SHARP CONSTANTS IN THE HARDY-RELLICH INEQUALITIES

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Abstract

Optimal constants are found in Hardy-Rellich inequalities containing derivatives of arbitrary (not necessarily integer) order l. Some new inequalities of this type are also obtained.

1. INTRODUCTION

Our goal is to find optimal constants in the two following Hardy-Rellich inequalities. If $l < d/2$, then

$$\int_{\mathbb{R}^d} |x|^{-2l} |u(x)|^2 dx \leq C_l \int_{\mathbb{R}^d} |\xi|^{2l} |\hat{u}(\xi)|^2 d\xi. \tag{1.1}$$

If

$$l > d/2, \quad l - d/2 \notin \mathbb{Z} \quad \text{and} \quad n = [l - d/2], \tag{1.2}$$

then

$$\int_{\mathbb{R}^d} |x|^{-2l} |u(x) - \sum_{|\alpha| \leq n} (\alpha!)^{-1} (\partial^\alpha u)(0) x^\alpha|^2 dx \leq C_l \int_{\mathbb{R}^d} |\xi|^{2l} |\hat{u}(\xi)|^2 d\xi. \tag{1.3}$$

Here $\hat{u} = \Phi u$ is of course the Fourier transform of a function u. Note that both inequalities (1.1) and (1.3) are invariant with respect to dilations $u(x) \mapsto u_\kappa(x) = u(\kappa x)$, $\kappa > 0$. Clearly, (1.1) and (1.3) coincide if, say, $u \in C_0^\infty(\mathbb{R}^d \setminus \{0\})$.

We show that inequalities (1.1) and (1.3) hold at least on the Sobolev class $H^l(\mathbb{R}^d)$ with the estimating constants

$$C_l = 2^{-2l} \max\{ \frac{\Gamma^2((d/2 - l)/2)}{\Gamma^2((d/2 + l)/2)}, \frac{\Gamma^2((d/2 - l + 1)/2)}{\Gamma^2((d/2 + l + 1)/2)} \} \tag{1.4}$$

(Γ is the Gamma-function) and these constants cannot be improved even on functions $u \in C_0^\infty(\mathbb{R}^d \setminus \{0\})$. In the case $l < d/2$, the maximum in (1.4) can be replaced by the first number, but in the case (1.2) both numbers in (1.4) can be dominating. For example, if $l - d/2$ is close to an integer even number, then the first number is bigger than the second. On the contrary, the second number is bigger if $l - d/2$ is close to an odd number.

If l is an integer and $l \geq d/2$, then condition (1.2) requires that d be odd. For integer l, the maximum in (1.4) always equals the first number, that is

$$C_l = 2^{2l} (d - 2l)^{-2} (d - 2l + 4)^{-2} \cdots (d + 2l - 4)^{-2}. \tag{1.5}$$

In particular, for $l = 1$ when $d \neq 2$ and for $l = 2$ when $d \neq 2, 4$ we recover the classical inequalities of Hardy [7] and Rellich [9], respectively. For other integer $l < d/2$, inequality (1.1) with constant (1.5) can be found in [1] (see also [5], for more general results of this type).

If $l < d/2$ is not an integer, then (1.1) with some estimating constant and its different generalizations (in particular, to weighted L_p-norms for $p \neq 2$) were also discussed extensively in the literature. Actually, the functions $|x|^{-2l}$ and $|\xi|^{2l}$ can be replaced in (1.1) by more general functions (see [4] and references therein). Inequality (1.1) can also be considered as a particular case of the Stein and Weiss inequality (see e.g. [8] and references therein) which is formulated as the boundedness of an integral operator with kernel $|x-y|^{-s}$ in different L_p-spaces with power weights. On the other hand, in the case $l > d/2$, we have not found inequality (1.3) (even with an implicit estimating constant) for non-integer l in the literature.

Our proof of inequalities (1.1) and (1.3) is based on the following scheme. We consider them separately on each of the subspaces $\mathfrak{H}_m$, $m = 0, 1, \ldots$, spanned by spherical functions of order m. On the subspace $\mathfrak{H}_m$ these inequalities reduce to inequalities on the half-axis $\mathbb{R}_+$ and are equivalent to the boundedness of some integral operator $A_l^{(m)}$ in the space $L_2(\mathbb{R}_+)$. We diagonalize this operator explicitly with the help of the Mellin transform and, in particular, calculate its norm. Finally, it turns out that all norms $||A_l^{(m)}||$ are estimated either by $||A_l^{(0)}||$ or by $||A_l^{(1)}||$. The numbers $||A_l^{(0)}||^2$ and $||A_l^{(1)}||^2$ equal, respectively, the first and the second terms in the right-hand side of (1.4). "Individual" estimates (see (2.4) and (2.15)) for each m are possibly of independent interest.

We first prove inequalities (1.1) and (1.3) on the class $\hat{u} \in C_0^\infty(\mathbb{R}^d \setminus \{0\})$ but then extend them to, in some sense, maximal classes of functions. Our approach automatically gives the best possible constants in (1.1) and (1.3). We emphasize that the corresponding "maximizing" sequences of functions are either spherically symmetric or belong to the "next" subspace $\mathfrak{H}_1$. The equalities in (1.1) and (1.3) are never realized because the operators $A_l^{(m)}$ do not have eigenvalues.

In the case $l \geq d/2$, the number $l - d/2$ is integer if either d is even and l is an integer or d is odd and l is a half-integer. In these exceptional cases the operators $A_l^{(m)}$ are unbounded for $m = l - d/2, l - d/2 - 2, \ldots$. Thus, inequalities (1.1) and (1.3) break down (even for functions $u \in C_0^\infty(\mathbb{R}^d \setminus \{0\})$) on the corresponding subspaces $\mathfrak{H}_m$ but remain valid if these subspaces are removed. For example, if $l = d = 2$ (this case was treated in [9]), functions u should be orthogonal to $\mathfrak{H}_1$. It turns out (see e.g. [3]) that on the removed subspaces inequalities of the type (1.1) still hold if the logarithmic factor $\ln^{-2}|x|$ is added in its left-hand side. We calculate explicitly estimating constants in all these situations. Note that the case of integer $l \geq d/2$ and even d was considered earlier in [10], where the technique of [6] was used. In this case the results of [10] and ours essentially coincide. The author is grateful to M. Z. Solomyak for useful discussions.

2. THE MAIN RESULT

2.1. Let us decompose the space $L_2(\mathbb{R}^d)$ with respect to spherical functions. Let $\mathfrak{h}_m \subset$

$L_2(\mathbb{S}^{d-1})$ be the subspace of spherical functions $Y_m(\omega)$, $\omega\in\mathbb{S}^{d-1}$, of order m, let $\mathcal{K}=L_2(\mathbb{R}_+;r^{d-1})$ be the L_2-space with weight r^{d-1} of functions defined on $\mathbb{R}_+=(0,\infty)$ and let $\mathfrak{H}_m=\mathcal{K}\otimes\mathfrak{h}_m$. To put it differently, $\mathfrak{H}_m\subset\mathbb{R}^d$ is the subspace spanned by functions

$$u(x)=f(r)Y_m(\omega),\quad r=|x|,\quad \omega=x|x|^{-1},\quad \delta=(d-1)/2, \tag{2.1}$$

where $f\in\mathcal{K}$ and $Y_m\in\mathfrak{h}_m$. Then

$$L_2(\mathbb{R}^d)=\bigoplus_{m=0}^{\infty}\mathfrak{H}_m,\quad \mathfrak{H}_m=\mathcal{K}\otimes\mathfrak{h}_m, \tag{2.2}$$

and every $\mathfrak{H}_m$ is invariant with respect to the Fourier operator Φ which reduces (see e.g. [11]) to the Fourier-Bessel transform Φ_m on this subspace. This means that for function (2.1)

$$(\Phi u)(\xi)=i^{-m}g(s)Y_m(\theta),\quad s=|\xi|,\quad \theta=\xi|\xi|^{-1},$$

where

$$g(s)=(\Phi_m f)(s)=s^{-\delta}\int_0^\infty (rs)^{1/2}\mathcal{I}_{m+(d-2)/2}(rs)r^\delta f(r)dr \tag{2.3}$$

and $\mathcal{I}_{m+(d-2)/2}$ is the Bessel function. The operator $\Phi_m=\Phi_m^*$ is of course unitary on $\mathcal{K}$ and hence $f=\Phi_m g$. In the case $d=1$ equality (2.2) reduces to the decomposition $L_2(\mathbb{R})=\mathfrak{H}_0\oplus\mathfrak{H}_1$ of $L_2(\mathbb{R})$ into the subspaces of even and odd functions, so Φ_0 is the cosinus and Φ_1 is the sinus Fourier transforms.

