Navier-Stokes limit for a thermal stochastic lattice gas.

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Abstract: We study a stochastic particle system on the lattice whose particles move freely according to simple exclusion process and change velocities during collisions preserving energy and momentum. In the hydrodynamic limit, under diffusive space-time scaling, the local velocity field u satisfies the incompressible Navier-Stokes equation, while the temperature field θ solves the heat equation with drift u. The results are also extended to include a suitably rescaled external force.

Résumé : Nous étudions un système de particules qui évoluent sur $\mathbb{Z}^d$ selon le processus d'exclusion simple et qui changent leurs vitesses lors de collisions préservant l'énergie et le moment. Dans la limite hydrodynamique, obtenue par un changement d'échelle d'espace-temps diffusif, le champ local des vitesses u vérifie l'équation de Navier-Stokes incompressible, tandis que la température θ résout l'équation de la chaleur avec u pour dérive. Ces résultats sont étendus pour inclure une force extérieure convenablement renormalisée.

1. Introduction and results

In last few years a big development in the study of the hydrodynamical limit of stochastic particle systems has been obtained thanks to the introduction of new powerful probabilistic tools ([GPV], [Y], [V]). In particular, the analysis of the simple exclusion process and related models ([X], [EMY1], [LOY1], [LOY2], [LY], [EMY3], [QY], [VY]) on the diffusive space-time scale was made possible by the introduction of the non-gradient method due to Varadhan [V], which also permits to obtain very clean variational formulas for the diffusion coefficients, providing mathematical support to the classical heuristic Green-Kubo formulas.

In this paper we consider the following model: particles of several colors, labeled by vectors $v \in \mathcal{V} \subset \mathbb{R}^3$ are considered. The cardinality of $\mathcal{V}$ is finite and set to N. The particles of the color v move on the lattice $\mathbb{Z}^3$ according to the simple exclusion process,

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i.e. they can only jump, at independent exponential times, to nearest neighbors (provided that the target site in not already occupied by a particle of the same color) with rates such that the drift is v. The vectors $v \in \mathcal{V}$ are called *velocities* of the particles. The set of velocities $\mathcal{V}$ is assumed invariant under rotations and permutations of the coordinate axes.

Particles change velocity during collisions: in sites where couples of particles are present, they undergo collisions at exponential independent times, subject to the only restriction that the total outgoing momentum and energy equal the incoming ones.

Our model differs from the one introduced in [EMY3] because in the latter model velocities have all the same modulus so that the energy and the mass coincide, while our model introduce two different species of particles corresponding to different kinetic energies, so that thermal effects in the hydrodynamic equations can be obtained.

In [EMY3] and [QY] space and time are rescaled diffusively with a scale parameter ε and the local velocity is assumed of order ε (low Mach numbers limit). If the space dimension d is greater or equal to three, under some assumptions on the initial data, a law of large numbers is proved for the mass and momentum density and the limiting fields satisfy an incompressible Navier-Stokes type equation. For $d = 3$ this becomes exactly the Navier-Stokes equation for a suitable choice of the set of velocities $\mathcal{V}$. Some anisotropy in the viscosity matrix is present, due to the lattice structure.

The model we consider in this paper will show a similar behavior on the diffusive scale, but, since the energy is non trivial, a heat equation is also obtained in the scaling limit. The main problem for achieving this result is to find a model with the local ergodic property which also produces hydrodynamical equations of the usual form.

The simplest three dimensional example we were able to construct is the following model where we have exactly two values of $|v|$: $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$ with $\mathcal{V}_1 = \{(\pm 1, \pm 1, \pm 1)\}$ and $\mathcal{V}_2 = \{(\pm 1, \pm 1 \pm \varpi)\}$, up to permutations, with $\varpi \neq 1$. If ϖ^2 is irrational it is possible to check (Proposition 2.1) that the only invariants for this model are mass, momentum and energy.

The structure of the invariants allows to introduce the Gibbs states for the system. Due to the absence of interactions between the particles, but for the exclusion, they are product states parameterized by the chemical potentials λ_α, $\alpha = 0, \dots, 3$, conjugate to the conserved quantities. Let $I_0(x)$ denote the total mass in the site x, $I_j(x)$, for $j = 1, \dots, 3$ the components of the total momentum in x and $I_4(x)$ the total energy in x. Then a Gibbs state in the finite box Λ is a measure whose density, with respect to the uniform distribution of the configurations, is given by

$$Z^{-1} \prod_{x \in \Lambda} \exp \sum_{\alpha=0}^{3} \lambda_\alpha I_\alpha(x)$$

with Z the normalization factor.

Because of the presence of a non trivial energy it is possible to introduce for the system a notion of temperature. As usual when dealing with models with a discrete set of velocities (see [Ce], [ED]), this can be done in several ways. We will follow the rule of defining the temperature with the help of the Gibbs states. The chemical potential associated to the energy, λ_4, can be interpreted as the inverse of the temperature of the system and we take this as definition of temperature. Since the velocities are discrete, this temperature does not coincide with the variance of the velocity distribution. However, we remark that, since we will only consider deviations of order ε of the temperature from a constant profile, in order to fulfill the low Mach number assumption, the various possible definitions of temperature coincide up to higher orders in ε and we can disregard the ambiguity of the definition.

As usual (see [S], [EM], [EMY3]) a formal argument, based on the local equilibrium assumption, can be given to guess the structure of the limiting equations. The local equilibrium assumption means that the non equilibrium distribution of the system is assumed to be close to a Gibbs state with chemical potentials $\lambda^\varepsilon_\alpha$ slowly varying with x. We also assume low Mach numbers, which corresponds to take

$$\lambda^\varepsilon_\alpha(x) = \lambda^{(0)}_\alpha + \varepsilon\lambda^{(1)}_\alpha(\varepsilon x) + \varepsilon^2\lambda^{(2)}_\alpha(\varepsilon x), \quad \alpha = 0, \dots, d$$

with $\lambda^{(1)}_\alpha, \lambda^{(2)}_\alpha$ smooth functions of the macroscopic variables and $\lambda^{(0)}_j = 0$ for $j = 1, \dots, d$. We will call *equilibrium measure* the Gibbs measure with parameters $\lambda^{(0)}_\alpha$.

We assume $d = 3$ and the initial distribution of the system to be of local equilibrium with chemical potentials satisfying the low Mach numbers assumption. We prove that during the evolution, in the diffusive scaling the non equilibrium distribution is close, in the sense of the relative entropy ([Y], [OVY]) to a local equilibrium, up to a time $\bar t$ such that there is a smooth solution to the limiting equations

$$\begin{aligned} &\mathrm{div} u = 0 \\ &\partial_t u + \nabla p + K\, u \cdot \nabla u = \sum_{\alpha=1}^{3} D_\alpha \partial_\alpha^2 u \\ &\frac{\partial}{\partial t} T + H u \cdot \nabla T = \sum_{\alpha=1}^{3} \mathcal{K}_\alpha \partial_\alpha^2 T. \end{aligned} \tag{1.1}$$

Here u_j, $j = 1, \dots, 3$ are the components of the velocity field, proportional to the chemical potentials $\lambda^{(1)}_j$, T is the temperature, related to the chemical potential $\lambda^{(1)}_4$, p is the second order correction to the pressure, D the viscosity, $\mathcal{K}$ the heat conductivity. K and H are two constants depending only on the parameters of the equilibrium measure and on the set $\mathcal{V}$. Finally the first correction to the density is related to the first correction to the temperature T by the Boussinesq condition which ensure the constancy of the pressure up to the second order.

About (1.1) we have several remarks: first we note that in general, as in [EMY3], non-isotropic Euler terms are present in the limiting equations, but we can get rid of them by an appropriate choices of ϖ and of the parameters of the equilibrium measure. Moreover, the constants K and H have not a definite sign. Suitable choices of the temperature and density of the equilibrium measure provide such a positivity (Proposition 2.2). We also note that in the usual hydrodynamic equations $K = 1$ and $H = 1$. This is not the case in the present model. We can get rid of one of them by rescaling time, but not of both. This is usual when dealing with discrete velocity models ([Fr]). Finally, as in [EMY3], the diffusion matrix is not completely isotropic. For similar phenomena in cellular automata we refer to [Fr]. Moreover, the contribution of the asymmetric motion to the heat conductivity $\mathcal{K}$ is given by a complicated combination of diffusion coefficients which we cannot check to be positive in general, but for zero inverse temperature β. Therefore, in order to ensure positivity of $\mathcal{K}$ and hence well posedness of the initial value problem associated to (1.1), we choose in the case $\beta \neq 0$ the symmetric part of the exclusion process be suitably large.

Once the local ergodicity (Proposition 2.1) and positivity properties (Proposition 2.2 and related remarks) are proved, above results are obtained via the entropy method and the non-gradient method already used in [EMY3], suitably extended to take into account the extra conservation law and the extra corrections of order ε in the initial datum, $\lambda_0^{(1)}$ and $\lambda_4^{(1)}$. We observe that the non gradient method used in [EMY3] works without modifications also in this case since it is based on the properties of the symmetric simple exclusion and on the fact that the collision operator conserves only the quantities I_α. The heuristics and the proof of Propositions 2.1 and 2.2 are given in Section 2, after a precise definition of the model. Section 3 is devoted to the entropy method and Section 4 to the diffusion matrix. In particular, we give the expression for the diffusion matrix and prove that the Green-Kubo formulas for this model are not only formal, but rigorously equivalent to that expression.

To conclude, we mention that hydrodynamical equations, especially in the presence of thermal phenomena, are particularly interesting when an external force F is present. The Benard problem of the motion of a fluid in a slab heated from below in a gravitational field is a typical example where convective instabilities arise. For this reason in Section 5 we propose a method to get hydrodynamical equations containing a force. This requires some care because we are dealing with discrete velocity models, where the notion of acceleration is not well defined. We simulate the presence of a force by adding to our model a birth and death process, organized in such a way that the mass and energy conservation still holds, but during the process a change of momentum proportional to the force is provided to the system. This new process introduces extra difficulties because the equilibrium measure we consider in the absence of force is no more invariant under the full process, including the force. An immediate consequence is that the relative entropy w.r.t. the equilibrium measure is no more decreasing in time. Fortunately, in the low Mach number limit we have to rescale the force as $\varepsilon^3 F$ so that the generator of this new process is slowed down as ε^3 and this allows to treat its effect by perturbative arguments. In this way we prove the convergence to equations of the form (1.1) with the second of them replaced by

$$\partial_t u + \nabla p + K\, u \cdot \nabla u = \sum_{\alpha=1}^{3} D_\alpha \partial_\alpha^2 u + F \tag{1.2}$$

The force that enters in the Benard problem is conservative and to get the right hydrodynamical equations for this case one has to rescale the force as $\varepsilon^2 F$. This scaling is borderline because the symmetric part of the birth-death process appears in the limiting equation in the form of an additional viscous force. At the moment we do not know how to avoid this term, which would destroy the convective instabilities arising in the classical Benard problem. Moreover, this kind of phenomena are driven by the presence of a boundary. The problem of including boundary conditions in the set up of entropy and non-gradient methods is still open and we plan to discuss it in a future paper.

2. Model and heuristics

2.1 *Description of the model and notations.*

Particles evolve on the sublattice $\Lambda_L = \{-L, \dots, L\}^d$ with periodic boundary conditions. We call $\varepsilon = L^{-1}$. Each particle has a velocity $v \in \mathcal{V}$, where $\mathcal{V}$ is a finite subset of $\mathbb{R}^d$ which is invariant under any coordinate permutation (**IP**) and invariant under reflexions (**IR**) with respect to the orthogonal planes of the coordinate vectors e_i, $i = 1, \cdots, d$ with components

$$(e_i)_j = \delta_{i,j} \tag{2.1}$$

We define $\mathcal{D} = \{\pm e_i,\ i = 1, \dots, d\}$ as the set of the $2d$ different directions for moving on the lattice.

