SHAPE AND MORSE THEORY OF ATTRACTORS

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Introduction

There are many different notions of attractors used in the literature. The notion that we consider in this paper is the same as *the global minimal B-attractor* of [Ladyzhenskaya, 1987], *the global uniform attractor* of [Hale, 1987], and *the maximal attractor* of [Babin & Vishik, 1992]. The attractors we study appear in the context of nonlinear dissipative evolution equations as the minimal compact sets that attract all bounded sets of initial data. The theory of attractors and examples can be found in the literature cited above and, also, in [Kapitanski & Kostin, 1990] and the Appendix in [Kapitanski, 1995]. A brief account of the relevant general theory is given below in the section entitled *Attractors*.

In an abstract setting, consider a continuous semiflow $\mathcal{E}^t$, $t \geq 0$, on a complete metric space X. The "dissipativity" of the system $\{X, \mathcal{E}\}$ can be modelled by the hypothesis that there exisits a bounded (diameter$< \infty$) set $\mathfrak{B} \subset X$ that absorbes every point of X in the sense that for every $x \in X$, there is time $t = t(x) \geq 0$ such that $\mathcal{E}^t(x) \in \mathfrak{B}$. In addition to the existence of such an absorbing set $\mathfrak{B}$, assume that the evolution $\mathcal{E}$ is "compact" in the sense that any sequence of the form $\{\mathcal{E}^{t_n}(x_n)\}$ is precompact provided $\{x_n\}$ is a bounded sequence, and $t_n \to +\infty$. It turns out that these two conditions ("dissipativity" and "compactness") are necessary and sufficient for the existence of a compact set $\mathfrak{F}$ that attracts every bounded subset B of X, i.e.,

$$\operatorname{dist}(\mathcal{E}^t(B), \mathfrak{F}) = \sup_{x \in B} \operatorname{dist}(\mathcal{E}^t(x), \mathfrak{F}) \to 0, \text{ as } t \to \infty.$$

If there exists an attracting compact $\mathfrak{F}$, then there exists the smallest among all such compact sets, say, $\mathfrak{A}$, and this is *the global attractor* of the system $\{X, \mathcal{E}\}$.

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The global attractor contains all stationary points, all periodic trajectories and all almost-periodic trajectories. In fact, the attractor is weaved by all bounded global trajectories, i.e., the trajectories that can be extended to negative t and stay in a bounded region for all t. Also, the attractor can be characterized as the union of the ω-limit sets of all bounded sets. The attractor $\mathfrak{A}$ is invariant under the flow: $\mathcal{E}^t(\mathfrak{A}) = \mathfrak{A}$, for all $t > 0$. In fact, it is the largest compact invariant subset of X. If the absorbing set is connected, then the attractor is connected as well. If, for some $\tau > 0$, $\mathcal{E}^\tau$, when restricted to $\mathfrak{A}$, is 1-to-1, then the semigroup $\mathcal{E}^t$ can be extended to a continuous group of transformations of $\mathfrak{A}$, thus defining a dynamical system on $\mathfrak{A}$. These properties practically exhaust our knowledge of the structure of attractors in the general case.

In most examples coming from different dissipative evolution problems (such as the initial boundary value problem for the Navier-Stokes equations in a 2-dimentional bounded domain with Dirichlet or periodic boundary conditions), one can show in addition that the attractor has a finite Hausdorff (or fractional, or informational) dimension, but this is only an estimate. It would be too good if the attractors were manifolds. However, given an infinite-dimensional system with a finite-dimensional attractor $\mathfrak{A}$, it is reasonable to seek a finite dimensional manifold $\mathcal{M}$ that contains the attractor and is *positively invariant*, i.e., $\mathcal{E}^t(\mathcal{M}) \subset \mathcal{M}$, for all $t > 0$. Indeed, should such a manifold exist, the analysis of the long-term behavior of the original system will be reduced to the finite-dimensional case. This question is addressed by the theory of inertial manifolds, but so far the theory applies only to a limited class of problems, see [Constantin et al., 1989] for details.

Even in the finite-dimensional situation the geometric structure of an attractor may be very complicated, as is the case with the Lorenz attractor, [Sparrow, 1982]. Because of potentially pathological geometry, it may not be productive to ask about the homotopy type of an attractor. Instead, it is more reasonable to work with Borsuk's concept of shape that replaces homotopy type for "bad" spaces, see [Borsuk, 1975], [Fox 1972, 1973], and section entitled *Shape* below for the definition and the relevant properties of shape. The standard example showing the difference between the homotopy type and shape is the Warsaw circle $\mathcal{C}$, defined as follows. Consider the graph of the function $\sin(1/x)$ for $0 < x \leq 1/\pi$, add to this set the vertical segment $\{x = 0, -1 \leq y \leq 1\}$, and an arc (hanging below the graph of $\sin(1/x)$) joining the points $\{x = 0, y = 0\}$ and $\{x = 1/\pi, y = 0\}$. The resulting geometric figure is $\mathcal{C}$. Now, $\mathcal{C}$ is not homotopy equivalent to the standard circle $S^1 = \{x^2 + y^2 = 1\}$, but $\mathcal{C}$ and S^1 have the same shape. Note that, although it is usually not difficult to prove the connectedness of the attractor, it is hopeless to show that it is arc-wise connected, simply because it may be like the Warsaw circle. Shape seems to ignore such pathologies in the structure.

There are several papers concerning the shape of what we call a *neighborhood attractor* of a finite-dimensional (semi)flow. By a *neighborhood attractor* (sometimes called asymptotic attractor) in a semidynamical system $\{X,\, \mathcal{E}\}$ we mean a compact set $\mathfrak{F} \subset X$, which is an ω-limit set of one of its bounded neighborhoods, $\mathfrak{F} = \omega(U)$; in other words, $\mathfrak{F}$ is the global attractor of the semidynamical system induced on the closure of the positive trajectory of the neighborhood U.

The situation considered in [Hastings, 1979] is as follows. X is a compact n-dimensional submanifold of $\mathbb{R}^n$ with boundary, and $\mathcal{E}^t$ is a C^0-semiflow such that the trajectories through the boundary of X enter the interior of X. Under these assumptions it is proved that the inclusion of the corresponding attractor $\mathfrak{A}$ into X is a shape equivalence.

In [Bogatyj & Gutsu, 1989], the space X is a smooth finite-dimensional closed manifold, and the evolution $\mathcal{E}^t$ is defined for $-\infty < t < \infty$ by the *flow* corresponding to a C^1 vector field on X. Bogatyj and Gutsu prove that any neighborhood attractor $\mathfrak{F}$ in $\{X,\, \mathcal{E}\}$ is a movable pointed fundamental neighborhood retract (FANR), and has the shape of a finite polyhedron. In particular, the Čhech homologies $\check{H}_q(\mathfrak{F})$ and cohomologies $\check{H}^q(\mathfrak{F})$ are finitely generated[1] (and vanish for large q). Similar results in the slightly more general case of X being a finite-dimensional topological manifold and $\mathcal{E}^t$, $-\infty < t < \infty$, a continuous flow on X, were rediscovered in [Günther & Segal, 1993]. Günther and Segal also prove that a finite-dimensional compactum can be an attractor of a continuous flow on a manifold if and only if it has the shape of a finte polyhedron[2]. Later, in [Günther, 1995], it was shown that the flow can be of class C^r with arbitrarily large r.

One of our first concerns here will be how to find the shape of an attractor (and, hence, a neighborhood attractor) in the non-locally-compact case. The answer turns out to be quite simple: *the global attractor has the same shape as the state space.* We prove this under very mild assumptions that the state space, X, is a complete metric space and the evolution, $\mathcal{E} : X \times \mathbb{R}_+ \to X$, is a continuous semiflow.

This result, for example, implies that any (global) attractor in $\mathbb{R}^n$, or a Banach space, or any metric space that has the shape of a point, has the shape of a point itself, any

[1] Concerning the fact that the Čech (co)homologies of a finite-dimensional neighborhood attractor are fintely generated see also [Pilyugin, 1985].

[2] That on a two-dimensional surface every continuum with finitely generated Čech cohomologies can be a neighborhood attractor for an appropriate flow, was proved in [Fokin, 1981]

attractor on a cylinder $S^m \times \mathbb{R}$ has the shape of S^m, any attractor in a punctured n-dimensional ball has the shape of S^{n-1}, etc.[3]

Next, we show that if X is an absolute neighborhood retract (ANR), then $\mathfrak{A}$ is a, so-called, pointed FANR and, hence, has the shape of a polyhedron. Being compact, $\mathfrak{A}$ is shape dominated by a finite polyhedron. In particular, the Čech homologies $\check{H}_q(\mathfrak{A})$ and cohomologies $\check{H}^q(\mathfrak{A})$ are finitely generated and vanish for large q. If X is locally compact, then $\mathfrak{A}$ does have the shape of a finite polyhedron. We prove this under weaker restrictions than in [Bogatyj & Gutsu, 1989] and [Günther & Segal, 1993]. We also address the non-locally-compact case. In that case, we also prove that $\mathfrak{A}$ has the shape of a finite polyhedron, but for this we need the additional assumption that X is simply connected.

We believe that our results will have many applications to PDEs, functional differential equations, and more general processes. Here we use some of our results on the shape of attractors to develop an elementary Morse theory of an attractor. It should be mentioned that there exists an extensive literature on the Morse theory for attractors which is based on Conley's index. In fact, the development and generalizations of Conley's index have been in many respects motivated by justification of the Morse equations, see [Conley, 1978], [Conley & Zehnder, 1984], [Rybakowski, 1982, 1983, 1987], [Kurland, 1982], [Rybakowski & Zehnder, 1985], [Robbin & Salamon, 1988], to mention a few. It seems that our approach is simpler. In vague language, the simplification comes from replacing homotopy equivalence by less restrictive shape equivalence, which transfers into isomorphism in Čech (co)homology. In addition to its simplicity our approach allows us to handle more general metric spaces and evolutions. Also, we do not need to assume that certain Betti numbers are finite as in the applications of Conley's index, see [Rybakowski, 1987, Theorem III.3.5]. In the situations of interest we have the finiteness of the appropriate Betti numbers proven via shape theory. An example of a semi-dynamical system for which our theory works while Conley's is not applicable is shown on a picture in the end of the introduction.