We first consider inequalities (1.1) and (1.3) for $\hat{u}\in C_0^\infty(\mathbb{R}^d\setminus\{0\})$. Since (1.1) can be verified on each of the subspaces $\mathfrak{H}_m$ separately, we formulate

Lemma 2.1 *The estimate* (1.1) *for $\hat{u}\in C_0^\infty(\mathbb{R}^d\setminus\{0\})$ is equivalent to the set of estimates*

$$\int_0^\infty r^{d-1-2l}|(\Phi_m g)(r)|^2dr\le C_l^{(m)}\int_0^\infty s^{d-1+2l}|g(s)|^2ds,\quad g\in C_0^\infty(\mathbb{R}_+), \tag{2.4}$$

for all $m=0,1,\ldots$. Moreover,

$$C_l=\sup_{m\ge0}C_l^{(m)}. \tag{2.5}$$

Let us consider in the space $L_2(\mathbb{R}_+)$ an integral operator defined by the equality

$$(A_l^{(m)}h)(r)=\int_0^\infty a_l^{(m)}(rs)h(s)ds, \tag{2.6}$$

where

$$a_l^{(m)}(t)=t^{-l+1/2}\mathcal{I}_{m+(d-2)/2}(t). \tag{2.7}$$

Set for $h\in C_0^\infty(\mathbb{R}_+)$

$$g(s)=s^{-l-\delta}h(s). \tag{2.8}$$

Then

$$r^{\delta-l}(\Phi_m g)(r)=(A_l^{(m)}h)(r) \tag{2.9}$$

and

$$\int_0^\infty s^{d-1+2l}|g(s)|^2ds=||h||^2,\quad \int_0^\infty r^{d-1-2l}|(\Phi_m g)(r)|^2dr=||A_l^{(m)}h||^2. \tag{2.10}$$

This gives us the following

Lemma 2.2 *The estimate* (2.4) *is equivalent to the boundedness in the space* $L_2(\mathbb{R}_+)$ *of the integral operator* $A_l^{(m)}$ *defined on* $C_0^\infty(\mathbb{R}_+)$ *by equality* (2.6) *with kernel* (2.7). *Moreover,*

$$C_l^{(m)} = ||A_l^{(m)}||^2. \tag{2.11}$$

We emphasize that in Lemma 2.2 as well as in Lemma 2.4 and Proposition 2.8 below there are no restrictions on l and d.

To rewrite (1.3) in terms of functions

$$f_m(r) = \int_{\mathbb{S}^{d-1}} u(r\omega)\overline{Y_m(\omega)}d\omega, \tag{2.12}$$

we take into account that $\mathfrak{h}_m$ is orthogonal in $L_2(\mathbb{S}^{d-1})$ to all homogeneous polynomials of degree k if either $k < m$ or $k = m + 2j + 1,\ j = 0, 1, \ldots$. Therefore for a smooth function u (say, from the Schwartz class $\mathcal{S} = \mathcal{S}(\mathbb{R}^d)$), functions (2.12) satisfy the relations $f_m^{(k)}(0) = 0$ if either $k < m$ or $k = m + 2j + 1$ and admit the following expansion as $r \to 0$:

$$f_m(r) = \sum_{j=0}^{\infty}(m + 2j)!^{-1} f_m^{(m+2j)}(0)r^{m+2j}. \tag{2.13}$$

It follows for the function

$$\mathbf{u}^{(n)}(x) = u(x) - \sum_{|\alpha|\leq n}(\alpha!)^{-1}(\partial^\alpha u)(0)x^\alpha \tag{2.14}$$

that

$$\int_{\mathbb{S}^{d-1}} \mathbf{u}^{(n)}(r\omega)\overline{Y_m(\omega)}d\omega = f_m(r) - \sum_{j=0}^{[(n-m)/2]}(m + 2j)!^{-1} f_m^{(m+2j)}(0)r^{m+2j},$$

for $m = 0, 1, \ldots, n$ and

$$\int_{\mathbb{S}^{d-1}} \mathbf{u}^{(n)}(r\omega)\overline{Y_m(\omega)}d\omega = f_m(r)$$

for $m \geq n + 1$. Clearly,

$$[(n - m)/2] = [(l - d/2 - m)/2] =: \kappa_{l,m}.$$

Thus, we have

Lemma 2.3 *The estimate* (1.3) *for* $\hat{u} \in C_0^\infty(\mathbb{R}^d \setminus \{0\})$ *is equivalent to the estimates*

$$\int_0^\infty r^{d-1-2l}|f_m(r) - \sum_{j=0}^{\kappa_{l,m}}(m + 2j)!^{-1} f_m^{(m+2j)}(0)r^{m+2j}|^2 dr$$
$$\leq C_l^{(m)}\int_0^\infty s^{d-1+2l}|g(s)|^2 ds, \quad m = 0, 1, \ldots, n, \quad g \in C_0^\infty(\mathbb{R}_+), \quad f_m = \Phi_m g, \tag{2.15}$$

and to the estimates (2.4) *for* $m \geq n + 1$. *The estimating constants are related by* (2.5).

Recall that

$$t^{1/2}\mathcal{I}_{m+(d-2)/2}(t)=\sum_{j=0}^{\infty}\sigma_m^{(j)}t^{m+\delta+2j},$$

where

$$\sigma_m^{(j)}=(-1)^j 2^{-2j-m-(d-2)/2}(\Gamma(j+m+d/2)j!)^{-1}.$$

Consequently, functions $f_m=\Phi_m g$, $m=0,1,\dots,n$, satisfy equalities

$$f_m(r)-\sum_{j=0}^{\kappa_{l,m}}(m+2j)!^{-1}f_m^{(m+2j)}(0)r^{m+2j}$$

$$=r^{-\delta}\int_0^{\infty}\Big((rs)^{1/2}\mathcal{I}_{m+(d-2)/2}(rs)-\sum_{j=0}^{\kappa_{l,m}}\sigma_m^{(j)}(rs)^{m+\delta+2j}\Big)s^{\delta}g(s)ds.$$

Therefore, similarly to Lemma 2.2, we obtain

Lemma 2.4 *The estimate* (2.15) *is equivalent to the boundedness in the space* $L_2(\mathbb{R}_+)$ *of the integral operator* $A_l^{(m)}$ *defined on* $C_0^{\infty}(\mathbb{R}_+)$ *by equality* (2.6) *with kernel*

$$a_l^{(m)}(t)=t^{-l}\Big(t^{1/2}\mathcal{I}_{m+(d-2)/2}(t)-\sum_{j=0}^{\kappa_{l,m}}\sigma_m^{(j)}t^{m+\delta+2j}\Big). \tag{2.16}$$

Moreover, relation (2.11) *holds.*

2.2. An integral operator A in the space $L_2(\mathbb{R}_+)$ with kernel $a(rs)$ can be diagonalized with the help of the Mellin transform $\mathbf{M}$ defined by the equality

$$(\mathbf{M}h)(\lambda)=(2\pi)^{-1/2}\int_0^{\infty}r^{-1/2-i\lambda}h(r)dr.$$

The operator $\mathbf{M}:L_2(\mathbb{R}_+)\to L_2(\mathbb{R})$ is unitary. The proof of the following result can be found in [12].

