$ [39]. Choose a matrix $M \in \mathrm{GL}(n, \mathbf{R})$ sufficiently close to the identity matrix (in particular, subject to the condition $M^{-1}Q_m \subset T_{\rho^2/2}$) and such that Mc is proportional to some integer vector $k \in \mathbf{Z}^n \setminus \{0\}$. Set

$$\Omega_{\mathrm{new}}(\tau) = \frac{\partial}{\partial\tau}F(M^{-1}\tau) = (M^{-1})^{\mathrm{t}}\Omega(M^{-1}\tau) \tag{7}$$

(the superscript t means transposing), then $\langle k, \Omega_{\mathrm{new}}(\tau)\rangle \equiv 0$. We will look for function h_m in the form

$$h_m = [\ F(M^{-1}\tau) - F(\tau) + \delta\cos\langle k, \varphi\rangle\ ]\psi, \tag{8}$$

where $\delta \in \mathbf{R} \setminus \{0\}$. Having chosen matrix M to be sufficiently close to the identity matrix and then number $\delta \neq 0$ to be sufficiently small, one can satisfy the inequalities

$$\left|\frac{\partial^{|\kappa|}h_m(p,q)}{\partial p_1^{\kappa_1}\cdots\partial p_n^{\kappa_n}\partial q_1^{\kappa_{n+1}}\cdots\partial q_n^{\kappa_{2n}}}\right| \leq \frac{1}{m} \quad \text{for all} \quad \kappa \in \mathbf{Z}_+^{2n}, \ \ 0 \leq |\kappa| \leq m \tag{9}$$

in $P^{-1}(Q_m)$. These inequalities imply that the function $\sum_{m=1}^{\infty} h_m$ is flat at 0.

In $P^{-1}(S_m)$, the Hamilton function $H_0 + h_m$ is equal to $F(M^{-1}\tau) + \delta\cos\langle k, \varphi\rangle$ and affords the system of differential equations

$$\dot{\tau} = \delta k \sin\langle k, \varphi\rangle, \qquad \dot{\varphi} = \Omega_{\mathrm{new}}(\tau),$$

for which

$$\frac{d}{dt}\langle k, \varphi\rangle = \langle k, \Omega_{\mathrm{new}}(\tau)\rangle \equiv 0.$$

Suppose that the system with Hamilton function $H_0 + h_m$ admits a nonexceptional invariant set lying wholly in $P^{-1}(K_m)$. On that set, choose a point (τ^0, φ^0) for which $\sin\langle k, \varphi^0\rangle \neq 0$. Along the trajectory passing through this point, one has

$$\dot{\tau} = \delta k \sin\langle k, \varphi^0\rangle = \mathrm{const} \neq 0.$$

The contradiction obtained proves the first part of theorem 1. Note that we nowhere used that $\psi \equiv 1$ in $P^{-1}(S_m)$; it sufficed for our purposes that $\psi \equiv 1$ in $P^{-1}(K_m)$.

Let now the function $F(\tau) = \langle\omega, \tau\rangle$ be linear, so that $\Omega = \mathrm{const} = \omega$. Choose a sufficiently small vector $\vartheta \in \mathbf{R}^n$ such that the vector $\omega + \vartheta$ is proportional to an integer one: $\omega + \vartheta = \Lambda L$ where $\Lambda > 0$ and $L \in \mathbf{Z}^n$. Let $\langle k, L\rangle = 0$ for some integer vector $k \in \mathbf{Z}^n \setminus \{0\}$ (recall that $n \geq 2$). Then $\langle k, \omega + \vartheta\rangle = 0$. We will look for function h_m in the form

$$h_m = [\ \langle\vartheta, \tau\rangle + \delta\cos\langle k, \varphi\rangle\ ]\psi, \tag{10}$$

where $\delta \in \mathbf{R} \setminus \{0\}$. Having chosen vector ϑ and number $\delta \neq 0$ to be sufficiently small, one can satisfy the inequalities (9). Fix a number $\sigma > 0$ such that $\tau' \in S_m$ whenever $\tau \in K_m$

and $|\tau' - \tau| \le \sigma$, and impose the additional condition that

$$|\delta| \le \frac{\Lambda\sigma}{2\pi|k|}. \tag{11}$$

In $P^{-1}(S_m)$, the Hamilton function $H_0 + h_m$ is equal to $\langle \omega + \vartheta, \tau \rangle + \delta \cos\langle k, \varphi \rangle$ and affords the system of differential equations

$$\dot{\tau} = \delta k \sin\langle k, \varphi \rangle, \qquad \dot{\varphi} = \omega + \vartheta,$$

for which

$$\frac{d}{dt}\langle k, \varphi \rangle = \langle k, \omega + \vartheta \rangle \equiv 0.$$