The kinetic energy of a particle with velocity v is $|v|^2/2$, where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^d$. In our case the particles can have different energies.

On each site of the lattice there is at most one particle with a given velocity. The integer $\eta(x, v) \in \{0, 1\}$ denotes the number of particles on $x \in \Lambda_L$ with the velocity $v \in \mathcal{V}$, $\eta_x = \{\eta(x, v),\ v \in \mathcal{V}\}$ and Ω is the set of all the configurations $\eta = \{\eta_x,\ x \in \Lambda_L\}$.

Infinitesimal generators

The dynamics of the particles is driven by jumps and collisions: we consider on Ω the generator $\mathcal{L}$ defined as

$$\mathcal{L} = \mathcal{L}^{\rm ex} + \mathcal{L}^{\rm c} \tag{2.2}$$

Since there is only hard core interaction between the particles with the same velocities, the operator $\mathcal{L}^{\rm ex}$ is the generator of the exclusion process for particles with different colors (here velocities): for any function f on Ω and any configuration $\eta \in \Omega$

$$\mathcal{L}^{\rm ex} f(\eta) = \sum_{v \in \mathcal{V}} \sum_{e \in \mathcal{D}} \sum_{x \in \mathbb{Z}^d} p(x, x+e, v, \eta) \left[f(\eta^{x,x+e,v}) - f(\eta) \right] \tag{2.3}$$

The configuration $\eta^{x,x+e,v}$ is obtained from η by exchanging the number of particles with velocity v on x and $x+e$

$$\eta^{x,y,v}(z,w) = \begin{cases} \eta(y,v) & \text{if } z = x \text{ and } w = v \\ \eta(x,v) & \text{if } z = y \text{ and } w = v \\ \eta(z,w) & \text{otherwise} \end{cases} \tag{2.4}$$

The jump rate $p(x, x+e, v, \eta)$ is defined by

$$p(x, x+e, v, \eta) = \eta(x,v)p(e,v) \tag{2.5}$$

and the intensity $p(e,v)$ is such that the mean displacement of a particle with velocity v is

$$\sum_{e \in \mathcal{D}} p(e,v)e = v \tag{2.6}$$

It is given by

$$p(e,v) = \chi + \frac{1}{2}\, e \cdot v \tag{2.7}$$

where the constant χ is chosen large enough to prevent the rate from being negative (e.g. $2\chi > \max\{|v|,\, v \in \mathcal{V}\}$).

Two body collisions between particles are allowed provided that the momentum and the kinetic energy are conserved and provided the exclusion rule is preserved. A collision between two particles with incoming velocities v, w and outgoing velocities v', w' is denoted by the quadruple $q = (v, w, v', w')$. The set of allowed collisions is therefore

$$\mathcal{Q} = \Big\{ (v, w, v', w') \in \mathcal{V}^4 \,:\, v + w = v' + w',\ |v|^2 + |w|^2 = |v'|^2 + |w'|^2 \Big\} \tag{2.8}$$

The operator $\mathcal{L}^c$ is the generator of the collisions: for any function f on Ω and any configuration $\eta \in \Omega$

$$\mathcal{L}^c f(\eta) = \sum_{x \in \mathbb{Z}^d} \sum_{q \in \mathcal{Q}} \tilde{p}(x,q) \left[f\left(\eta^{x,q}\right) - f(\eta) \right] \tag{2.9}$$

If $q = (v, w, v', w')$, the collision rate $\tilde{p}(x,q)$ is defined as

$$\tilde{p}(x,q) = \eta(x,v)\eta(x,w)\big(1 - \eta(x,v')\big)\big(1 - \eta(x,w')\big) \tag{2.10}$$

and $\eta^{x,q}$ is the configuration resulting from η by the collision q at site x

$$\eta^{x,q}(y,u) = \begin{cases} 0 & \text{if } \tilde{p}(x,q) = 1 \text{ and } y = x \text{ and } (u = v \text{ or } u = w) \\ 1 & \text{if } \tilde{p}(x,q) = 1 \text{ and } y = x \text{ and } (u = v' \text{ or } u = w') \\ \eta(y,u) & \text{otherwise} \end{cases} \tag{2.11}$$

Remark that denoting $Q_x^q f(\eta) = f(\eta^{x,q}) - f(\eta)$, (2.9) can be rewritten as

$$\mathcal{L}^{c} f(\eta) = \sum_{x\in\mathbb{Z}^d} \sum_{q\in\mathcal{Q}} Q_x^q f(\eta) \tag{2.12}$$

For $\ell < L$, we call Ω_ℓ the set of the particle configurations in the box $\Lambda_\ell = \{-\ell, \dots, \ell\}^d \subset \Lambda_L$. We denote by $\mathcal{L}_\ell$ the restriction of $\mathcal{L}$ to Λ_ℓ. More precisely $\mathcal{L}_\ell = \mathcal{L}_\ell^{\text{ex}} + \mathcal{L}_\ell^{\text{c}}$ where $\mathcal{L}_\ell^{\text{c}}$ is obtained from $\mathcal{L}^{\text{c}}$ considering only collisions at sites $x \in \Lambda_\ell$ and $\mathcal{L}_\ell^{\text{ex}}$ is obtained from $\mathcal{L}^{\text{ex}}$ considering only the jumps over bonds $(x, x+e) \subset \Lambda_\ell$.

Conserved quantities

The total mass, the total momentum and the total energy at the site x are

$$I_0(\eta_x) = \sum_{v\in\mathcal{V}} \eta(x,v) \tag{2.13}$$

$$I_\alpha(\eta_x) = \sum_{v\in\mathcal{V}} (v\cdot e_\alpha)\eta(x,v)\,, \quad \alpha = 1,\dots,d \tag{2.14}$$

$$I_4(\eta_x) = \sum_{v\in\mathcal{V}} \frac{1}{2}|v|^2 \eta(x,v) \tag{2.15}$$

These quantities are conserved by the collision operator since $\mathcal{L}_\ell^{\text{c}}\Big[g\big(I_\beta(\eta_x)\big)\Big] = 0$ for any function g on $\mathbb{R}$, for any $x \in \Lambda_\ell$ and for $\beta = 0, \dots, d+1$.

Moreover, the jump operator $\mathcal{L}_\ell^{\text{ex}}$ conserves the total number of particles with a given velocity. So, defining $N_\beta(\eta) = \sum_{x\in\Lambda_\ell} I_\beta(\eta_x)$ $(\beta = 0, \dots, d+1)$, for any function g

$$\mathcal{L}_\ell\Big[g\big(N_\beta(\eta)\big)\Big] = 0 \tag{2.16}$$

We require the velocity set $\mathcal{V}$ to be such that $N_\beta(\eta)$, $\beta = 0, \dots, d+1$, are the only quantities conserved by the dynamics. In other words the Markov chain associated to $\mathcal{L}_\ell$ on the set $\Omega_{\ell,m}$ $(m \in \mathbb{R}^{d+2})$ of configurations $\eta \in \Omega_\ell$ with $N_\beta(\eta) = m_\beta(2\ell+1)^d$, $\beta = 0, \dots, d+1$ should be ergodic. This property is called *local ergodicity* (**LE**).

2.2 *An example of velocity set.*

In this subsection we provide an example of velocity set $\mathcal{V}$ satisfying the conditions (**IP**), (**IR**), (**LE**) and (2.17) (see below) in the case $d = 3$.

The set $\mathcal{V}$ is made of two species of velocities with different energies. The first species $\mathcal{V}_1$ contain the following 8 velocities

$$(\pm 1, \pm 1, \pm 1) \tag{s$_1$}$$

and the second species $\mathcal{V}_2$ contain 24 velocities given up to permutation by

$$(\pm\varpi, \pm 1, \pm 1) \tag{s$_2$}$$

where ϖ is a real different from ± 1. We require for the moment that ϖ has to be irrational.

For reasons which will be explained in the heuristic derivation of the Navier-Stokes equations (see subsection 2.5 below), an extra assumption is made on $\mathcal{V}$. Denoting by v_α, $\alpha = 1, \ldots, 3$ the components of v, we suppose that

$$\sum_{v \in \mathcal{V}} \overline{h}_2(v) \big[v_1^4 - 3 v_1^2 v_2^2 \big] = 0 \tag{2.17}$$

with $\overline{h}_2$ is defined in (2.63), depending on some parameters r, θ and ϖ. One can see that there are couple (r, θ) such that ϖ determined by (2.17) is irrational and larger than 1. For example, for $\theta = 0$, and for any r we get $\varpi^2 = 5\sqrt{3}$.

Local ergodicity

Proposition 2.1 *The finite Markov chain associated to $\mathcal{L}_\ell$ on $\Omega_{\ell,m}$, $m \in \mathbb{R}^5$ is ergodic.*

Proof. Let η and ζ in Ω be two configurations with the same total mass, momentum and energy:

$$N_\beta(\eta) = N_\beta(\zeta)\,, \;\; \beta = 0, \ldots, 4 \tag{2.18}$$

We have to prove that η and ζ communicate ($\eta \leftrightarrow \zeta$), i.e. using a sequence of jumps and collisions one can transform η into ζ.

We denote by $N_\alpha^{+1}(\eta)$, $\alpha = 1, \ldots, 3$, the number of particles in the configuration η with velocity v such that $v_\alpha = +1$. We define in the same way $N_\alpha^{-1}(\eta)$, $N_\alpha^{+\varpi}(\eta)$ and $N_\alpha^{-\varpi}(\eta)$. We put $\delta f = f(\eta) - f(\zeta)$. We also denote by $\eta^{(i)}$, $i = 1, 2$, the configuration η restricted to the particles with velocity of species $\mathcal{V}_i$, more precisely $\eta^{(i)}(x, v) = \eta(x, v)\, \mathbb{1}_{\{v \in \mathcal{V}_i\}}$ where $\mathbb{1}$ is the indicator function. We also put $\delta^{(i)} f = f(\eta^{(i)}) - f(\zeta^{(i)})$.

In [EMY3] (Theorem 3.3) it is proved that if $\delta^{(2)} N_\beta = 0$ for $\beta = 0, \ldots, 3$, then $\eta^{(2)} \leftrightarrow \zeta^{(2)}$.

