On absolute continuity of spectra of periodic elliptic operators

Peter Kuchment*
Mathematics and Statistics Department
Wichita State University
Wichita, KS 67260-0033, USA

Sergei Levendorski
Mathematics Department
Rostov State Academy of Economics
Rostov-on-Don, Russia

Abstract

The paper contains a brief description of a simplified version of A. Sobolev's proof of absolute continuity of spectra of periodic magnetic Schrödinger operators. This approach is applicable to all periodic elliptic operators known to be of interest for math physics (including Maxwell), and in all these cases leads to the same model problem of complex analysis. The full account of this approach will be provided elsewhere.

1 Introduction

Elliptic differential operators with periodic coefficients arise naturally in many areas of mathematical physics. One can mention quantum solid state theory, where the main operators of interest are the stationary Schrödinger operator

$$-\Delta + V(x) \tag{1}$$

and the magnetic Schrödinger operator

$$(i\partial - A(x))^2 + V(x). \tag{2}$$

*The work of both authors was partially supported by an NRC COBASE Grant. The work of the first author was also supported in part by the NSF Grant DMS 9610444 and by a DEPSCoR Grant.

[0] *1991 Mathematics Subject Classification.* Primary 35P99, Secondary 35J10

Key words and phrases. Periodic operator, elliptic operator, absolutely continuous spectrum, Schrödinger operator, magnetic and electric potential

Here the scalar electric potential $V(x)$ and the vector magnetic potential $A(x)$ are assumed to be periodic with respect to a lattice in $\mathbf{R^d}$ [1]. Another example, which gained importance due to recent advances in the photonic crystals theory [7] is the periodic Maxwell operator $\nabla \times \varepsilon(x)^{-1}\nabla\times$ defined on zero-divergence vector fields in $\mathbf{R^3}$. Here the periodic function $\varepsilon(x)$ represents the electric permittivity of the medium. Scalar counterparts of this operator (arising also in acoustics) are the operators $-\nabla\cdot\varepsilon(x)^{-1}\nabla$ and $-\varepsilon(x)^{-1}\Delta$. Other examples are the anisotropic divergent type operators $\sum \partial_i a_{ij}(x)\partial_j$, where the magnetic and electric potential terms can also be included, and the periodic Dirac operator ([2], [4], [11]).

For all these operators, the structure of the spectrum is of major interest. Consider a periodic Schrödinger operator (1) in $\mathbf{R^d}$. Under mild conditions on the real potential the operator is self-adjoint [12]. It has been clear to physicists for a long time that the spectrum of this operator in $L_2(\mathbf{R^d})$ does not contain any eigenvalues. One can easily prove absence of eigenvalues of finite multiplicity [5], however the proof of the general statement had to wait for a long time until the celebrated Thomas' theorem [16] (see also its improved version in [12], [14]). The paper [16] contained the proof of a more general statement: the spectrum of a selfadjoint periodic operator (1) is absolutely continuous. Absence of singular continuous spectrum for periodic elliptic operators holds in a very general situation and is a rather straightforward consequence of Floquet theory ([8], [12], [13]).

The question arises on whether absence of eigenvalues is shared by all periodic elliptic differential operators. It is known, however [8] that this is not true in general for elliptic operators of order four. Still, the common belief is that periodic elliptic operators of second order do not possess any eigenvalues. The talks presented at this conference by M. Birman and T. Suslina described some of such results. There are non-standard situations of mesoscopic physics and photonic crystal theory where one can encounter point spectrum of periodic second order differential problems on graphs [9]. This, however, does not influence our belief in absolute continuity of spectra of such problems in $\mathbf{R^d}$. Let us list the known results. The Thomas' theorem [16] was extended to a broader class of periodic Schrödinger operators in the M. Reed and B. Simon's book [12]. L. Danilov [4] proved absolute continuity of the spectrum for the case of the Dirac operator with a periodic scalar potential. The result for the magnetic Schrödinger operator (2) was obtained by R. Hempel and I. Herbst in [6] for the case of small magnetic potentials. Finally, in the remarkable papers by M. Birman and T. Suslina [2] and A. Sobolev [15] the full strength statement about the magnetic Schrödinger operator was proven. The elegant algebraic approach of the paper [2] works only in dimension two. The paper [2] lead A. Sobolev to an ingenious proof in arbitrary dimension [15]. A generalization of the result of [4] was obtained in [3]. A recent very interesting preprint by A. Morame [11] treats an anisotropic divergent type operator in $2D$.