Lemma 2.5 *Suppose that a real function* $a(t)$ *is locally bounded on* $(0,\infty)$ *and that the integral*

$$\int_0^{\infty}a(t)t^{-1/2-i\lambda}dt=:\alpha(\lambda) \tag{2.17}$$

converges at $t=0$ *and* $t=\infty$ *uniformly in* $\lambda\in\mathbb{R}$ *(in this case* $\alpha(-\lambda)=\overline{\alpha(\lambda)}$ *and* α *is a continuous function of* λ*). Then for any function* $h\in C_0^{\infty}(\mathbb{R}_+)$

$$\int_0^{\infty}\int_0^{\infty}a(rs)h(s)\overline{h(r)}drds=\int_{-\infty}^{\infty}\alpha(\lambda)(\mathbf{M}h)(-\lambda)\overline{(\mathbf{M}h)(\lambda)}d\lambda. \tag{2.18}$$

In particular, the spectrum of an integral operator A *with kernel* $a(rs)$ *in the space* $L_2(\mathbb{R}_+)$ *consists of the union* $[-p,-q]\cup[q,p]$*, where*

$$p=||A||=\max_{\lambda\in\mathbb{R}}|\alpha(\lambda)|,\quad q=\min_{\lambda\in\mathbb{R}}|\alpha(\lambda)|.$$

Moreover, A *does not have eigenvalues if* $\alpha(\lambda)$ *is not constant on any interval.*

Let us apply Lemma 2.5 to the operators $A_l^{(m)}$. First we compute integrals (2.17) for functions $a_l^{(m)}$ defined by equalities (2.7) or (2.16). According to formula (19), section 7.7 of [2], vol. 2,

$$\int_0^\infty t^{-z}\mathcal{I}_{m+(d-2)/2}(t)dt = 2^{-z}\Gamma((m+d/2-z)/2)\Gamma^{-1}((m+d/2+z)/2) \tag{2.19}$$

for $-1/2 < \operatorname{Re} z < m + d/2$. Calculating the analytic continuation of the integral in the left-hand side, we see that

$$\int_0^\infty t^{-z}\Big(\mathcal{I}_{m+(d-2)/2}(t) - \sum_{j=0}^{k}\sigma_m^{(j)}t^{m+\delta-1/2+2j}\Big)dt = 2^{-z}\frac{\Gamma((m+d/2-z)/2)}{\Gamma((m+d/2+z)/2)} \tag{2.20}$$

for

$$m + d/2 + 2k < \operatorname{Re} z < m + d/2 + 2k + 2, \quad k = 0, 1, \ldots .$$

Let functions $\alpha_l^{(m)}$ be defined by equality (2.17) where $a = a_l^{(m)}$. Using formulas (2.19) or (2.20) for $z = l + i\lambda$ and $k = \kappa_{l,m}$, we obtain that

$$\alpha_l^{(m)}(\lambda) = 2^{-l-i\lambda}\Gamma((m+d/2-l-i\lambda)/2)\Gamma^{-1}((m+d/2+l+i\lambda)/2). \tag{2.21}$$

By the Stirling formula, $\alpha_l^{(m)}(\lambda) \to 0$ as $|\lambda| \to \infty$. Moreover, it follows from the infinite product (see e.g. formula (3), section 1.1 of [2], vol. 1) for the Γ-function that

$$\Big|\frac{\Gamma(a+i\lambda)}{\Gamma(b+i\lambda)}\Big|^2 = e^{2\gamma(b-a)}\frac{b^2+\lambda^2}{a^2+\lambda^2}\prod_{n=1}^{\infty}\frac{(b+n)^2+\lambda^2}{(a+n)^2+\lambda^2}e^{2(a-b)/n}, \tag{2.22}$$

where γ is the Euler constant. This function is analytic in λ^2. In particular, for $a = (m+d/2-l)/2$, $b = (m+d/2+l)/2$ this implies analyticity of the function $|\alpha_l^{(m)}(\lambda)|^2$, which, consequently, is not constant on any interval. This yields

Lemma 2.6 *Let functions $a_l^{(m)}(t)$ be given by formulas (2.7) or (2.16) and let functions $\alpha_l^{(m)}(\lambda)$ be defined by formula (2.21). Then the norm of the operator (2.6) satisfies*

$$||A_l^{(m)}|| = \max_{\lambda\in\mathbb{R}} |\alpha_l^{(m)}(\lambda)|. \tag{2.23}$$

Moreover, the spectrum of the operator $A_l^{(m)}$ coincides with the interval $[-||A_l^{(m)}||, ||A_l^{(m)}||]$ and does not contain eigenvalues.

2.3. Let us show that the maximum in (2.23) is attained at the point $\lambda = 0$.

Lemma 2.7 *For any m, functions (2.21) satisfy*

$$|\alpha_l^{(m)}(\lambda)| \le |\alpha_l^{(m)}(0)|. \tag{2.24}$$

Proof. Remark first that

$$|\Gamma(a+i\lambda)\Gamma^{-1}(b+i\lambda)| \le |\Gamma(a)\Gamma^{-1}(b)| \tag{2.25}$$

for $b > 0$, $a \in [-b, b]$ and $\lambda \in \mathbb{R}$. Indeed, according to (2.22) it suffices to check that

$$((b+n)^2 + \lambda^2)((a+n)^2 + \lambda^2)^{-1} \leq (b+n)^2(a+n)^{-2}, \quad n = 0, 1, \ldots,$$

which is equivalent to

$$(b-a)(b+a+2n) \geq 0$$

and hence it is fulfilled for $a \in [-b, b]$. According to (2.21), to conclude the proof it suffices to take into account that numbers $a = (m + d/2 - l)/2$, $b = (m + d/2 + l)/2$ satisfy the conditions $b > 0$, $a \in (-b, b)$. $\square$

Combining this assertion with Lemmas 2.2, 2.4 and 2.6, we obtain inequalities (2.4) and (2.15) with the constants

$$C_l^{(m)} = 2^{-2l} \frac{\Gamma^2((m + d/2 - l)/2)}{\Gamma^2((m + d/2 + l)/2)}. \tag{2.26}$$

Proposition 2.8 *If $l < d/2$, then inequalities* (2.4) *are satisfied for all m. In the case $l \geq d/2$, inequalities* (2.4) *hold for $m \geq [l - d/2] + 1$ and inequalities* (2.15) *hold for $m \leq [l - d/2]$ if $l - d/2 - m$ is not a non-negative even number.*

Now inequalities (1.1) and (1.3) with the constant C_l defined by (2.5), (2.26) follow directly from Proposition 2.8 and Lemmas 2.1 and 2.3, respectively. To simplify the expression for C_l, remark that according to the identity $\Gamma(z+1) = z\Gamma(z)$ functions (2.21) satisfy

$$|\alpha_l^{(m+2)}(\lambda)| \leq |\alpha_l^{(m)}(\lambda)|. \tag{2.27}$$

Putting together (2.24) and (2.27), we get that

$$C_l = \max\{C_l^{(0)}, C_l^{(1)}\}.$$

This yields expression (1.4) for C_l.

Now we consider two cases when the maximum in (1.4) can be replaced by the first number, that is

$$\frac{\Gamma^2((d/2 + l)/2)}{\Gamma^2((d/2 - l)/2)} \leq \frac{\Gamma^2((d/2 + l + 1)/2)}{\Gamma^2((d/2 - l + 1)/2)}. \tag{2.28}$$

Suppose first that $l < d/2$. Since the function $\Gamma'(x)\Gamma^{-1}(x)$ is increasing for $x > 0$, we have that

$$\Big(\Gamma(1/2 + x)\Gamma^{-1}(x)\Big)' > 0, \quad x > 0,$$

and, consequently,

$$\Gamma(1/2 + x)\Gamma^{-1}(x) \leq \Gamma(1/2 + y)\Gamma^{-1}(y), \quad 0 < x \leq y.$$

Choosing $x = (d/2 - l)/2$, $y = (d/2 + l)/2$, we arrive at (2.28).