Step 1. We first show that the result of [EMY3] also holds for the particles of the species $\mathcal{V}_1$: if $\delta^{(1)} N_\beta = 0$ for $\beta = 0, \ldots, 3$, then $\eta^{(1)} \leftrightarrow \zeta^{(1)}$. By assumption

$$\delta^{(1)} N_\alpha^{+1} + \delta^{(1)} N_\alpha^{-1} = \delta^{(1)} N_0 = 0 \quad \text{and} \quad \delta^{(1)} N_\alpha^{+1} - \delta^{(1)} N_\alpha^{-1} = \delta^{(1)} N_\alpha = 0 \tag{2.19}$$

so that $\delta^{(1)} N_\alpha^{+1} = \delta^{(1)} N_\alpha^{-1} = 0$, for $\alpha = 1, \ldots, 3$. Therefore, on a given component, the velocities of the particles in the configuration $\zeta^{(1)}$ are obtained from those of $\eta^{(1)}$ by doing permutations. So it suffices to show that if the velocities of the particles in $\zeta^{(1)}$ are obtained from those of $\eta^{(1)}$ exchanging the coordinates in one component of two given velocities, then $\eta^{(1)} \leftrightarrow \zeta^{(1)}$. Suppose that, for instance, the velocities of the particles

$\eta^{(1)}$ are the same as those of $\zeta^{(1)}$ except that $\eta^{(1)}$ contains the two particles p_1 and p_2 with velocities $(1, a_1, b_1)$ and $(-1, a_2, b_2)$ and $\zeta^{(1)}$ contains the two particles p_1' and p_2' with velocities $(-1, a_1, b_1)$ and $(1, a_2, b_2)$. With a sequence of jumps, one can move p_2 (or another particle in η with the same velocity) to the same site as p_1, then p_1 and p_2 can collide

$$(1, a_1, b_1) + (-1, a_2, b_2) \longrightarrow (-1, a_1, b_1) + (1, a_2, b_2) \tag{2.20}$$

If we denote by $\overline{\eta}^{(1)}$ the new configuration, then $\overline{\eta}^{(1)} \leftrightarrow \eta^{(1)}$. Moreover the particles in $\overline{\eta}^{(1)}$ have the same velocities as the particles in $\zeta^{(1)}$, so using only jumps $\overline{\eta}^{(1)} \leftrightarrow \zeta^{(1)}$. $\square$

The proof of the local ergodicity will consist in showing that if η and ζ satisfy (2.18), then there are two configurations $\overline{\eta}$ and $\overline{\zeta}$ such that $\eta \leftrightarrow \overline{\eta}$, $\zeta \leftrightarrow \overline{\zeta}$ and for $\beta = 0, \ldots, 3$, $N_\beta(\overline{\eta}^{(2)}) = N_\beta(\overline{\zeta}^{(2)})$. That is enough since from (2.18), [EMY3] and Step 1, it would imply that both $\eta^{(1)} \leftrightarrow \zeta^{(1)}$ and $\eta^{(2)} \leftrightarrow \zeta^{(2)}$ hold and therefore $\eta \leftrightarrow \zeta$.

Step 2. We now prove that if η and ζ satisfy (2.18), then $N_0(\eta^{(i)}) = N_0(\zeta^{(i)})$ for $i = 1, 2$. From the conservation of mass and energy,

$$\delta^{(1)} N_0 + \delta^{(2)} N_0 = 0 \quad \text{and} \quad (\varpi^2 + 2)\delta^{(1)} N_0 + 3\delta^{(2)} N_0 = 0 \tag{2.21}$$

and since $\varpi \neq \pm 1$ we get $\delta^{(1)} N_0 = \delta^{(2)} N_0 = 0$. $\square$

Let $\ell_\alpha(\eta) = N_\alpha^{+1}(\eta) - N_\alpha^{-1}(\eta)$ and $k_\alpha(\eta) = N_\alpha^{+\varpi}(\eta) - N_\alpha^{-\varpi}(\eta)$.

Step 3. We claim that if η and ζ satisfy (2.18), then $k_\alpha(\eta) = k_\alpha(\zeta)$, $\ell_\alpha(\eta) = \ell_\alpha(\zeta)$ and both $\delta^{(1)}\ell_\alpha$ and $\delta^{(2)}\ell_\alpha$ are even. We have $N_\alpha(\eta) = k_\alpha(\eta)\varpi + \ell_\alpha(\eta)$. Since ϖ is not a rational number, it results from (2.18) that $\delta k_\alpha = 0$ and $\delta^{(1)}\ell_\alpha + \delta^{(2)}\ell_\alpha = \delta\ell_\alpha = 0$. Moreover $\delta^{(1)}\ell_\alpha = \delta^{(2)}\ell_\alpha \bmod 2$, $\delta^{(2)}\ell_\alpha = \delta^{(2)}\ell_\alpha + \delta k_\alpha$, $\delta^{(2)}\ell_\alpha + \delta k_\alpha = \delta^{(2)} N_0 \bmod 2$ and from Step 2 $\delta^{(2)} N_0 = 0$. $\square$

Step 4. Construction of $\overline{\eta}$ and $\overline{\zeta}$. Suppose that, for instance, there are more velocities of species $\mathcal{V}_2$ with ± 1 in the α-th component for η than for ζ

$$\delta^{(2)} N_\alpha^{+1} + \delta^{(2)} N_\alpha^{-1} \geq 0 \tag{2.22}$$

and suppose that $\ell_\alpha(\eta^{(2)}) \neq \ell_\alpha(\zeta^{(2)})$.

First case: $\ell_\alpha(\eta^{(2)}) > \ell_\alpha(\zeta^{(2)})$.
Summing this inequality with (2.22), we obtain $N_\alpha^{+1}(\eta^{(2)}) > N_\alpha^{+1}(\zeta^{(2)})$. In particular, there is at least one particle in η with a velocity $v \in \mathcal{V}_2$ such that $v_\alpha = +1$. By contradiction suppose that $N_\alpha^{-1}(\eta^{(1)}) = 0$, then

$$N_0(\eta^{(1)}) + \ell_\alpha(\eta^{(2)}) = \ell_\alpha(\eta) = \ell_\alpha(\zeta) = \ell_\alpha(\zeta^{(1)}) + \ell_\alpha(\zeta^{(2)}) < N_0(\eta^{(1)}) + \ell_\alpha(\eta^{(2)}) \tag{2.23}$$

So there is at least one particle in η with a velocity $w \in \mathcal{V}_1$ such that $w_\alpha = -1$. Let $\overline{\eta}$ be a configuration obtained from η after jumps by a collision between two particles of species

$\mathcal{V}_1$ and $\mathcal{V}_2$, with incoming velocities v and w and outgoing velocities v' and w' given by $v'_\alpha = -1$, $w'_\alpha = +1$ and $v_\beta = v'_\beta$, $w_\beta = w'_\beta$ for $\beta \neq \alpha$. Then $\eta \leftrightarrow \overline{\eta}$, $\ell_\alpha(\overline{\eta}^{(2)}) = \ell_\alpha(\eta^{(2)}) - 2$ and $\ell_\beta(\overline{\eta}^{(2)}) = \ell_\beta(\eta^{(2)})$ for $\beta \neq \alpha$.

Second case: $\ell_\alpha(\eta^{(2)}) < \ell_\alpha(\zeta^{(2)})$.
Then multiplying this inequality by -1 and adding (2.22), we get $N_\alpha^{-1}(\eta^{(2)}) > N_\alpha^{-1}(\zeta^{(2)})$. With the same argument as in the first case (using -1 instead of $+1$ and vice-versa), one can build a configuration $\overline{\eta}$ such that $\eta \leftrightarrow \overline{\eta}$, $\ell_\alpha(\overline{\eta}^{(2)}) = \ell_\alpha(\eta^{(2)}) + 2$ and $\ell_\beta(\overline{\eta}^{(2)}) = \ell_\beta(\eta^{(2)})$ for $\beta \neq \alpha$.

If the inequality (2.22) is reversed, then we modify ζ (instead of η) in the same way. In any case, we obtain two configurations $\overline{\eta}$ and $\overline{\zeta}$ such that $\eta \leftrightarrow \overline{\eta}$, $\zeta \leftrightarrow \overline{\zeta}$, $|\ell_\alpha(\overline{\eta}^{(2)}) - \ell_\alpha(\overline{\zeta}^{(2)})| = |\ell_\alpha(\eta^{(2)}) - \ell_\alpha(\zeta^{(2)})| - 2$ and $\ell_\beta(\overline{\eta}^{(2)}) - \ell_\beta(\overline{\zeta}^{(2)}) = \ell_\beta(\eta^{(2)}) - \ell_\beta(\zeta^{(2)})$ for $\beta \neq \alpha$.

Arguing as in the second part of Step 3, we iterate this procedure and we repeat it for all the three components as far as we get two configurations $\overline{\eta}$ and $\overline{\zeta}$ such that $\eta \leftrightarrow \overline{\eta}$, $\zeta \leftrightarrow \overline{\zeta}$ and $\ell_\alpha(\overline{\eta}^{(2)}) = \ell_\alpha(\overline{\zeta}^{(2)})$ for $\alpha = 1, \dots, 3$. Since (2.18) still holds for $\overline{\eta}$ and $\overline{\zeta}$, it follows from Step 2 that $N_0(\overline{\eta}^{(i)}) = N_0(\overline{\zeta}^{(i)})$ for $i = 1, 2$. Moreover, from the first part of Step 3, $\ell_\alpha(\eta) = \ell_\alpha(\zeta)$ and $k_\alpha(\eta) = k_\alpha(\zeta)$, so $I_\alpha(\overline{\eta}^{(i)}) = I_\alpha(\overline{\zeta}^{(i)})$ for $\alpha = 1, \dots, 3$ and $i = 1, 2$. $\square$

2.3 *Gibbs states, currents.*

Since the quantities $\sum_{x \in \Lambda_L} I_\beta(\eta_x)$, $\beta = 0, \dots, 4$, are conserved by the dynamics, the following grand canonical measures are invariant for the generator $\mathcal{L}$

$$\mu_{L,n}(\eta) = Z_{L,n}^{-1} \prod_{x \in \Lambda_L} \exp\Big\{ \sum_{\beta=0}^{d+1} n_\beta I_\beta(\eta_x) \Big\} \tag{2.24}$$

where $n = (n_0, \dots, n_4) \in \mathbb{R}^5$ and $Z_{L,n}$ is a normalization constant. All these measures are absolutely continuous with respect to the measure $\mu_{L,r,\theta}$ obtained by taking $n = (r, 0, 0, 0, \theta)$:

$$\mu_{L,r,\theta}(\eta) = Z_{L,r}^{-1} \prod_{x \in \Lambda_L} \exp\{ r I_0(\eta_x)\} + \theta I_4(\eta_x)\} \tag{2.25}$$

Notice that the collision generator $\mathcal{L}^{\mathrm{c}}$ is symmetric with respect to $\mu_{L,n}$, but the jump generator $\mathcal{L}^{\mathrm{ex}}$ is not. The adjoint operator $\mathcal{L}^*$ of $\mathcal{L}$ with respect to $\mu_{L,n}$ is then $\mathcal{L}^* = \mathcal{L}^{\mathrm{ex}\,*} + \mathcal{L}^{\mathrm{c}}$ where for any function f on Ω and any configuration η

$$\mathcal{L}^{\mathrm{ex}\,*} f(\eta) = \sum_{v \in \mathcal{V}} \sum_{e \in \mathcal{D}} \sum_{x \in \mathbb{Z}^d} p^*(x, x+e, v, \eta) \left[f(\eta^{x,x+e,v}) - f(\eta) \right] \tag{2.26}$$

and

$$p^*(x, x+e, v, \eta) = \eta(x, v)\big[\chi - \frac{1}{2}\, e \cdot v\big] \tag{2.27}$$

If $n(x) = (n_0(x), \dots, n_4(x))$ are functions on Λ_L, we also call $\mu_{L,n}$ the measure with varying chemical potentials n

$$\mu_{L,n}(\eta) = Z_{L,n}^{-1} \prod_{x \in \Lambda_L} \exp\Big\{\sum_{\beta=0}^{4} n_\beta(x) I_\beta(\eta_x)\Big\} \tag{2.28}$$

Then the density of particles with velocity $v \in \mathcal{V}$ on site $x \in \Lambda_L$ is

$$f(x, v, n) = E^{\mu_{L,n}}[\eta(x, v)] = \frac{e^{n_0(x) + \underline{n} \cdot v + \frac{1}{2}|v|^2 n_{d+1}(x)}}{1 + e^{n_0(x) + \underline{n} \cdot v + \frac{1}{2}|v|^2 n_{d+1}(x)}} \tag{2.29}$$

where $\underline{n}(x) = (n_1(x), \dots, n_3(x))$.

