On the thermodynamic limit for a one-dimensional sandpile process

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Abstract

Considering the standard abelian sandpile model in one dimension, we construct an infinite volume Markov process corresponding to its thermodynamic (infinite volume) limit. The main difficulty we overcome is the strong non-locality of the dynamics. However, using similar ideas as in recent extensions of the standard Gibbs formalism for lattice spin systems, we can identify a set of 'good' configurations on which the dynamics is effectively local. We prove that every configuration converges in a finite time to the unique invariant measure.

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1 Introduction.

The standard abelian sandpile model was introduced in 1988, [1]. This model lives on a finite graph (e.g. a square on $\mathbb{Z}^2$) in which to each vertex is associated a height-variable (representing the height of a sand column at that site or, the averaged difference in sandlevels with neighboring sites). A vertex is picked at random and its height is increased by one. If the height variable exceeds some critical value, then the vertex becomes "unstable" and "topples", i.e. gives an equal portion of its sand to each of its neighbours (adjacent vertices) which in their turn, can become "unstable" and "topple", and so on, until every vertex has again a subcritical height-value. Some vertices are connected to a sink (the boundary) where sand disappears. An unstable vertex thus creates an "avalanche" which can cause the toppling of many vertices around it. The number of vertices involved in or affected by one avalanche can be arbitrarily large (dependent on the configuration).

Over the last decade, sandpile processes and various versions thereof have been intensively studied as simple lattice models exhibiting the phenomenon of self-organized criticality. While it is still a subject of debate what is the precise nature of the phenomenon and under exactly what conditions it obtains, sandpiles have become widely appreciated as simple threshold dynamics showing scale invariance of various response functions with (critical) exponents of an astounding universality. Not surprisingly, a vast amount of computer simulations have been performed involving sandpile processes and their acclaimed self-organized criticality. Yet, both controlled laboratory or real nature experiments and a rigorous mathematical analysis have remained rather limited. We refer to [13] and the references therein for a number of experiments validating some of the universal aspects of the sandpile paradigm. In this paper we do not address questions of self-organized criticality nor do we refer to the most interesting physical realizations of sandpile-like models. Rather we turn to the simplest mathematical questions one can ask from the point of view of the theory of interacting particle systems for the simplest one-dimensional abelian sandpile process.

For the abelian sandpiles the first mathematical results have been obtained by D. Dhar, see [5, 6, 4] and the review paper [7]. As further discussed also in [16], the main question there was the characterization of the invariant measure and the recurrent configurations of the finite Markov chain defined in the sandpile process. In one dimension, the answers are rather simple. A detailed study can be found in [15]. Our questions concern the so called thermodynamic limit, i.e. how to construct a *bona fide* infinite volume limit dynamics and how to characterize its main properties. In [14], Priezzhev was able to compute exactly the single-site marginal of the invariant measure of the sandpile process on a square in the limit as the square tends to cover the whole plane. Various other mathematical results have been obtained but, for all we know, the problem of the thermodynamic limit has remained widely open so far. In the present paper we give a complete solution to this problem for the standard abelian sandpile process in one dimension.

It should be realized that this one-dimensional model has as such rather limited interest from the physics point of view. Yet, some of the mathematical questions pertaining to the thermodynamic limit remain very non-trivial and, arguably have answers which are expected to be not so different in higher dimensions. It is therefore unavoidable to start in one dimension and clarify the situation there. The main mathematical difficulty remains unchanged and has to do with the strong non-locality of the dynamics. It is built in in the definition of the sandpile process and it seems to be essential for some of its most interesting aspects. The effect of adding a sand particle at one site can affect the whole system. This implies that the standard machinery to construct a Feller process (Hille-Yoshida theorem, martingale problem) does not work. Similar problems were encountered in the study of long range exclusion processes, see [9], [8], [2]. Here the situation is even more exciting as will be seen from the convergence to the stationary measure in a finite time and from the relation between the process and its formal generator. This relation is only 'pointwise' for a certain class of 'decent' configurations (in particular the process does not have a generator in the standard sense). One finds analogies with some recent work on weakly and almost Gibbsian measures where the interaction energy is only absolutely summable on a full measure set of configurations but not uniformly, see [11, 12].

Our paper is organized as follows. In Section 2 we introduce the model and the formal generator of the sandpile process. In Section 3 we prove how to construct a Markov semigroup, and thus a Markov process which corresponds to this formal generator. The main technical tool in this construction is monotonicity. In Section 4 we show the precise relation between the semigroup and the formal generator. In Section 5 we study the invariant measure of the process and show that every initial configuration converges in a finite time to this invariant measure.

2 Definitions.

2.1 Configurations.

The state space of the one-dimensional sandpile process is $\Omega = \{1,2\}^{\mathbb{Z}}$. Elements of Ω are called configurations and are denoted by η, ξ, ζ. The state space is a compact metric space in the product topology. The value $\eta(x) = 2$

is called the *threshold value*. Sites for which $\eta(x) = 2$ are called *critical sites* in the configuration η. The set of all finite subsets of $\mathbb{Z}$ is denoted by $\mathcal{S}$. We define Ω_f to be the set of configurations with a finite number of critical sites, i.e.

$$\Omega_f := \{\eta \in \Omega : \eta^{-1}(\{2\}) \in \mathcal{S}\}. \tag{2.1}$$

There is a one-to-one correspondence between $\mathcal{S}$ and Ω_f via $A \to \eta_A$, where η_A denotes the configuration whose set of critical sites is A, i.e. $\eta_A^{-1}(\{2\}) = A$. The configurations in Ω_f will play an important role in the construction of the process (cf. below). A function $f : \Omega \to \mathbb{R}$ is called local if it depends only on a finite number of coordinates. Every local function is continuous and every continuous function is a uniform limit of local functions.

2.2 Toppling transformation T_i.

Given a configuration $\eta \in \Omega_f$ and a lattice site $i \in \mathbb{Z}$, $T_i\eta$ represents the configuration obtained by adding one sand particle at site i and letting the system *"topple"* until a (stable) configuration (i.e. an element of Ω_f) is obtained. This toppling is described in words as follows: If $\eta(i) = 2$ and one sand particle is added at site i, then from that site two particles will be removed, one to the left neighbour $i-1$ and one to the right neighbour $i+1$. If say $\eta(i+1)$ was already 2, then site $i+1$ will also "topple", giving one particle to i, and one to $i+2$. This goes on until a configuration in which $\eta(j) \le 2$ for all $j \in \mathbb{Z}$ is obtained. This succession of topplings at different sites caused by adding one particle to a critical site is called an *avalanche*, and thus $T_i\eta$ represents the configuration after the avalanche. If $\eta(i) = 1$, then $T_i\eta$ differs from η at site i only ($T_i\eta(i) = 2$), but if $\eta(i) = 2$, then $T_i\eta$ differs from η at three lattice sites which depend on i and η in a non-local way. This corresponds exactly to the dynamics of the standard abelian sandpile process in a finite lattice region.