Inequality (2.28) holds also in the case of odd d and integer $l > d/2$ when it can be rewritten as

$$(d-2l)^2(d-2l+4)^2 \cdots (d+2l-4)^2 \leq (d-2l+2)^2(d-2l+6)^2 \cdots (d+2l-2)^2. \tag{2.29}$$

If $d = 1$, then both sides here are equal $(2l-1)!!^2$. If $d > 1$, then the left-hand side of (2.29) is

$$(2l-d)!!^2(2l-d+2)^2(2l-d+6)^2\cdots(2l+d-4)^2$$

and the right-hand side is

$$(2l-d)!!^2(2l-d+4)^2(2l-d+8)^2\cdots(2l+d-2)^2.$$

These expressions contain the common factor $(2l-d)!!^2$ and, since $2l > d$, every factor $(2l-d+2+2k)^2$, $k = 0, 2, \ldots, d-3$, of the first expression is smaller than the corresponding factor $(2l-d+4+2k)^2$ of the second expression. Thus for integer l, the estimating constants in (1.1) and (1.3) are always given by (1.5).

Let us now formulate the results obtained.

Theorem 2.9 *Let $\hat{u} \in C_0^\infty(\mathbb{R}^d \setminus \{0\})$. If $l < d/2$, then inequality* (1.1) *holds with*

$$C_l = 2^{-2l}\Gamma^2((d/2-l)/2)\Gamma^{-2}((d/2+l)/2).$$

If l satisfies condition (1.2)*, then inequality* (1.3) *holds with C_l defined by* (1.4)*. In both cases for integer l the constant C_l is given by* (1.5).

3. MISCELLANEOUS

3.1. Let us extend inequalities (1.1) and (1.3) to larger (and in some sense maximal) classes of functions. The constants C_l will of course be the same as in Theorem 2.9. We start with inequality (1.1) for the case $2l < d$ and show that it is preserved for all functions $\hat{u}$ such that its right-hand side is finite. Thus we formulate conditions in terms of Fourier transforms $w = \hat{u}$ rather than of functions u themselves. Let $\mathcal{S}' = \mathcal{S}'(\mathbb{R}^d)$ be the Schwartz class of distributions. Since $L_2(\mathbb{R}^d; |\xi|^{2l}) \subset \mathcal{S}'$ for $2l < d$, the (inverse) Fourier transform $u = \Phi^* w$ is well-defined as an element of $\mathcal{S}'$.

If $w \in L_2(\mathbb{R}^d; |\xi|^{2l})$, then there exists a sequence $w_k \in C_0^\infty(\mathbb{R}^d \setminus \{0\})$ such that

$$\lim_{k\to\infty} \int_{\mathbb{R}^d} |\xi|^{2l} |w_k(\xi) - w(\xi)|^2 d\xi = 0. \tag{3.1}$$

It follows from estimate (1.1) applied to the functions $u_k - u_j = \Phi^* w_k - \Phi^* w_j$ that

$$\lim_{k,j\to\infty} \int_{\mathbb{R}^d} |x|^{-2l} |u_k(x) - u_j(x)|^2 dx = 0 \tag{3.2}$$

and, consequently,

$$\lim_{k\to\infty} \int_{\mathbb{R}^d} |x|^{-2l} |u_k(x) - v(x)|^2 dx = 0 \tag{3.3}$$

for some function $v \in L_2(\mathbb{R}^d; |x|^{-2l})$. Passing to the limit $k \to \infty$ in inequality (1.1) for functions u_k we see that

$$\int_{\mathbb{R}^d} |x|^{-2l} |v(x)|^2 dx \le C_l \int_{\mathbb{R}^d} |\xi|^{2l} |w(\xi)|^2 d\xi. \tag{3.4}$$

It remains to check that $v = u$ as elements of $\mathcal{S}'$. For any test function $\varphi \in \mathcal{S}$, we have that

$$< u_k, \varphi > = < w_k, \hat{\varphi} >, \quad \hat{\varphi} = \Phi\varphi. \tag{3.5}$$

It follows from (3.1) that

$$\lim_{k\to\infty} < w_k, \hat{\varphi} > = < w, \hat{\varphi} > \tag{3.6}$$

and from (3.3) that

$$\lim_{k\to\infty} < u_k, \varphi > = < v, \varphi > . \tag{3.7}$$

Therefore (3.5) implies that

$$< v, \varphi > = < w, \hat{\varphi} > \tag{3.8}$$

and hence $v = \Phi^* w$. Let us formulate the result obtained.

Proposition 3.1 *In the case $2l < d$, inequality* (1.1) *holds for all functions $u \in \mathcal{S}'$ such that $\hat{u} \in L_2(\mathbb{R}^d; |\xi|^{2l})$ and, in particular, for all $u \in H^l(\mathbb{R}^d)$.*

3.2. In the case (1.2) we first extend inequality (1.3) to the class $H^l(\mathbb{R}^d)$. Note that by the Sobolev theorem for $u \in H^l(\mathbb{R}^d)$ all functions $u^{(\alpha)}(x)$, $|\alpha| \leq n$, are continuous so that the left-hand side of (1.3) is well-defined. Let us choose a sequence $\hat{u}_k \in C_0^\infty(\mathbb{R}^d \setminus \{0\})$ such that

$$\lim_{k\to\infty} ||u_k - u||_{H^l(\mathbb{R}^d)} = 0 \tag{3.9}$$

and consider inequality (1.3) for u_k. Its right-hand side tends of course to the right-hand side of (1.3) for u. Using notation (2.14) we deduce from estimate (1.3) for $u_k - u_j$ that

$$\lim_{k\to\infty} \int_{\mathbb{R}^d} |x|^{-2l} |\mathbf{u}_k^{(n)}(x) - v(x)|^2 dx = 0 \tag{3.10}$$

for some function $v \in L_2(\mathbb{R}^d; |x|^{-2l})$. This yields inequality

$$\int_{\mathbb{R}^d} |x|^{-2l} |v(x)|^2 dx \leq C_l \int_{\mathbb{R}^d} |\xi|^{2l} |\hat{u}(\xi)|^2 d\xi. \tag{3.11}$$

It follows from (3.9) and the Sobolev theorem that $\mathbf{u}_k^{(n)} \to \mathbf{u}^{(n)}$ as $k \to \infty$ in $L_2^{(loc)}(\mathbb{R}^d)$. This is compatible with (3.10) only if $v = \mathbf{u}^{(n)}$. Therefore (3.11) implies (1.3). Thus we have proven

Proposition 3.2 *If l satisfies condition* (1.2)*, then inequality* (1.3) *holds for all $u \in H^l(\mathbb{R}^d)$.*

Next we consider inequality (1.1) in the case (1.2) and show that it can be extended to all functions for which both sides of (1.1) are finite. However since $L_2(\mathbb{R}^d; |\xi|^{2l}) \not\subset \mathcal{S}'$ for $2l > d$, a relation between u and $\hat{u}$ has to be clarified. Let $\mathcal{S}_n \subset \mathcal{S}$ be a subset of test functions $\varphi(x)$ such that $\varphi^{(\alpha)}(0) = 0$ for all $|\alpha| \leq n$. Similarly, $\hat{\mathcal{S}}_n$ consist of functions $\varphi \in \mathcal{S}$ such that $\hat{\varphi}^{(\alpha)}(0) = 0$, $|\alpha| \leq n$. A function $w \in L_2(\mathbb{R}^d; |\xi|^{2l})$ is well-defined as a distribution on the class $\hat{\mathcal{S}}_n$. Consider the set $\mathcal{F}_n(w)$ of $v \in \mathcal{S}'$ such that (3.8) is satisfied for all $\varphi \in \hat{\mathcal{S}}_n$. Clearly, all distributions $v \in \mathcal{F}_n(w)$ differ by a polynomial of degree n.

A proof of the following auxiliary assertion can be found e.g. in [12].