Currents

We denote by $w^\beta_{x,\alpha}$ the current in the direction e_α ($\alpha = 1, \dots, 3$) at site $x \in \Lambda_L$ for the conserved quantity I_β ($\beta = 0, \dots, 4$). They are defined by

$$\mathcal{L}\big[I_\beta(\eta_x)\big] = \sum_{\alpha=1}^{3} \nabla_\alpha^- w^\beta_{x,\alpha} \tag{2.30}$$

where, if g is a function on Λ_L,

$$\nabla_\alpha^- g(x) = (\nabla_\alpha g)(x - e_\alpha) \quad \text{and} \quad \nabla_\alpha g(x) = g(x + e_\alpha) - g(x) \tag{2.31}$$

Since the collision operator conserves the quantities $I_\beta(\eta_x)$ for any $x \in \Lambda_L$, there is no contribution to the currents coming from $\mathcal{L}^{\mathrm{c}}$

$$\mathcal{L}\big[I_\beta(\eta_x)\big] = \mathcal{L}^{\mathrm{ex}}\big[I_\beta(\eta_x)\big] \tag{2.32}$$

Similarly we define the currents $w^{*,\beta}_{x,\alpha}$ for the adjoint operator $\mathcal{L}^*$

$$\mathcal{L}^*\big[I_\beta(\eta_x)\big] = \sum_{\alpha=1}^{3} \nabla_\alpha^- w^{*,\beta}_{x,\alpha} \tag{2.33}$$

The currents $w_{x,\alpha}(v)$ and $w^*_{x,\alpha}(v)$ related to the density of particles with the velocity $v \in \mathcal{V}$ for the asymmetric simple exclusions $\mathcal{L}^{\mathrm{ex}}$ and $\mathcal{L}^{\mathrm{ex}\,*}$ are given by

$$\mathcal{L}^{\mathrm{ex}} \eta(x, v) = \sum_{\alpha=1}^{3} \nabla_\alpha^- w_{x,\alpha}(v), \qquad \mathcal{L}^{\mathrm{ex}\,*} \eta(x, v) = \sum_{\alpha=1}^{3} \nabla_\alpha^- w^*_{x,\alpha}(v) \tag{2.34}$$

with

$$\begin{aligned} w_{x,\alpha}(v) &= \frac{1}{2}\big(p(e_\alpha,v)+p(-e_\alpha,v)\big)\nabla_\alpha\eta(x,v)+w^{(a)}_{x,\alpha}(v) \\ w^*_{x,\alpha}(v) &= \frac{1}{2}\big(p(e_\alpha,v)+p(-e_\alpha,v)\big)\nabla_\alpha\eta(x,v)-w^{(a)}_{x,\alpha}(v) \\ w^{(a)}_{x,\alpha}(v) &= \big(p(e_\alpha,v)-p(-e_\alpha,v)\big)b_{x,\alpha}(v) \end{aligned} \tag{2.35}$$

In this formula $p(e_\alpha, v)$ is the intensity of the jump in the direction e_α (see (2.5)) and

$$b_{x,\alpha}(v) = \eta(x+e_\alpha,v)\eta(x,v) - \frac{1}{2}\big(\eta(x+e_\alpha)+\eta(x,v)\big) \tag{2.36}$$

Remark that from (2.7)

$$\frac{1}{2}\big(p(e_\alpha,v)+p(-e_\alpha,v)\big)=\chi \quad \text{and} \quad p(e_\alpha,v)-p(-e_\alpha,v)=e_\alpha\cdot v \tag{2.37}$$

Using (2.32), we can now compute the currents related to the conserved quantities for the generator $\mathcal{L}$. They can be written as a sum of a symmetric part and an antisymmetric part. For $\alpha = 1,\dots,3$ and $\beta = 0,\dots,4$

$$w^\beta_{x,\alpha} = \chi\nabla_\alpha I_\beta(\eta_x) + w^{(a),\beta}_{x,\alpha} \tag{2.38}$$

the asymmetric mass current is

$$w^{(a),0}_{x,\alpha} = \sum_{v\in V} w^{(a)}_{x,\alpha}(v) = \sum_{v\in\mathcal{V}}(e_\alpha\cdot v)b_{x,\alpha}(v) \tag{2.39}$$

the asymmetric momentum current is, for $\alpha,\beta = 1,\dots,d$

$$w^{(a),\beta}_{x,\alpha} = \sum_{v\in V}(e_\beta\cdot v)w^{(a)}_{x,\alpha}(v) = \sum_{v\in\mathcal{V}}(e_\alpha\cdot v)(e_\beta\cdot v)b_{x,\alpha}(v) \tag{2.40}$$

and the asymmetric energy current is, for $\alpha = 1,\dots,d$

$$w^{(a),d+1}_{x,\alpha} = \sum_{v\in V}\frac{1}{2}|v|^2 w^{(a)}_{x,\alpha}(v) = \sum_{v\in\mathcal{V}}\frac{1}{2}(e_\alpha\cdot v)|v|^2 b_{x,\alpha}(v) \tag{2.41}$$

In the same way the currents associated to the adjoint operator $\mathcal{L}^*$ are given by

$$w^{*,\beta}_{x,\alpha} = \chi\nabla_\alpha I_b(\eta_x) - w^{(a),\beta}_{x,\alpha} \tag{2.42}$$

with $\alpha = 1,\dots,3$ and $\beta = 0,\dots,4$.

Time reversal invariance.

Define for any configuration η the configuration $S\eta$ as

$$S\eta = \{\eta(x,-v), x\in\Lambda_L, v\in\mathcal{V}\}$$

Define the operator S that flip the velocities acting on functions of η as

$$Sf(\eta) = f(S\eta)$$

We claim that

$$\mathcal{L}S = S\mathcal{L}^* \tag{2.43}$$

This is equivalent to show that

$$\mathcal{L}_a S = -S\mathcal{L}_a, \quad \mathcal{L}_s S = \mathcal{L}_s S$$

where $\mathcal{L}_a$ and $\mathcal{L}_s$ are the antisymmetric and symmetric part of the generator $\mathcal{L}$.

This property is easily seen by direct inspection

$$\left(\mathcal{L}^{\mathrm{ex}}{}_a Sf\right)(\eta) = \left(\mathcal{L}^{\mathrm{ex}}{}_a f\right)(S\eta) = \sum_{v\in\mathcal{V}}\sum_{e\in\mathcal{D}}\sum_{x\in\mathbb{Z}^d}(e\cdot v)\eta(x,v)\left[f\left((S\eta)^{x,x+e,v}\right) - f(S\eta)\right]$$

$$\begin{aligned}
\left(S\mathcal{L}^{\mathrm{ex}}{}_a f\right)(\eta)) &= \sum_{v\in\mathcal{V}}\sum_{e\in\mathcal{D}}\sum_{x\in\mathbb{Z}^d}(e\cdot v)S\eta(x,v)\left[f\left(S(\eta^{x,x+e,v})\right) - f(S\eta)\right] \\
&= \sum_{v\in\mathcal{V}}\sum_{e\in\mathcal{D}}\sum_{x\in\mathbb{Z}^d}(e\cdot v)\eta(x,-v)\left[f\left((S\eta)^{x,x+e,-v}\right) - f(S\eta)\right] \\
&= -\sum_{v\in\mathcal{V}}\sum_{e\in\mathcal{D}}\sum_{x\in\mathbb{Z}^d} e\cdot v\eta(x,v)\left[f\left((S\eta)^{x,x+e,v}\right) - f(S\eta)\right]
\end{aligned}$$

We have used that

$$[S(\eta^{x,x+e,v})](z,w) = (S\eta)^{x,x+e,-v}(z,w) \tag{2.44}$$

In fact,

$$[S(\eta^{x,x+e,v})](z,w) = (\eta^{x,x+e,v})(z,-w)$$

and

$$(S\eta)^{x,x+e,-v}(z,w) = \begin{cases} (S\eta)(x+e,-v) = \eta(x+e,v) & \text{if } z = x \text{ and } w = -v \\ (S\eta)(x,-v) = \eta(x,v) & \text{if } z = x+e \text{ and } w = -v \\ (S\eta)(z,w) = \eta(z,-w) & \text{otherwise} \end{cases} \tag{2.45}$$

By comparison of (2.45) with

$$\eta^{x,x+e,v}(z,-w) = \begin{cases} \eta(x+e,v) & \text{if } z = x \text{ and } -w = v \\ \eta(x,v) & \text{if } z = x+e \text{ and } -w = v \\ \eta(z,-w) & \text{otherwise} \end{cases}$$

we get

$$(\eta^{x,x+e,v})(z,-w) = (S\eta)^{x,x+e,-v}(z,w)$$

and then (2.44).

It is obvious that $\mathcal{L}^{\mathrm{ex}}{}_s$ commute with S. It is also true that $\mathcal{L}^c$ commute with S. In fact, if in a site x there is a collision with ingoing velocities v_1, v_2 and outgoing v_1', v_2' the collision with reversed velocities is still admissible since conserve momentum and energy.

The property (2.43) says that the dynamics is invariant under time-reversal.

Hydrodynamical equations on the Euler scale

We start the process from the local equilibrium measure (2.28) with chemical potentials $n(\varepsilon x, 0)$ slowly varying in space. Then we look at the system on Euler time scale, that is we consider the generator $\varepsilon^{-1}\mathcal{L}$ instead of $\mathcal{L}$. We denote by f_t the density of the process at time t with respect to the reference measure $\mu_{L,r,\theta}$. The microscopic conservation laws imply that for any smooth test function J on the torus and for $\beta = 0, \dots, 4$

$$\frac{d}{dt}\varepsilon^3 \sum_x J(\varepsilon x) E^{f_t \mu_{L,r,\theta}}\left[I_\beta(\eta_x)\right] = \varepsilon^2 \sum_x \sum_{\alpha=1}^{3} J(\varepsilon x) \nabla_\alpha^- E^{f_t \mu_{L,r,\theta}}[w_{x,\alpha}^\beta] \tag{2.46}$$

We assume that the measure $f_t \mu_{L,r,\theta}$ is well approximated by the local equilibrium measure with chemical potentials $n_t(x) = (\lambda_0(\varepsilon x, t), \dots, \lambda_4(\varepsilon x, t))$. Then the macroscopic quantities corresponding to the conserved quantities I_β are the mass density ρ

$$\rho(\varepsilon x, t) = E^{\mu_{L,n_t}}\left[I_0(\eta_x)\right] = \sum_{v \in \mathcal{V}} f(x, v, n_t) \tag{2.47}$$

the momentum density u_β for $\beta = 1, \dots, 3$

$$u_\beta(\varepsilon x, t) = E^{\mu_{L,n_t}}\left[I_\beta(\eta_x)\right] = \sum_{v \in \mathcal{V}} (e_\beta \cdot v) f(x, v, n_t) \tag{2.48}$$

and the energy density

$$\mathcal{E}(\varepsilon x, t) = E^{\mu_{L,n_t}}\left[I_4(\eta_x)\right] = \sum_{v \in \mathcal{V}} \frac{1}{2}|v|^2 f(x, v, n_t) \tag{2.49}$$

The macroscopic currents, for $\alpha = 1, \dots, 3$, are the mass current

$$j_\alpha(\varepsilon x, t) = E^{\mu_{L,n_t}}\left[w_{x,\alpha}^0\right] = \chi \nabla_\alpha \rho(\varepsilon x, t) - \sum_{v \in \mathcal{V}} (e_\alpha \cdot v) h(x, v, n_t) \tag{2.50}$$

where

$$h(x, v, n_t) = f(x, v, n_t) - f^2(x, v, n_t) \tag{2.51}$$

the stress tensor

$$\pi_{\alpha,\beta}(\varepsilon x, t) = E^{\mu_{L,n_t}}\left[w_{x,\alpha}^\beta\right] = \chi \nabla_\alpha u_\beta(\varepsilon x, t) - \sum_{v \in \mathcal{V}} (e_\alpha \cdot v)(e_\beta \cdot v) h(x, v, n_t) \tag{2.52}$$

and the energy current

$$g_\alpha(\varepsilon x, t) = E^{\mu_{L,n_t}}\left[w^4_{x,\alpha}\right] = \chi \nabla_\alpha \mathcal{E}(\varepsilon x, t) - \sum_{v\in\mathcal{V}} \frac{1}{2}(e_\alpha \cdot v)|v|^2 h(x, v, n_t) \tag{2.53}$$

From the conservation laws (2.46), we obtain the following hydrodynamical equations

$$\begin{aligned} &\frac{\partial}{\partial t}\rho + \sum_{\alpha=1}^{d} \partial_\alpha j_\alpha = 0 \\ &\frac{\partial}{\partial t}u_\beta + \sum_{\alpha=1}^{d} \partial_\alpha \pi_{\alpha,\beta} = 0 \\ &\frac{\partial}{\partial t}\mathcal{E} + \sum_{\alpha=1}^{d} \partial_\alpha g_\alpha = 0 \end{aligned} \tag{2.54}$$

where ∂_α is the partial derivative with respect to the macroscopic coordinate z_α.