More precisely we introduce

$$k^+(i, \eta) := \inf\{j \ge 0 : \eta(i + j) = 1\}, \tag{2.2}$$

and

$$k^-(i, \eta) := \inf\{j > 0 : \eta(i - j) = 1\}, \tag{2.3}$$

where $\inf \emptyset := +\infty$. We denote by e_i the mapping from $\mathbb{Z}$ to $\{0, 1\}$ which is one at site i and zero at all other site. We distinguish five cases:

1. $k^+(i, \eta) = 0$, i.e. $\eta(i) = 1$, then

$$T_i\eta = \eta + e_i. \tag{2.4}$$

2. $k^+(i,\eta) > 0$, $k^+(i,\eta) \vee k^-(i,\eta) < \infty$, then

$$T_i\eta = \eta + e_{i+k^+(i,\eta)} + e_{i-k^-(i,\eta)} - e_{i+k^+(i,\eta)-k^-(i,\eta)} \tag{2.5}$$

In what follows we abbreviate

$$i^\eta := i + k^+(i,\eta) - k^-(i,\eta). \tag{2.6}$$

3. $k^+(i,\eta) = \infty, k^-(i,\eta) < \infty$, then

$$T_i\eta = \eta + e_{i-k^-(i,\eta)} \tag{2.7}$$

4. $k^+(i,\eta) < \infty, k^-(i,\eta) = \infty$, then

$$T_i\eta = \eta + e_{i+k^+(i,\eta)} \tag{2.8}$$

5. $k^+(i,\eta) = k^-(i,\eta) = \infty$ (i.e. $\eta \equiv 2$), then

$$T_i\eta = \eta. \tag{2.9}$$

Cases 3,4,5 above are just conventions to define $T_i\eta$ for all $\eta \in \Omega$, and to write down the formal generator, but they are not relevant in the construction of the sandpile process. That these choices are 'correct' will be seen later once it is shown that the configuration $\eta \equiv 2$ is invariant for the constructed process (cf. Proposition 5.1).

The explicit and simple expression of $T_i\eta$ is typical for the model in dimension one. In higher dimensions $T_i\eta$ will become a much more complicated function of η. Note that the transformations T_i and T_j commute, i.e.

$$T_i(T_j(\eta)) = T_j(T_i(\eta)),\ \forall i,j \in \mathbb{Z},\ \forall \eta \in \Omega. \tag{2.10}$$

For this reason the sandpile model has been called *abelian* in the physics literature, cf. [5].

2.3 Monotonicity.

In the construction of the sandpile process, monotonicity will play a crucial role, just as in the construction of the long-range exclusion process, cf. [9], [8]. For more details on monotonicity, we refer to [10], Chapter 2, Section 2.

For $\eta,\xi \in \Omega$ we define $\eta \le \xi$ if $\eta(x) \le \xi(x)$ for all $x \in \mathbb{Z}$. A function $f : \Omega \to \mathbb{R}$ is called *monotone* if $\eta \le \xi$ implies $f(\eta) \le f(\xi)$, for all $\eta,\xi \in \Omega$. We denote by $\mathcal{M}$ the class of all monotone bounded Borel measurable functions. A Markov process $\{\eta_t : t \ge 0\}$ on Ω with Markov semigroup $\{S(t) : t \ge 0\}$ is called monotone if $f \in \mathcal{M}$ implies $S(t)f \in \mathcal{M}$. Usually one proves monotonicity of a Markov process by explicitly constructing a coupling of path space measures such that for $\eta \le \xi$ one has $P^{\eta,\xi}(\eta_t \le \xi_t,\ \forall t \ge 0) = 1$.

2.4 Formal generator.

In sandpile processes, the avalanches are immediate. Once the vertex i is chosen, the transformation $\eta \to T_i\eta$ happens instantaneously. This is what one calls *infinite separation of time scales* in the physics literature, and it has been argued as essential to have a so-called self-organized critical state.

The formal generator of the sandpile process has therefore to be written as

$$Lf(\eta) = \sum_{i\in\mathbb{Z}} [f(T_i\eta) - f(\eta)]. \tag{2.11}$$

Since L is of course not the generator of a Feller semigroup, we shall have to give a precise meaning to this description of the dynamics, and to clarify how to "associate" a process to L.

We can rewrite the formal generator

$$Lf(\eta) = L_b f(\eta) + L_a f(\eta), \tag{2.12}$$

where

$$L_b f(\eta) = \sum_{i\in\mathbb{Z}} \chi(\eta(i) = 1)[f(\eta + \delta_i) - f(\eta)] \text{ is the "birth part"}, \tag{2.13}$$

$$L_a f(\eta) = \sum_{i\in\mathbb{Z}} \chi(\eta(i) = 2)[f(T_i\eta) - f(\eta)] \text{ is the "avalanche part"}, \tag{2.14}$$

and χ denotes the indicator function. The operator L_b corresponds to a pure birth process (independent births with rate one at each lattice site) and thus is a well-defined generator of a Feller semigroup. The avalanche part however gives technical problems related to the nonlocality of the transformation T_i. One cannot run the standard Hille-Yoshida program, which would give a Feller semigroup associated to $L_b + L_a$. Indeed, the process we shall associate to L will be constructed by monotonicity and turn out not to be Feller.

Remark. The processes we consider here run in continuous time while it is a standard procedure to describe the finite volume sandpile process in discrete time. However, since both updating mechanisms are sequential, the difference is unimportant.

3 Construction of the sandpile process.

We proceed in three steps:

1. Definition of a process associated to L_a (only avalanches) on Ω_f. Construction of a coupling showing that this process is monotone.

2. Definition of a process associated to L_a + births in the finite interval $[-n,n]$ on Ω_f. Construction of a coupling showing that this process is still monotone. This will give us a semigroup defined on $f \in \mathcal{M}$ by $S_n(t)f(\eta) = \lim_{\eta' \in \Omega_f,\ \eta' \uparrow \eta} S_n(t)f(\eta')$, where $\lim_{\eta' \uparrow \eta}$ denotes the limit along sequences $\{\eta'_n, n \in \mathbb{N}\} \subset \Omega_f$ such that ${\eta'}_n \leq {\eta'}_{n+1}$.

3. Monotonicity of the semigroups $S_n(t)$ in n, i.e., for all $f \in \mathcal{M}$, $\eta \in \Omega$, $t \geq 0$, $S_{n+1}(t)f(\eta) \geq S_n(t)f(\eta)$.

