Singular continuous spectrum for a class of substitution Hamiltonians*

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Abstract

We consider discrete one-dimensional Schrödinger operators with potentials generated by primitive substitutions. Purely singular continuous spectrum with probability one is established provided that the potentials have a local four-block structure.

1 Introduction

The discovery of quasicrystals by Shechtman *et al.* in 1984 [20] has motivated continuing interest by both physicists and mathematicians in adequate models describing these structures. A class of models that has attracted particular attention in this context is provided by one-dimensional Schrödinger operators with potentials generated by so-called substitution sequences. A considerable amount of knowledge about the spectral properties of these operators has since been accumulated, and, from a mathematical point of view, their study is also motivated by the fact that they exhibit rather unusual properties such as purely singular continuous spectral measures which are supported on Cantor sets of Lebesgue measure zero. In particular the apparent tendency of the spectral measures to be purely singular continuous seems to reflect that substitution sequences give rise to potentials being intermediate between periodic (leading to absolutely continuous spectrum) and random (leading to pure point spectrum). In this respect, it is important to note that, while ergodicity plays an important role in most of the previous approaches (e.g., [1, 2, 3, 4, 5, 13, 17, 26]), many of the previous results on the absence of point spectrum of substitution operators do not hold with probability one with respect to the ergodic measure but rather on Baire typical sets which could well be of measure zero. Our goal here is to present a class of substitution Hamiltonians exhibiting purely singular continuous spectrum with probability one.

We are now going to describe the models we will be dealing with. Let $A = \{a_1, \ldots, a_s\}$ be a finite set, called the *alphabet*. The a_i are called *symbols* or *letters*, the elements of $A^* \equiv \bigcup_{k \geq 1} A^k$ are called *words*. We denote by $|v|$ the *length* of a word $v \in A^*$ (i.e., $|v| = l$ iff $v \in A^l$). If v_1, v_2 are words, we denote by $\#_{v_1} v_2$ the number of occurences of v_1 in v_2. A *substitution* S is a map $S : A \to A^*$. S can be extended homomorphically to A^* (resp., $A^{\mathbf{N}}$) by $S(b_1 \ldots b_n) \equiv S(b_1) \ldots S(b_n)$ (resp., $S(b_1 b_2 b_3 \ldots) \equiv S(b_1) S(b_2) S(b_3) \ldots$). A fixed point $u \in A^{\mathbf{N}}$ of S is called *substitution sequence*. The existence of such a fixed point follows from the following conditions.

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1. There exists a letter $a \in A$ such that the first letter of $S(a)$ is a.
2. $\lim_{n\to\infty} |S^n(a)| = \infty$.

In this case, $u \equiv \lim_{n\to\infty} S^n(a)$ exists and is a substitution sequence. If the substitution S is *primitive*, which means that there exists $k \in \mathbf{N}$ such that, for every $a \in A$, $S^k(a)$ contains every symbol from A, then these conditions are satisfied in the sense that they hold for a suitably chosen power of S (which is primitive, too). Put discrete topology on A. Extend u to the left arbitrarily giving $\hat{u} \in A^{\mathbf{Z}}$ and define Ω to be the set of accumulation points of the translates of $\hat{u}$ with respect to pointwise convergence,

$$\Omega \equiv \{\omega \in A^{\mathbf{Z}} \; : \; \omega = \lim T^{n_i}\hat{u}, \; n_i \to \infty\},$$

where the shift T on $A^{\mathbf{Z}}$ is defined by $(T\tau)_n = \tau_{n+1}$, $\tau \in A^{\mathbf{Z}}$. By definition, Ω is a closed (and therefore compact) subset of $A^{\mathbf{Z}}$ that is invariant under T. The dynamical system (Ω, T) is called the *substitution dynamical system associated to* S. If S is primitive, this system has the following well-known properties [19]. It is uniquely ergodic, that is, there is exactly one ergodic invariant probability measure μ on Ω. Moreover, it is minimal, that is, the orbit of every element $\omega \in \Omega$ is dense in Ω. Finally, every word occuring in *some* $\omega \in \Omega$ in fact occurs in *every* $\omega \in \Omega$. In order to define the associated operators, choose an arbitrary function $f : A \to \mathbf{R}$. The potentials V_ω defined by $V_\omega(n) = f(\omega_n)$, $\omega \in \Omega$, give rise to discrete one-dimensional Schrödinger operators

$$(H_\omega \phi)(n) = \phi(n+1) + \phi(n-1) + V_\omega(n)\phi(n)$$

in $l^2(\mathbf{Z})$. It is this ergodic family of operators we shall study below. To make things non-trivial we shall always assume that S, f are such that the potentials V_ω are aperiodic.