The theory applies to a special class of attractors which admit the so-called Morse filtration. The Morse filtration may be available due to special properties of the semi-dynamical system such as the existence of a Lyapunov function. Consider first the case of a *good* Lyapunov function on an attractor. By a *good* Lyapunov function on the attractor $\mathfrak{A}$ we mean any continuous function $\mathcal{L} : \mathfrak{A} \to \mathbb{R}$ that strictly decreases along

[3] That a global attractor in a Banach space has the shape of a point was proved in [Garay, 1991].

every trajectory except for when the trajectory is a point (a stationary point). If $\mathcal{L}$ is a *good* Lyapunov function on $\mathfrak{A}$, and $\mathfrak{A}$ is not a single point, then $\mathfrak{A}$ contains at least two stationary points, namely, the point at which $\mathcal{L}$ achieves its maximum, and the one where $\mathcal{L}(x)$ is minimal. If, in addition, it is known that the set of all stationary points of the flow $\mathcal{E}$ is finite, then the attractor $\mathfrak{A}$ consists of the stationary points and all the global trajectories connecting them; in other words, for any nonstationary global trajectory $\{x(t),\, t \in \mathbb{R}\}$ in $\mathfrak{A}$, the limits $\lim_{t\to\pm\infty} x(t)$ do exist and are stationary points. The Lyapunov function assumes a finite number of values, $\ell_0 < \cdots < \ell_N$, on the finite set $\mathcal{Z}$ of (all) stationary points. Denote by $\mathcal{Z}_j$ the set of stationary points on which the value of $\mathcal{L}$ is ℓ_j. The description of the attractor prompts the following filtration:

$$A_0 \subset \dots A_j \subset A_{j+1} \subset \cdots \subset A_m = \mathfrak{A},$$

where $A_0 = \mathcal{Z}_0$, and A_{j+1} is the union of A_j with the unstable "manifolds" of the stationary points from $\mathcal{Z}_{j+1}$, where by the unstable "manifold" of a stationary point z we mean the set of all points from $\mathfrak{A}$, for which z is their common α-limit set in $\mathfrak{A}$. Note, that each A_j is a neighborhood attractor, where a neighborhood can be any ε-neighborhood of A_j with ε sufficiently small. Also, A_j is the attractor of the systems $\{\mathcal{L}^a,\, \mathcal{E}\}$ with $\ell_j < a < \ell_{j+1}$, where $\mathcal{L}^a$ denotes the set of all points of X with $\mathcal{L}(x) \le a$.

A simple model that illustrates the situation with a *good* Lyapunov function and a finite number of stationary points is provided by a gradient system

$$\dot{x} + \nabla f(x) = 0$$

on a closed, finite-dimensional Riemannian manifold X, where $f : X \to \mathbb{R}$ is a C^1 function with a finite number of critical points. The corresponding evolution is defined by the solution operator $\mathcal{E}^t : x^0 \to \mathcal{E}^t(x^0) = x(t)$, where $x(t)$ solves the equation with the initial condition $x(0) = x^0$. Of course, $\mathcal{L} = f$ is the distinguished *good* Lyapunov function. The attractor $\mathfrak{A}$ of $\{X,\, \mathcal{E}\}$ is the manifold X itself (since X is the maximal invariant compact set in X). Each of A_j in the filtration of $\mathfrak{A} = X$, being a neighborhood attractor in the ANR X, has the shape of a finite CW complex. The Čhech homologies and cohomologies of A_j are finitely generated. [We choose coefficients so that all of the standard Eilenberg-Steenrod axioms are satisfied.] Now, it is an easy exercise to check that the following Morse equations hold:

$$\sum_{j=0}^{r} \sum_{q=0}^{\infty} t^q \dim \check{H}_q(A_j, A_{j-1}) = \sum_{q=0}^{\infty} t^q \dim \check{H}_q(A_r) + (1+t)\, Q_r(t),$$

for $r = 1, \ldots, m$, where $A_{-1} = \emptyset$ and $Q_r(t)$ are some polynomials with non-negative integer coefficients. [That the relative homologies $\check{H}_q(A_j, A_{j-1})$ have finite rank follows from the homology exact sequence of the pair (A_j, A_{j-1}).] Similar equations hold for cohomologies as well. The left side of the Morse equations can be replaced by the sum

$$\sum_{j=0}^{r} \sum_{q=0}^{\infty} t^q \dim \check{H}_q(\mathcal{L}^{\ell_j}, \mathcal{L}^{\ell_j - \varepsilon})$$

with a sufficiently small $\varepsilon > 0$. It can be also replaced by

$$\sum_{j=0}^{r} \sum_{z \in \mathcal{Z}_j} \sum_{q=0}^{\infty} t^q \dim \check{H}_q \left(\mathcal{W}^u_\varepsilon(z), \eth\mathcal{W}^u_\varepsilon(z)\right) ,$$

where $\mathcal{W}^u_\varepsilon(z)$ is the part of the unstable "manifold" $\mathcal{W}^u(z)$ of the stationary point z, on which the values of f are between $f(z) - \varepsilon$ and $f(z)$, and $\eth\mathcal{W}^u_\varepsilon(z)$ is the intersection of $\mathcal{W}^u(z)$ with the level set $f = f(z) - \varepsilon$.

The homology $\check{H}_* \left(\mathcal{W}^u_\varepsilon(z), \eth\mathcal{W}^u_\varepsilon(z)\right)$ can be viewed as the Morse index of the invariant (one-point) set z. If all the stationary points are hyperbolic (i.e., f is C^2 and the critical points of f are non-degenerate), then each $\mathcal{W}^u_\varepsilon(z)$ is a ball of dimension of the index of z, and the left side of the Morse equations assumes the familiar form

$$\sum_{j=0}^{r} \sum_{z \in \mathcal{Z}_j} t^{\,\mathrm{ind}\,(z)} \quad .$$

In the case when there is no good Lyapunov function available, but there is a non-trivial Morse filtration of the global attractor,

$$A_0 \subset \ldots A_j \subset A_{j+1} \subset \cdots \subset A_m = \mathfrak{A},$$

we *construct* the corresponding Morse decomposition, i.e., a finite disjoint union of the form

$$\mathfrak{A} \;=\; M_0 \cup W(A_0, M_1) \cup M_1 \cup \cdots \cup\; M_{m-1} \cup W(A_{m-1}, M_m) \cup M_m \, ,$$

where $M_0, \ldots, M_m$ are compact invariant sets and the sets $W(A_{k-1}, M_k)$ are composed of trajectories connecting M_k with the lower level neighborhood attractor

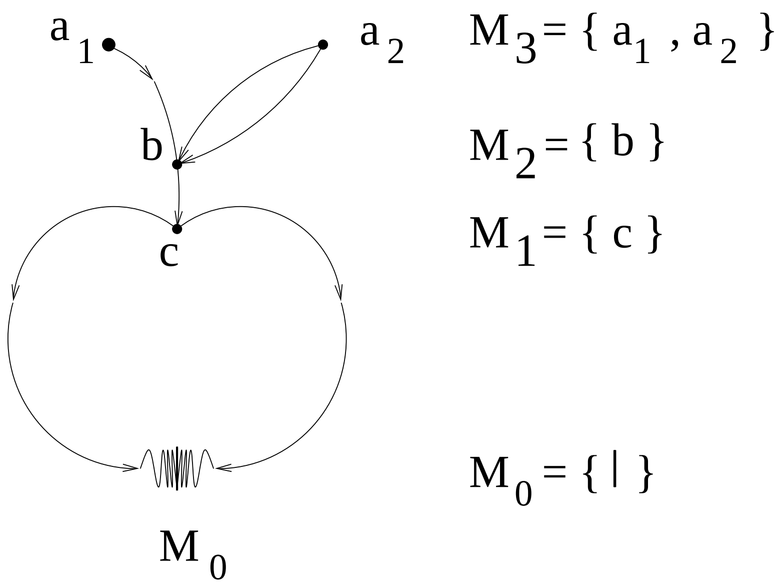
$$A_{k-1} \;=\; M_0 \cup W(A_0, M_1) \cup M_1 \cup \cdots \cup\; M_{k-2} \cup W(A_{k-2}, M_{k-1}) \cup M_{k-1} \, ,$$

where we set $A_0 = M_0$. It is with the Morse decomposition that we associate now the Morse equations. They take the form

$$\sum_{j=0}^{r} \sum_{q=0}^{\infty} t^q \dim \check{H}_q\left(\mathcal{W}_\varepsilon(M_j), \eth\mathcal{W}_\varepsilon(M_j)\right) = \sum_{q=0}^{\infty} t^q \dim \check{H}_q(A_r) + (1+t)\, Q_r(t).$$

Here $\mathcal{W}_\varepsilon(M_j)$ is a piece of the "unstable manifold" of the invariant set M_j and $\eth\mathcal{W}_\varepsilon(M_j)$ is its "boundary", or "exit set". Again, $\check{H}_*\left(\mathcal{W}_\varepsilon(M_j), \eth\mathcal{W}_\varepsilon(M_j)\right)$ is the right Morse index of the invariant set M_j, and, of course, it is intimately related to Conley's homological index of M_j in the case the latter is well-defined. Note, however, that there are situations where the existing theory of Conley's index is not applicable.

As a peculiar example, consider the following semi-dynamical system (a semiflow on the contour of the "little apple"):



The dimple in the bottom of the "apple" is filled with $\sin(1/x)$-like curves and their "limit" M_0, the vertical interval, as in the Warsaw circle (see [Borsuk]). The points a_1, a_2, b, c, and all the points of M_0 are stationary. The global attractor $\mathfrak{A}$ coincides with the whole "graph" and, thus, has the shape of the bouquet $\mathbb{S}^1 \vee \mathbb{S}^1$. Note that the homotopy types of $\mathfrak{A}$ and $\mathbb{S}^1 \vee \mathbb{S}^1$ are different.

The Poincaré polynomial of $\mathfrak{A}$, i.e., $P(\mathfrak{A}) = \sum_{q=0}^{\infty} t^q \dim \check{H}_q(\mathfrak{A})$, is $1 + 2t$. The Poincaré polynomials corresponding to the invariant sets M_j, $j = 0,1,2,3$, i.e., the polynomials $P(M_j) = \sum_{q=0}^{\infty} t^q \dim \check{H}_q\left(\mathcal{W}_\varepsilon(M_j), \eth\mathcal{W}_\varepsilon(M_j)\right)$, are easy to find from the picture:

$$P(M_0) = 1, \quad P(M_1) = t, \quad P(M_2) = 0, \quad P(M_3) = 0 + t.$$

The Morse equation $\sum_j P(M_j) = P(\mathfrak{A}) + (1+t)Q(t)$ takes the form of the following identity

$$1 + 2t = 1 + 2t + (1+t)Q(t),$$

where $Q(t) = 0$.

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ATTRACTORS

In our sketch of the theory of attractors of dissipative systems we follow, in general, the approach in [Ladyzhenskaya], though, in exposition, we rely on [Kapitanski & Kostin].

1. Semi-Dynamical Systems. To describe a semi-dynamical system one needs to tell what its state space, time scale and evolution are. Here we adopt the following definitions.