2.4 *Formal derivation of the Navier-Stokes equations.*

We now consider the incompressible limit: first we choose as initial state the equilibrium measure (2.28) with chemical potential $n(x) = (n_0(x), \dots, n_4(x))$ given by

$$n_\beta(x) = \lambda^{(0)}_\beta + \varepsilon\lambda^{(1)}_\beta(\varepsilon x) + \varepsilon^2\lambda^{(2)}_\beta(\varepsilon x) \tag{2.55}$$

where $\lambda^{(1)}_\beta$, $\lambda^{(2)}_\beta$ are smooth periodic functions and where $\lambda^{(0)}_\beta = r\delta_{\beta,0} + \theta\delta_{\beta,4}$. We consider a diffusive scaling, that is $\mathcal{L}$ is replaced by $\varepsilon^{-2}\mathcal{L}$. Then the conservation laws (2.46) become

$$\frac{d}{dt}\varepsilon^3 \sum_x J(\varepsilon x) E^{f_t\mu_{L,r,\theta}}\left[I_\beta(\eta_x)\right] = \varepsilon \sum_x \sum_{\alpha=1}^{3} J(\varepsilon x) \nabla^-_\alpha E^{f_t\mu_{L,r,\theta}}[w^\beta_{x,\alpha}] \tag{2.56}$$

where $f_t\mu_{L,r,\theta}$ is the law of the process at time t.

On this time scale the local equilibrium is not enough and some correction to the local equilibrium is needed. The form of this correction can be guessed by ε-expansion arguments as in [EM] and will give rise to the dissipative term in the limiting equations not coming from the symmetric part of the exclusion process. The diffusion coefficients are given by expressions like the Green-Kubo formulas. We do not give here the explicit expression of the correction, that we call R and is of order ε^2, because it will be given in the entropy argument in the next section. In conclusion we assume that the non-equilibrium measure is well approximated by the measure with chemical potentials $n(x, t)$ given by

$$n_\beta(x, t) = \lambda^{(0)}_\beta + \varepsilon\lambda^{(1)}_\beta(\varepsilon x, t) + \varepsilon^2\lambda^{(2)}_\beta(\varepsilon x, t), \quad \beta = 0, \dots, 4 \tag{2.57}$$

and a correction of order ε^2. In this case the density of particles $f(x,v,n_t)$ (see (2.29)) with velocity $v\in\mathcal{V}$ on site $x\in\Lambda_L$ at time t has the following Taylor expansion

$$
\begin{aligned}
&f_0+\varepsilon f_1\Big[\lambda_0^{(1)}+\underline{\lambda}^{(1)}\cdot v+\frac{1}{2}|v|^2\lambda_4^{(1)}\Big](\varepsilon x,t)+\varepsilon^2 f_2\Big[\lambda_0^{(2)}+\underline{\lambda}^{(2)}\cdot v+\frac{1}{2}|v|^2\lambda_4^{(2)}\Big](\varepsilon x,t)\\
&+\varepsilon^2\overline{f}_2\Big[(\underline{\lambda}^{(1)}\cdot v)^2+(\lambda_0^{(1)})^2+\frac{1}{4}|v|^4(\lambda_4^{(1)})^2\\
&\qquad+2(\underline{\lambda}^{(1)}\cdot v)\lambda_0^{(1)}+(\underline{\lambda}^{(1)}\cdot v)\lambda_4^{(1)}+|v|^2\lambda_0^{(1)}\lambda_4^{(1)}\Big](\varepsilon x,t)+o(\varepsilon^2)
\end{aligned}
\tag{2.58}
$$

where $\underline{\lambda}^{(i)}=(\lambda_1^{(i)},\dots,\lambda_3^{(i)})$ and

$$
f_0=\frac{e^{r+\theta\frac{v^2}{2}}}{1+e^{r+\theta\frac{v^2}{2}}},\qquad f_1=\frac{e^{r+\theta\frac{v^2}{2}}}{(1+e^{r+\theta\frac{v^2}{2}})^2}=f_0(1-f_0).
\tag{2.59}
$$

$f_1=f_2$ and $\overline{f}_2=\frac{1}{2}f_0(1-f_0)(1-2f_0)$.
Here and below $o(\varepsilon^k)$ and $O(\varepsilon^k)$ denote quantities going to 0 faster than ε^k and as ε^k respectively.

For any function h on $\mathcal{V}$, we let $\langle h\rangle=\sum_{v\in\mathcal{V}}h(v)$ and we denote by $N=\langle 1\rangle$ the cardinality of $\mathcal{V}$. Remark that since $\mathcal{V}$ satisfies (**IR**) the identity $\langle\underline{\lambda}^{(1)}\cdot v\rangle=0$ holds. Then using (2.47), (2.48) and (2.49), the mass, momentum and energy densities have the following first order Taylor expansions

$$
\begin{aligned}
\rho(\varepsilon x,t)&=\rho^{(0)}+\varepsilon\rho^{(1)}(\varepsilon x,t)+o(\varepsilon)\\
u_\beta(\varepsilon x,t)&=\varepsilon u_\beta^{(1)}(\varepsilon x,t)+o(\varepsilon),\quad \beta=1,\dots,3\\
\mathcal{E}(\varepsilon x,t)&=\mathcal{E}^{(0)}+\varepsilon\mathcal{E}^{(1)}(\varepsilon x,t)+o(\varepsilon)
\end{aligned}
\tag{2.60}
$$

where $\rho^{(0)}=\langle f_0\rangle$, $\mathcal{E}^{(0)}=\frac{1}{2}\langle f_0|v|^2\rangle$ and

$$
\begin{aligned}
\rho^{(1)}&=\langle f_1\rangle\lambda_0^{(1)}+\frac{1}{2}\langle f_1|v|^2\rangle\lambda_4^{(1)}\\
u_\beta^{(1)}&=\frac{1}{3}\langle f_1|v|^2\rangle\lambda_\beta^{(1)},\quad \beta=1,\dots,3\\
\mathcal{E}^{(1)}&=\frac{1}{2}\langle f_1|v|^2\rangle\lambda_0^{(1)}+\frac{1}{4}\langle f_1|v|^4\rangle\lambda_4^{(1)}
\end{aligned}
\tag{2.61}
$$

The second order Taylor expansion of $h(x,v,n_t)$ (see (2.51)) is

$$
\begin{aligned}
&h_0+\varepsilon h_1\Big[\lambda_0^{(1)}+\underline{\lambda}^{(1)}\cdot v+\frac{1}{2}|v|^2\lambda_4^{(1)}\Big](\varepsilon x,t)\\
&+\varepsilon^2 h_2\Big[\lambda_0^{(2)}+\underline{\lambda}^{(2)}\cdot v+\frac{1}{2}|v|^2\lambda_4^{(2)}\Big](\varepsilon x,t)\\
&+\varepsilon^2\overline{h}_2\Big[(\underline{\lambda}^{(1)}\cdot v)^2+(\lambda_0^{(1)})^2+\frac{1}{4}|v|^4(\lambda_4^{(1)})^2\\
&\qquad+2(\underline{\lambda}^{(1)}\cdot v)\lambda_0^{(1)}+(\underline{\lambda}^{(1)}\cdot v)|v|^2\lambda_4^{(1)}+|v|^2\lambda_0^{(1)}\lambda_4^{(1)}\Big](\varepsilon x,t)+o(\varepsilon^2)
\end{aligned}
\tag{2.62}
$$

where

$$
\begin{aligned}
h_0 &= f_1 = f_0(1-f_0),\\
h_1 &= h_2 = f_1(1-2f_0),\\
\overline{h}_2 &= \frac{1}{2} f_0(1-f_0)(1-2f_0)\big(1-6f_0(1-f_0)\big)
\end{aligned}
\tag{2.63}
$$

From (2.50), (2.52) and (2.53), we can compute the second order Taylor expansions of the mass current, the stress tensor and energy current. There is also the second order in ε contribution coming from the correction R to the equilibrium measure. These terms are combination of gradients of the conserved quantities with suitable constant coefficients, which are indeed the diffusion coefficients $D^{\alpha,\beta}_{\gamma,\delta}$, $\alpha,\beta = 0,\cdots,4$, $\gamma,\delta = 1,2,3$. Their properties are discussed in Section 4. Call $D = \chi\mathbb{I} + \bar{D}$.

$$
\begin{aligned}
j_\alpha(\varepsilon x,t) &= \varepsilon\chi\nabla_\alpha\rho(\varepsilon x,t) - \varepsilon j^{(1)}_\alpha(\varepsilon x,t) - \varepsilon^2 j^{(2)}_\alpha(\varepsilon x,t)\\
&\quad + \varepsilon\sum_{\beta=1}^{3}\nabla_\beta[\bar{D}^{0,0}_{\alpha,\beta}\rho + \bar{D}^{0,4}_{\alpha,\beta}\mathcal{E}] + o(\varepsilon^2)\\
\pi_{\alpha,\beta}(\varepsilon x,t) &= \varepsilon\chi\nabla_\alpha u_\beta(\varepsilon x,t) - \pi^{(0)}_{\alpha,\beta} - \varepsilon\pi^{(1)}_{\alpha,\beta}(\varepsilon x,t) - \varepsilon^2\pi^{(2)}_{\alpha,\beta}(\varepsilon x,t)\\
&\quad + \varepsilon\sum_{\beta,\delta=1}^{3}\bar{D}^{\beta,\gamma}_{\alpha,\beta}\nabla_\delta u_\gamma + o(\varepsilon^2), \quad \beta = 1,\dots,d\\
g_\alpha(\varepsilon x,t) &= \varepsilon\chi\nabla_\alpha\mathcal{E}(\varepsilon x,t) - \varepsilon g^{(1)}_\alpha(\varepsilon x,t) - \varepsilon^2 g^{(2)}_\alpha(\varepsilon x,t) +\\
&\quad \varepsilon\sum_{\beta=1}^{3}\nabla_\beta[\bar{D}^{4,0}_{\alpha,\beta}\rho + \bar{D}^{4,4}_{\alpha,\beta}\mathcal{E}] + o(\varepsilon^2)
\end{aligned}
\tag{2.64}
$$