Lemma 3.3 *Let $\mathcal{H}$ be a Hilbert space and let $f_1, \dots, f_N$ be a finite family of linear functionals defined on a set $\mathcal{D}$ dense in $\mathcal{H}$. If all functionals $f_1, \dots, f_N$ are unbounded, then the set $\mathcal{D}_0 \subset \mathcal{D}$ of their common zeros is dense in $\mathcal{H}$.*

Since the functionals $f_\alpha(w) = (\Phi^* w)^{(\alpha)}(0)$, $|\alpha| \le n$, defined on $\mathcal{S}$ are not bounded in the space $L_2(\mathbb{R}^d; |\xi|^{2l})$, the set $\mathcal{S}_n$ is dense in this space by virtue of Lemma 3.3.

The next assertion can be viewed as an extension of Proposition 3.1 to the case $2l > d$.

Proposition 3.4 *Let condition* (1.2) *be satisfied. For a function $w \in L_2(\mathbb{R}^d; |\xi|^{2l})$, there exists the unique distribution $v \in \mathcal{F}_n(w) \cap L_2(\mathbb{R}^d; |x|^{-2l})$. For this function v estimate* (3.4) *holds.*

Proof. Uniqueness of v is obvious because all distributions $v \in \mathcal{F}_n(w)$ differ by a polynomial $\mathcal{P}$ of degree n and $\mathcal{P} \notin L_2(\mathbb{R}^d; |x|^{-2l})$. Since the set $\mathcal{S}_n$ is dense in the space $L_2(\mathbb{R}^d; |\xi|^{2l})$, for a given $w \in L_2(\mathbb{R}^d; |\xi|^{2l})$, there exists a sequence $w_k \in \mathcal{S}_n$ such that relation (3.1) holds. It follows from inequality (1.3) (which coincides with (1.1) on the class $\mathcal{S}_n$) applied to the functions $u_k - u_j = \Phi^* w_k - \Phi^* w_j$ that relations (3.2) and, consequently, (3.3) for some function $v \in L_2(\mathbb{R}^d; |x|^{-2l})$ are satisfied. Passing to the limit $k \to \infty$ in inequality (1.1) for functions u_k we obtain estimate (3.4).

It remains to check that $v \in \mathcal{F}_n(w)$. Let $\varphi \in \hat{\mathcal{S}}_n$. We deduce from (3.1) that relation (3.6) is satisfied and from (3.3) that relation (3.7) is satisfied. Now (3.8) follows directly from equality (3.5). $\square$

Corollary 3.5 *Let $u \in L_2(\mathbb{R}^d; |x|^{-2l})$ and let its Fourier transform $\hat{u}$ (in the sense of $\mathcal{S}'$) belong to the space $L_2(\mathbb{R}^d; |\xi|^{2l})$. Then inequality* (1.1) *holds.*

Remark Inequality (1.3) for $u \in H^l(\mathbb{R}^d)$ can also be deduced from Proposition 3.4. Indeed, it gives estimate (3.11) for some $v \in \mathcal{F}_n(\hat{u})$. Therefore $u - v$ is a polynomial of degree n. This polynomial is necessarily $\sum_{|\alpha| \le n} \alpha!^{-1} u^{(\alpha)}(0) x^\alpha$ since otherwise v would not belong to the space $L_2(\mathbb{R}^d; |x|^{-2l})$.

3.3. For integer l we recover the standard formulation.

Proposition 3.6 *Let l be integer and let either $l < d/2$ or $l > d/2$ but $l - d/2 \notin \mathbb{Z}$. Then*

$$\int_{\mathbb{R}^d} |x|^{-2l} |u(x)|^2 dx \le C_l \int_{\mathbb{R}^d} |(\nabla^l u)(x)|^2 dx \tag{3.12}$$

for all $u \in H^l_{loc}(\mathbb{R}^d)$ such that both sides of (3.12) *are finite.*

Proof. For $|\alpha| = l$, set $u_\alpha = D^\alpha u \in L_2(\mathbb{R}^d)$. Note that

$$< u, D^{\alpha+\beta}\varphi > = < u_\alpha, D^\beta \varphi > = < \xi^\beta \hat{u}_\alpha, \hat{\varphi} >, \quad |\alpha| = |\beta| = l,$$

for any $\varphi \in \mathcal{S}$ and all multi-indices α, β. Thus, the functions $\xi^\beta \hat{u}_\alpha(\xi)$ coincide for almost all $\xi \in \mathbb{R}^d$. This implies that the functions $\xi^{-\alpha} \hat{u}_\alpha(\xi)$ do not depend on α, $|\alpha| = l$, and justifies the definition

$$w(\xi) = \xi^{-\alpha} \hat{u}_\alpha(\xi). \tag{3.13}$$

Since $\hat{u}_\alpha \in L_2(\mathbb{R}^d)$, this function belongs to the space $L_2(\mathbb{R}^d; |\xi|^{2l})$ and

$$\int_{\mathbb{R}^d} |w(\xi)|^2 |\xi|^{2l} d\xi = \int_{\mathbb{R}^d} |(\nabla^l u)(x)|^2 dx. \tag{3.14}$$

By Propositions 3.1 or 3.4, estimate (3.4) holds, where $v = \Phi^* w$ if $2l < d$ and $v \in \mathcal{F}_n(w)$ in the case (1.2). In particular, for any $\psi \in \mathcal{S}$ and $|\alpha| = l$ we have that

$$< v, D^\alpha \psi >=< w, \widehat{D^\alpha \psi} > . \tag{3.15}$$

By definition (3.13),

$$< u, D^\alpha \psi >=< u_\alpha, \psi >=< \widehat{u_\alpha}, \hat{\psi} >=< \xi^\alpha w, \hat{\psi} >=< w, \widehat{D^\alpha \psi} > . \tag{3.16}$$

Comparing (3.15) and (3.16), we see that $< u - v, D^\alpha \psi >= 0$ for any $\psi \in \mathcal{S}$ and all α, $|\alpha| = l$. Hence $u - v$ is a polynomial of degree $l - 1$. This polynomial is necessarily zero since both u and v belong to the space $L_2(\mathbb{R}^d; |x|^{-2l})$. Now estimate (3.12) follows from (3.4) and (3.14). $\square$

3.4. Here we show that the constants C_l in Theorem 2.9 cannot be improved. Actually, we shall check that the constants C_l are optimal even on the "narrow" class $u \in C_0^\infty(\mathbb{R}^d \setminus \{0\})$ when inequality (1.3) reduces to (1.1). First similarly to subsections 3.1 and 3.2, we extend (1.1) to a larger class of functions.

Lemma 3.7 *Suppose that* (1.1) *holds with some constant* C_l *for all* $u \in C_0^\infty(\mathbb{R}^d \setminus \{0\})$. *Then* (1.1) *is satisfied with the same constant* C_l *for all* $u \in H^l(\mathbb{R}^d)$ *if* $2l < d$ *and for all* $u \in H^l(\mathbb{R}^d)$ *such that* $u^{(\alpha)}(0) = 0$, $|\alpha| \leq n$, *in the case* (1.2).

Proof. For a given u, choose functions $u_k \in C_0^\infty(\mathbb{R}^d \setminus \{0\})$ such that (3.9) is satisfied. Then estimates (1.1) for u_k imply estimate (3.11) where the function v obeys (3.3) and consequently (3.7) (for any $\varphi \in \mathcal{S}$). Therefore $v = u$ and (3.11) yields (1.1). $\square$

Let us check that constants (2.26) are optimal for inequalities (2.4). Let $\mathcal{D} = \mathcal{D}_q$ consist of functions $h \in C_0^\infty(\mathbb{R}_+)$ such that

$$\int_0^\infty s^{\delta+m-l+2j} h(s) ds = 0, \quad j = 0, 1, \ldots, q. \tag{3.17}$$

Since functionals defined by the left-hand side of (3.17) are not bounded in $L_2(\mathbb{R}_+)$, the set $\mathcal{D}$ is dense (for any q) in this space according to Lemma 3.3. By definition (2.6), (2.7) of the operator $A_l^{(m)}$,

$$(A_l^{(m)} h)(r) = \int_0^\infty (rs)^{-l+1/2} \mathcal{I}_{m+(d-2)/2}(rs) h(s) ds \tag{3.18}$$

for any $h \in C_0^\infty(\mathbb{R}_+)$ if $2l < d$ or $m \geq n + 1$. Moreover, according to (2.16) in the case $2l > d$, $m \leq n$, (3.18) holds for any $h \in \mathcal{D}_q$ if $q \geq \kappa_{l,m}$.