State space is a complete metric space. We use the same notation, say X, for the space and the space with a metric $\operatorname{dist}(x,\, y)$.

The choice of the time scale depends on whether the evolution is continuous or discrete. In general, if $\mathcal{T}$ is any subgroup of $\mathbb{R}$, then $\mathcal{T}_+ = \{t \in \mathcal{T},\, t \geq 0\}$ can be a *time scale*. In most applications $\mathcal{T}_+$ is either $\mathbb{R}_+$ or $\mathbb{Z}_+$.

By *evolution* we mean a mapping $\mathcal{E} : X \times \mathcal{T}_+ \to X$ with the following two properties:

1) For every $t \in \mathcal{T}_+$, the mapping $\mathcal{E}(t,\cdot) : x \mapsto \mathcal{E}(t,x)$ from X to X is continuous.
2) $\mathcal{E}$ satisfies the semigroup property: $\mathcal{E}(0,x) = x$, and $\mathcal{E}(t_1, \mathcal{E}(t_2,x)) = \mathcal{E}(t_1 + t_2, x)$, for all $x \in X$ and all $t_1, t_2 \in \mathcal{T}_+$.

A *semi-dynamical system* is a triple $\{X, \mathcal{T}_+, \mathcal{E}\}$ (we omit $\operatorname{dist}(\cdot,\,\cdot)$ in this notation). Sometimes we write $\mathcal{E}^t$ instead of $\mathcal{E}(t,\cdot)$. We say that $\{X, \mathcal{T}_+, \mathcal{E}\}$ is a system with *continuous evolution* if $\mathcal{T}_+ = \mathbb{R}_+$ and the mapping $\mathcal{E} : X \times \mathbb{R}_+ \to X$ is continuous.

2. Attractors. Given a semi-dynamical system $\{X, \mathcal{T}_+, \mathcal{E}\}$, a **compact set** $\mathfrak{A} \subset X$ **is** called **a global attractor** if $\mathfrak{A}$ attracts all bounded subsets of X and $\mathfrak{A}$ is the minimal (compact) set with this property. (In [Ladyzhenskaya] such a set is called a minimal, compact, global B-attractor.) Attracts all bounded subsets means that

$$\operatorname{dist}(\mathcal{E}^t(B),\, \mathfrak{A}) \;=\; \sup_{y \in B} \operatorname{dist}(\mathcal{E}^t(y),\, \mathfrak{A}) \;\to\; 0, \qquad t \to \infty,$$

for any bounded (= with finite diameter) set $B \subset X$.

The following result can be extracted from several sources, cf. [Ladyzhenskaya], [Hale], [Kapitanski & Kostin].

3. Existence and Uniqueness of a Global Attractor. *For a semi-dynamical system* $\{X, \mathcal{T}_+, \mathcal{E}\}$ *to possess a global attractor it is necessary and sufficient that the following two conditions be satisfied.*

i) *For every bounded sequence* $x_n \in X$ *and every increasing to* $+\infty$ *sequence* $t_n \in \mathcal{T}_+$*, the set* $\{\mathcal{E}(t_n, x_n)\}$ *is precompact.*

ii) *There exists a bounded set* B_0 *such that, for every* $x \in X$*,* $\mathcal{E}(t_x, x) \in B_0$ *at some moment* $t_x \in \mathcal{T}_+$*.*

The global attractor $\mathfrak{A}$ *is minimal in the sence that if* $\mathfrak{B}$ *is any closed, bounded set that attracts all bounded subsets of* X*, then* $\mathfrak{A} \subseteq \mathfrak{B}$*.*

4. Basic Properties of Global Attractors. *If* $\mathfrak{A}$ *is the global attractor of* $\{X, \mathcal{T}_+, \mathcal{E}\}$*, then*

1) $\mathfrak{A}$ *is an invariant set, i.e.,* $\mathcal{E}(t, \mathfrak{A}) = \mathfrak{A}$ *for all* $t \in \mathcal{T}_+$*.*
2) $\mathfrak{A}$ *is maximal among all bounded, invariant subsets of* X*.*
3) $\mathfrak{A}$ *is the union of the* ω*-limit sets of all bounded sets* $B \subset X$*.*
4) *Through every point* $x_0 \in \mathfrak{A}$ *passes a complete trajectory* $\{x(t)\}_{t\in\mathcal{T}} \subset \mathfrak{A}$*, and* $\mathfrak{A}$ *is the union of all bounded, complete trajectories.*
5) *If* X *is connected then* $\mathfrak{A}$ *is connected.*
6) *If the maps* $\mathcal{E}^t$ *are one-to-one on* $\mathfrak{A}$*, then, for all* $t > 0$*, the inverse maps* $\mathcal{E}^{-t} := (\mathcal{E}^t)^{-1}$ *are well-defined and continuous, and* $\{\mathcal{E}^t, t \in \mathcal{T}\}$ *is a group of transformations of* $\mathfrak{A}$*.*

Recall that the ω-limit set of a set B, denoted $\omega(B)$, is the set of all limits of the form $y = \lim \mathcal{E}(t_n, x_n)$, where $x_n \in B$ and $t_n \to \infty$. Recall, also, that $\{x(t)\}_{t\in\mathcal{T}}$ being a complete trajectory means $x(t+\tau) = \mathcal{E}(\tau, x(t))$, for every $t \in \mathcal{T}$ and every $\tau \in \mathcal{T}_+$. A complete trajectory $\{x(t)\}_{t\in\mathcal{T}}$ passes through x_0 if $x_0 = x(t_0)$ for some t_0.

5. Good Lyapunov Functions. One can say more about the structure of the attractor if there is a good Lyapunov function.

Let $\{X, \mathcal{T}_+, \mathcal{E}\}$ *be a semi-dynamical system with a global attractor* $\mathfrak{A}$*. Assume there is a good Lyapunov function defined on* $\mathfrak{A}$*, i.e., a continuous function* $L : \mathfrak{A} \to \mathbb{R}$ *such that* $L(\mathcal{E}(t, x)) < L(x)$ *for all* $t > 0$ *and all* $x \in \mathfrak{A} \setminus \{\text{stationary points}\}$*. Then* $\mathfrak{A}$ *has the following additional properties.*

1) *The set of stationary points,* Z*, is nonempty (of course,* $Z \subset \mathfrak{A}$*).*

2) *For every* $x_0 \in \mathfrak{A}$, $\omega(x_0) \cap Z \neq \emptyset$.

3) *For every* $x_0 \in \mathfrak{A}$ *and a complete trajectory* $\gamma = \{x(t)\}$ *passing through it and staying in* $\mathfrak{A}$, $\alpha_\gamma(x_0) \cap Z \neq \emptyset$, *where* $\alpha_\gamma(x_0)$ *is the* α*-limit set of* x_0 *along* $\gamma = \{x(t)\}$, $\alpha_\gamma(x_0) = \{y \in \mathfrak{A} \big| \, y = \lim_{t_n \to -\infty} x(t_n)\}$.

If, in addition, it is known that the number of stationary points is finite, then

4) *for every* $x_0 \in \mathfrak{A}$ *and a complete trajectory* $\{x(t)\} \subset \mathfrak{A}$ *passing through it, the limits* $\lim_{t \to \pm\infty} x(t) = x_{\pm\infty}$ *exist and the limit values are stationary points,* $x_{+\infty}, x_{-\infty} \in Z$; *thus, in this case, the attractor is the union of stationary points and all (complete) trajectories connecting them.*

6. Remark. In all the results mentioned so far there was no difference between continuous and discrete systems. However, our applications of the shape theory work only for continuous systems. For brevity, we write $\{X, \mathcal{E}\}$ instead of $\{X, \mathbb{R}_+, \mathcal{E}\}$. In addition, we have to assume that the evolution, $\mathcal{E} : X \times \mathbb{R}_+ \to X$, is continuous.

7. Primitive Lyapunov Function. Let $\{X, \mathcal{E}\}$ be a semi-dynamical system with continuous evolution. Assume that it has a global attractor $\mathfrak{A}$. Define the function, [Lyapunov, 1892]:

$$\phi(x) := \sup_{t \geq 0} \operatorname{dist}\left(\mathcal{E}^t(x), \mathfrak{A}\right), \qquad x \in X.$$

It is not hard to show that ϕ has the following properties: ϕ is a continuous function from X to $\mathbb{R}_+$; $\phi(x) = 0$ if and only if $x \in \mathfrak{A}$; ϕ decreases to 0 along any trajectory outside of the attractor, i.e., for any $x \in X \setminus \mathfrak{A}$, the function $\tilde{\phi}(t) = \phi(\mathcal{E}^t(x))$ decreases, and $\lim \tilde{\phi}(t) = 0$ as $t \to \infty$.

Now define, [Bhatia & Szegö, 1967],

$$\Phi(x) := \int_0^\infty e^{-t} \, \phi(\mathcal{E}^t(x)) \, dt.$$

This function is continuous, $\Phi(\mathcal{E}^t(x)) < \Phi(x)$ for all $t \geq 0$ and all x outside of the attractor, $\Phi(x) = 0$ for $x \in \mathfrak{A}$.

We call Φ *the primitive Lyapunov function* for system $\{X, \mathcal{E}\}$.

Note that for many purposes the Lyapunov function $\phi(x)$ is sufficient.

8. Neighborhood Attractors. Let $\{X, \mathcal{E}\}$ be a semi-dynamical system. A compact set $A \subset X$ is called a *neighborhood attractor* if A is the ω-limit set of some open, bounded neighborhood $U \supset A$. Such a neighborhood is then called *domain of attraction* for A.

The notion of a neighborhood attractor is useful in the analysis of the fine structure of global attractors, in particular, Morse decompositions.

9. Lemma. *Let $\{X, \mathcal{E}\}$ be a semi-dynamical system with continuous evolution. A compact set $A \subset X$ is a neighborhood attractor if and only if there exists a positively invariant open neighborhood $V \supset A$ with $\omega(V) = A$.*

Proof. Let U be an open neighborhood of A with $\omega(U) = A$. Denote by Y the closure of the positive trajectory of U, $Y := \overline{\cup_{t \in \mathcal{T}_+} \mathcal{E}(t, U)}$. It is clear that A is the global attractor of the semi-dynamical system $\{Y, \mathcal{E}\big|_Y\}$. Let Φ be the primitive Lyapunov function associated with this system. For sufficiently small $\varepsilon > 0$, the set $V = \{y \in Y \,\big|\, \Phi(y) < \varepsilon\}$ and its closure, $\bar{V}$, both lie in U. (It is not hard to see that $\bar{V} = \{y \in Y \,\big|\, \Phi(y) \leq \varepsilon\}$.) In addition, both V and $\bar{V}$ are positively invariant.[4] It remains to notice that V is open in Y, $V \subset U \subset Y$, U is open in X, and therefore V is open in X.