We shall finally define the sandpile process as the process associated to the semigroup

$$S(t)f(\eta) = \lim_{n \uparrow \infty} \lim_{\eta' \in \Omega_f, \eta' \uparrow \eta} S_n(t)f(\eta'). \tag{3.15}$$

3.1 Step 1: Avalanche part of the generator.

For $\eta \in \Omega_f$, $A(\eta) := \eta^{-1}(\{2\}) \in \mathcal{S}$ and we can construct the process with generator L_a from the non-exploding continuous time Markov chain $\{A_t : t \geq 0\}$ on $\mathcal{S}$ with generator

$$\hat{L_a} f(A) := \sum_{i \in A} \left[f(A \setminus \{i^\eta\} \cup \{i + k^+(i,\eta)\} \cup \{i - k^-(i,\eta)\}) - f(A) \right], \tag{3.16}$$

for $f : \mathcal{S} \to \mathbb{R}$ bounded. Indeed, $\mathbb{E}^A |A_t| = |A| e^t$ (where $|A|$ denotes the cardinality of A) so that $A_t \in \mathcal{S}$, for all $t \geq 0$, P^A-a.s. The process with generator L_a (avalanche part) is thus defined on Ω_f by $\eta_t := \eta_{A_t}$, where $\{A_t : t \geq 0\}$ starts from $A_0 = \eta^{-1}(\{2\})$. We show that this process is monotone via the coupling of η_t, ξ_t (for $\eta, \xi \in \Omega_f$ with $\xi \leq \eta$) described as follows. When a sand particle is added in η at site $i \in \mathbb{Z}$ where $\eta(i) = 2$, an avalanche creates a 1 at site $i^\eta = i + k^+(i,\eta) - k^-(i,\eta)$. Define now $\phi(i,\eta,\xi)$ as the unique site on $\mathbb{Z}$ such that a sand particle added at that site in configuration ξ would create a 1 in ξ at the same site i^η. More precisely, if $\xi(i^\eta) = 2$,

$$\phi(i,\eta,\xi) = i^\eta + k^+(i^\eta,\xi) - k^-(i^\eta,\xi), \tag{3.17}$$

and if $\xi(i^\eta) = 1$, we define $\phi(i,\eta,\xi) = \infty$ and $T_\infty \xi = \xi$. We now write the coupling generator on $\Omega_f \times \Omega_f$ for initial configurations (η,ξ) with $\eta \geq \xi$ as

$$L_a^c f(\eta,\xi) = \sum_{i \in \mathbb{Z}} \chi(\eta(i) = 2)[f(T_i \eta, T_{\phi(i,\eta,\xi)} \xi) - f(\eta,\xi)]. \tag{3.18}$$

Proposition 3.1 *1. On $\Omega_f \times \Omega_f$ the generator L_a^c defines a Markov process. For $\eta \geq \xi$, the path space measure $P^{\eta,\xi}$ is a coupling of P^η and P^ξ (the path space measures of the processes with generator L_a starting respectively from η and ξ).*

2. *For $\eta \geq \xi$ we have $P^{\eta,\xi}(\eta_t \geq \xi_t,\ \forall t \geq 0) = 1$.*

3. *The process with generator L_a is monotone on Ω_f. For the semigroup $S_a(t) := \exp(tL_a)$ associated to L_a, $f \in \mathcal{M}$ implies $S_a(t)f \in \mathcal{M}$.*

Proof: To check that the process with generator L_a^c has the right second marginal for $\eta \geq \xi$, remark that for every $j \in \mathbb{Z}$ such that $\xi(j) = 2$, there exists a unique $i \in \mathbb{Z}$ such that $\phi(i, \eta, \xi) = j$ (L_a^c does not define a coupling of P^η and P^ξ for general couples (η, ξ) but only for the ordered couples, i.e. (η, ξ) with $\eta \geq \xi$ and $\eta, \xi \in \Omega_f$). For points 2 and 3, note that by construction any transition in the coupled process preserves the order. ∎

3.2 Step 2: Avalanche + birth in a finite interval.

We now include the birth part of the formal generator L, only on the interval $[-n, n]$. We have to prove that the associated process on Ω_f is still monotone, since births and avalanches do not commute. For this, the generator

$$
\begin{aligned}
L^{(n)} f(\eta) &= \sum_{i \in \mathbb{Z}} \chi(\eta(i) = 2)[f(T_i \eta) - f(\eta)] \\
&+ \sum_{i=-n}^{n} \chi(\eta(i) = 1)[f(T_i \eta) - f(\eta)] \qquad (3.19)
\end{aligned}
$$

defines a continuous time Markov chain on $\mathcal{S}$, and thus a process on Ω_f. We now couple two versions of this process starting from $\eta, \xi \in \Omega_f$ such that $\eta \geq \xi$, with the generator

$$
\begin{aligned}
L^{c,(n)} f(\eta, \xi) &= \sum_{i} \chi(\eta(i) = 2)\chi(\xi(i^\eta) = 2)[f(T_i \eta, T_{\phi(i,\eta,\xi)}\xi) - f(\eta, \xi)] \\
&+ \sum_{i \in [-n,n]} \chi(\eta(i) = 2)\chi(\xi(i) = 1)[f(\eta, T_i \xi) - f(\eta, \xi)] \\
&+ \sum_{i \in [-n,n]} \chi(\eta(i) = 1)\chi(\xi(i) = 1)[f(T_i \eta, T_i \xi) - f(\eta, \xi)] \\
&+ \sum_{i} \chi(\eta(i) = 2)\chi(\xi(i^\eta) = 1)[f(T_i \eta, \xi) - f(\eta, \xi)]. \qquad (3.20)
\end{aligned}
$$

This coupling is described as follows: To each site $i \in \mathbb{Z}$ we associate two independent Poisson clocks, C and C'. C indicates the "avalanche event-times" and C' the "birth event-times". When C rings at i and $\eta(i) = 2$, we simultaneously add a sand particle at i in η and at $\phi(i, \eta, \xi)$ in ξ (if $\phi(i, \eta, \xi) = \infty$, we have no transition in ξ). When C' rings at i and $\eta(i) = 1$, then, since $\eta \geq \xi$, $\xi(i) = 1$ and we add a sand particle in both ξ and η

simultaneously. If however $\eta(i) = 2$ and $\xi(i) = 1$, then we only add in ξ at site i. Like in Section 3.1, this construction implies

Proposition 3.2 *1. The generator $L^{c,(n)}$ defines a Markov process on $\Omega_f \times \Omega_f$. For $\eta \geq \xi$ its path space measure $P_n^{(\eta,\xi)}$ is a coupling of P_n^η and P_n^ξ, the path space measures of the processes with generator $L^{(n)}$ starting respectively from η and ξ.*

2. If $\eta \geq \xi$, then $P_n^{(\eta,\xi)}(\eta_t \geq \xi_t,\ \forall t \geq 0) = 1$.

3. The process with generator $L^{(n)}$ is monotone on Ω_f. For the associated semigroup $S_n(t) := \exp(tL^{(n)})$, $f \in \mathcal{M}$ implies $S_n(t)f \in \mathcal{M}$.