Using the (**IP**) and (**IR**) properties of the velocity set $\mathcal{V}$, we get for $\alpha = 1,\dots,3$

$$
\begin{aligned}
j^{(1)}_\alpha &= \frac{1}{3}\langle h_1|v|^2\rangle\lambda^{(1)}_\alpha\\
j^{(2)}_\alpha &= \frac{1}{3}\Big[\langle h_2|v|^2\rangle\lambda^{(2)}_\alpha + 2\langle\overline{h}_2|v|^2\rangle\lambda^{(1)}_\alpha\lambda^{(1)}_0 + \langle\overline{h}_2|v|^4\rangle\lambda^{(1)}_\alpha\lambda^{(1)}_4\Big]\\
\pi^{(0)}_{\alpha,\beta} &= \delta_{\alpha,\beta}\frac{1}{3}\langle h_0|v|^2\rangle\\
\pi^{(1)}_{\alpha,\beta} &= \delta_{\alpha,\beta}\frac{1}{3}\Big[\langle h_1|v|^2\rangle\lambda^{(1)}_0 + \frac{1}{2}\langle h_1|v|^4\rangle\lambda^{(1)}_4\Big]\\
\pi^{(2)}_{\alpha,\beta} &= \delta_{\alpha,\beta}\frac{1}{3}\Big[\langle h_2|v|^2\rangle\lambda^{(2)}_0 + \frac{1}{2}\langle h_2|v|^4\rangle\lambda^{(2)}_4\Big]\\
&\quad + \delta_{\alpha,\beta}\frac{1}{3}\Big[\langle\overline{h}_2|v|^2(v\cdot w)^2\rangle + \langle\overline{h}_2|v|^2\rangle(\lambda^{(1)}_0)^2\\
&\quad + \frac{1}{4}\langle\overline{h}_2|v|^6\rangle(\lambda^{(1)}_4)^2 + \langle\overline{h}_2|v|^4\rangle\lambda^{(1)}_0\lambda^{(1)}_4\Big]
\end{aligned}
\tag{2.65}
$$

$$g_\alpha^{(1)} = \frac{1}{6}\langle h_1|v|^4\rangle\lambda_\alpha^{(1)}$$
$$g_\alpha^{(2)} = \frac{1}{3}\Big[\frac{1}{2}\langle h_2|v|^4\rangle\lambda_\alpha^{(2)} + \langle\overline{h}_2|v^4|\rangle\lambda_0^{(1)}\lambda_\alpha^{(1)} + \langle\overline{h}_2|v^6|\rangle\lambda_4^{(1)}\lambda_\alpha^{(1)}\Big]$$

Therefore, up to a term $o_\varepsilon(1)$, the conservation laws (2.56) with (2.60) give rise to

$$\begin{aligned}\partial_t\rho^{(1)} =& -\varepsilon^{-1}a_0\,\mathrm{div}\underline{\lambda}^{(1)} - \underline{\lambda}^{(1)}\cdot\nabla\Big[A_0\lambda_0^{(1)} + B_0\lambda_4^{(1)}\Big] - C_0\,\mathrm{div}\underline{\lambda}^{(2)}\\ &+\sum_{\alpha=1}^{3}[\chi + \bar{D}^{0,0}_{\alpha,\alpha}]\partial_\alpha^2\rho^{(1)} + \sum_{\alpha=1}^{3}\bar{D}^{0,4}_{\alpha,\alpha}\partial_\alpha^2\mathcal{E}^{(1)}\end{aligned} \tag{2.66}$$

where $\mathrm{div}\underline{\lambda}^{(i)} = \sum_{\alpha=1}^3\partial_\alpha\lambda_\alpha^{(i)}$, if J is a smooth function on $\mathbb{R}^3$, $\nabla J = (\partial_1 J,\ldots,\partial_3 J)$ and

$$a_0 = \frac{1}{3}\langle h_1|v|^2\rangle,\quad A_0 = \frac{2}{3}\langle\overline{h}_2|v|^2\rangle,\quad B_0 = \frac{1}{3}\langle\overline{h}_2|v|^4\rangle\quad\text{and}\quad C_0 = \frac{1}{3}\langle h_2|v|^2\rangle \tag{2.67}$$

In (2.66) we have used the property (4.13) of the matrix D stated in Theorem 4.5. The momentum density satisfies the PDE (up to a term $o_\varepsilon(1)$)

$$\begin{aligned}\partial_t u_\beta^{(1)} =& -\varepsilon^{-1}\partial_\beta\Big[a_1\lambda_0^{(1)} + b_1\lambda_4^{(1)}\Big] - \partial_\beta p - A_1\partial_\beta\big((\lambda_\alpha^{(1)})^2\big) - B_1\sum_{\alpha=1}^{3}\lambda_\alpha^{(1)}\partial_\alpha\lambda_\beta^{(1)}\\ &+\sum_{\alpha,\gamma=1}^{3}\partial_\alpha^2[\chi + \bar{D}^{\beta,\gamma}_{\alpha,\alpha}]u_\gamma^{(1)}\end{aligned} \tag{2.68}$$

where

$$a_1 = \frac{1}{3}\langle h_1|v|^2\rangle,\quad b_1 = \frac{1}{6}\langle h_1|v|^4\rangle,\quad A_1 = \langle\overline{h}_2(v_1^4 - 3v_1^2v_2^2)\rangle\quad\text{and}\quad B_1 = 2\langle\overline{h}_2v_1^2v_2^2\rangle \tag{2.69}$$

and the pressure p is defined by

$$\begin{aligned}p =& \frac{1}{3}\langle h_2|v|^2\rangle\lambda_0^{(2)} + \frac{1}{6}\langle h_2|v|^4\rangle\lambda_4^{(2)} + \frac{1}{3}\langle\overline{h}_2|v|^2\rangle\big(\lambda_0^{(1)}\big)^2 + \frac{1}{12}\langle\overline{h}_2|v|^6\rangle\big(\lambda_4^{(1)}\big)^2\\ &+\frac{1}{3}\langle\overline{h}_2|v|^2\rangle\lambda_0^{(1)}\lambda_4^{(1)} + \langle\overline{h}_2v_1^2v_2^2\rangle|\underline{\lambda}^{(1)}|^2\end{aligned} \tag{2.70}$$

The energy equation is

$$\begin{aligned}\partial_t\mathcal{E}^{(1)} =& -\varepsilon^{-1}a_2\,\mathrm{div}\underline{\lambda}^{(1)} - \underline{\lambda}^{(1)}\cdot\nabla\Big[A_2\lambda_0^{(1)} + B_2\lambda_4^{(1)}\Big] - C_2\,\mathrm{div}\underline{\lambda}^{(2)}\\ &+\sum_{\alpha=1}^{3}[\chi + \bar{D}^{4,4}_{\alpha,\alpha}]\partial_\alpha^2\mathcal{E}^{(1)} + \sum_{\alpha=1}^{3}D^{4,0}_{\alpha,\alpha}\partial_\alpha^2\rho^{(1)}\end{aligned} \tag{2.71}$$

with

$$a_2 = \frac{1}{6}\langle h_1|v|^4\rangle, \quad A_2 = \frac{1}{3}\langle \overline{h}_2|v|^4\rangle, \quad B_2 = \frac{1}{6}\langle \overline{h}_2|v|^6\rangle \quad \text{and} \quad C_2 = \frac{1}{6}\langle h_1|v|^4\rangle \tag{2.72}$$

We have to choose the chemical potentials $\lambda^{(1)}$ such that the term of order ε^{-1} vanish in (2.66), (2.68) and (2.71), that is

$$\text{div}\underline{\lambda}^{(1)} = \sum_{\alpha=1}^{3} \lambda_\alpha^{(1)} = 0 \tag{2.73}$$

and

$$\langle h_1|v|^2\rangle\lambda_0^{(1)} + \frac{1}{2}\langle h_1|v|^4\rangle\lambda_4^{(1)} = \text{const} \tag{2.74}$$

From a physical point of view, this last condition can be derived from the Boussinesq condition (**BC**): the first order term in the Taylor expansion of the pressure, that is the diagonal part of the stress tensor, is constant.

Now we can derive the Navier-Stokes equations. We choose r, θ and ϖ irrational in such a way that the coefficient in front of the unusual term $\partial_\beta(\lambda^{(1)})^2)$ in (2.68) vanishes. For example, $\theta = 0$ and $\varpi^2 = 5\sqrt{3}$ for any r satisfy $A_1 = 0$ (see (2.69)). Moreover from (2.61), we have $\langle f_1|v|^2\rangle\lambda_\beta^{(1)} = 3\,u_\beta^{(1)}$, so using (2.73) and (2.74), the momentum equation (2.68) becomes

$$\begin{aligned} &\text{div}u^{(1)} = 0 \\ &\partial_t u_\beta^{(1)} + \partial_\beta p + K\, u^{(1)} \cdot \nabla u_\beta^{(1)} = \sum_{\alpha=1}^{3} D_{\alpha,\beta}\partial_\alpha^2 u_\beta^{(1)} \end{aligned} \tag{2.75}$$

where

$$K = 18\frac{\langle v_1^2 v_2^2 \overline{h}_2\rangle}{\langle h_0|v|^2\rangle^2} \tag{2.76}$$

and

$$D_{\alpha,\beta} = B_1 + B_2\delta_{\alpha,\beta} \tag{2.77}$$

(see (4.12)).

The equation corresponding to the energy conservation law is usually written as an equation for the temperature. For a real particle system the temperature is the inverse of the coefficient β in front of the Hamiltonian in the Gibbs measure. Hence it coincides with the variance of the velocity distribution, which is in this case a Gaussian. For models with discrete velocities that is no longer true, and the right choice [ED], [C] is the first one. In our notation

$$T = \lambda_4^{-1}$$

As a consequence the internal energy is in general a complicated non linear function of T. Nevertheless, the first correction to the temperature $T^{(1)}$, defined as $T = T^{(0)} + \varepsilon T^{(1)} + O(\varepsilon^2)$, is easily find as

$$T^{(1)} = -\frac{1}{\theta^2}\lambda_4^{(1)}$$

where $\theta = \lambda_4^{(0)}$ is the main term in the Taylor expansion of λ_4. Hence both $\rho^{(1)}$ and $\mathcal{E}^{(1)}$ are affine functions of $T^{(1)}$

$$\begin{aligned}\rho^{(1)} &= c\,\langle h_0\rangle + \frac{\theta^2\,\Phi_1}{2\,\langle h_1|v|^2\rangle}T^{(1)} \\ \mathcal{E}^{(1)} &= c\frac{\langle h_0|v|^2\rangle}{2} + \frac{\theta^2\,\Phi_2}{4\,\langle h_1|v|^2\rangle}T^{(1)}\end{aligned} \tag{2.78}$$

where $c = \frac{const}{\langle h_1|v|^2\rangle}$ and $const$ stands for the constant appearing in the Boussinesq condition (2.74) and

$$\begin{aligned}\Phi_1 &= \langle h_1|v|^4\rangle\langle h_0\rangle - \langle h_1|v|^2\rangle\langle h_0|v|^2\rangle \\ \Phi_2 &= \langle h_0|v|^4\rangle\langle h_1|v|^2\rangle - \langle h_1|v|^4\rangle\langle h_0|v|^2\rangle\end{aligned} \tag{2.79}$$

Notice that $T^{(1)}$ can be written in terms of $\rho^{(1)}$ and $\mathcal{E}^{(1)}$ as

$$T^{(1)} = \frac{2}{\theta^2\,\bar{\Phi}}\Big(\langle h_0|v|^2\rangle\,\rho^{(1)} - 2\,\langle h_0\rangle\,\mathcal{E}^{(1)}\Big) \tag{2.80}$$

with

$$\bar{\Phi} = \langle h_0|v|^4\rangle\,\langle h_0\rangle - \langle h_0|v|^2\rangle^2 \tag{2.81}$$