Suppose that the system with Hamilton function $H_0 + h_m$ admits an invariant torus of the form $\tau = f(\varphi)$ intersecting $\operatorname{Int} P^{-1}(K_m)$. On this torus, choose a point $(f(\varphi^0), \varphi^0)$ inside $P^{-1}(K_m)$ for which $\sin\langle k, \varphi^0 \rangle \ne 0$. Due to inequality (11), the trajectory $(\tau(t), \varphi(t))$ starting at this point remains in $P^{-1}(S_m)$ for $|t| \le 2\pi/\Lambda$. One has:

$$\begin{aligned} &\varphi(0) = \varphi^0, && \varphi(2\pi/\Lambda) = \varphi^0 + 2\pi L = \varphi^0 \bmod 2\pi, \\ &\tau(0) = f(\varphi^0), && \tau(2\pi/\Lambda) = f(\varphi^0) + 2\pi\Lambda^{-1}\delta k \sin\langle k, \varphi^0 \rangle \ne f(\varphi^0). \end{aligned}$$

The contradiction obtained proves the second part of theorem 1.

Proof of theorem 2 runs entirely similarly to the proof of theorem 1, and we will point out just the necessary changes in our construction. The condition that the image of mapping (4) lies in some affine hyperplane in $\mathbf{R}^n$ means that $\langle c, \Omega(\tau) \rangle \equiv c_0$ for some $c \in \mathbf{R}^n \setminus \{0\}$, $c_0 \in \mathbf{R}$. For a fixed $m \ge 1$, choose a matrix $M \in \mathrm{GL}(n, \mathbf{R})$ sufficiently close to the identity matrix and such that the pair (Mc, c_0) is proportional to some pair $(k, 2\pi k_0)$ with $k \in \mathbf{Z}^n \setminus \{0\}$, $k_0 \in \mathbf{Z}$. Introduce the mapping $\Omega_{\text{new}}(\tau)$ by formula (7) and the function h_m by formula (8). Then $\langle k, \Omega_{\text{new}}(\tau) \rangle \equiv 2\pi k_0$. The desired symplectic diffeomorphism A is determined by the generating function $F + \sum_{m=1}^{\infty} h_m$, i.e., in $P^{-1}(Q_m)$

$$A : (\tau, \varphi) \mapsto (\tau', \varphi'),$$

where

$$\tau = \tau' + \frac{\partial h_m(\tau', \varphi)}{\partial \varphi}, \qquad \varphi' = \varphi + \Omega(\tau') + \frac{\partial h_m(\tau', \varphi)}{\partial \tau'}.$$

One can prove by the standard methods [3, 40] that A is well defined for sufficiently small h_m. In particular, for $\tau' \in S_m$ we have

$$\tau' = \tau + \delta k \sin\langle k, \varphi \rangle, \qquad \varphi' = \varphi + \Omega_{\text{new}}(\tau'),$$

and

$$\langle k, \varphi' \rangle = \langle k, \varphi \rangle + \langle k, \Omega_{\mathrm{new}}(\tau') \rangle = \langle k, \varphi \rangle + 2\pi k_0 = \langle k, \varphi \rangle \bmod 2\pi.$$

One easily sees that A cannot admit nonexceptional invariant sets lying wholly in $P^{-1}(K_m)$.

Let now the function $F(\tau) = \langle \omega, \tau \rangle$ be linear, so that $\Omega = \mathrm{const} = \omega$. Choose a sufficiently small vector $\vartheta \in \mathbf{R}^n$ such that the vector $\omega + \vartheta$ has the form $\omega + \vartheta = 2\pi L / L_0$ with $L \in \mathbf{Z}^n$ and $L_0 \in \mathbf{N}$. Let $\langle k, L \rangle = L_0 k_0$ for some $k \in \mathbf{Z}^n \setminus \{0\}$ and $k_0 \in \mathbf{Z}$. Then $\langle k, \omega + \vartheta \rangle = 2\pi k_0$. We will look for function h_m in the form (10) imposing the additional condition

$$|\delta| \le \frac{\sigma}{L_0 |k|}. \tag{12}$$

In $A^{-1}[P^{-1}(S_m)]$, the mapping $A : (\tau, \varphi) \mapsto (\tau', \varphi')$ has the form

$$\tau' = \tau + \delta k \sin\langle k, \varphi \rangle, \qquad \varphi' = \varphi + \omega + \vartheta.$$

If a point (τ^0, φ^0) lies in $P^{-1}(K_m)$ then its image (τ^*, φ^*) under the L_0-th iteration A^{L_0} of diffeomorphism A lies in $P^{-1}(S_m)$ due to inequality (12) and

$$\tau^* = \tau + L_0 \delta k \sin\langle k, \varphi^0 \rangle, \qquad \varphi^* = \varphi^0 + 2\pi L = \varphi^0 \bmod 2\pi.$$

Hence, A cannot admit invariant tori of the form $\tau = f(\varphi)$ that intersect $\operatorname{Int} P^{-1}(K_m)$.