3.3 Step 3: Avalanche + births in different intervals.

This final step is the easiest: By allowing births on a larger interval, we can only obtain larger configurations. We couple the processes with generators $L^{(n+1)}$ and $L^{(n)}$ through the following generator, where $\eta, \xi \in \Omega_f$, $\eta \geq \xi$.

$$\begin{aligned}
L^{c,n+1,n} f(\eta,\xi) &= \sum_i \chi(\eta(i)=2)\chi(\xi(i^\eta)=2))[f(T_i\eta, T_{\phi(i,\eta,\xi)}) - f(\eta,\xi)] \\
&+ \sum_{i\in[-n,n]} \chi(\eta(i)=2)\chi(\xi(i)=1)[f(\eta, T_i\xi) - f(\eta,\xi)] \\
&+ \sum_{i\in[-n,n]} \chi(\eta(i)=1)\chi(\xi(i)=1)[f(T_i\eta, T_i\xi) - f(\eta,\xi)] \\
&+ \sum_{i\in\{-n-1,n+1\}} \chi(\eta(i)=1)\chi(\xi(i)=1)[f(T_i\eta,\xi) - f(\eta,\xi)] \\
&+ \sum_i \chi(\eta(i)=2)\chi(\xi(i^\eta)=1)[f(T_i\eta,\xi) - f(\eta,\xi)]. \quad (3.21)
\end{aligned}$$

Proposition 3.3 *1. The generator $L^{c,n+1,n}$ defines a Markov process on $\Omega_f \times \Omega_f$. For $\eta \geq \xi$ the path space measure $P_{n+1,n}^{\eta,\xi}$ is a coupling of P_{n+1}^η and P_n^ξ.*

2. For $\eta \geq \xi$, $P_{n+1,n}^{\eta,\xi}(\eta_t \geq \xi_t,\ \forall t \geq 0) = 1$.

3. For $f \in \mathcal{M}$, $\eta \in \Omega_f$, we have $S_{n+1}(t)f(\eta) \geq S_n(t)f(\eta)$.

Proof: By construction any transition in the process with generator $L^{c,n+1,n}$ preserves the order. To see point 3, for $f \in \mathcal{M}$

$$\begin{aligned}
S_{n+1}(t)f(\eta) &= \mathbb{E}_{(n+1)}^\eta f(\eta_t) = \mathbb{E}_{n+1,n}^{\eta,\eta} f(\eta_t^1) \geq \mathbb{E}_{n+1,n}^{\eta,\eta} f(\eta_t^2) \\
&= \mathbb{E}_n^\eta f(\eta_t) = S_n(t)f(\eta). \quad (3.22)
\end{aligned}$$

∎

3.4 Construction of the process.

We can now proceed to the definition of the sandpile process as a well-defined Markov process with formal generator L. For functions $f \in \mathcal{M}$, on Ω we define the semigroup associated to L_n by

$$S_n(t)f(\eta) = \lim_{\eta' \in \Omega_f, \eta' \uparrow \eta} S_n(t)f(\eta'). \tag{3.23}$$

In (3.23) the limit is taken along the increasing sequence η'_n defined by $\eta'_n(x) = \eta(x)$ for $|x| \le n$, and $\eta'_n(x) = 1$ otherwise, but any other increasing sequence $\eta'_n \uparrow \eta$ gives the same limit. Then, for $f \in \mathcal{M}$ and $\eta \in \Omega$,

$$S_{n+1}(t)f(\eta) \ge S_n(t)f(\eta). \tag{3.24}$$

Since $S_n(t)f(\eta) \le \|f\|_\infty$ (where $\|f\|_\infty = \sup_{\eta\in\Omega} |f(\eta)|$), for $f \in \mathcal{M}$ and $\eta \in \Omega$ we can define

$$S(t)f(\eta) = \lim_{n\uparrow\infty} S_n(t)f(\eta). \tag{3.25}$$

Because each $S_n(t)$ is a semigroup on functions on Ω_f, we have, using (3.24), that $S(t)1 = 1$ and that $S(t)f \ge 0$ if $f \ge 0$. Moreover, for $f \in \mathcal{M}$,

$$\begin{aligned} S(t+s)f &= \lim_{n\uparrow\infty} S_n(t+s)f = \lim_{n\uparrow\infty} S_n(t)S_n(s)f \\ &\le \lim_{n\uparrow\infty} S_n(t)(S(s)f) = S(t)S(s)f, \end{aligned} \tag{3.26}$$

and for all $m \in \mathbb{N}$

$$\begin{aligned} S(t+s)f &= \lim_{n\uparrow\infty} S_n(t)S_n(s)f \ge \liminf_{n\uparrow\infty} S_m(t)(S_n(s)f) \\ &\ge S_m(t)(\liminf_{n\uparrow\infty} S_n(s)f) = S_m(t)S(s)f, \end{aligned} \tag{3.27}$$

so by taking the limit as $m \uparrow \infty$ we derive the semigroup property for $S(t)$. This semigroup defines the finite dimensional distributions of a Markov process; by Kolmogorov's theorem, there is a unique Markov process $\{\eta_t : t \ge 0\}$ such that $S(t)f(\eta) = \mathbb{E}^\eta f(\eta_t)$. We call this process the *one-dimensional sandpile process* (SP). We denote its path space measure (i.e. a measure on $\Omega^{[0,\infty)}$) starting from η by P^η_{SP}.

Remarks.

1. We cannot conclude that P^η_{SP} has a version concentrating on $D([0,\infty),\Omega)$ (the set of cadlag paths on Ω) for all initial η (cf. Corollary 5.1 below).

This question is related to the following one: For which $\eta \in \Omega$ do we have

$$\lim_{t \downarrow 0} S(t) f(\eta) = f(\eta) \tag{3.28}$$

for all local f? It is not even clear that this limit exists. In contrast, this is automatic for long range exclusion: But in our case, because the values of a configuration in a finite volume are influenced by transitions outside this volume, the estimate (2.3) of [9] does not hold.