Lemma 3.8 *Suppose that* $l - d/2 - m$ *is not a non-negative even number and let* $C_l^{(m)}$ *be defined by* (2.26). *For any* $N > l - d/2$, *there exists a sequence* $g_k \in C_0^\infty(\mathbb{R}_+)$ *such that*

$$(\Phi_m g_k)(r) = O(r^N), \quad r \to 0, \tag{3.19}$$

and

$$\int_0^\infty s^{d-1+2l} |g_k(s)|^2 ds = 1, \quad \lim_{k\to\infty} \int_0^\infty r^{d-1-2l} |(\Phi_m g_k)(r)|^2 dr = C_l^{(m)}. \tag{3.20}$$

Proof. Let $2q \geq N-m-2$. Since the set $\mathcal{D}_q$ is dense in $L_2(\mathbb{R}_+)$, there exists a sequence $h_k \in \mathcal{D}_q$ such that

$$||h_k|| = 1, \quad \lim_{k\to\infty} ||A_l^{(m)} h_k|| = ||A_l^{(m)}||. \tag{3.21}$$

Let us define functions g_k by formula (2.8). According to (3.18), the functions $\Phi_m g_k$ are related to $A_l^{(m)} h_k$ by formula (2.9). In view of (2.10), equalities (3.21) imply (3.20). Finally, by (3.18), conditions (3.17) guarantee (3.19). $\square$

Below we put $N = n+1$ in the case (1.2) and neglect condition (3.19) if $2l < d$. Let us use Lemma 3.8 for $m = 0$ or $m = 1$ and set $u_k(x) = (\Phi_0 g_k)(|x|)$ if $C_l^{(0)} \geq C_l^{(1)}$ or $u_k(x) = (\Phi_1 g_k)(|x|)Y_1(\omega)$, $\omega = x|x|^{-1}$, if $C_l^{(0)} \leq C_l^{(1)}$. Then $\hat{u}_k(\xi) = g_k(|\xi|)$ in the first case and $\hat{u}_k(\xi) = g_k(|\xi|)Y_1(\theta)$, $\theta = \xi|\xi|^{-1}$, in the second. The sequence $u_k \in \mathcal{S}$ satisfies the condition $u_k^{(\alpha)}(0) = 0$ for $|\alpha| \leq n$ (if $2l > d$) and

$$\int_{\mathbb{R}^d} |\xi|^{2l} |\hat{u}_k(\xi)|^2 d\xi = 1, \quad \lim_{k\to\infty} \int_{\mathbb{R}^d} |x|^{-2l} |u_k(x)|^2 dx = C_l. \tag{3.22}$$

This shows that the constant C_l cannot be improved on the class of functions introduced in Lemma 3.7 and consequently, by this lemma, even for functions $u \in C_0^\infty(\mathbb{R}^d \setminus \{0\})$. Thus, we have proven

Proposition 3.9 *Let C_l be the same constant as in Theorem* 2.9*. Then there exists a sequence $u_k \in C_0^\infty(\mathbb{R}^d \setminus \{0\})$ such that relations* (3.22) *are satisfied.*

3.5. Now we check that equality in (1.1) is never attained, that is, in terminology of the paper [8], a maximizing function does not exist even on the largest possible class of functions. More precisely, we have the following assertion.

Proposition 3.10 *Let $l < d/2$ or $l > d/2$ but $l - d/2 \notin \mathbb{Z}$. Suppose that*

$$\int_{\mathbb{R}^d} |x|^{-2l} |u(x)|^2 dx = C_l \int_{\mathbb{R}^d} |\xi|^{2l} |w(\xi)|^2 d\xi < \infty \tag{3.23}$$

and

$$\int_{\mathbb{R}^d} u(x)\overline{\varphi(x)} dx = \int_{\mathbb{R}^d} w(\xi)\overline{\hat{\varphi}(\xi)} d\xi, \quad \forall \varphi \in \hat{\mathcal{S}}_q, \tag{3.24}$$

for some $q > l - d/2 - 1$. Then $u = w = 0$.

As always, we use decomposition (2.2) which allows us to consider functions u of the form (2.1) and $w(\xi) = g(|\xi|)Y_m(\theta)$. Therefore (3.23) implies that at least for one m (necessarily $m = 0$ if $C_l^{(0)} \geq C_l^{(1)}$ and $m = 1$ if $C_l^{(1)} \geq C_l^{(0)}$)

$$\int_0^\infty r^{d-1-2l} |f(r)|^2 dr = C_l^{(m)} \int_0^\infty s^{d-1+2l} |g(s)|^2 ds < \infty. \tag{3.25}$$

Choose $\varphi(x) = \phi(r)Y_m(\omega)$ where $\phi \in C_0^\infty(\mathbb{R}_+)$ and

$$\int_0^\infty r^{d-1+m+2j} \phi(r) dr = 0, \quad j = 0, 1, \ldots, q_m,\ 2q_m \geq q - m - 1. \tag{3.26}$$

Then $\varphi \in \hat{\mathcal{S}}_q$ and (3.24) means that

$$\int_0^\infty r^{d-1} f(r)\overline{\phi(r)} dr = \int_0^\infty s^{d-1} g(s)\overline{(\Phi_m \phi)(s)} ds. \tag{3.27}$$

Thus in order to prove Proposition 3.10, it suffices to check

Lemma 3.11 *Let $l-d/2-m\neq 0,2,4,\dots$. Suppose that* (3.25) *and* (3.27) *hold for all ϕ satisfying* (3.26). *Then $f=g=0$.*

Proof. Let us set $h(s)=s^{\delta+l}g(s)$ (then $h\in L_2(\mathbb{R}_+)$) and check that $r^{\delta-l}f(r)=(A_l^{(m)}h)(r)$. Since $A_l^{(m)}$ is self-adjoint and bounded in the space $L_2(\mathbb{R}_+)$, it suffices to show that

$$\int_0^\infty r^{\delta-l}f(r)\overline{\psi(r)}dr=\int_0^\infty h(r)\overline{(A_l^{(m)}\psi)(r)}dr \tag{3.28}$$

for any function ψ from some dense set. Put $\psi(r)=r^{\delta+l}\phi(r)$ so that $\psi\in\mathcal{D}_q$. Comparing (2.3) and (3.18) we see that $r^{\delta-l}(\Phi_m\phi)(r)=(A_l^{(m)}\psi)(r)$ and consequently (3.28) reduces to equality (3.27). In terms of h, condition (3.25) means that $||A_l^{(m)}h||=||A_l^{(m)}||\,||h||$. This implies that $h=0$ because, by Lemma 2.6, the operator $A_l^{(m)}$ does not have eigenvalues. Consequently, $f=g=0$. $\square$

3.6. Although equalities in (1.1) and (1.3) are never attained, we can construct improper elements u for which, in some natural sense, both sides coincide being of course infinite. We emphasize that considerations of this subsection are of heuristic nature.