This expression suggests to consider the conserved quantity I_4', which is a linear combination of the others given by

$$I_4' = \frac{2}{\theta^2\,\bar{\Phi}}\Big(\langle h_0|v|^2\rangle\,I_0 - 2\,\langle h_0\rangle\,I_4\Big) \tag{2.82}$$

such that $E^{\mu_{L,n_t}}\left[I_4'\right] = A_T + T^{(1)} + O(\varepsilon^2)$, where the constant is

$$A_T := \frac{2}{\theta^2\,\bar{\Phi}}\Big(\langle h_0|v|^2\rangle < f_0 > -2\,\langle h_0\rangle < f_0|v|^2 >\Big)$$

Then (2.71) (by eliminating the term $\mathrm{div}\underline{\lambda}^{(2)}$) and (2.78) give the equation for the temperature

$$\frac{\partial}{\partial t}T^{(1)} + H\underline{u}^{(1)}\cdot\nabla T^{(1)} = \sum_{\alpha=1}^{3}\mathcal{K}_\alpha(\partial_\alpha^2 T^{(1)}) \tag{2.83}$$

where

$$\mathcal{K}_\alpha = \chi + \frac{\Phi_1}{\Phi_2 + C\,\Phi_1}(C\bar{D}^{0,0}_{\alpha,\alpha} - D^{4,0}_{\alpha,\alpha}) + \frac{\Phi_2}{\Phi_2 + C\,\Phi_1}(\bar{D}^{4,4}_{\alpha,\alpha} - C\,D^{0,4}_{\alpha,\alpha}) \tag{2.84}$$

$$H = \frac{1}{\langle h_0 |v|^2\rangle} \frac{\Psi_1 - 2C\Psi_2}{\Phi_2 + C\Phi_1} \tag{2.85}$$

with

$$\begin{aligned}
\Psi_1 &= \langle \overline{h}_2 |v|^6\rangle\langle h_1 |v|^2\rangle - \langle \overline{h}_2 |v|^4\rangle\langle h_1 |v|^4\rangle \\
\Psi_2 &= \langle \overline{h}_2 |v|^4\rangle\langle h_1 |v|^2\rangle - \langle \overline{h}_2 |v|^2\rangle\langle h_1 |v|^4\rangle \\
C &= \frac{1}{2}\frac{\langle h_1 |v|^4\rangle}{\langle h_1 |v|^2\rangle}
\end{aligned} \tag{2.86}$$

Moreover, the term $\mathrm{div}\underline{\lambda}^{(2)}$ in (2.71) is determined, by using also (2.66), in terms of the $\lambda^{(1)}$'s and their derivatives as

$$\begin{aligned}
-(C_0 + C_2)\,\mathrm{div}\underline{\lambda}^{(2)} &= \partial_t(\rho^{(1)} + \mathcal{E}^{(1)}) - \sum_{\alpha=1}^{3}\sum_{\delta=0,4}\left[\bar{D}^{\delta,0}_{\alpha,\alpha}\partial^2_\alpha\rho^{(1)} + \bar{D}^{\delta,4}_{\alpha,\alpha}\partial^2_\alpha\mathcal{E}^{(1)})\right] \\
&\quad + \underline{\lambda}^{(1)}\cdot\nabla\Big[(A_0 + A_2)\lambda^{(1)}_0 + (B_0 + B_2)\lambda^{(1)}_4\Big]
\end{aligned} \tag{2.87}$$

The Navier-Stokes equations (2.75) and (2.83) differ in many aspects from the usual ones. First of all the coefficient K in (2.75) is different from 1. This is a minor point since it can be eliminated by scaling the time, but it has to be positive. For example, we can choose $r > 0$ in (2.59) so that $f_0(|v|) > \frac{1}{2}$ for any $v \in \mathcal{V}$ which implies $1 - 2f_0 < 0$ and if $f_0 \in (1/2, z)$, $z = \frac{3+\sqrt{3}}{6}$ so that $1 - 6f_0(1 - f_0) < 0$, we get for $\overline{h}_2$ in (2.63)

$$\overline{h}_2 = \frac{1}{2} f_0(1 - f_0)(1 - 2f_0)(1 - 6f_0(1 - f_0)) > 0$$

One possible choice is $r \in [1/2, \bar{r}], \bar{r} = \log\frac{z}{1-z}$, $\theta = 0$, $\varpi^2 = 5\sqrt{3}$.

Moreover, the viscosity is determined by two different constants, namely is anisotropic as in all the known cellular automata models. In particular, the model in [EMY3] has the same lack of isotropy.

In the equation for the temperature the constant H is different from 1 and we cannot use time scaling again to get rid of it. Moreover, its positivity is not evident and has to be checked.

Proposition 2.2 (Positivity properties of the coefficients) *The inequality $\Phi_2 \geq 0$ holds. If $f_0 \geq \frac{1}{2}$, then $\Phi_1 \leq 0$ and $\Psi_2 \leq 0$. If moreover $f_0 \in [\frac{1}{2}, \frac{3+\sqrt{3}}{6}]$ then $\Psi_1 \leq 0$.*

Proof. We start by Φ_1

$$\Phi_1 = \langle h_1|v|^4\rangle\langle h_0\rangle - \langle h_1|v|^2\rangle\langle h_0|v|^2\rangle$$

$$= \sum_{v,v'} h_0(v)h_0(v')b(v')|v'|^2[|v|^2 - |v'|^2]$$
$$= \frac{1}{2}\sum_{v,v'} h_0(v)h_0(v')[b(v')|v'|^2 - b(v)|v|^2][|v|^2 - |v'|^2]$$

where $b(v) = 2f_0(v) - 1 > 0$ (for $f_0 > \frac{1}{2}$). Therefore $|v|^2 b(v)$ is an increasing function of v and we get that $\Phi_1 \leq 0$.

We now examine Φ_2.

$$\Phi_2 = \langle h_0|v|^4\rangle\langle h_1 v^2\rangle - \langle h_1|v|^4\rangle\langle h_0|v|^2\rangle$$
$$= \frac{1}{2}\sum_{v,v'} h_0(v)h_0(v')[b(v) - b(v')]|v|^4|v'|^2$$
$$= \frac{1}{2}\sum_{v,v'} h_0(v)h_0(v')[b(v) - b(v')][|v|^2 - |v'|^2]|v'|^2|v|^2$$

Since $b(v)$ is increasing in v, Φ_2 is positive.

Now we study Ψ_1 and Ψ_2.

$$\Psi_1 = \langle \overline{h}_2|v|^6\rangle\langle h_1|v|^2\rangle - \langle \overline{h}_2|v|^4\rangle\langle h_1|v|^4\rangle$$
$$= \sum_{v,v'} h_1(v)h_1(v')g(v)|v|^4|v'|^2[|v|^2 - |v'|^2]$$
$$= \frac{1}{2}\sum_{v,v'} h_1(v)h_1(v')|v'|^2|v|^2[|v|^2 g(v) - |v'|^2 g(v')][|v|^2 - |v'|^2]$$

where $g(v) = 1 - 6f_0(1 - f_0)$. We have that $g(v) \leq 0$ if $f_0 \in [\frac{1}{2}, \frac{3+\sqrt{3}}{6}]$, so that by choosing suitably r we get $g(v) \leq 0$. On the other hand $g(v)$ is increasing since

$$d(|v|) - d(|v'|) = [f_0(|v|) - f_0(|v'|)][f_0(|v|) + f_0(|v'|) - 1] > 0$$

for $|v| > |v'|$ and $f_0 > \frac{1}{2}$. Hence $\Psi_1 \leq 0$.

The constant Ψ_2 is dealed with in an analogous way.

$$\Psi_2 = \langle \overline{h}_2|v|^4\rangle\langle h_1|v|^2\rangle - \langle \overline{h}_2|v|^2\rangle\langle h_1|v|^4\rangle$$
$$= \frac{1}{2}\sum_{v,v'} h_1(v)h_1(v')[g(|v|) - g(|v'|)][|v|^2 - |v'|^2]$$

Hence $\Psi_2 \geq 0$. $\square$

Finally we remark that, since $f_0(v)$ is independent of v at $\theta = 0$, Φ_2 and Ψ_2 vanish at $\theta = 0$. Moreover it is easy to see that $\Phi_1 < 0$ and $\Psi_1 < 0$ at $\theta = 0$ (and $r \in (0, \bar{r})$) , so that we have also $H > 0$. By continuity arguments H remains positive for small θ. We recall here that, with this choice for r and θ, $\overline{h}_2 > 0$ so that also the coefficient K in the momentum equation is positive.

The last remark is about the conductivity. We observe that for $\beta = 0$ we have that

$$C = \frac{(\bar{\Theta}^{-1})^{4,4}}{(\bar{\Theta}^{-1})^{0,4}}, \qquad \Phi_2 = 0$$

where $\bar{\Theta}$ is the 4×4 compressibility matrix defined in (4.9). Hence we can rewrite $\mathcal{K}$ as

$$\mathcal{K}_\alpha = \chi + (\bar{\Theta}^{-1})^{0,4}\Big[\frac{\Phi_1}{\Phi_2 + C\Phi_1}(\Theta^{-1}D)^{0,0}_{\alpha,\alpha} - \frac{\Phi_2}{\Phi_2 + C\Phi_1}(\Theta^{-1}D)^{0,4}_{\alpha,\alpha}\Big]$$

where again Θ is defined in (4.9).

We prove in Section 4 that $(\Theta^{-1}D)$ is non negative as a quadratic form and symmetric. This implies that the diagonal elements are non negative. Therefore, for $\beta = 0$ we have $\mathcal{K} - \chi \geq 0$. Were we able to prove the strict positivity of $(\Theta^{-1}D)$, then for $\beta \neq 0$ but small by continuity in β of the diffusion matrix [LOY2] the difference $\mathcal{K} - \chi$ would remain positive.

We write for future use the equations for the chemical potentials $\lambda^{(1)}_\alpha$, $\alpha = 0, \cdots, 4$ associated to the Navier-Stokes equations (2.75) and (2.83): $\underline{\lambda}^{(1)} = (\lambda^{(1)}_1, \ldots, \lambda^{(1)}_3)$, $\lambda^{(1)}_4$ and $p' = p\frac{3}{\langle h_0|v|^2\rangle}$ are solutions of

$$\begin{aligned}
&\mathrm{div}\underline{\lambda}^{(1)} = 0 \\
&\partial_t \lambda^{(1)}_\beta + \partial_\beta p' + K'\,\underline{\lambda}^{(1)} \cdot \nabla \lambda^{(1)}_\beta = \sum_{\alpha=1}^{3} D_\alpha \partial^2_\alpha \lambda^{(1)}_\beta, \quad \beta = 1, \ldots, 3 \\
&\frac{\partial}{\partial t}\lambda^1_4 + H'\underline{\lambda}^{(1)} \cdot \nabla \lambda^{(1)}_4 = \sum_{\alpha=1}^{3} \mathcal{K}_\alpha \partial^2_\alpha \lambda^{(1)}_4
\end{aligned} \tag{2.88}$$

where

$$K' = 6\frac{\langle \overline{h}_2 v_1^2 v_2^2\rangle}{\langle h_0|v|^2\rangle} \quad \text{and} \quad H' = H\frac{\langle |v|^2 h_0\rangle}{3} \tag{2.89}$$

$\mathcal{K}_\alpha$ and H are defined in (2.84) and (2.85).