2. We cannot expect that the formal generator L is "really" the generator of the process. However we do prove in Theorem 4.1 that for some class of "decent" configurations and local f

$$\lim_{t \downarrow 0} \frac{1}{t}(S(t)f(\eta) - f(\eta)) = Lf(\eta). \tag{3.29}$$

This situation of pointwise convergence for a set of "decent" configurations, rather than uniform convergence, can be compared to the situation of "weakly Gibbsian measures" where the potential is only convergent on a set of good configurations, cf. [11],[12]. In the following section we will deal with (3.29) and show that for short times (depending on η) and for "decent" configurations (a set of measure one for every ergodic probability measure on Ω, except δ_2, the measure concentrating on $\eta \equiv 2$) we actually have

$$S(t)f(\eta) = \sum_{n=0}^{\infty} \frac{t^n}{n!} L^n f(\eta), \tag{3.30}$$

which is of course much stronger than (3.29) (but note that we do have (3.30) for the pure birth process).

3. In order to find an invariant measure for the sandpile process, we cannot proceed in a standard way. Since the process is not Feller (see Corollary 5.1 below), a weak limit point of $\frac{1}{T}\int_0^T \mu S(t) dt$ does not need to be invariant, and $\int Lf d\mu = 0$ does not imply that μ is invariant. In Section 5 we shall show that the only invariant measure is δ_2 and that every initial measure converges to δ_2 *in a finite time.* This explosive convergence to the invariant measure is related to the non-locality of the dynamics and to the fact that we continue to add sand at rate one at each lattice site (cf. [2]).

4 Decent Configurations.

As we mentioned in Section 2, the transformation T_i is non-local. Nevertheless we still have some kind of locality: Let $f : \Omega \to \mathbb{R}$ be a local function depending on coordinates in $A \in \mathcal{S}$. Put $A^- := \min(A)$, $A^+ := \max(A)$, then we have

$$T_i f(\eta) - f(\eta) = 0,\ \forall i \in \mathbb{Z} \setminus [A^- - k^-(A^-, \eta), A^+ + k^+(A^+, \eta)]. \tag{4.31}$$

This means that the effect of adding a sand particle is only felt in the region bounded by the first one to the left and the first one to the right of the dependence set of f. Loosely speaking, $T_i f$ has a dependence set which is finite but dependent on the configuration.

We now formalize these statements. Let Ω_1 denote the set of configurations with an infinite number of ones to the right and to the left of the origin, i.e.,

$$\Omega_1 := \{\eta \in \Omega : \sum_{i<0}(2 - \eta(i)) = \sum_{i>0}(2 - \eta(i)) = +\infty\}. \tag{4.32}$$

For $\eta \in \Omega_1$ we order the sites $i \in \mathbb{Z}$ for which $\eta(i) = 1$ as follows

$$\eta^{-1}(\{1\}) := \{X_i(\eta) : i \in \mathbb{Z}\} \tag{4.33}$$

where $X_0(\eta) := \min\{i \geq 0 : \eta(i) = 1\}$ and $i < j$ implies $X_i(\eta) < X_j(\eta)$. Then we define

$$\begin{aligned} I_0 &= (X_{-1}, X_0] \cap \mathbb{Z} \\ I_1 &= (X_0, X_1] \cap \mathbb{Z} \\ I_{-1} &= (X_{-2}, X_{-1}] \cap \mathbb{Z}, \quad \text{etc.} \end{aligned} \tag{4.34}$$

In this way we view configuration η as a sequence of intervals, where each interval is bounded by two sites $i, j \in \mathbb{Z},\ i < j$, for which $\eta(i) = \eta(j) = 1$. A function $f : \Omega_1 \to \mathbb{R}$ will be called $N-$*local* around $k \in \mathbb{Z}$ if

$$f(\eta) = f\left(\eta(i),\ i \in \bigcup_{j=-N}^{N} I_{j+k}(\eta)\right). \tag{4.35}$$

We say that f is N-local if it is N-local around the origin.

Examples.

1. $f(\eta) = \eta(i)$ is $0-$local around i, and every local function is N-local for some N.

2. $f(\eta) = |I_k(\eta)|$ is $k-$local for every k, but is not local.

3. $f(\eta) = \sum_{x\in \mathbb{Z}} e^{-|x|}\eta(x)$, while continuous, is not N-local for any N.

Proposition 4.1 *Let $f : \Omega_1 \to \mathbb{R}$ be $N-$local then*

$$f(T_i\eta) - f(\eta) = 0 \textit{ for all } i \in \mathbb{Z}\setminus \bigcup_{j=-N-1}^{N+1} I_j(\eta), \tag{4.36}$$

and Lf is $(N+1)$-local.

Proof: If $i \in \cup_{j=-N-1}^{N+1} I_j(\eta)$, then, for f N-local, $f(T_i\eta)$ is $(N+1)$-local. Otherwise $T_i\eta = \eta$. ∎

Proposition 4.2 *Let $f : \Omega_1 \to \mathbb{R}$ be $N-$local and bounded, then we have*

$$|L^n f(\eta)| \le (|I_{-N-n}(\eta)| + \ldots + |I_{n+N}(\eta)|)^n\, 2^n \|f\|_\infty. \tag{4.37}$$

Proof: We put $N = 0$ for the sake of simplicity, the case $N > 0$ is treated in the same way. We proceed by induction. For $n = 1$

$$Lf(\eta) = \sum_i [f(T_i\eta) - f(\eta)], \tag{4.38}$$

and, by Proposition 4.1, this sum runs over $i \in I_{-1}(\eta) \cup I_0(\eta) \cup I_1(\eta)$, Lf is $1-$local, and

$$|Lf(\eta)| \le (|I_{-1}(\eta)| + |I_0(\eta)| + |I_1(\eta)|)2\|f\|_\infty. \tag{4.39}$$

Suppose now that for $n \ge 1$, $L^n f$ is $n-$local and

$$|L^n f(\eta)| \le (|I_{-n}(\eta)| + \ldots + |I_n(\eta)|)^n 2^n \|f\|_\infty. \tag{4.40}$$

Then we have

$$L^{n+1} f(\eta) = \sum_i [L^n f(T_i\eta) - L^n f(\eta)], \tag{4.41}$$

where the sum runs over $i \in I_{-n-1}(\eta) \cup \ldots \cup I_{n+1}(\eta)$, since $L^n f$ is $n-$local. Therefore, for such an i,

$$\begin{aligned} |L^n f(T_i\eta)| &\le [|I_{-n}(T_i\eta)| + \ldots + |I_n(T_i\eta)|]^n\, 2^n \|f\|_\infty \\ &\le [|I_{-n-1}(\eta)| + \ldots + |I_{n+1}(\eta)|]^n\, 2^n \|f\|_\infty. \end{aligned} \tag{4.42}$$

Thus

$$|L^{n+1} f(\eta)| \le [|I_{-n-1}(\eta)| + \ldots + |I_{n+1}(\eta)|]^{n+1}\, 2^{n+1} \|f\|_\infty. \tag{4.43}$$

∎

The following lemma is a direct consequence of Cauchy's formula for the radius of convergence of a power series.