First we calculate eigenfunctions of the continuous spectrum of the operator $A=A_l^{(m)}$. According to (2.18) $A=\mathbf{M}^*\mathcal{A}\mathbf{M}$ where

$$(\mathcal{A}w)(\lambda)=\alpha(\lambda)w(-\lambda)$$

and the function $\alpha(\lambda)=\alpha_l^{(m)}(\lambda)$ is defined by equality (2.21). Since $\alpha(-\lambda)=\overline{\alpha(\lambda)}$, eigenfunctions of the continuous spectrum of the operator $\mathcal{A}$ are given by the equality

$$w_{\lambda_0}^{(\pm)}(\lambda)=\alpha(\lambda_0)\delta(\lambda-\lambda_0)\pm|\alpha(\lambda_0)|\delta(\lambda+\lambda_0),\quad \lambda_0>0, \tag{3.29}$$

(here $\delta(\cdot)$ is the Dirac-function) so that

$$\mathcal{A}w_{\lambda_0}^{(\pm)}=\pm|\alpha(\lambda_0)|w_{\lambda_0}^{(\pm)}.$$

Now eigenfunctions $h_{\lambda_0}^{(\pm)}$ of the operator A can be obtained by the inverse Mellin transform $h_{\lambda_0}^{(\pm)}=(2\pi)^{1/2}\mathbf{M}^*w_{\lambda_0}^{(\pm)}$. According to (3.29) this yields

$$h_{\lambda_0}^{(\pm)}(r)=r^{-1/2}\int_{-\infty}^\infty r^{i\lambda}w_{\lambda_0}^{(\pm)}(\lambda)d\lambda=r^{-1/2}\Big(\alpha(\lambda_0)r^{i\lambda_0}\pm|\alpha(\lambda_0)|r^{-i\lambda_0}\Big).$$

Passing to the limit $\lambda_0\to 0$, we obtain also eigenfunctions

$$h_0^{(+)}(r)=r^{-1/2},\quad h_0^{(-)}(r)=r^{-1/2}(\operatorname{Im}\alpha'(0)+2\alpha(0)\ln r), \tag{3.30}$$

corresponding to the upper $|\alpha(0)|$ and lower $-|\alpha(0)|$ edges of the spectrum of the operator A.

The improper elements $f^{(\pm)}=\Phi_m g^{(\pm)}$ which realize equalities in estimates (2.4) and (2.15) are related to eigenfunctions (3.30) by equations (2.8), (2.9). The second of them implies that for any m

$$f^{(+)}(r)=r^{l-d/2},\quad f^{(-)}(r)=r^{l-d/2}(\operatorname{Im}\alpha'(0)+2\alpha(0)\ln r).$$

Returning to estimates (1.1) and (1.3), we see that according to (2.1) they reduce to the "equalities" on functions $u^{(\pm)}(x) = f^{(\pm)}(r)$ if $C_l^{(1)} \leq C_l^{(0)}$ (in particular, for $l < d/2$ and for integer l) or on functions

$$u^{(+)}(x) = r^{l-d/2} Y_1(\omega), \quad u^{(-)}(x) = r^{l-d/2} (\operatorname{Im} \alpha'(0) + 2\alpha(0) \ln r) Y_1(\omega), \quad \alpha = \alpha_l^{(1)},$$

$(r = |x|,\ \omega = x/r)$ if $C_l^{(0)} \leq C_l^{(1)}$.

4. THE EXCEPTIONAL CASE

4.1. Here we consider the case $l = d/2 + n$ where $n = 0, 1, \ldots$, so l is integer if d is even and l is half-integer if d is odd. Now we cannot hope to have inequality (1.1) even on the class $u \in C_0^\infty(\mathbb{R}^d \setminus \{0\})$. Actually, inequalities (2.4) are violated for all m such that

$$n - m = 2q, \quad q = 0, 1, \ldots. \tag{4.1}$$

For the proof, it suffices to repeat essentially the arguments of subsection 3.4. Let the operator $A_l^{(m)}$ be defined by formula (3.18) on domain $\mathcal{D}_q$. This operator satisfies equality (2.18) where $h \in \mathcal{D}_q$ and the corresponding function $\alpha_l^{(m)}$ is given by (2.21). However, since $\alpha_l^{(m)}(\lambda) \to \infty$ as $\lambda \to 0$, the operator $A_l^{(m)}$ is unbounded. Similarly to Lemma 3.8, this allows us to prove the following

Lemma 4.1 *In the case* (4.1) *there exists a sequence* $g_k \in C_0^\infty(\mathbb{R}_+)$ *satisfying* (3.19) *for* $N > l - d/2$ *such that*

$$\int_0^\infty s^{d-1+2l} |g_k(s)|^2 ds = 1, \quad \lim_{k\to\infty} \int_0^\infty r^{d-1-2l} |(\Phi_m g_k)(r)|^2 dr = \infty.$$

Choosing any m obeying (4.1) and defining functions $u_k(x)$ by the formula

$$u_k(x) = (\Phi_m g_k)(|x|) Y_m(\omega),$$

we obtain a sequence $u_k \in \mathcal{S}$ such that $u_k^{(\alpha)}(0) = 0$ for $|\alpha| \leq l - d/2$ and

$$\int_{\mathbb{R}^d} |\xi|^{2l} |\hat{u}_k(\xi)|^2 d\xi = 1, \quad \lim_{k\to\infty} \int_{\mathbb{R}^d} |x|^{-2l} |u_k(x)|^2 dx = \infty. \tag{4.2}$$

Finally, Lemma 3.7 shows that such a sequence can be chosen from the class $C_0^\infty(\mathbb{R}^d \setminus \{0\})$.

Proposition 4.2 *If* $l - d/2 = 0, 1, \ldots$, *then there exists a sequence* $u_k \in C_0^\infty(\mathbb{R}^d \setminus \{0\})$ *such that relations* (4.2) *are satisfied.*

4.2. On the other hand, by Proposition 2.8, estimates (2.4) hold for $m \geq n + 1$ and estimates (2.15) hold for $m = n - 1, n - 3, \ldots$. In both cases the estimating constants are given by equality (2.26). If n is even, it follows from (2.27) that

$$\max_{\substack{m=1,3,\ldots,n-1 \\ m \geq n+1}} \{C_l^{(m)}\} = \max\{C_l^{(1)}, C_l^{(n+2)}\}. \tag{4.3}$$

Similarly, if n odd, then

$$\max_{\substack{m=0,2,\ldots,n-1\\ m\geq n+1}} \{C_l^{(m)}\} = \max\{C_l^{(0)}, C_l^{(n+2)}\}. \tag{4.4}$$

Let us show that the maxima in (4.3) and (4.4) are given by $C_l^{(1)}$ and $C_l^{(0)}$, respectively. By virtue of (2.26), we need to check that

$$\Gamma^2(-p+1/2)\Gamma^{-2}(-p+1/2+l) \geq \Gamma^{-2}(l+1), \quad p = [(n+1)/2]. \tag{4.5}$$

If l is integer, then (4.5) is equivalent to

$$(-p+1/2)^2(-p+3/2)^2\cdots(l-p-1/2)^2 \leq l!^2. \tag{4.6}$$

The first p factors in the left-hand side

$$(-p+1/2)^2(-p+3/2)^2\cdots(-1/2)^2 < l^2(l-1)^2\cdots(l-p+1)^2 = l!^2(l-p)!^{-2}$$

because $p-1/2 < l$. The remaining $l-p$ factors in the left-hand side

$$(1/2)^2(3/2)^2\cdots(l-p-1/2)^2 < (l-p)!^2.$$

Comparing the last two estimates we arrive at (4.6).

If l is half-integer, then

$$\Gamma(l+1) = \Gamma(1/2)(1/2)(3/2)\cdots l, \quad \Gamma(-p+1/2+l) = (l-p-1/2)!.$$

Since

$$\Gamma(-p+1/2) = \Gamma(1/2)(-1/2)^{-1}(-3/2)^{-1}\cdots(-p+1/2)^{-1},$$

inequality (4.5) is equivalent to

$$(1/2)(3/2)\cdots(p-1/2)(l-p-1/2)! \leq \pi(1/2)(3/2)\cdots l \tag{4.7}$$

or

$$(l-p-1/2)! \leq \pi(p+1/2)(p+3/2)\cdots l.$$

Clearly, $k < k+p+1/2$ for every $k = 1, 2, \ldots, l-p-1/2$. Moreover, the right-hand side contains the extra factor $\pi(p+1/2) \geq \pi/2$. This yields (4.7).