Our initial goal was to get good grasp of the proof presented in [15]. We found that the initial proof can be significantly simplified, which in particular leads to improvements in the result and to the possibility of broad generalizations. Our aim is mostly methodological: to provide a simplified and unified approach that treats in an uniform way all the operators mentioned above and, we hope, clarifies the situation. It also leads to some improvements in the result of [15] (weaker assumptions on the potential, a stronger estimate from below, and validity in non-selfadjoint case). The interesting observation is that in all situations known to the authors the same model $\overline{\partial}$ problem arises, good understanding of which would lead to progress in all cases that are not treated yet (Maxwell, divergent, etc.). Most of the ideas of the proof were already contained in a more obscured form in papers [2] and [15]. This paper contains only a brief description of the results. The complete account will be given elsewhere.

2 The magnetic Schrödinger operator

Our main object of study is the magnetic Schrödinger operator in $\mathbf{R^d}$

$$H = (D + A(x))^2 + V(x)$$

with a scalar electric potential $V(x)$ and a vector magnetic potential $A(x)$. We will assume for simplicity of presentation that both potentials are periodic with respect to the integer lattice $\mathbf{Z^d}$; the case of general lattices does not require any significant changes in the proofs. According to Thomas [16] (see also [12], [8]), absence of eigenvalues will be proven if one is able to show existence of a *quasimomentum* $k \in \mathbf{C^d}$ such that the operator

$$H(k) = (D + k + A(x))^2 + V$$

has zero kernel on the torus $\mathbf{T^d} = \mathbf{R^d}/\mathbf{Z^d}$. Our key statement is the following theorem:

Theorem 1 *Let $A \in \left[H^s(\mathbf{T^d})\right]^d$ for some $s > 3d/2 - 1$. Then there exist constants $C > 0$, $\beta > 0$ and vector $e \in \mathbf{R^d}$, such that for sufficiently large $\rho \in \mathbf{R}$ and any $u \in H^2(\mathbf{T^d})$*

$$\left|\left|(D + k + A(x))^2 u\right|\right|_{L_2} \geq C(|\rho| \, ||u||_{L_2} + ||u||_{H^1}),$$

where $k = (\beta + i\rho)e \in \mathbf{C^d}$.

An immediate consequence of this estimate is the following statement.

Theorem 2 *Assume that A is like in the previous theorem and the electric potential V is such that*

$$||Vu||_{L_2(\mathbf{T^d})} \leq C_\varepsilon \, ||u||_{L_2(\mathbf{T^d})} + \varepsilon \, ||u||_{H^1(\mathbf{T^d})} \tag{3}$$

for arbitrary $\varepsilon > 0$ and $u \in H^1$ (for instance, $||Vu||_{L_2(\mathbf{T^d})} \leq C \, ||u||_{H^\alpha(\mathbf{T^d})}$ for some $\alpha < 1$). Then the periodic magnetic Schrödinger operator $H = (D + A(x))^2 + V(x)$ has no point spectrum in $L_2(\mathbf{R^d})$.

The condition (3) can be verified for different classes of potentials, as it was done in the previously mentioned works on this subject (the details will be added in the complete version).

In the case of real potentials the operator is self-adjoint, and one obtains the following result.

Theorem 3 *The spectrum of the periodic magnetic Schrödinger operator* $H = (D + A(x))^2 + V(x)$ *in* $L_2(\mathbf{R^d})$ *is absolutely continuous.*

3 The scheme of the proof

One can consider the case of zero electric potential, since a standard procedure enables one to include the electric potential (see for instance [8], [12], [15]).