Lemma 4.1 *Let $\{a_n : n \geq 0\}$ be a sequence of positive real numbers such that $\limsup_{n\to\infty} a_n/n = a < \infty$. Then the series $\sum_{n=0}^{\infty} t^n a_n^n/n!$ converges for $|t| < \frac{1}{a.e}$.*

We now define the set of "decent" configurations:

$$\Omega_{\text{dec}} := \{\eta \in \Omega_1 : a(\eta) := \limsup_{n\to\infty} \frac{|I_{-n}(\eta)| + \ldots + |I_n(\eta)|}{2n} < \infty\}. \tag{4.44}$$

Observe that for $\eta \in \Omega_f$, if $|i|$ is large enough, $|I_i(\eta)| = 1$, hence $a(\eta) = 1$. Therefore $\Omega_f \subset \Omega_{\text{dec}}$. More generally, if the density of ones

$$\rho(\eta) = \lim_{n\to\infty} \frac{\sum_{i=-n}^{n}(2-\eta(i))}{2n+1} \tag{4.45}$$

exists, then $a(\eta) = 1/\rho(\eta)$. This implies that for every probability measure μ ergodic under spatial translations and for which $\mu(2-\eta(0)) > 0$, the set Ω_{dec} has μ-measure 1. It is also clear that Ω_{dec} is a translation invariant set in the tail field, i.e., if $\eta \in \Omega_{\text{dec}}$ and ζ differs from η in a finite number of lattice sites, then $\zeta \in \Omega_{\text{dec}}$.

Theorem 4.1 *Let f be $N-$local for some $N \in \mathbb{N}$ and $\eta \in \Omega_{dec}$, then for $t < \frac{1}{4ea(\eta)}$ the series $\sum_{n=0}^{\infty} \frac{t^n L^n f(\eta)}{n!}$ converges absolutely and equals $S(t)f(\eta)$. In particular, L is the "pointwise generator" of the semigroup, i.e.*

$$\lim_{t\downarrow 0} \frac{S(t)f(\eta) - f(\eta)}{t} = Lf(\eta). \tag{4.46}$$

Proof: The absolute convergence of the series follows from Proposition 4.2, Lemma 4.1 and (4.44). By definition, for an N-local $f \in \mathcal{M}$

$$S(t)f(\eta) = \lim_{n\uparrow\infty} \lim_{\eta'\in\Omega_f, \eta'\uparrow\eta} S_n(t)f(\eta'). \tag{4.47}$$

For $\eta' \in \Omega_f$, $a(\eta') = 1$ and

$$S_n(t)f(\eta') = \sum_{k=0}^{\infty} t^k/k!(L_n)^k f(\eta'). \tag{4.48}$$

Moreover, for every $k \in \mathbb{N},\ \eta \in \Omega$:

$$\lim_{n\uparrow\infty} \lim_{\eta'\in\Omega_f, \eta'\uparrow\eta} (L_n)^k f(\eta') = L^k f(\eta). \tag{4.49}$$

We can bring in the limits $\lim_{n\uparrow\infty} \lim_{\eta'\uparrow\eta}$ into the sum of (4.48) by the dominated convergence theorem together with Lemma 4.1 and the inequalities

$$\begin{aligned} |L_n^k f(\eta')| &\leq (|I_{-N-k}(\eta')| + \ldots + |I_{N+k}(\eta')|)^k\, 2^k \|f\|_\infty \\ &\leq (|I_{-N-k}(\eta) + \ldots + |I_{N+k}(\eta)|)^k\, 2^k \|f\|_\infty, \end{aligned} \tag{4.50}$$

for $\eta' \leq \eta,\ \eta' \in \Omega_f$. ■

Remark. Theorem 4.1 shows that the infinite volume sandpile process is indeed a natural extension of the finite volume standard abelian sandpile model. More precisely, if P_n denotes the transition probability matrix of the discrete-time sandpile model in the finite volume $\Lambda_n = [-n, n] \cap \mathbb{Z}$, then we have for $\eta \in \Omega_{\text{dec}}$ and t small enough (depending on η),

$$\lim_{n\uparrow\infty} P_n^{\lfloor nt \rfloor} f(\eta^n) = S(t) f(\eta), \tag{4.51}$$

where η^n denotes the restriction of η to Λ_n.

5 Invariant measure for the sandpile process.

In this section we show that after a finite time, the process reaches δ_2, which is therefore its only invariant measure. Nevertheless we first prove Proposition 5.1 to illustrate some "pathological" behavior of the process (cf. Corollary 5.1).

Proposition 5.1 *δ_2 is invariant for the sandpile process.*

Proof: Denote by $\hat{2}$ the configuration $\eta \equiv 2$. Notice that this is not a decent configuration. Thus we have to prove that

$$S(t) f(\hat{2}) = f(\hat{2}). \tag{5.52}$$

Without loss of generality we put $f(\eta) = \eta(0) - 1$. Clearly, $S(t)f(\hat{2}) \leq f(\hat{2})$, so we are left to show the opposite inequality. By definition (3.25),

$$\begin{aligned} S(t) f(\hat{2}) &= \lim_{n\to\infty} \lim_{\eta' \in \Omega_f, \eta' \uparrow \eta} S_n(t) f(\eta') \\ &\geq \lim_{\eta' \in \Omega_f, \eta' \uparrow \eta} S_a(t) f(\eta') \\ &= \lim_{n\uparrow\infty} P^{A_n}(0 \in A(t)). \end{aligned} \tag{5.53}$$

Here P^{A_n} denotes the Markov measure on $\mathcal{S}^{[0,\infty)}$ associated to the process with generator $\hat{L}_a$ (only avalanches), starting from the set $A_n := [-n, n] \cap \mathbb{Z}$. The set $A(t)$ will always be of the form $I(t) \setminus \{x\}$ where $I(t)$ is an interval which grows in time. The probability that $\{x\} = \{0\}$ will thus be of the order $(|A_n| + f(t))^{-1}$ where $f(t)$ is some increasing function of t. Let us make this precise. Denote by $\tau_1, \tau_2, \ldots, \tau_n$ the successive event times of the Markov chain $\{A(t) : t \geq 0\}$. Then we have

$$\begin{aligned} P^{A_n}(0 \notin A(\tau_1)) &= P^{A_n}(x \notin A(\tau_1)),\ \forall x \in A_n \\ &= \frac{1}{|A_n|}, \\ P^{A_n}(0 \notin A(\tau_2)) &= \sum_{B \in \mathcal{S}} \frac{1}{|B|} P^{A_n}(A(\tau_1) = B) = \frac{1}{|A_n| + 1}, \end{aligned} \tag{5.54}$$

and analogously

$$P^{A_n}(0 \notin A(\tau_n)) = \frac{1}{|A_n| + n - 1}. \tag{5.55}$$

Hence we conclude that for all $t \geq 0$

$$P^{A_n}(0 \notin A(t)) \leq \frac{1}{|A_n|} = \frac{1}{2n+1}. \tag{5.56}$$

Therefore

$$\limsup_{n \uparrow \infty} P^{A_n}(0 \notin A(t)) = 0 \tag{5.57}$$

which proves the claim. ■

Corollary 5.1 *The sandpile process is not Feller, and for some initial configurations the process has no right-continuous version.*