10. Morse Filtration and Morse Decomposition of an Attractor. The terminology we use below is not universally accepted. The construction of the Morse decomposition from the Morse filtration is known, though maybe not exactly in the generality we need, [Rybakowski, 1987]; we include it here for the convenience of the reader. The result of ¶13 (a neighborhood attractor within the global attractor is a neighborhood attractor in the whole space) we have not seen in the literature.

A *Morse filtration* of an attractor $\mathfrak{A}$ is a finite collection of compact invariant subsets $A_j \subset \mathfrak{A}$, $j = 0, \ldots, m-1$, such that

$$A_0 \subset A_1 \subset \ldots \subset A_{m-1} \subset A_m := \mathfrak{A},$$

and, for $j = 0, \ldots, m-1$,

$$A_j \text{ is a neighborhood attractor in } A_{j+1} \text{ for } \{A_{j+1}, \mathcal{E}\big|_{A_{j+1}}\} .$$

A *Morse decomposition* of $\mathfrak{A}$ is a finite disjoint union of the form

$$\mathfrak{A} = M_0 \cup W(A_0, M_1) \cup M_1 \cup \cdots \cup M_{m-1} \cup W(A_{m-1}, M_m) \cup M_m ,$$

where $M_0, \ldots, M_m$ are compact invariant sets and the sets $W(A_{k-1}, M_k)$ are comprised of trajectories connecting M_k with

$$A_{k-1} = M_0 \cup W(A_0, M_1) \cup M_1 \cup \cdots \cup M_{k-2} \cup W(A_{k-2}, M_{k-1}) \cup M_{k-1} ,$$

[4] That V is positively invariant means $\mathcal{E}(t, V) \subset V$ for all $t \in \mathcal{T}_+$.

where we set $A_0 = M_0$. More precisely,

$$W(A_{k-1}, M_k) = \{x \in A_k \setminus (A_{k-1} \cup M_k) \,\big|\, \omega(x) \subset A_{k-1},\ \omega_k^*(x) \subset M_k\}$$

where

$$\omega_k^*(x) \;=\; \{y \in A_k \,\big|\, y = \lim y_n,\ y_n \in \mathcal{E}^{-t_n}(x),\ t_n \to \infty\}.$$

Note that every Morse decomposition gives rise to a Morse filtration. It turns out that under very mild assumptions the reverse is also true. Here is the basic technical lemma that allows us to prove the general result by induction.

11. Lemma. *Let $\{X, \mathcal{E}\}$ be a semi-dynamical system with continuous evolution. Assume that this system has a compact attractor A_1. Assume that $A_0 \subsetneqq A_1$ is a neighborhood attractor of the restricted system $\{A_1, \mathcal{E}\big|_{A_1}\}$. Then there exists a unique invariant compact set $M_1 \subset A_1$ such that $M_1 \cap A_0 = \emptyset$ and*

$$A_1 \;=\; A_0 \,\cup\, W(A_0, M_1) \,\cup\, M_1,$$

where $W(A_0, M_1)$ is the set of all trajectories connecting M_1 and A_0,

$$W(A_0, M_1) \;=\; \{x \in A_1 \setminus (A_0 \cup M_1) \,\big|\, \omega(x) \subset A_0,\ \omega^*(x) \subset M_1\}\,.$$

Proof. Denote $A_0^* := \{x \in A_1 \,|\, \omega(x) \cap A_0 = \emptyset\}$. This set is non-empty because otherwise we would have $A_1 = A_0$. Also, A_0^* is closed (compact) and invariant[5]. Set $M_1 := A_0^*$. It remains to show that $\omega^*(x) \subset M_1$ for every $x \in A_1 \setminus (A_0 \cup M_1)$. By definition, $\omega^*(x) = \{y \in A_1 \,|\, y = \lim y_n,\ y_n \in \mathcal{E}^{-t_n}(x),\ t_n \to \infty\}$. Note that $\omega^*(x)$ is a compact invariant set. Hence, $\omega(z) \subset \omega^*(x)$ for every $z \in \omega^*(x)$. Thus, it suffices to show that $\omega^*(x) \cap A_0 = \emptyset$. Assume that $B = \omega^*(x) \cap A_0 \neq \emptyset$. Then B is compact and invariant. Pick $y \in B$. On the one hand, $y = \lim y_n$, where $y_n \in \mathcal{E}^{-t_n}(x) \cap A_1$ and $t_n \to \infty$. On the other hand, $y \in A_0$. Since A_0 is a neighborhood attractor in A_1, we have a system of neighborhoods $U_\varepsilon = \{x \in A_1 \,|\, \Phi(x) < \varepsilon\} \supset A_0$, all sufficiently small ε, such that any trajectory, once in U_ε, stays there forever. Thus, given $\varepsilon > 0$, we have $y_n \in U_\varepsilon$ and $\mathcal{E}^{t_n} y_n = x \in U_\varepsilon$. This means that $x \in A_0$, which does not agree with our assumtion. □

Here is the general result.

[5] The pair (A_0^*, A_0) is a repeller-attractor pair in A_1.

12. Morse Decomposition of an Attractor. *Let $\{X, \mathcal{E}\}$ be a semi-dynamical system with continuous evolution. Assume that this system has a compact attractor $\mathfrak{A}$, and*

$$A_0 \subset A_1 \subset \ldots \subset A_{m-1} \subset A_m := \mathfrak{A}$$

is its Morse filtration. Then there exists a unique Morse decomposition associated with the given filtration. Namely, there exist compact invariant sets $M_1 \subset A_1, \ldots, M_m \subset A_m$ such that each of the attractors $A_1, \ldots, A_m$ is uniquely represented as a disjoint union

$$A_j = M_0 \cup W(A_0, M_1) \cup M_1 \cup \cdots \cup M_{j-1} \cup W(A_{j-1}, M_j) \cup M_j ,$$

where $M_0 = A_0$,

$$W(A_{k-1}, M_k) = \{x \in A_k \setminus (A_{k-1} \cup M_k) \,\big|\, \omega(x) \subset A_{k-1},\ \omega_k^*(x) \subset M_k\}$$

and

$$\omega_k^*(x) = \{y \in A_k \,\big|\, y = \lim y_n,\ y_n \in \mathcal{E}^{-t_n}(x),\ t_n \to \infty\}.$$

In particular, the whole attractor $\mathfrak{A}$ admits the following decomposition:

$$\mathfrak{A} = M_0 \cup W(A_0, M_1) \cup M_1 \cup \cdots \cup M_{m-1} \cup W(A_{m-1}, M_m) \cup M_m .$$

Regarding the neighborhood attractors within the global attractor, the following result sheds light on the situation.

13. Lemma. *Let $\{X, \mathcal{E}\}$ be a semi-dynamical system with continuous evolution. Assume that this system has a compact attractor $\mathfrak{A}$. Assume that $A \subset \mathfrak{A}$ is a neighborhood attractor of the restricted system $\{\mathfrak{A}, \mathcal{E}\big|_{\mathfrak{A}}\}$. Then A is a neighborhood attractor of $\{X, \mathcal{E}\}$.*

Proof. Assume that, on the contrary, A is not a neighborhood attractor of $\{X, \mathcal{E}\}$. Then there exists a sequence $x_n \in X \setminus \mathfrak{A}$ such that $x_n \to x \in A$ but $\operatorname{dist}(\mathcal{E}^{t_n}(x_n), A) > \delta > 0$, where $t_n \to \infty$. We may assume that $\mathcal{E}^{t_n}(x_n) \to y$ as $n \to \infty$. Since y lies in the ω-limit set of a compact set $K = \{x, x_1, \ldots, x_n, \ldots\}$, the closure of the whole trajectory $\gamma(y)$ must be a subset of $\omega(K)$. In particular, $\omega^*(y) \subset \omega(K)$. Thus, given $z \in \omega^*(y)$, there exist $T_n \to \infty$ and a subsequence x_n such that $\mathcal{E}^{T_n}(x_n) \to z$. Define

$$s_n := \sup\{t \le T_n :\ \operatorname{dist}(\mathcal{E}^t(x_n), A) \le \varepsilon\}$$

where $\varepsilon > 0$ is less than the distance between A and the invariant compact M in the Morse decomposition $\mathfrak{A} = A \cup W(A, M) \cup M$. Note that $\operatorname{dist}(\mathcal{E}^{s_n}(x_n), A) = \varepsilon$. Without loss of generality, we will assume that $\mathcal{E}^{s_n}(x_n) \to p \in \mathfrak{A}$. Note that $s_n \to \infty$, because otherwise we would be able to pass to the limit (along a subsequence) in $\operatorname{dist}(\mathcal{E}^{s_n}(x_n), A) = \varepsilon$ and arrive at $\operatorname{dist}(\mathcal{E}^{s_*}(x), A) = \varepsilon$, which is absurd. We can assume that $\mathcal{E}^{s_n}(x_n) \to p$, where $p \in \mathfrak{A}$ and then $\operatorname{dist}(p, A) = \varepsilon$. We have $p \in W(A, M)$, where $W(A, M)$ is the set of trajectories connecting M with A. If $T_n - s_n$ were bounded, we would pick a subsequence so that $\lim (T_n - s_n) = \tau \geq 0$, and then have, on the one hand, $\mathcal{E}^{T_n}(x_n) = \mathcal{E}^{T_n - s_n} \cdot \mathcal{E}^{s_n}(x_n) \to \mathcal{E}^{\tau}(p)$, and on the other hand, $\mathcal{E}^{T_n}(x_n) \to z \in \omega^*(y) \subset M$. A contradiction. Thus, $T_n - s_n \to \infty$. From the definition of s_n it follows that $\operatorname{dist}(\mathcal{E}^{s_n + t}(x_n), A) > \varepsilon$ for any $t > 0$. Then $\operatorname{dist}(\mathcal{E}^{t}(p), A) \geq \varepsilon$ for all $t > 0$ in contradiction with the fact that $\omega(p) \subset A$. This contradiction proves Lemma. $\square$

Shape of Attractor

14. Homotopy Type. Two (topological) spaces X and Y have the same *homotopy type* if there exist continuous maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f$ is homotopic to the identity map $\mathrm{id}_X : X \to X$ and $f \circ g$ is homotopic to $\mathrm{id}_Y : Y \to Y$,

$$g \circ f \simeq \mathrm{id}_X, \qquad f \circ g \simeq \mathrm{id}_Y .$$

A continuous map $f : X \to Y$, for which these homotopy relations hold true for some $g : Y \to X$, is called a *homotopy equivalence*.