Let $H(k) = (D+k+A(x))^2$, and Λ_ρ be the operator that multiplies the mth Fourier coefficient of a periodic function by $(\rho^2 + m^2)^{1/2}$. These are operators on the torus $\mathbf{T^d}$. Let

$$k = 2\pi(i\rho + \beta)e \in \mathbf{C^d},$$

where $\beta \in \mathbf{R}$ is fixed, $\rho \in \mathbf{R}$ is arbitrarily large, and $e \in \mathbf{R^d}$. We introduce the principal symbol of the operator $H(k)$ as

$$H_0(k, m) = (2\pi m + k)^2 = 4\pi^2\left[(m + \beta e)^2 - \rho^2 + 2i\rho e \cdot (m + \beta e)\right].$$

Notice that we include some lower order differential terms with parameter k into the principal symbol, which is rather standard when working with pseudodifferential operators with parameters. The zero set of this symbol is

$$Z_\rho = \left\{m |\ (m + \beta e)^2 = \rho^2,\, e \cdot (m + \beta e) = 0\right\}.$$

We choose a suitable finite multiplicity covering of the dual space $\mathbf{R^d_\xi}$ by sets $U_{\rho j}$ of diameter ρ^δ for an appropriately chosen $\delta \in (0, 1)$. The goal is to produce a set of local approximate inverse operators $R_{\rho,j}$ such that on functions whose Fourier series are supported in $U_{\rho j}$ the following properties are satisfied: $||R_{\rho,j}|| \leq C$, $R_{\rho,j}H(k)\Lambda_\rho^{-1} = I + T_{\rho,j}$, and $||T_{\rho,j}|| \to 0$ uniformly with respect to j when $\rho \to \infty$. Then a suitable partition of unity in the dual space finishes the job. Namely, an operator

$$R_\rho = \sum_j \phi_j(D) R_{\rho,j} \psi_j(D)$$

is constructed in such a way that

$$R_\rho H(k)\Lambda_\rho^{-1} = I + T_\rho,$$

where

$$\lim_{\rho\to\infty} ||T_\rho|| = 0. \tag{4}$$

Here $\phi_j(D)$ and $\psi_j(D)$ are operators that multiply Fourier coefficients of a function by cut-off functions ϕ_j and ψ_j. Existence of such an operator R_ρ proves the Theorem 1.

Now we see that the main task is to construct the local approximate inverse operators $R_{\rho j}$. The situation looks differently away from the set Z_ρ and closely to it. Namely, when the distance from $U_{\rho j}$ to Z_ρ is more than ρ^δ, the principal part will dominate the magnetic one, which analogously to the Thomas' case leads to invertibility. However, close to Z_ρ the magnetic potential part becomes of comparable strength with the principal part. In each of these open sets the operator $H(k)\Lambda_\rho^{-1}$ can be reduced to a model first order differential operator by linearizing its symbol at one point. This model operator happens to be of a generalized Cauchy-Riemann type

$$\frac{\partial}{\partial \overline{z}} + g(z) \tag{5}$$

on a complex plane, where the plane arises as a rational plane in $\mathbf{R^d}$ spanned by two integer vectors l and n, and the function $g(z)$ is periodic. The goal is to invert such an operator on the torus with controlled norm of the inverse. A. Sobolev invented a trick that does this [15]. Namely, one can gauge away most of the magnetic potential A, leaving only a small part of it, which allows to construct a controllable inverse operator. The construction of the gauge transform amounts to solving the Cauchy-Riemann equation $\overline{\partial} f = g(z) f$ for an invertible periodic function f on the plane. In general there are obstructions (some Fourier coefficients of g must vanish), but a clever choice of an $l-n$ plane guarantees that these obstructions are moved into a tail of the Fourier series of the potential. This allows one to gauge away most of the potential and to construct an approximate inverse with controllable norm.