Some prominent examples of primitive substitutions (in each case $A = \{a, b\}$) are

$a \mapsto ab,\ b \mapsto a$	Fibonacci substitution,
$a \mapsto ab,\ b \mapsto aa$	Period doubling substitution,
$a \mapsto ab,\ b \mapsto aaa$	Binary non-Pisot substitution,
$a \mapsto ab,\ b \mapsto ba$	Thue-Morse substitution.

Let us now be explicit about previous results concerning singular continuity of the spectral measures. First of all, in case of aperiodic V_ω it follows from Kotani [17] and Last-Simon [18] that the absolutely continuous spectrum of H_ω is always empty. One is therefore left with the task to exclude eigenvalues. Results of generic type, that is, empty point spectrum for a dense G_δ set of ω's, are, by minimality and Simon [21], in general obtained by excluding eigenvalues for only *one* element ω_0. This was done for the Fibonacci case by Sütö [25], for the period doubling case by Bellissard *et al.* [1], and for the Thue-Morse case by Delyon-Peyrière [11]. Moreover, two classes of primitive substitutions leading to generic singular continuous spectrum were exhibited by Bovier-Ghez [4, 5] and Hof *et al.* [14]. The first singular continuity result for substitution operators holding with probability one was obtained by Kaminaga [16] for the Fibonacci case. He used the quasiperiodicity of the potential and extended an approach due to Delyon-Petritis [10] in order to include the Fibonacci case. The first result of almost sure type for a non-quasiperiodic substitution operator was obtained in [7] for the period doubling Hamiltonian. In our present study we extend the idea of [7] and provide a class of primitive substitutions leading to purely singular continuous spectrum with probability one.

Theorem 1 *Let S be a primitive substitution and let $u \in A^{\mathbf{N}}$ be a substitution sequence associated to S. Suppose there exists a word $v \in A^*$ such that $vvvv$ occurs in u. Then, for almost every $\omega \in \Omega$, H_ω has no eigenvalues.*

Recall that we assumed the V_ω's to be aperiodic. Thus, combining this result with Kotani [17], we get

Corollary 1 *If S obeys the assumptions of Theorem 1, then, for almost every ω, H_ω has purely singular continuous spectrum.*

Remarks.

1. The binary non-Pisot substitution satisfies the assumptions of Theorem 1 since $S^3(a) = abaaaababab$. More generally, Theorem 1 applies to the substitutions $a \mapsto ab$, $b \mapsto a^n$, $n \geq 3$. Note, however, that almost sure absence of eigenvalues also holds for $n = 1, 2$ as these cases correspond to the Fibonacci and period doubling substitution, respectively. We thus wind up with the conclusion that the entire family $a \mapsto ab$, $b \mapsto a^n$, $n \geq 1$ exhibits purely singular continuous spectrum with probability one.

2. Further examples are given by $a \mapsto a^n b$, $b \mapsto a$, $n \geq 3$. These examples correspond to a subclass of the quasiperiodic potentials studied by Delyon-Petritis [10] and Kaminaga [16]. We therefore provide a different proof of almost sure absence of eigenvalues for these models which relies on their substitution generated nature. However, the spectrum is again purely singular continuous with probability one for *all* $n \geq 1$ as shown by Kaminaga [16].

3. The proof of Theorem 1 is based on a three-block argument. The word $vvvv$ can be thought of as a local symmetry of order one which, by self-similarity, generates symmetries of higher orders. The first three blocks of these symmetries are necessary for appropriate cubes to occur, whereas the fourth block is somehow necessary for sufficiently many different cubes to occur. It will be seen from the proof that, in this regard, the assumption of Theorem 1 is naturally tied to our use of the Gordon three-block method.

4. However, the period doubling case *does not* obey the assumptions of Theorem 1, even though the proof of almost sure absence of eigenvalues in [7] is also based on the three-block method. We shall deal with this curiosity in the final section of this paper where it will be shown that, in some sense, the period doubling substitution is of “almost-four-block type”.