The notion of homotopy type is especially well adapted for the study of "nice" spaces, namely, absolute neighborhood retracts (ANRs; see below). For not so nice spaces, e.g. metric compacta, this notion turns out to be too strict. Borsuk's shape theory and its generalizations apply comfortably to a wider class of spaces. For our purposes, we do not need the most general shape theory and are completely content with the treatment suggested in [Fox, 1972]. Fox's theory applies directly to metric space. Metric spaces can be viewed as subsets of ANRs. Fox's approach (very roughly) is then to replace those subsets by the (inverse) systems of their neighborhoods (open subsets of ANRs are ANRs!), replace maps by systems of maps (mutations), and then define the shapes as the "homotopy types" of such systems of neighborhoods, see [Fox, 1972] for details.

Note also that homotopy groups and singular (co)homologies have their analogues in shape theory in the form of *pro*-homotopy groups and Čech (co)homologies.

In what follows we give the relevant definitions and facts concerning shape that will allow us to prove new results on the shape of attractors.

15. NB. *All spaces are assumed to be metrizable and all maps are continuous.*

16. ANRs. We call P an absolute neighborhood retract (ANR) provided, for any embedding[6] $\imath : P \to P'$ of P into a (metrizable) space P', there exists a neighborhood U of $\imath(P)$ in P' and a map $r : U \to \imath(P)$ such that $r(x) = x$ for $x \in \imath(P)$, i.e., $\imath(P)$ is a retract of the neighborhood U. Banach spaces and their open subsets are examples of ANRs.

If P is an ANR, then for any (metric) space X, any closed subset $A \subset X$, and any (continuous) map $f : A \to P$, there exists a continuous extension of f to some neighborhood of A in X. (This important property of ANRs can be used as the definition.)

Every metric space can be embedded into an ANR. In fact, by the Kuratowski - Wojdysławski theorem, every metric space can be embedded into an ANR as a *closed* subspace.

17. Mutations. Let Y and Z be two metric spaces, and we assume that Y is a subset of an ANR P and Z is a subset[7] of an ANR Q. Denote by $\mathbf{U}(Y, P)$ (respectively, $\mathbf{U}(Z, Q)$) the set of all open neighborhoods of Y in P (respectively, of Z in Q).

Let $\underline{f} = \{f : U \to V\}$ be a collection of continuous maps from the neighborhoods $U \in \mathbf{U}(Y, P)$ to $V \in \mathbf{U}(Z, Q)$. We call $\underline{f}$ *a mutation* from $\mathbf{U}(Y, P)$ to $\mathbf{U}(Z, Q)$ if the following conditions are fulfilled:

1) For every $V \in \mathbf{U}(Z, Q)$ there exists a map $f \in \underline{f}$ such that $f(U) \subset V$, for some $U \in \mathbf{U}(Y, P)$;
2) If $f : U \to V$ is in the collection $\underline{f}$, then the restriction $f\big|_{U_1} : U_1 \to V_1$ is also in $\underline{f}$, for any neighborhood $U_1 \subset U$ and for any neighborhood $V_1 \supset V$;
3) If the two maps $f, f' : U \to V$ are in $\underline{f}$, then there exists a neighborhood $U_1 \subset U$ such that the restrictions of f and f' to U_1 are *homotopic*.

An example of a mutation is a trivial mutation $\underline{\mathbf{id}}_{\,\mathbf{U}(Y,P)}$ comprised of the identity maps $\mathrm{id} : U \to U$.

18. Shape. The sets Y and Z have the same shape (notation: $Sh(Y) = Sh(Z)$), if there exist mutations $\underline{f} : \mathbf{U}(Y, P) \to \mathbf{U}(Z, Q)$ and $\underline{g} : \mathbf{U}(Z, Q) \to \mathbf{U}(Y, P)$ such that the compositions $\underline{g} \circ \underline{f} : \mathbf{U}(Y, P) \to \mathbf{U}(Y, P)$ and $\underline{f} \circ \underline{g} : \mathbf{U}(Z, Q) \to \mathbf{U}(Z, Q)$ are homotopic

[6] $\imath : P \to P'$ is an embedding if $\imath : P \to \imath(P)$ is a homeomorphism.

[7] Here we do not assume that Y is closed in P and Z is closed in Q, see [Hyman, 1972].

to the trivial mutations $\underline{\mathbf{id}}_{\mathbf{U}(Y,P)} : \mathbf{U}(Y,P) \to \mathbf{U}(Y,P)$ and $\underline{\mathbf{id}}_{\mathbf{U}(Z,Q)} : \mathbf{U}(Z,Q) \to \mathbf{U}(Z,Q)$, respectively.

That the composition $\underline{g} \circ \underline{f}$ is homotopic to the trivial mutation $\underline{\mathbf{id}}_{\mathbf{U}(Y,P)}$ (notation: $\underline{g} \circ \underline{f} \simeq \underline{\mathbf{id}}_{\mathbf{U}(Y,P)}$) means that, whenever $f : U \to V \in \underline{f}$ and $g : V \to U' \in \underline{g}$, there exists $U_0 \subset U \cap U'$ so that $g \circ f|_{U_0} : U_0 \to U'$ is homotopic to the identity map.

It is known that the equivalence of the shapes of the sets depends neither on the ANRs they are embedded in, nor on the embeddings.

That the notion of shape is a generalization of that of the homotopy type can be seen from the following results (see [Fox, 1972]).

- Spaces that belong to the same homotopy type have the same shape.
- ANRs have the same shape if and only if they belong to the same homotopy type.

19. Shape Domination. In analogy with Whitehead's notion of homotopy domination, we say that the space Y is shape dominated by Z if there exist mutations $\underline{f} : \mathbf{U}(Y,P) \to \mathbf{U}(Z,Q)$ and $\underline{g} : \mathbf{U}(Z,Q) \to \mathbf{U}(Y,P)$ such that $\underline{g} \circ \underline{f} \simeq \underline{\mathbf{id}}_{\mathbf{U}(Y,P) \to \mathbf{U}(Y,P)}$. That Y is shape dominated by Z will be denoted $Sh(Y) \leq Sh(Z)$.

20. Shape Equivalence. Let Y and Z be two metric spaces, and we assume that Y is a closed subset of an ANR P and Z is a closed subset of an ANR Q. Given a map $f : Y \to Z \subset Q$, its extensions (to neighborhoods of Y in P) give rise to a mutation $\underline{f}$ from $\mathbf{U}(Y,P)$ to $\mathbf{U}(Z,Q)$. We call f *shape equivalence* if there exists a mutation $\underline{g}$ from from $\mathbf{U}(Z,Q)$ to $\mathbf{U}(Y,P)$ such that

$$\underline{g} \circ \underline{f} \simeq \underline{\mathbf{id}}_{\mathbf{U}(Y,P)}, \qquad \underline{f} \circ \underline{g} \simeq \underline{\mathbf{id}}_{\mathbf{U}(Z,Q)}.$$

If $f : Y \to Z$ is a shape equivalence, then, of course, $Sh(Y) = Sh(Z)$.

21. Shape of a Deformation Retract. If B is a deformation retract of X then the inclusion map $\imath : B \hookrightarrow X$ is a shape equivalence. This fact is well-known ([Borsuk]), but we prove it here to facilitate the understanding of the proof of Main Lemma below.

Proof. Recall that B is a *retract* of X if B is a (closed) subset of X and there exists a *retraction* of X to B, i.e., a continuous map $r : X \to B$ such that $r(x) = x$ for all $x \in B$. Recall, also, that B is a *deformation retract* of X if there exists a retraction $r : X \to B$ homotopic to the identity map $\mathrm{id}_X : X \to X$. Thus, assume that $r : X \to B$ is the retraction and $\phi : X \times [0,1] \to X$ is a homotopy, $\phi(x,0) = x$ and $\phi(x,1) = r(x)$. Let P be an ANR in which X is a closed subset. We define a mutation $\underline{r}$ from $\mathbf{U}(X,P)$ to

$\mathbf{U}(B,P)$ as consisting of all extensions of the retraction $r : X \to B$ viewed as a mapping from X to P. That property 3) of the mutation is satisfied follows from Homotopy Extension Lemma (stated below). Next, define a mutation $\underline{\imath}$ from $\mathbf{U}(B,P)$ to $\mathbf{U}(X,P)$ as consisting of all identity maps $\mathrm{id}_U : U \to U$ for $U \in \mathbf{U}(B,P)$. Pick any $\tilde{r} \in \underline{r}$, $\tilde{r} : V \to P$. Let $\tilde{\imath} \in \underline{\imath}$, $\tilde{\imath} : U \to U$ be such that $\tilde{\imath} \circ \tilde{r}$ is well-defined. Then, of course, $\tilde{\imath} \circ \tilde{r} = \tilde{r} : V \to U$. Since $\tilde{r}$ restricted to X is r and r is homotopic to the identity map on X, by Homotopy Extension Lemma $\tilde{r}$ is also homotopic to the identity map when restricted to some neighborhood $V' \subset V$ of X in P. This proves that $\underline{\imath} \circ \underline{r} \simeq \underline{\mathbf{id}}_{\,\mathbf{U}(X,P)}$. Now, if $\tilde{r} \circ \tilde{\imath}$ is defined, then $\tilde{r} \circ \tilde{\imath} = \tilde{r} : U \to \tilde{U}$ for some $U, \tilde{U} \in \mathbf{U}(B,P)$. When restricted to B, $\tilde{r} = r$ is the identity map, and therefore, by Homotopy Extension Lemma, $\tilde{r}\big|_{U_1}$ is homotopic to $\mathrm{id}|_{U_1}$. This implies $\underline{r} \circ \underline{\imath} \simeq \underline{\mathbf{id}}_{\,\mathbf{U}(B,P)}$ and completes the proof of the fact that a deformation retract of a metric space is shape equivalent to this space.

The Homotopy Extension Lemma we have used and will use later can be stated as follows.

22. Homotopy Extension Lemma. *Let X be a metric space and A be a closed subset of X. Let $h_0, h_1 : X \to P$, where P is an ANR. If the restrictions of h_0 and h_1 to A are homotopic, then there exists an extension of this homotopy to a homotopy between $h_0\big|_U$ and $h_1\big|_U$ for some neighborhood U of A.* $\square$

[See [Mardešić & Segal] or any textbook on homotopy theory.]

We now turn to the study of shapes of attractors.

23. Main Lemma. *Let B be a bounded metric space and $\{\mathcal{E}^t\}_{t\geq 0}$ – a semigroup of continuous transformations of B. Assume that the semi-dynamical system $\{B, \mathcal{E}\}$ has a global attractor $\mathfrak{A} \subset B$. Then the inclusion map $\imath : \mathfrak{A} \hookrightarrow B$ is a shape equivalence.*

Proof. We will assume that B is embedded into some ANR P. Denote by $\mathcal{U}$ (respectively, $\mathcal{V}$) the system of open neighborhoods of $\mathfrak{A}$ (respectively, B) in P. Define the mutation $\underline{f} : \mathcal{U} \to \mathcal{V}$ as consisting of all extensions of the inclusion $\imath : \mathfrak{A} \hookrightarrow P$ to open neighborhoods of $\mathfrak{A}$ in P.