4 Concluding remarks and acknowledgments

This scheme works uniformly in all cases of interest. It enables one to obtain in the same way (and with the same model local operator) the results of [2], [3], [4], [15], and [16]. The scheme also works in all cases where complete results are not yet known: the Maxwell operator, Dirac operator with a general matrix potential, and anisotropic divergent type operators. In all these three situations one discovers that the local model operator on Z_ρ (5) has a matrix "potential" g. One ends up with the following question: given a periodic matrix function $g(z)$ on the complex plane one needs to find an invertible periodic matrix function $f(z)$ such that $\overline{\partial} f = g(z) f$. So, we have to deal with a non-commutative version of the model problem that was discussed above. This problem has been studied in complex analysis [10]. The function g defines in some natural way a holomorphic vector bundle, and a required function f exists if and only if this bundle is trivial. What one needs to know for the spectral problem that we discuss is whether there is any "non-commutative" analog of Sobolev's trick that moves the obstruction into a long tail of the Fourier expansion and gauges away the rest of the series.

The authors express their gratitude to Professors M. Birman, A. Sobolev, T. Suslina, A. Tumanov, R. Novikov, A. Pankov, and V. Palamodov for information and useful discussions. This research was partly sponsored by the NRC through a COBASE Grant, NSF through the Grant DMS 9610444, and by the Department of Army, Army Research Office, through a DEPSCoR Grant. The authors thank the NRC, NSF, and the ARO for this support. The content of this paper does not necessarily reflect the position or the policy of the federal government, and no official endorsement should be inferred.

References

[1] N.W. Ashcroft and N.D. Mermin, *Solid State Physics,* Holt, Rinehart and Winston, New York-London 1976.

[2] M. Sh. Birman and T. A. Suslina, The two-dimensional periodic magnetic Hamiltonian is absolutely continuous. (Russian) Algebra i Analiz 9(1997), no. 1, 32–48; translation in St. Petersburg Math. J. 9(1998), no. 1, 21–32.

[3] M. Sh. Birman and T. A. Suslina, The periodic Dirac operator is absolutely continuous, Preprint 1998.

[4] L. Danilov, Spectrum of the Dirac operator in $\mathbf{R^n}$, Theor. Math. Phys. 85(1990), no.1, 41-53.

[5] M. S. P. Eastham, *The Spectral Theory of Periodic Differential Equations,* Scottish Acad. Press, Edinburgh-London, 1973.

[6] R. Hempel and I. Herbst, Strong magnetic fields, Dirichlet boundaries, and spectral gaps, Comm. Math. Phys. 164(1995), 237-259.

[7] J.D. Joannopoulos, R.D. Meade, and J. N. Winn, *Photonic Crystals. Molding the Flow of Light,* Princeton Univ. Press, Princeton, NJ, 1995.

[8] P. Kuchment, *Floquet Theory for Partial Differential Equations,* Birkhäuser Verlag, Basel 1993.

[9] P. Kuchment and L. Kunyansky, Spectral Properties of High Contrast Band-Gap Materials and Operators on Graphs, to appear in *Experimental Mathematics*, Part II in preparation

[10] B. Malgrange, *Lectures on the theory of functions of several complex variables*, Tata Institute of Fundamental Research, Bombay, Springer-Verlag, Berlin 1958.

[11] A. Morame, Absence of singular spectrum for a perturbation of a two-dimensional Laplace-Beltrami operator with periodic electro-magnetic potential, Preprint 1998.

[12] M. Reed, B. Simon, *Methods of Modern Mathematical Physics, Vol.IV: Analysis of Operators,* Academic Press, New York, 1978.

[13] J. Sjostrand, Microlocal analysis for the periodic magnetic Schrödinger equation and related questions, in : Microlocal Analysis and Applications, Lect. Notes in Phys., v.1495, 237-332, Springer-Verlag, Berlin, 1991.

[14] Z. Shen. On absolute continuity of the periodic Schrödinger operators, Preprint ESI 597 (1998), Vienna

[15] A. Sobolev, Absolute continuity of the periodic magnetic Schrödinger operator, Preprint, 1997.

[16] L. E. Thomas, Time dependent approach to scattering from impurities in a cristal, Comm. Math. Phys. 33(1973), 335-343.