5. In a similar context, one of the key ideas in our proof of Theorem 1, namely, that an additional block of repetition enables one to consider cyclic permutations, was also employed in [6].

6. Singular continuous Schrödinger operator spectra have become an objective of intense study in the past few years. In a series of papers [21, 9, 15, 8, 24, 22, 23], Simon and co-workers have revealed the generic aspect of this spectral type. Moreover, they have found many concrete operators with purely singular continuous spectrum. In this regard, Corollary 1 adds to this list.

The paper is organized as follows. Section 2 recalls basic properties of substitution sequences arising from primitive substitutions. The proof of Theorem 1 is given in Section 3 while the last section deals with the inclusion of the period doubling case into the framework of the general approach.

2 Basic properties of substitution sequences arising from primitive substitutions

In this section we collect some results from the general theory of primitive substitutions which are necessary for the proof of Theorem 1 in the next section. The material is taken from the book by Queffélec [19].

Let S be a primitive substitution. The *substitution matrix* $M(S) = (m_{ij})_{1 \leq i,j \leq s}$ is defined by $m_{ij} \equiv \#_{a_i} S(a_j)$. The action of the substitution is compatible with matrix product, in the sense that $M(S^k) = M^k(S)$. This shows that primitivity of S is equivalent to primitivity of $M(S)$ (i.e., there exists $k \in \mathbf{N}$ such that all entries of $M^k(S)$ are strictly positive). The Perron-Frobenius theorem yields the existence of a dominant eigenvalue θ which can be regarded as an asymptotic blow-up factor:

Proposition 2.1 *Let S be a primitive substitution and let $M = M(S)$ be the associated substitution matrix. Then, M has an eigenvalue θ with the following properties:*

1. *θ is strictly positive.*

2. *For any other eigenvalue ζ of M, we have $\theta > |\zeta|$.*

3. *θ has multiplicity one.*

4. *The corresponding eigenvector can be chosen strictly positive.*

5. *For every $b \in A$, $\lim_{n\to\infty} \frac{|S^{n+1}(b)|}{|S^n(b)|} = \theta$.*

6. *For every $v \in A^*$, there exists $c(v) > 0$ such that $\lim_{n\to\infty} \frac{|S^n(v)|}{\theta^n} = c(v)$.*

Another key property of substitution sequences arising from primitive substitution is the existence of frequencies of words. Moreover, they are always strictly positive:

Proposition 2.2 *Let S be a primitive substitution. Then, for any $v \in A^*$ which occurs in a fixed point of S and any $b \in A$, the limits*

$$d(v) \equiv \lim_{n\to\infty} \frac{\#_v S^n(b)}{|S^n(b)|} > 0$$

exist and are independent of b.

As the final central result, we want to mention the connection between the frequencies of words and the unique ergodic measure μ on Ω. The Borel σ-algebra is generated by the cylinder sets

$$[b_0 \dots b_{l-1}]_{[m,m+l-1]} \equiv \{\omega \in \Omega \;:\; w_{m+i} = b_i,\; 0 \le i \le l-1\}.$$

The μ-measure of such a set is simply given by the frequency of the word $b_0 \dots b_{l-1}$:

Proposition 2.3 *For every word $b_0 \dots b_{l-1}$ occuring in some fixed point of a primitive substitution S, we have*

$$\mu([b_0 \dots b_{l-1}]_{[m,m+l-1]}) = d(b_0 \dots b_{l-1}).$$

3 Discovering plenty of three-block structures

Throughout this section we assume that S is a primitive substitution satisfying the assumptions of Theorem 1. Let $u, \Omega, d, \mu, (H_\omega)_{\omega\in\Omega}$ be induced by S as described above. Define

$$C \equiv \{\omega \;:\; \sigma_{pp}(H_\omega) = \emptyset\}.$$

Our goal is to prove $\mu(C) = 1$. The following proposition provides a general method how to accomplish this.

Proposition 3.1 *Suppose there exist Borel sets $G(n)$, $n \in \mathbf{N}$, satisfying*

1. $\limsup G(n) \subseteq C$,

2. $\limsup \mu(G(n)) > 0$.

Then, $\mu(C) = 1$.