To define the mutation $\underline{g} : \mathcal{V} \to \mathcal{U}$, we use the maps $\mathcal{E}^t$. For every $U \in \mathcal{U}$ there exists $T_U > 0$ such that $\mathcal{E}^t(B) \subset U$, for all $t \geq T_U$. Each map $\mathcal{E}^t$ can be extended to a map $\tilde{\mathcal{E}}^t : V \to U$, for some $V \in \mathcal{V}$. To prove that the collection of such extensions $\{\tilde{\mathcal{E}}^t,\, t \geq 0\}$ is a mutation, it suffices to check that any two extensions $\tilde{\mathcal{E}}^t$, $\tilde{\mathcal{E}}^\theta : V \to U$ are homotopic, perhaps, after a restriction to a smaller neighborhood $V' \subset V$. Here $\tilde{\mathcal{E}}^t$

and $\tilde{\mathcal{E}}^\theta$ may be either the extensions of the maps $\mathcal{E}^t$ and $\mathcal{E}^\theta$ corresponding to t and $\theta \neq t$ (different times), or, if $t = \theta$, just different extensions of the same map.

The homotopy follows from Homotopy Extension Lemma and the fact that $\mathcal{E}^{(1-s)t+s\theta}: B \to U$, $0 \leq s \leq 1$, is a homotopy on B.

We will prove now that $\underline{g} \circ \underline{f} \simeq \underline{\mathbf{id}}_{\,\mathcal{U}}$ and $\underline{f} \circ \underline{g} \simeq \underline{\mathbf{id}}_{\,\mathcal{V}}$.

Let $U \in \mathcal{U}$, $V \in \mathcal{V}$, $f : U \hookrightarrow V$, and let $g : V \to U_1$ be one of the extensions of $\mathcal{E}^t$, $t \geq T_{U_1}$. Denote $h = g \circ f$. Since the restriction of h to $\mathfrak{A}$ is $\mathcal{E}^t$, and $\mathcal{E}^t$ is homotopic to the identity map on $\mathfrak{A}$, we see that, for some neighborhood U', $h : U' \to U_1$ is homotopic to the identity.

Consider next $h' = f \circ g$, where $g : V \to U$ is an extension of $\mathcal{E}^t$, $t \geq T_U$, and $f : U \to V'$ is an extension of the inclusion $\mathfrak{A} \subset V'$. Let U_1 be a neighborhood of $\mathfrak{A}$ such that $U_1 \subset U \cap V'$, and let f_1 be the inclusion map $U_1 \subset V'$. Since $f = f_1$ on $\mathfrak{A}$, there exists a smaller neighborhood $\tilde{U} \in \mathcal{U}$ such that f and f_1 are homotopic on $\tilde{U}$. Next, take a $\tau \geq T_{\tilde{U}}$, so that $\mathcal{E}^\tau(B) \subset \tilde{U}$, and let g_1 be an extension of $\mathcal{E}^\tau : B \to \tilde{U}$ to some $V_1 \subset V$. Since g and g_1 are homotopic on B, this homotopy can be extended to some neighborhood $\tilde{V} \subset V_1$.

Thus, $f \circ g$ and $f_1 \circ g_1$ are homotopic as the mappings from $\tilde{V}$ to V'. Now, $f_1 \circ g_1 = g_1$ on $\tilde{V}$, and g_1 is nothing but an extension of $\mathcal{E}^\tau$ from B to $\tilde{V}$. Obviously, $\mathcal{E}^\tau$ is homotopic to the identity on B. This homotopy can be extended to a homotopy between g_1 and the identity on some neighborhood $V_2 \subset \tilde{V}$. Thus, we have shown that $f \circ g : V_2 \to V'$ is homotopic to the identity, and lemma is proved. The following diagram may help to visualize the sets and maps used in the proof of the homotopy $\underline{f} \circ \underline{g} \simeq \underline{\mathbf{id}}_{\,\mathcal{V}}$.

$$
\begin{array}{ccccccccccccccc}
 & & V & & \supset & & V_1 & \supset & \tilde{V} & = & \tilde{V} & \supset & V_2 & \supset & B \\
 & & \Big\downarrow{\scriptstyle g=\tilde{\mathcal{E}}^t} & & & & \Big\downarrow{\scriptstyle g_1=\tilde{\mathcal{E}}^\tau} & & \Big\downarrow{\scriptstyle g_1\simeq g} & & & & & & \\
 & & U & \supset & U_1 & \supset & \tilde{U} & = & \tilde{U} \supset \mathfrak{A} & & \Big\downarrow{\scriptstyle f_1\circ g_1 = g_1 = \tilde{\mathcal{E}}^\tau} & & \Big\downarrow{\scriptstyle g_1\simeq \mathrm{id}} & & \\
 & & \Big\downarrow{\scriptstyle f=\tilde{\imath}} & & \Big\downarrow{\scriptstyle f_1=\imath} & & \Big\downarrow{\scriptstyle f_1\simeq f} & & \Big\downarrow{\scriptstyle f_1\simeq f} & & & & & & \\
B & \subset & V' & = & V' & = & V' & = & V' & = & V' & = & V' & &
\end{array}
$$

$\square$

24. Main Theorem on the Shape of an Attractor. *Let $\{X, \mathcal{E}\}$ be a semi-dynamical system with continuous evolution and let $\mathfrak{A}$ be its global attractor. Then the inclusion map $\imath : \mathfrak{A} \hookrightarrow X$ is a shape equivalence. As a result of this, $Sh(\mathfrak{A}) = Sh(X)$, and $\imath$ induces the isomorphism between Čech homologies, $\imath_* : \check{H}_q(\mathfrak{A}, G) \to \check{H}_q(X, G)$, and the* *isomorphism between Čech cohomologies,* $\imath^* : \check{H}^q(X,G) \to \check{H}^q(\mathfrak{A},G)$.

Proof. Let Φ be the primitive Lyapunov function associated with $\{X,\mathcal{E}\}$ (see previous section). Pick a positive $\varepsilon < \sup_X \Phi(x)$ and define the set

$$B := \{x \in X \,\big|\, \Phi(x) \le \varepsilon\}.$$

This set is bounded. In fact, B lies in the closed ε-neighborhood of $\mathfrak{A}$. Indeed, $\Phi(x) \le \varepsilon$ implies $\operatorname{dist}(\mathcal{E}^t(x), \mathfrak{A}) \le \varepsilon$, for all $t \ge 0$.

Next, B is invariant under the flow in the sense that $\mathcal{E}^t(B) \subset B$, for all $t \ge 0$. [We can say that B is an absorbing set.] Obviuosly, $\mathfrak{A}$ is the attractor of the system $\{B,\mathcal{E}\}$. Hence, by the Main Lemma, $\imath : \mathfrak{A} \hookrightarrow B$ is a shape equivalence.

Let us show now that B is a deformation retract of X.

For any $x \in X \setminus B$ there is a unique moment $t = t_x$ such that $\Phi(\mathcal{E}^t(x)) = \varepsilon$. Define:

$$T(x) := \begin{cases} t_x\,, & x \in X \setminus B \\ 0\,, & x \in B \end{cases}$$

We claim that $T : X \to \mathbb{R}_+$ is continuous. Indeed, let $x \in X \setminus B$. Choose any δ, $0 < \delta < T(x)$. We have

$$\Phi(\mathcal{E}^{T(x)+\delta}(x)) \;<\; \Phi(\mathcal{E}^{T(x)}(x)) = \varepsilon \;<\; \Phi(\mathcal{E}^{T(x)-\delta}(x)).$$

Because of continuity of Φ and $\mathcal{E}^t$, there is an open neighborhood U of x such that $U \subset X \setminus B$ and

$$\Phi(\mathcal{E}^{T(x)+\delta}(y)) \;<\; \Phi(\mathcal{E}^{T(y)}(y)) = \varepsilon \;<\; \Phi(\mathcal{E}^{T(x)-\delta}(y)), \quad \forall y \in U.$$

This implies $T(x)+\delta \,>\, T(y) \,>\, T(x)-\delta$, for all $y \in U$, and hence, T is continuous at x.

If $x \in B$, the worst case is when x lies on the boundary, i.e., $\Phi(x) = \varepsilon$. Assume that there is a sequence of points $x_n \in X \setminus B$ which converges to x but $T(x_n)$ does not converge to $T(x) = 0$. Switching to a subsequence, if necessary, we may assume that $T(x_n) \ge \varkappa > 0$, for all n. Then $\Phi(\mathcal{E}^{\varkappa/2}(x_n)) > \varepsilon$, for all n. Passing to the limit we obtain $\Phi(\mathcal{E}^{\varkappa/2}(x)) \ge \varepsilon$, which is impossible since $x \in X \setminus \mathfrak{A}$, $\Phi(x) = \varepsilon$, and $\Phi(\mathcal{E}^{\varkappa/2}(x)) < \Phi(x)$. The contradiction proves the continuity of T.

Now, define the map $r : X \to B$ as follows:

$$r(x) := \mathcal{E}^{T(x)}(x).$$

This is a retraction of X to B, and it is easy to see that r is homotopic to the identity map. In fact, the homotopy is given by the formula

$$X \times [0,1] \ni (x, \lambda) \mapsto \mathcal{E}^{\lambda\, T(x)}(x) \in X.$$

We have proved that each of the inclusions, $\mathfrak{A} \hookrightarrow B$ and $B \hookrightarrow X$, is a shape equivalence, and hence, the through map $\mathfrak{A} \hookrightarrow X$ is a shape equivalence as well. $\square$

Now we apply the Main Theorem to obtain information about the shape of a neighborhood attractor.