Remark 9. For Lagrangian tori of the form $\tau = f(\varphi)$ invariant under a symplectic diffeomorphism, Herman has obtained, in some cases, estimates of the derivatives $\partial f_i / \partial \varphi_j$ in terms of the entries of the Jacobi matrix of the diffeomorphism [41, 42]. The dynamics on such tori was studied by Herman in papers [43, 44].

Theorems 1 and 2 (as well as corollaries 1 and 2) can be carried over to reversible differential equations and reversible diffeomorphisms. The reversing involution is $G : (p, q) \mapsto (p, -q)$, or $G : (\tau, \varphi) \mapsto (\tau, -\varphi)$. We will not formulate the reversible counterparts of theorems 1 and 2 here and will just point out briefly the key changes in the proofs. In the sequel, as before, M denotes a matrix in $\mathrm{GL}(n, \mathbf{R})$ close to the identity matrix.

In the case of flows, one has a G-reversible system of differential equations

$$\dot{\tau} = 0, \qquad \dot{\varphi} = \Omega(\tau),$$

where now the mapping $\Omega : T_{\rho^2/2} \to \mathbf{R}^n$ is, generally speaking, not a gradient. Let $\langle c, \Omega(\tau) \rangle \equiv 0$ for some $c \in \mathbf{R}^n \setminus \{0\}$, and let Mc be proportional to $k \in \mathbf{Z}^n \setminus \{0\}$. Set

$$\Omega_{\mathrm{new}}(\tau) = (M^{-1})^{\mathrm{t}} \Omega(\tau), \qquad \langle k, \Omega_{\mathrm{new}}(\tau) \rangle \equiv 0.$$

The perturbed G-reversible system of differential equations in $P^{-1}(Q_m)$ has the form

$$\dot{\tau} = \eta[\, \sin\langle k, \varphi\rangle \,]\psi, \qquad \dot{\varphi} = \Omega(\tau) + [\, \Omega_{\rm new}(\tau) - \Omega(\tau) \,]\psi,$$

where $\eta \in \mathbf{R}^n \setminus \{0\}$ is small while a buffer function $\psi(\tau, \varphi)$ is even in φ. In $P^{-1}(S_m)$, this system takes the form

$$\dot{\tau} = \eta \sin\langle k, \varphi\rangle, \qquad \dot{\varphi} = \Omega_{\rm new}(\tau).$$

In the case of mappings, one has a G-reversible diffeomorphism

$$A_0 : (\tau, \varphi) \mapsto (\tau, \varphi + \Omega(\tau)).$$

Let $\langle c, \Omega(\tau)\rangle \equiv c_0$ for $c \in \mathbf{R}^n \setminus \{0\}$, $c_0 \in \mathbf{R}$, and let the pair (Mc, c_0) be proportional to a pair $(k, 2\pi k_0)$ with $k \in \mathbf{Z}^n \setminus \{0\}$, $k_0 \in \mathbf{Z}$. Set

$$\Omega_{\rm new}(\tau) = (M^{-1})^{\rm t}\Omega(\tau), \qquad \langle k, \Omega_{\rm new}(\tau)\rangle \equiv 2\pi k_0.$$

The perturbed G-reversible diffeomorphism A in $P^{-1}(Q_m)$ is determined as

$$A = GWGW^{-1},$$

where the diffeomorphism $W : (\tau, \varphi) \mapsto (\tilde{\tau}, \tilde{\varphi})$ has the form

$$\begin{aligned} \tilde{\tau} &= \tau - \tfrac{1}{2}\eta[\, \sin\langle k, \tilde{\varphi}\rangle \,]\psi(\tau, \tilde{\varphi}), \\ \tilde{\varphi} &= \varphi - \tfrac{1}{2}\Omega(\tau) - \tfrac{1}{2}[\, \Omega_{\rm new}(\tau) - \Omega(\tau) \,]\psi(\tau, \varphi) \end{aligned}$$

with small $\eta \in \mathbf{R}^n \setminus \{0\}$. It is not hard to verify that for $\psi \equiv 1$, the diffeomorphism $A : (\tau, \varphi) \mapsto (\tau', \varphi')$ takes the form

$$\tau' = \tau + \eta \sin\langle k, \varphi\rangle, \qquad \varphi' = \varphi + \Omega_{\rm new}\left(\tfrac{1}{2}(\tau + \tau')\right).$$

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