Proof: As in the preceding proof, we have that for all $\eta \in \Omega$ such that $\eta^{-1}(\{1\}) \in \mathcal{S}$ (finite number of ones)

$$P^{\eta}_{SP}(\eta_t(x) \neq 2) = 0,\ \forall t > 0. \tag{5.58}$$

Therefore there is no right-continuous version of the process starting from these configurations (cf. Remark 1 of Section 3.4). ■
The following theorem will be proven at the end of this section.

Theorem 5.1 *In the sandpile process there exists a finite $T > 0$ such that for all $t \geq T$, $\eta \in \Omega$,*

$$P^{\eta}_{SP}[\eta_t(0) = 1] = 0. \tag{5.59}$$

This result seems intuitive: Since we continue adding sand at each lattice site with rate 1, after a time of order unity, every site will have received an extra sand particle. Once the configuration is full, it remains full. Nevertheless, this intuitive picture is somewhat dangerous, since we are working in infinite volume, and it could be possible that sand particles "disappear" at infinity. Indeed, the same statement is not true anymore in higher dimensions: Although we still expect convergence to the invariant measure in a finite time, this invariant measure will not be the Dirac measure concentrating on the "full configuration".

5.1 The finite volume sandpile process.

For the finite volume sandpile process in $[-n, n]$, we only add sand particles in $[-n, n]$, and at the boundaries we put fixed ones (i.e., sand falls off the boundary and disappears when the boundary site topples). More formally, for $\eta \in \Omega$, we denote by η^n the configuration which is equal to η in $[-n, n]$ and is identically 1 outside that interval. Define

$$T_{n,i}\eta := [T_i \eta^n]^n , \tag{5.60}$$

$$\Omega_{f,n} := \{\eta \in \Omega : \eta(j) = 1, \ \forall j \in \mathbb{Z} \setminus [-n, n]\}. \tag{5.61}$$

Note that $\Omega_f = \cup_n \Omega_{f,n}$. The finite volume sandpile process (FVSP) is the pure jump process on $\Omega_{f,n}$ with generator

$$\mathcal{L}_n f(\eta) = \sum_{i=-n}^{n} [f(T_{n,i}\eta) - f(\eta)] \tag{5.62}$$

5.2 Poisson representation.

Let $\{N_t : t \geq 0\} := \{N_t^i, i \in \mathbb{Z}, t \geq 0\}$ be a collection of independent rate one Poisson processes. For $\Xi \in \mathbb{N}_0^{\mathbb{Z}}$ we define $\theta_n(\Xi)$ to be the configuration obtained from "toppling" in $[-n, n]$ and fixing ones outside, i.e.,

$$\theta_n(\Xi) = \left[\prod_{i=-n}^{n} T_{n,i}^{\Xi(i)-1}\right] \hat{1} \tag{5.63}$$

where $\hat{1}$ denotes the configuration $\eta \equiv 1$, and $T_{n,i}^{\Xi(i)-1}$ means $\Xi(i)-1$ iterations of $T_{n,i}$.

Proposition 5.2 *The process $\{\theta_n(N_t + \eta) : t \geq 0\}$ is a version of the FVSP starting from $\eta \in \Omega_{f,n}$ ($N_t + \eta$ means coordinatewise addition).*

Proof: Notice that the transformations $T_{n,i}$ commute for different i. Therefore, first adding all the particles and then letting the system topple is equivalent to adding the particles one by one and letting the system topple each time. More formally, using (5.63),

$$\left(\frac{d}{dt}\mathbb{E}f(\theta_n(N_t+\eta))\right)_{t=0} = \mathcal{L}_n f(\eta) \tag{5.64}$$

for $\eta \in \Omega_{f,n}$. ■

5.3 Invariant measure for the FVSP.

Since the FVSP is a pure jump process on the finite set $\Omega_{f,n}$, it converges exponentially fast to its unique stationary measure μ_n.

$$\mu_n = \frac{1}{|\mathcal{R}_n|}\sum_{\eta\in\mathcal{R}_n}\delta_\eta \tag{5.65}$$

is the uniform measure on the set $\mathcal{R}_n$ of recurrent configurations:

$$\mathcal{R}_n = \{\eta \in \Omega_{f,n} : \sum_{i=-n}^{n}(2-\eta(i)) \leq 1\}, \tag{5.66}$$

i.e., those configurations in $\Omega_{f,n}$ containing at most one site $i \in [-n,n]$ such that $\eta(i) = 1$. For all $i \in [-n,n]\cap\mathbb{Z}$, $\mu_n T_{n,i} = \mu_n$. More precisely for every pair $(\eta,\zeta) \in \mathcal{R}_n \times \mathcal{R}_n$, $\eta \neq \zeta$ there exists a unique lattice site $i \in [-n,n]$ such that $T_{n,i}\eta = \zeta$. Therefore,

$$\begin{aligned}\int(\mathcal{L}_n f)g\,d\mu_n &= \sum_{\eta\in\mathcal{R}_n}\frac{1}{|\mathcal{R}_n|}\sum_{i=-n}^{n}\left[f(T_{n,i}\eta)-f(\eta)\right]g(\eta)\\ &= \frac{1}{|\mathcal{R}_n|}\sum_{\eta,\zeta\in\mathcal{R}_n,\eta\neq\zeta}\left[f(\zeta)-f(\eta)\right]g(\eta)\\ &= \int f(\mathcal{L}_n g)\,d\mu_n. \end{aligned}\tag{5.67}$$

so that μ_n is actually reversible for $\mathcal{L}_n$.

5.4 Convergence to equilibrium.

The key inequality in the proof of Theorem 5.1 is: There exists $T > 0$ such that for all $t \geq T$

$$P(\theta_n(N_t+\eta)(0) = 1) \leq Ce^{-n\lambda} + C'/n. \tag{5.68}$$

Indeed, after a finite time T in the Poisson process we have for large n

$$\sum_{i=-n}^{n} N_T^i \approx 2nT. \tag{5.69}$$

For T large enough this represents a serious "excess" at most lattice sites in $[-n, n]$. If we let this configuration topple, then we will obtain a recurrent configuration, as we show in the following lemma.