25. Shape of a Neighborhood Attractor. *Let $\{X, \mathcal{E}\}$ be a semi-dynamical system with continuous evolution. Let $A \subset X$ be a (compact) neighborhood attractor. Then A has a bounded, open neighborhood, say, V, such that the inclusions of A into V and into its closure $\bar{V}$ are shape equivalences.*

Proof. Let U be the domain of attraction for A, $\omega(U) = A$. Denote by Y the closure of the positive trajectory of U, $Y := \overline{\cup_{t \geq 0} \mathcal{E}^t(U)}$. It is clear that A is the global attractor of the semi-dynamical system $\{Y, \mathcal{E}\big|_Y\}$. Let Φ be the primitive Lyapunov function associated with this system. For sufficiently small $\varepsilon > 0$, the set $V = \{y \in Y \big| \, \Phi(y) < \varepsilon\}$ and its closure, $\bar{V}$, both lie in U. (It is not hard to see that $\bar{V} = \{y \in Y \big| \, \Phi(y) \leq \varepsilon\}$.) In addition, both V and $\bar{V}$ are positively invariant. Applying Main Theorem to the systems $\{V, \mathcal{E}\big|_V\}$ and $\{\bar{V}, \mathcal{E}\big|_{\bar{V}}\}$, we conclude that $\imath : A \hookrightarrow V$ and $\imath : A \hookrightarrow \bar{V}$ are shape equivalences. It remains to notice that, since V is open in Y, $V \subset U \subset Y$, and U is open in X, so is V. $\square$

Corollaries

An important feature of the shape theory is that the Čech homologies $\check{H}_*(\cdot, G)$ and Čech cohomologies $\check{H}^*(\cdot, G)$ are shape invariants (we refer to [Eilenberg & Steenrod] and [Spanier] for basic fact about Čech (co)homologies, and to [Mardešić & Segal, Ch. II] for Čech (co)homologies and shape theory). It is known that if $Sh(Y) = Sh(Z)$, then, for any abelian group G, and any integer $q \geq 0$, the homology groups $\check{H}_q(Y, G)$ and $\check{H}_q(Z, G)$ are isomorphic, and the cohomology groups $\check{H}^q(Y, G)$ and $\check{H}^q(Y, G)$ are isomorphic as well. In addition, if $f : Y \to Z$ is a shape equivalence, then the induced maps $f_* : \check{H}_q(Y, G) \to \check{H}_q(Z, G)$ and $f^* : \check{H}^q(Z, G) \to \check{H}^q(Y, G)$ are isomorphisms. Hence, we have immediately the following corollary of Main Theorem.

26. Čech (co)homology of an Attractor. *Let $\{X, \mathcal{E}\}$ be a semi-dynamical system with continuous evolution, and let this system have a global attractor, $\mathfrak{A}$. Then the inclusion $\imath : \mathfrak{A} \to X$ induces the isomorphism $\imath_* : \check{H}_q(\mathfrak{A}, G) \to \check{H}_q(X, G)$ in Čech homologies and the isomorphism $\imath^* : \check{H}^q(X, G) \to \check{H}^q(\mathfrak{A}, G)$ in Čech cohomologies.*

□

A similar result can be stated for neighborhood attractors; we skip it.

We can say more about the attractor if we know of some additional properties of the state space X. A natural assumption is that X is shape dominated by an ANR (equivalently: a polyhedron, a CW-complex) If a polyhedron is not locally finite, then we assume that it is equipped with Whitehead's (or CW-) topology: a set U is open iff, for every closed simplex σ, the intersection $\sigma \cap U$ is an open set of σ.).

27. Finiteness of Čech (co)homologies. *Let $\{X, \mathcal{E}\}$ be a semi-dynamical system with continuous evolution and let this system have a global attractor, $\mathfrak{A}$. Assume that the state space X is shape dominated by an ANR. Then $\mathfrak{A}$ is shape dominated by a finite polyhedron, the Čech homologies and cohomologies of $\mathfrak{A}$ are finitely generated, and $\check{H}_q(\mathfrak{A}, G)$ and $\check{H}^q(\mathfrak{A}, G)$ are trivial for all sufficiently large q.*

Proof. Since $Sh(\mathfrak{A}) = Sh(X)$ and X is shape dominated by an ANR, then $\mathfrak{A}$ is shape dominated by an ANR. This implies that $\mathfrak{A}$ is a FANR and, hence, $\mathfrak{A}$ is shape dominated by a finite polyhedron, [Mardešić & Segal, Ch.II, §9.5]. The latter implies the finiteness of Čech (co)homologies. □

28. Remark. *If in addition to the assumptions of the previous result we know that X is connected, then $\mathfrak{A}$ is a pointed FANR (for any choice of base point), $(\mathfrak{A}, *)$ is movable, the shape dimension of $\mathfrak{A}$ is finite, and all of its homotopy pro-groups pro-$\pi_k(\mathfrak{A}, *)$ are stable (=isomorphic to homotopy groups of a polyhedron).*

All these facts follow from the results described in detail in [Mardešić & Segal, Ch.II, §9.5].

In general, the fact that a compact set is shape dominated by a finite polyhedron does not imply that the set has the shape of a finite polyhedron, see [Edwards and Geoghegan, 1975]. This is closely related to the Wall obstruction, [Wall, 1965]. However, with additional assumptions it is possible to show that the attractor has the shape of a finite polyhedron.

29. When an Attractor has the Shape of a Finite Polyhedron. *Let $\{X, \mathcal{E}\}$ be a semi-dynamical system with continuous evolution and let $A \subset X$ be a neighborhood attractor.*

A has the shape of a finite polyhedron if either of the following conditions is satisfied.

1) *A has a domain of attraction which is a locally compact ANR.*
2) *The system $\{X, \mathcal{E}\}$ has a stationary solution, say, z, in A. The attractor A has a positively invariant, connected domain of attraction, U, which is an ANR, and pro-$\pi_1(U, z) = 0$.*

Proof. Assume 1). We have already shown in ¶25 that $Sh(A) = Sh(V)$, where $V = \{y \in Y \,|\, \Phi(y) < \varepsilon\}$ and Φ is the primitive Lyapunov function defined on $Y =$ the closure of the positive trajectory of the domain of attraction of A. For sufficiently small ε, V is a true open neighborhood of A and, as an open subset of an ANR U, it is an ANR. Because X is locally compact, every closed subset of V is compact. Since A is the global attractor of the semi-dynamical system $\{V, \mathcal{E}\big|_V\}$, we can show, as in the proof of the Main Theorem, that the set $V_{\leq 1/2} := \{y \in Y \,|\, \Phi(y) \leq \varepsilon/2\}$ is a deformation retract of V. Hence, $V_{\leq 1/2}$ is an ANR. Since a compact ANR has the homotopy type of a finite polyhedron, [West], and $Sh(A) = Sh(V) = Sh(V_{\leq 1/2})$, A has the shape of a finite polyhedron. [The argument invoking West's theorem is borrowed from [Günther & Segal, 1993].]

Assume now 2). In view of ¶28, A is a connected FANR, and hence, by [Hastings & Heller, 1980], (A, z) is a pointed FANR. We know already that all Čech homology groups $\check{H}_q(A; \mathbb{Z})$ are finitely generated. Also, since $Sh(A, z) = Sh(U, z)$, we have pro-$\pi_1(A, z) =$ pro-$\pi_1(U, z) = 0$ (about pro-homotopy groups see ¶32 below). Being a pointed ANR, (U, z) is homotopy equivalent to a pointed CW-complex, $(P, *)$, say. Hence, $Sh(U, z) = Sh(P, *)$. Consequently, $0 =$ pro-$\pi_1(P, *) = \pi_1(P, *)$, and $\check{H}_*(P; \mathbb{Z})$ are finitely generated. Then it follows, [Lundell & Weingram, 1969; Theorem IV.7.2], that $(P, *)$ has the homotopy type of a compact, finite CW-complex. Thus, (A, z) is shape equivalent to a compact, finite CW-complex. □

30. Corollary. *Let $\mathfrak{A}$ be the global attractor of system $\{X, \mathcal{E}\}$ with continuous evolution. Then $\mathfrak{A}$ has the shape of a finite polyhedron if X is a locally compact ANR, or if X is a connected ANR, and pro-$\pi_1(X, z) = 0$, for some stationary point $z \in \mathfrak{A}$.*

31. Remark. *Note, that if X is connected, then $(\mathfrak{A} \times \mathbb{S}^1, *)$ always has the shape of a finte polyhedron. Indeed, since $(\mathfrak{A}, *)$ is a connected, pointed FANR, the result follows* *from* [Dydak & Orłowski, 1977].

Another set of important shape invariants consists of homotopy pro-groups (see [Mardešić & Segal, Ch. II]). Strictly speaking, the homotopy pro-groups are defined for pointed spaces, and there are examples of spaces for which certain homotopy pro-groups depend on the choice of the base point ([Mardešić & Segal, page 132]). We do not know at the moment whether or not our results on the shape of attractors, without additional assumptions on a semi-dynamical system, can be extended to the category of pointed spaces. Here is a simple example (with additional assumption) where the extention is straightforward.

32. Homotopy pro-Groups of an Attractor. *Let a semi-dynamical system* $\{X, \mathcal{E}\}$ *with continuous evolution possess a compact attractor* $\mathfrak{A}$. *Assume that the system has an equilibrium,* $z \in \mathfrak{A}$. *Then the inclusion* $\imath : (\mathfrak{A}, z) \to (X, z)$ *induces shape equivalence of pointed spaces* $(\mathfrak{A}, z)$ *and* (X, z). *The homotopy pro-groups of* $(\mathfrak{A}, z)$ *and* (X, z) *are isomorphic: pro-*$\pi_k(X, z) \simeq$ *pro-*$\pi_k(\mathfrak{A}, z)$, *for* $k = 0, 1, \dots$ *The shape groups are also isomorphic:* $\check{\pi}_k(X, z) \simeq \check{\pi}_k(\mathfrak{A}, z)$.

Proof. The proof of the Main Theorem works without any changes because $\mathcal{E}^t(z) = z$ for all t. The conclusions about the homotopy pro-groups follow from general results, see [Mardešić & Segal, Ch. II, §3.3]. $\square$

Morse's Equations

Let $\{X, \mathcal{E}\}$ be a semi-dynamical system with continuous evolution. Assume that this system has a global attractor $\mathfrak{A}$, and $\mathfrak{A}$ admits a (nontrivial) Morse filtration:

$$A_0 \subset A_1 \subset \dots \subset A_{m-1} \subset A_m = \mathfrak{A}.$$

Once there is a filtration, there are certain general Morse equations. Here is what we mean.

Set $A_{-1} := \emptyset$. Let H_* be any homology theory for which the long homology sequence of any pair (A_k, A_{k-1}),

$$\cdots \to H_{q+1}(A_k, A_{k-1}) \xrightarrow{\partial_{q+1,k}} H_q(A_{k-1}) \xrightarrow{\imath_*}$$
$$H_q(A_k) \xrightarrow{\jmath_*} H_q(A_k, A_{k-1}) \xrightarrow{\partial_{q,k}} H_{q-1}(A_{k-1}) \to \cdots$$
$$\xrightarrow{\imath_*} H_1(A_k) \xrightarrow{\jmath_*} H_1(A_k, A_{k-1}) \xrightarrow{\partial_{1,k}} H_0(A_{k-1}) \xrightarrow{\imath_*} H_0(A_k) \xrightarrow{\jmath_*} H_0(A_k, A_{k-1}) \xrightarrow{\partial_{0,k}} 0$$

is exact. Then the following very general facts (that have nothing to do with attractors) take place.