Suppose that u is orthogonal to the subspaces $\mathfrak{H}_m$ for all $m = n, n-2, \ldots$. Then functions (2.12) satisfy estimates (2.4) for $m \geq n+1$ and estimates (2.15) for $m = n-1, n-3, \ldots$. As we have seen, the maximum of the corresponding estimating constants over all admissible m equals $C_l^{(1)}$ if n is even and $C_l^{(0)}$ if n is odd. Actually, these constants have been explicitly calculated. Let us also take into account that, according to expansions (2.13) for functions (2.12), $u^{(\alpha)}(0) = 0$ for $|\alpha| = n, n-2, \ldots$. Now we are able to formulate the results obtained.

Theorem 4.3 *Let $u \in H^l(\mathbb{R}^d)$ with $l = d/2$ and let u be orthogonal to the subspace $\mathfrak{H}_0$ of spherically symmetric functions. Then inequality (1.1) holds with $C_l = (2l-1)!!^{-2}$ if d is even and with $C_l = 2^{-2l}\pi(l-1/2)!^{-2}$ if d is odd.*

Let $u \in H^l(\mathbb{R}^d)$ with $l = d/2+n$, $n = 1, 2, \ldots$, and let u be orthogonal to the subspaces $\mathfrak{H}_m$ for all $m = n, n-2, \ldots$. Set $\tau = 0$ for n odd, $\tau = 1$ for n even and $p = [(n+1)/2]$. Then

$$\int_{\mathbb{R}^d} |x|^{-2l} |u(x) - \sum_{|\alpha|=\tau,\tau+2,\ldots,n-1} (\alpha!)^{-1}(\partial^\alpha u)(0)x^\alpha|^2 dx \le C_l \int_{\mathbb{R}^d} |\xi|^{2l} |\hat{u}(\xi)|^2 d\xi, \tag{4.8}$$

where

$$C_l = (2p-1)!!^{-2}(2l-2p-1)!!^{-2}$$

if d is even and

$$C_l = 2^{-2l+2p}\pi(2p-1)!!^{-2}(l-p-1/2)!^{-2}$$

if d is odd.

Let, in particular, $d = 1$. If $l = 1/2$, then inequality (1.1) holds on odd functions. If $l = 1/2+n$, $n = 1, 2, \ldots$, then inequality (4.8) holds on odd functions if n is even and it holds on even functions if n is odd.

Similarly to Section 3, we can extend these results to larger classes of functions, check that the constants in the inequalities are optimal but the equalities are never realized.

4.3. According to Lemma 4.1 in the case (4.1) estimate (2.4) is definitely violated. However a similar estimate still holds if a logarithmic factor is added. More precisely, we have the following

Theorem 4.4 *Let $f \in C_0^\infty(\mathbb{R}_+)$ and let $\alpha_l^{(m)}$ be defined by equality (2.21). Then for all $m = 0, 1, \ldots$ and any $l \ge 1$*

$$\int_0^\infty r^{d-1-2l} \ln^{-2} r |f(r) - r^{l-d/2} f(1)|^2 dr \le K_l^{(m)} \int_0^\infty s^{d-1+2l} |(\Phi_m f)(s)|^2 ds, \tag{4.9}$$

where

$$K_l^{(m)} = 4 \sup_{\lambda \in \mathbb{R}} (\lambda^2 |\alpha_l^{(m)}(\lambda)|^2). \tag{4.10}$$

Proof. Note first of all that $K_l^{(m)}$ is finite because $\alpha_l^{(m)}(\lambda)$ is a continuous function of λ, $\alpha_l^{(m)}(\lambda) = O(|\lambda|^{-l})$ as $|\lambda| \to \infty$ by the Stirling formula and $\alpha_l^{(m)}(\lambda) = O(|\lambda|^{-1})$ as $|\lambda| \to 0$.

We shall show that (4.9) reduces to inequality (1.3) for $l = 1$ and $d = 1$. Indeed, let us make in (4.9) and (2.3) the change of variables $r = e^\tau$, $s = e^{-\sigma}$ and introduce new functions

$$u(\tau) = e^{(d/2-l)\tau} f(e^\tau), \quad v(\sigma) = e^{-(d/2+l)\sigma} (\Phi_m f)(e^{-\sigma}).$$

Then (4.9) is equivalent to the estimate

$$\int_{-\infty}^\infty \tau^{-2} |u(\tau) - u(0)|^2 d\tau \le K_l^{(m)} \int_{-\infty}^\infty |v(\sigma)|^2 d\sigma, \tag{4.11}$$

and (2.3) is equivalent to the equality

$$v(\sigma)=\int_{-\infty}^{\infty} e^{(l+1)(\tau-\sigma)}\mathcal{I}_{m+(d-2)/2}(e^{\tau-\sigma})u(\tau)d\tau. \tag{4.12}$$

Let $z\in C_0^\infty(\mathbb{R})$ be an arbitrary function and let $\hat z$ be its Fourier transform. Let us consider l as the complex variable. If $\operatorname{Re} l\in(-m-d/2,1/2)$, then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{(l+1)(\tau-\sigma)}\mathcal{I}_{m+(d-2)/2}(e^{\tau-\sigma})u(\tau)\overline{z(\sigma)}d\tau d\sigma=\int_{-\infty}^{\infty} k_l^{(m)}(\lambda)\hat u(\lambda)\overline{\hat z(\lambda)}d\lambda, \tag{4.13}$$

where, by virtue of (2.19), (2.21),

$$k_l^{(m)}(\lambda)=\int_{-\infty}^{\infty} e^{-(l+1)\tau}\mathcal{I}_{m+(d-2)/2}(e^{-\tau})e^{-i\lambda\tau}d\tau=\int_0^{\infty} r^{l+i\lambda}\mathcal{I}_{m+(d-2)/2}(r)dr=\alpha_l^{(m)}(\lambda)^{-1}. \tag{4.14}$$

Since both sides of (4.13) are analytic in l, this equality extends to all $l>0$. According to (4.12), (4.14) $\hat v(\lambda)=k_l^{(m)}(\lambda)\hat u(\lambda)$, so, by definition (4.10),

$$4\int_{-\infty}^{\infty}\lambda^2|\hat u(\lambda)|^2d\lambda\le K_l^{(m)}\int_{-\infty}^{\infty}|\hat v(\lambda)|^2d\lambda.$$

Now (4.11) is a direct consequence of (1.3) for $l=1$ and $d=1$. $\square$

Although valid for all m, inequality (4.9) is of interest mainly in the case (4.1) when estimate (2.4) fails. In this case the constant (4.10) can be calculated explicitly. Indeed, by the identity $\Gamma(z+1)=z\Gamma(z)$, it follows from (2.21) that

$$2^l|\mu\alpha_l^{(m)}(2\mu)|=|\mu\Gamma(-q+i\mu)\Gamma^{-1}(l-q+i\mu)|=\beta_l^{(m)}(\mu)|\Gamma(1+i\mu)\Gamma^{-1}(l-q+i\mu)|, \tag{4.15}$$

where $\beta_l^{(m)}(\mu)=1$ if $q=0$ and

$$\beta_l^{(m)}(\mu)=|(1+i\mu)(2+i\mu)\cdots(q+i\mu)|^{-1},\quad q\ge 1.$$

Clearly, the maximum of $\beta_l^{(m)}(\mu)$ equals $q!^{-1}$. Since $l-q\ge 1$, it follows from inequality (2.25) that the maximum of the second factor in (4.15) is also attained at the point $\mu=0$ and it equals $\Gamma^{-1}(l-q)$. Thus, we find

$$K_l^{(m)}=2^{4-2l}(q!\Gamma(l-q))^{-2}.$$

Inequality (4.9) can of course be extended to a larger than $C_0^\infty(\mathbb{R}_+)$ class of functions but we shall not dwell upon it.

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