Proof. By assumption we have

$$\mu(C) \ge \mu(\limsup G(n)) \ge \limsup \mu(G(n)) > 0.$$

Now the assertion follows from shift-invariance of C and ergodicity of μ. $\square$

We define

$$G(n) \equiv \{\omega \;:\; V_\omega(-n+k) = V_\omega(k) = V_\omega(n+k),\; 1 \le k \le n\}.$$

It is easily seen that the $G(n)$ are Borel sets since they are finite unions of cylinder sets. The first crucial property of these sets is already part of the folklore.

Lemma 3.2 $\limsup G(n) \subseteq C$.

Proof. This is shown by a standard Gordon-type argument [12], see [10]. □

Define $n_k \equiv |S^k(v)|$, and, for $l \in \mathbf{N}$,

$$P_l \,:\, A^l \to A^l,\ b_1 b_2 \ldots b_l \mapsto b_2 \ldots b_l b_1.$$

We will obtain estimates for the μ-measure of $G(n)$ on the subsequence $(n_k)_{k\in\mathbf{N}}$. This is natural since, by self-similarity, the occurence of $vvvv$ in u yields existence of four-block structures of length $4n_k$ in u. Thus, we have plenty of three-block structures of length $3n_k$ if we consider cyclic permutations of $S^k(v)$. Define

$$r_k \equiv \min\{i \geq 1 \,:\, S^k(v) = P^i_{n_k}(S^k(v))\}.$$

Obviously, $1 \leq r_k \leq n_k$. Let us proceed with the following observation.

Lemma 3.3 *There exists $d_k \in \mathbf{N}$ such that $n_k = d_k \cdot r_k$.*

Proof. By definition we have

1. For every $i \geq r_k$, $P^i_{n_k}(S^k(v)) = P^{i-r_k}_{n_k}(S^k(v))$.
2. $S^k(v), P^1_{n_k}(S^k(v)), \ldots, P^{r_k-1}_{n_k}(S^k(v))$ are mutually different words.

Consider the expansion $n_k = d_k r_k + q_k$, where $0 \leq q_k < r_k$. By 1.,

$$S^k(v) = P^{n_k}_{n_k}(S^k(v)) = P^{q_k}_{n_k}(S^k(v)).$$

Thus, by 2., $q_k = 0$. □

Lemma 3.4 *For $0 \leq i < r_k$, we have*

$$d(P^i_{n_k}(S^k(v))P^i_{n_k}(S^k(v))P^i_{n_k}(S^k(v))) \geq \frac{d_k}{\theta^k} \cdot d(vvvv).$$

Proof. For every occurence of $vvvv$ in u there are d_k occurences of $P^i_{n_k}(S^k(v))P^i_{n_k}(S^k(v))P^i_{n_k}(S^k(v))$ in the corresponding occurence of $S^k(v)S^k(v)S^k(v)S^k(v)$ in $S^k(u) = u$. We therefore have

$$\begin{aligned} d(P^i_{n_k}(S^k(v))P^i_{n_k}(S^k(v))P^i_{n_k}(S^k(v))) &= \lim_{n\to\infty} \frac{\#_{P^i_{n_k}(S^k(v))P^i_{n_k}(S^k(v))P^i_{n_k}(S^k(v))}(S^n(a))}{|S^n(a)|} \\ &\geq \lim_{n\to\infty} \frac{d_k \cdot \#_{vvvv}(S^{n-k}(a))}{|S^n(a)|} \\ &= d_k \cdot \lim_{n\to\infty} \frac{\#_{vvvv}(S^{n-k}(a))}{|S^{n-k}(a)|} \cdot \frac{|S^{n-k}(a)|}{|S^n(a)|} \\ &= d_k \cdot d(vvvv) \cdot \frac{1}{\theta^k}. \end{aligned}$$

□

Lemma 3.5 $\limsup \mu(G(n_k)) \geq c(v) \cdot d(vvvv) > 0$.