Lemma 5.1 *Let $\eta \in \mathbb{N}_0^{\mathbb{Z}}$ be such that*

$$\sum_{i=-\lfloor n/2 \rfloor}^{\lfloor n/2 \rfloor} \eta(i) \geq 12n, \tag{5.70}$$

then $\theta_n(\eta) \in \mathcal{R}_n$.

Proof: First of all note that we put the "excess" of particles sufficiently far from $-n$ and n, so that not too much sand will be lost at the boundaries. We call a site $i \in [-n, n] \cap \mathbb{Z}$ an *excess site* if $\eta(i) > 2$. For the sake of simplicity we suppose n even. Put

$$\begin{aligned} m &= \min\{i \in [-n/2, n/2] \cap \mathbb{Z} : \eta(i) > 2\} \\ M &= \max\{i \in [-n/2, n/2] \cap \mathbb{Z} : \eta(i) > 2\}. \end{aligned} \tag{5.71}$$

Because of the monotonicity property

$$\eta, \zeta \in {\mathbb{N}_0}^{\mathbb{Z}},\ \zeta \geq \eta,\ \text{and } \theta_n(\eta) \in \mathcal{R}_n \text{ imply } \theta_n(\zeta) \in \mathcal{R}_n, \tag{5.72}$$

we can replace our configuration η by η' defined by: $\eta'(i) = \eta(i)$ for $i \in [-n/2, n/2] \cap \mathbb{Z}$ and $\eta'(i) = 1$ otherwise. We now let the system topple in the following way: All sites in (m, M) topple and the sand arriving at m or M will be stocked at these sites (i.e. sites m and M do not topple). We then end up with a configuration η'' with two excess sites at m and M and since no sand is lost in this toppling:

$$\eta''(m) + \eta''(M) \geq 10n. \tag{5.73}$$

We change again the configuration only retaining the site with the maximum excess, say M, and replacing all the rest by 1's, this configuration is denoted by η'''. The excess of η''' at M is not less than $5n$. So we are left to prove that a configuration with one excess site of this type topples to a recurrent configuration. The case $M = 0$ is trivial. Suppose $M > 0$. After $n - M$ topplings we get a configuration η^{iv} with an excess site at M (an excess not less than $3n$), $\eta^{iv}(i) = 2$ for all $i \in [2M-n, n]$, $\eta^{iv}(i) = 1$mbox for all $i \in \mathbb{Z} \setminus [2M-n, n]$. From that moment on, sand particles are lost at the boundary by further toppling. It is easy to see that the worst situation is when $M = n/2$. In that case, define the configuration η^v by $\eta^v(i) = 2$ for $i \in [0, n]$ and $\eta^v(i) = 1$ otherwise; since the excess at site $i = n/2$ in η^{iv} is not less than $3n$, it suffices to show that $(T_{n,n/2})^{3n}(\eta^v) \in \mathcal{R}_n$.

By adding sand particles at site $n/2$ in η^v, we shall always obtain a configuration in the set $\Gamma_n = \cup_{-n \le a < 0} \Gamma_{n,a}$, with

$$\Gamma_{n,a} = \{\eta \in \Omega_{f,n} : \ \eta(i) = 1 \ \forall i < a, \text{ and } \sum_{i=a}^{n} (2 - \eta(i)) \le 1\}. \tag{5.74}$$

That is, for $\eta \in \Gamma_{n,a}$, there exists a "leftmost" site $a < 0$ for which $\eta(a) = 2$ and the interval $[a, n]$ contains at most one site i for which $\eta(i) = 1$. When a is the position of the leftmost two in the configuration η, then by adding three sand particles at site $n/2$ and letting the system topple, the position of the leftmost two becomes $a-1$. I.e., for $a < 0$, $\eta \in \Gamma_{n,a}$ implies $(T_{n,n/2})^3 \in \Gamma_{n,a-1}$. By this last property, since $\Gamma_{n,-n} = \mathcal{R}_n$, $(T_{n,n/2})^{3n}\eta^v \in \mathcal{R}_n$. ∎

The following lemma is a standard large deviation estimate for sums of independent random variables.

Lemma 5.2 *For every strictly positive t and ϵ, there exists a constant $\lambda(\epsilon) > 0$ such that*

$$P\left(\sum_{x=-\lfloor n/2 \rfloor}^{\lfloor n/2 \rfloor} N_t^x \le n(t-\epsilon) \right) \le \exp(-n\lambda(\epsilon)). \tag{5.75}$$

Proof of Theorem 5.1: We show that the statement is true for $T = 14$. By a coupling argument as in Section 3, we get that the FVSP $\theta_n(N_t + \eta)$ is stochastically dominated by the sandpile process η_t (since we put fixed ones at the boundaries of the interval $[-n, n]$ in the FVSP). Put $0 < \delta < 1$. By Lemmas 5.2 and 5.1 we know that for $t > 14$, there exists $\lambda > 0$ such that

$$P(\theta_n(N_{t-\delta}) \notin \mathcal{R}_n) \le C \exp(-\lambda n). \tag{5.76}$$

Hence we obtain the inequalities

$$\begin{aligned} P_{SP}^{\eta}(\eta_t(0) = 1) &\le P(\theta_n(N_t + \eta)(0) = 1) \\ &\le P[\theta_n(N_t + \eta)(0) = 1 | \theta_n(N_{t-\delta} + \eta) \in \mathcal{R}_n] \\ &+ C \exp(-\lambda n). \end{aligned} \tag{5.77}$$

Conditioned on $\{\zeta := \theta(N_{t-\delta}+\eta) \in \mathcal{R}_n\}$, the distribution of $\theta_n(N_t+\eta)$ is uniform on a set of the form $\mathcal{R}_n \setminus \{\zeta'\}$ if there has been at least one Poisson event in $(t-\delta, t)$. Therefore, since there is only one $\eta \in \mathcal{R}_n$ with $\eta(0)=1$,

$$\begin{aligned} P & \ [\theta_n(N_t+\eta)(0) = 1 | \theta_n(N_{t-\delta}+\eta) i \in \mathcal{R}_n] \\ \leq & \ \frac{1}{2n+1} + (1-e^{-\delta})^{2n+1}. \end{aligned} \tag{5.78}$$

Combining (5.78) with (5.77) we arrive at

$$P^{\eta}_{SP}[\eta_t(0)=1] \leq \frac{1}{2n+1} + (1-e^{-\delta})^{2n+1} + Ce^{-n\lambda}, \tag{5.79}$$

which concludes the proof of the theorem. ∎

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