33. Morse's Equations.

(i) *If, for* $q = 0, 1, \ldots,$ *each of the groups* $H_q(A_k)$, $k = 0, \ldots, m$, *is finitely generated, then so is each of the groups* $H_q(A_k, A_{k-1})$, $k = 0, \ldots, m$.

(ii) *If, for* $q = 0, 1, \ldots,$ *each of the groups* $H_q(A_k, A_{k-1})$, $k = 0, \ldots, m$, *has a finite rank, then so does each of the groups* $H_q(A_k)$, $k = 0, \ldots, m$, *and vise versa.*

(iii) *The following equalities hold true*

$$\sum_{k=0}^{r} \sum_{q=0}^{\infty} t^q \dim H_q(A_k, A_{k-1}) = \sum_{q=0}^{\infty} t^q \dim H_q(A_r) + (1+t)\, Q_r(t), \quad r = 1, \ldots, m,$$

where

$$Q_r(t) = \sum_{k=0}^{r} \sum_{q=1}^{\infty} t^{q-1} \operatorname{rank} \partial_{q,k} .$$

Proof. The assertion (i) is a trivial consequence of the exactness of the long homology sequence. The exactness also implies the equalities:

1) $$\dim H_q(A_k, A_{k-1}) = \operatorname{rank} \jmath_* + \operatorname{rank} \partial_{q,k},$$

2) $$\dim H_q(A_{k-1}) = \operatorname{rank} \imath_* + \operatorname{rank} \partial_{q+1,k},$$

3) $$\operatorname{rank} \jmath_* + \operatorname{rank} \imath_* = \dim H_q(A_k).$$

By adding the first two equalities and invoking the third one, we obtain

4) $$\dim H_q(A_k, A_{k-1}) + \dim H_q(A_{k-1}) = \dim H_q(A_k) + (\operatorname{rank} \partial_{q,k} + \operatorname{rank} \partial_{q+1,k}).$$

If $\dim H_q(A_k) < \infty$ for all $q = 0, 1, \ldots$ and $k = 0, \ldots m$, then, as follows from 2), $\operatorname{rank} \partial_{q+1,k} < \infty$. Note, also, that $\operatorname{rank} \partial_{0,k} = 0$. Thus, the finiteness of the number $\dim H_q(A_k, A_{k-1})$ follows from 4). Conversely, assume that all $\dim H_q(A_k, A_{k-1})$, $k = 0, \ldots, m$, are finite (in particular, $\dim H_q(A_0, A_{-1}) = \dim H_q(A_0) < \infty$). If there existed a k such that $\dim H_q(A_k) = \infty$, then we would be able to find the smallest $j \geq 0$ with the property $\dim H_q(A_j) = \infty$. But then 4) would force $\dim H_q(A_{j-1})$ to be infinite also which contradicts the minimality of j. Therefore, all $\dim H_q(A_j)$, $j \geq 1$, must be finite as well.

Next, take the sum of equalities 4) for $k = 1, \dots, r$. This yields

$$\sum_{k=0}^{r} \dim\, H_q(A_k, A_{k-1}) = \dim\, H_q(A_r) + \sum_{k=0}^{r} (\mathit{rank}\, \partial_{q,k} + \mathit{rank}\, \partial_{q+1,k}),$$

and Morse's equations follow.

□

A similar result is valid for cohomologies. In particular, we have the following equality:

$$\sum_{k=0}^{r} \sum_{q=0}^{\infty} t^q \,\dim\, H^q(A_k, A_{k-1}) \;=\; \sum_{q=0}^{\infty} t^q \,\dim\, H^q(A_r) + (1+t)\, Q^r(t),$$

where

$$Q^r(t) \;=\; \sum_{k=0}^{r} \sum_{q=1}^{\infty} t^{q-1} \,(\mathit{rank}\, \delta^{q,k} + \mathit{rank}\, \delta^{q+1,k}).$$

34. There is not much to the Morse equations unless one can effectively evaluate the quantities $\dim H_q(A_k, A_{k-1})$ and $\dim H_q(A_m)$, or $\dim H^q(A_k, A_{k-1})$ and $\dim H^q(A_m)$. In our case, $A_m = \mathfrak{A}$ is the global attractor and $A_0,\ A_1,\ \dots,\ A_{m-1}$ are neighborhood attractors. For attractors, we advocate Čech (co)homologies. As we have shown, if the state space X is an ANR (which is almost always the case), then the Čech (co)homologies of attractors (global and neighborhood) are finitely generated and $\check{H}_q = 0$ for large q. Then the relative (co)homologies will have finite ranks as well. To give the means for evaluation of $\dim\, \check{H}_q(A_k, A_{k-1})$, we turn to the Morse decomposition of the attractor (see ¶12),

$$\mathfrak{A} \;=\; A_m \;=\; M_0 \cup W(A_0, M_1) \cup M_1 \cup \cdots \cup\; M_{m-1} \cup W(A_{m-1}, M_m) \cup M_m\,.$$

The principle is: *the dimensions* $\dim\, \check{H}_q(A_k, A_{k-1})$ *(or* $\dim\, \check{H}^q(A_k, A_{k-1})$*),* $q = 0,\ 1,$ *…, are determined by the appropriate index of the invariant set* M_k*.*

How to Evaluate $\dim\, \check{H}_q(A_k, A_{k-1})$**.** We can offer several ways to find the numbers $\dim\, \check{H}_q(A_k, A_{k-1})$ in terms of the invariant set M_k and its "unstable manifold" $W(A_{k-1}, M_k)$ in our general setting. (This is the "index" of M_k that we compute.)

35. *Case 1.* Let $\eth W_k$ be any closed subset of $W(A_{k-1}, M_k)$ with the property that it cuts the invariant set $W(A_{k-1}, M_k)$ into two disjoint parts, W_k and $\mathcal{E}^{(0,+\infty)}(\eth W_k)$, so that

$$W(A_{k-1}, M_k) \;=\; W_k \,\cup\, \eth W_k \,\cup\, \mathcal{E}^{(0,+\infty)}(\eth W_k),$$

and $\eth W_k \cap \mathcal{E}^{(0,+\infty)}(\eth W_k) = \emptyset$. Then we have

$$\check{H}_*(A_k, A_{k-1}) \simeq \check{H}_*(W_k, \eth W_k)\,.$$

Indeed, from the construction of the Morse decomposition in ¶12 it follows that A_{k-1} is the global attractor of the system restricted to either of the (positively invariant) sets $V = \mathcal{E}^{(0,+\infty)}(\eth W_k) \cup A_{k-1}$ and $\bar{V} = \eth W_k \cup \mathcal{E}^{(0,+\infty)}(\eth W_k) \cup A_{k-1}$. Then, by ¶24, the inclusion $\imath : A_{k-1} \hookrightarrow \bar{V}$ induces the isomorphism on homologies, $\check{H}_*(A_{k-1}) \simeq \check{H}_*(\bar{V})$. Hence, $\check{H}_*(A_k, A_{k-1}) \simeq \check{H}_*(A_k, \bar{V})$, [Spanier, Lemma 4.8.6]. By the excision property, we have $\check{H}_*(A_k, \bar{V}) \simeq \check{H}_*(A_k \setminus V, \bar{V} \setminus V)$, which proves our claim.

There are many possibilities to find the set $\eth W_k$. For example, we can proceed as follows. Define on A_k the following Lyapunov function:

$$\Phi_k(x) := \sup_{t \geq 0} \operatorname{dist}\left(\mathcal{E}^t(x),\ A_{k-1}\right), \quad \text{for } x \in A_k.$$

Note that $\Phi_k(x) = 0$ for $x \in A_{k-1}$, $\Phi_k(\mathcal{E}^t(x)) \leq \Phi_k(x)$ and $\Phi_k(\mathcal{E}^t(x)) \to 0$ for $x \in W(A_{k-1}, M_k)$, and $\Phi_k(\mathcal{E}^t(x)) \leq \operatorname{dist}(M_k,\ A_{k-1})$ for $x \in M_k$. If $\varepsilon < \operatorname{dist}(M_k,\ A_{k-1})$, then we can set $\eth W_k := \{x \in A_k \,|\, \Phi_k(x) = \varepsilon\}$.

If the semiflow on the attractor is reversible (as often happens), then, for the reversed flow, we can construct a Lyapunov function that vanishes on M_k, and take for $\eth W_k$ its level set where it equals ε for any sufficiently small ε.

36. *Case 2.* Assume that there exists a Lyapunov function L defined on the state space X such that it is constant on each of the invariant sets M_k, $L(x) = \ell_k$ for all $x \in M_k$, and strictly decreases under evolution outside of the invariant sets M_k, $L(\mathcal{E}^t(x)) < L(x)$ for all $x \in X \setminus (M_0 \cup \cdots \cup M_m)$. We call the constants $\ell_0 < \cdots < \ell_m$ the critical values of L. Denote $L^a := \{x \in X \,|\, L(x) \leq a\}$.

Claim. *If the interval $(a, b]$ contains only one of critical values of L, say, ℓ_k, then $\check{H}_*(A_k, A_{k-1}) \simeq \check{H}_*(L^b, L^a)$.*

Proof. Consider the commutative diagram

$$\begin{array}{ccccccccc}
\check{H}_q(A_{k-1}) & \xrightarrow{\imath_*} & \check{H}_q(A_k) & \xrightarrow{\jmath_*} & \check{H}_q(A_k, A_{k-1}) & \xrightarrow{\partial} & \check{H}_{q-1}(A_{k-1}) & \xrightarrow{\imath_*} & \check{H}_{q-1}(A_k) \\
{\scriptstyle \imath_*}\downarrow & & {\scriptstyle \imath_*}\downarrow & & {\scriptstyle \imath_*}\downarrow & & {\scriptstyle \imath_*}\downarrow & & {\scriptstyle \imath_*}\downarrow \\
\check{H}_q(L^a) & \xrightarrow{\imath_*} & \check{H}_q(L^b) & \xrightarrow{\jmath_*} & \check{H}_q(L^b, L^a) & \xrightarrow{\partial} & \check{H}_{q-1}(L^a) & \xrightarrow{\imath_*} & \check{H}_{q-1}(L^b)
\end{array}$$

The upper and lower rows are the exact homology sequences for the pairs (A_k, A_{k-1}) and (L^b, L^a). The vertical arrows are induced by the embeddings. Since A_k is the global attractor of $\{L^b, \mathcal{E}\}$ and A_{k-1} is the global attractor of $\{L^a, \mathcal{E}\}$, our Main Theorem implies that the induced maps $\imath_*$, i.e., the vertical arrows number 1, 2, 4, and 5, are the isomorphisms. Thus, our claim follows from the Five Lemma, [Spanier, Lemma IV.5.11]. $\square$

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