Proof. $G(n_k)$ contains the cylinder sets $[P^i_{n_k}(S^k(v))P^i_{n_k}(S^k(v))P^i_{n_k}(S^k(v))]_{[-n_k+1,2n_k]}$, $0 \le i < r_k$. Moreover, these cylinder sets are mutually disjoint. Hence, together with Proposition 2.3 and Lemma 3.4, we obtain

$$\begin{aligned}
\mu(G(n_k)) &\ge \sum_{i=0}^{r_k-1} \mu([P^i_{n_k}(S^k(v))P^i_{n_k}(S^k(v))P^i_{n_k}(S^k(v))]_{[-n_k+1,2n_k]}) \\
&= \sum_{i=0}^{r_k-1} d(P^i_{n_k}(S^k(v))P^i_{n_k}(S^k(v))P^i_{n_k}(S^k(v))) \\
&\ge \sum_{i=0}^{r_k-1} \frac{d_k}{\theta^k} d(vvvv) \\
&= \frac{n_k}{\theta^k} d(vvvv).
\end{aligned}$$

Thus, $\limsup \mu(G(n_k)) \ge c(v) \cdot d(vvvv)$, as desired. □

Proof of Theorem 1. By Lemma 3.2 and Lemma 3.5, the hypotheses of Proposition 3.1 are satisfied. □

4 Three-block structures in the period doubling case

In the previous section we have seen that the following mechanism allows a proof of almost sure absence of eigenvalues in self-similar potentials.

1. If a word vvv of three-block type occurs in the substitution sequence u, then, by self-similarity, all the words $S^k(v)S^k(v)S^k(v)$ occur in u. We therefore have three-block structures on an infinite number of levels. Thus, an application of a Gordon-type criterion in order to exclude eigenvalues is in principle possible.

2. In order for this criterion to be applicable on a set of positive measure, one has to deal with the obstacle that the frequencies of the higher level words $S^k(v)S^k(v)S^k(v)$ in general decay. To be precise, the decay will be of order θ^{-k}, where θ is the dominant eigenvalue of the substitution matrix.

3. Aiming at a compensation of this effect, one looks for three-block words other than $S^k(v)S^k(v)S^k(v)$ but having the same length.

4. Since the growth of $|S^k(v)|$ is of order θ^k, the number of required words (or repetitions) has to be of order $|S^k(v)|$. A natural way for this to hold is to require the occurence of a four-block word $vvvv$ in u and to consider cyclic permutations of $S^k(v)S^k(v)S^k(v)$ as was done in Section 3.

We are now going to consider the period doubling substitution S_{pd}. Recall that it is given by $a \mapsto ab$, $b \mapsto aa$. Iterating three times, we obtain the following word

$$S^3_{pd}(a) = abaaabab.$$

This word does not contain a four-block word. However, we are going to see that the word $aaab$, which is contained in $S^3_{pd}(a)$ and hence in $u = S^\infty_{pd}(a)$, can serve as a generator of sufficiently many higher order three-block symmetries. Namely, we shall show below that all the cyclic permutations of $S^k_{pd}(a)S^k_{pd}(a)S^k_{pd}(a)$ occur in $S^k_{pd}(a)S^k_{pd}(a)S^k_{pd}(a)S^k_{pd}(b)$. To this end, define the truncation operator $\mathcal{T}$ by

$$\mathcal{T} : A^l \to A^{l-1},\ b_1 \dots b_{l-1}b_l \mapsto b_1 \dots b_{l-1}.$$

Proposition 4.1 *For every $k \in \mathbf{N}$, we have $\mathcal{T}(S^k_{pd}(a)) = \mathcal{T}(S^k_{pd}(b))$.*

Proof by induction. The case $k = 1$ is obvious. For the induction step we note that

$$\mathcal{T}(S_{pd}^{k+1}(a)) = \mathcal{T}(S_{pd}^k(a)S_{pd}^k(b)) = S_{pd}^k(a)\mathcal{T}(S_{pd}^k(b)) = S_{pd}^k(a)\mathcal{T}(S_{pd}^k(a)) = \mathcal{T}(S_{pd}^k(a)S_{pd}^k(a)) = \mathcal{T}(S_{pd}^{k+1}(b)),$$

concluding the proof. $\square$

Hence, for every occurence of $S_{pd}^k(a)S_{pd}^k(a)S_{pd}^k(a)S_{pd}^k(b)$ in u there is a corresponding occurence of $P_{2^k}^i(S_{pd}^k(a))P_{2^k}^i(S_{pd}^k(a))P_{2^k}^i(S_{pd}^k(a))$, $0 \le i < 2^k$. The mechanism sketched above can now be used to prove almost sure absence of eigenvalues in the period doubling model, thus recovering the main result from [7].

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