Moreover $\lambda^{(1)}_0$ is determined by the Boussinesq condition (2.74), $\underline{\lambda}^{(2)}$ is chosen such that (2.87) is valid and finally, $\lambda^{(2)}_0$ and $\lambda^{(2)}_4$ are taken such that the pressure p defined by (2.70) satisfies $3p = \langle h_0|v|^2\rangle p'$.

We can now state the main theorem of the paper

Theorem 2.2 *Consider the velocity set $\mathcal{V}$ in Sect. 2.2 and assume that it satisfies (2.17). Let $u^{(1)}(z,t)$, $T^{(1)}(z,t)$, $t \in [0,t_0]$, $t_0 > 0$ be any smooth classical solution of the equations (2.75) and (2.83), with D_α's and $\mathcal{K}_\alpha$'s given by (4.11) and (2.84). We start the process $\eta_t(x,v)$ with generator $\varepsilon^{-2}\mathcal{L}$ from the measure $\mu_{L,n}$ defined in (2.28), with chemical potentials $n_\alpha(x)$ of the form (2.55), satisfying (2.73) and (2.74). We define the mass, momentum, energy and temperature empirical fields as*

$$
\begin{aligned}
\nu_0^\varepsilon(z,t) &= \varepsilon^2 \sum_{x\in\Lambda_L} \delta(z-\varepsilon x)\Big(I_0(\eta_x(t)) - \langle f_0\rangle\Big)\\
\nu_\beta^\varepsilon(z,t) &= \varepsilon^2 \sum_{x\in\Lambda_L} \delta(z-\varepsilon x) I_\beta(\eta_x(t)), \quad \beta = 1,\dots,3\\
\nu_4^\varepsilon(z,t) &= \varepsilon^2 \sum_{x\in\Lambda_L} \delta(z-\varepsilon x)\Big(I_4(\eta_x(t)) - \langle f_0|v|^2\rangle\Big)\\
(\nu')_4^\varepsilon(z,t) &= \varepsilon^2 \sum_{x\in\Lambda_L} \delta(z-\varepsilon x)\Big(I_4'(\eta_x(t)) - A_T\Big)
\end{aligned}
\tag{2.90}
$$

Then $(\nu_1(z,t),\dots,\nu_3(z,t))$ and $(\nu')_4(z,t)$ converge, for $t \le t_0$, weakly in probability to $u^{(1)}(z,t)dz$ and $T^{(1)}dz$. Moreover, $\nu_0(z,t)$ and $\nu_4^\varepsilon(z,t)$ converge to $\rho^{(1)}(z,t)dz$ and $\mathcal{E}^{(1)}(z,t)dz$, related to $T^{(1)}$ by (2.78).

3. Entropy

Let f_t be the density with respect to the reference measure $\mu_{L,r,\theta}$ of the process $\eta_t(x,v)$ with generator $\varepsilon^{-2}\mathcal{L}$ initially distributed with $\mu_{L,n}$ where the chemical potentials $n = (n_0,\dots,n_{d+1})$ satisfy

$$
n_\beta(x) = \lambda_\beta^{(0)} + \varepsilon\lambda_\beta^{(1)}(\varepsilon x) + \varepsilon^2\lambda_\beta^{(2)}(\varepsilon x) \tag{3.1}
$$

with $\lambda_\beta^{(0)} = r\delta_{\beta,0} + \theta\delta_{\beta,4}$. The density with respect to $\mu_{L,r,\theta}$ describing the local equilibrium up to the second order in ε is

$$
\Psi_t = Z_{L,n}^{-1} \exp\Big\{\varepsilon \sum_{x\in\Lambda_L}\sum_{\beta=0}^{4}\Big(\lambda_\beta^{(1)}(\varepsilon x,t) + \varepsilon\lambda_\beta^{(2)}(\varepsilon x,t)\Big) I_\beta(\eta_x)\Big\} \tag{3.2}
$$

This density is not suitable to describe the behavior of the system on the diffusive scale because the local equilibrium is not conserved in time. We need to consider a modified density including a suitable correction to the local equilibrium.

Given local functions F_α^β , we define

$$\Phi(\eta) = -\sum_{x\in\Lambda_L}\sum_{\alpha=1}^{3}\sum_{\beta=0}^{4}\partial_\alpha\lambda_\beta(\varepsilon x,t)(\hat\omega * \tau_y F_\alpha^\beta)_x \tag{3.3}$$

where $\hat\omega$ is some approximation of identity which will be defined later (see (3.16)).

We now modify the Ψ_t with second order terms given by (3.3):

$$\widetilde{\Psi}_t = \tilde{Z}_{L,n}^{-1}\exp\Big\{\varepsilon\sum_{x\in\Lambda_L}\sum_{\beta=0}^{4}\Big((\lambda_\beta^{(1)} * \hat\omega)(\varepsilon x,t) + \varepsilon(\lambda_\beta^{(2)} * \hat\omega)(\varepsilon x,t)\Big)I_\beta(\eta_x) + \varepsilon^2\Phi(\eta)\Big\} \tag{3.4}$$

where $\tilde{Z}_{L,n}$ is the normalization constant.

Assumptions on the chemical potentials

We choose the chemical potentials $\lambda_\beta^{(1)}(x,t), \beta = 1,\dots,4$ in (3.1), with initial values $\lambda_\beta^{(1)}(x)$, as smooth solutions of the equations (2.88) and $\lambda_0^{(1)}(x,t)$ determined by the Boussinesq condition (2.74). Moreover we require that $\lambda_\beta^{(2)}(x)$ satisfy (2.71) with $p = \frac{\langle h_0|v|^2\rangle}{3}p'$ and (2.87).

We will show that the non-equilibrium density f_t is near to the density $\widetilde{\Psi}_t$ in the sense of the relative entropy. If f and g are two densities with respect to $\mu_{L,r,\theta}$, the specific relative entropy of f and g is

$$s(f|g) = \varepsilon^d E\Big[f\log\frac{f}{g}\Big] \tag{3.5}$$

where $E[\cdot]$ denotes the expectation with respect to $\mu_{L,r,\theta}$. When $g = 1$, we simply denote $s(f|1)$ by $s(f)$.

The main result of this section is the following

Theorem 3.1 *Let $\widetilde{\Psi}_t$ be defined by (3.4) and the assumptions on the chemical potentials above fulfilled. Then there are functions F_j^i such that the specific relative entropy $s(f_t|\tilde\Psi_t)$ satisfies*

$$\lim_{\varepsilon\to 0}\varepsilon^{-2}s\big(f_t|\tilde\Psi_t\big) = 0 \tag{3.6}$$

This result, together with the entropy inequality, is enough to conclude the proof of Theorem 2.2. We recall that the entropy inequality states that for any random variable X and for any $\gamma > 0$

$$E^f[X] \leq \frac{\varepsilon^{-d}}{\gamma} s(f|g) + \gamma \log E^g[\exp(\gamma X)] \tag{3.7}$$

An a priori bound on entropy

Since the reference measure $\mu_{L,r,\theta}$ is invariant under the dynamics, the entropy $s(f_t)$ is a non increasing function of t. Moreover, a simple computation involving Taylor expansions shows that it is of order ε^2 at time 0, so that

$$\forall\, t \geq 0, \quad s(f_t) \leq c\varepsilon^2 \tag{3.8}$$

where c is a positive constant. More precisely, the time derivative of the entropy is bounded above by the Dirichlet form:

$$\frac{d}{dt} s(f_t) \leq -c\varepsilon D_{\Lambda_L}(\sqrt{f_t}) \tag{3.9}$$

where D_{Λ_L} denotes the Dirichlet form of the process. It can be written as $D_{\Lambda_L}(G) = D^{ex}_{\Lambda_L}(G) + D^{c}_{\Lambda_L}(G)$ where $D^{ex}_{\Lambda_L}$ is the Dirichlet form related to the jump part of the generator and $D^{c}_{\Lambda_L}$ is related to the collision part. They are given by

$$\begin{aligned} D^{ex}_{\Lambda_L}(G) &= \sum_{x \in \Lambda_L} \sum_{v \in \mathcal{V}} \sum_{\alpha=1}^{d} \chi \int \left[(\nabla^v_{x,\alpha} G)(\eta) \right]^2 d\mu_{L,r,\theta} \\ D^{c}_{\Lambda_L}(G) &= \sum_{x \in \Lambda_L} \sum_{q \in \mathcal{Q}} \int \left[(Q^q_x G)(\eta) \right]^2 d\mu_{L,r,\theta} \end{aligned} \tag{3.10}$$

where

$$(\nabla^v_{x,\alpha} G)(\eta) = G(\eta^{x,x+e_\alpha,v}) - G(\eta) \tag{3.11}$$

Theorem 3.1 will be proved by estimating the time derivative of the relative entropy. An important point in the proof of Theorem 3.1 is finding the functions F^i_j.

The strategy is to decompose the currents w^i_j into the sum of a gradient term and a term of the form $\mathcal{L}g$, i.e.

$$w^i_j - \sum_{k,\ell} D^{i,\ell}_{j,k} \nabla_k I_\ell - \mathcal{L} F^i_j = 0$$

The coefficients $D^{i,\ell}_{j,k}$ will be identified as the transport coefficients. This equation will be understood as an equation in a suitable Hilbert space and $D^{i,\ell}_{j,k}$ has a geometric interpretation as "the component of the currents in the gradient directions".

The transport coefficients will be discussed in Sect. 4. Here we state the main theorems we need in the proof of Theorem 3.1. These results have been proved in [EMY3]. Their proof is extended to the present model in a straightforward way and will not be given here.

Non gradient results.

We denote by $\overline{I}^+_\ell = (\overline{I}_{0,\ell}, \dots, \overline{I}_{4,\ell})$ the empirical averages of the conserved quantities over the block Λ_ℓ of length ℓ:

$$\overline{I}_{\beta,\ell} = \frac{1}{(2\ell+1)^3} \sum_{|y|\le \ell} I_\beta(\eta_y), \quad \beta = 0, \dots, 4 \tag{3.12}$$

The measure $\mu_{\ell,m}$, $m \in \mathbb{R}^5$ is defined as the canonical Gibbs state of $(2\ell+1)^3$ sites with parameters such that $\overline{I}^+_\ell = m$. It is the uniform probability on the set $\Omega_{\ell,m}$ of configurations on the block Λ_ℓ such that $\overline{I}^+_\ell = m$. The expectation of a local function G with respect to $\mu_{\ell,\overline{I}^+_\ell}$ is denoted by $\alpha_\ell(G)$, in other words, $\alpha_\ell(G)$ is the conditional expectation given the averages $\overline{I}^+_\ell$

$$\alpha_\ell(G) = E^{\mu_{\ell,r}}[G|\overline{I}^+_\ell] \tag{3.13}$$

We call $\mathcal{L}_{s,\ell}$ the symmetric part of the generator $\mathcal{L}$ restricted to the block Λ_ℓ, that is only jumps over bounds inside Λ_ℓ are allowed. From the Proposition 2.1, the measures $\mu_{\ell,m}$ are the only extremal invariant measures for $\mathcal{L}_{s,\ell}$. Therefore, we can define $\mathcal{L}^{-1}_{s,\ell}G$ for any function G such that $\alpha_\ell(G) = 0$. Given any local function G on Ω_ℓ, the finite volume "variance" $V_\ell(G,m)$ is

$$V_\ell(G,m) = \frac{1}{(2\ell_1+1)^3} \langle \Big[\sum_{|x|\le\ell_1} (\tau_x G - \alpha_\ell(G)\Big] (-\mathcal{L}_{s,\ell})^{-1} \Big[\sum_{|x|\le\ell_1} (\tau_x G - \alpha_\ell(G)\Big] \rangle_{\mu_{\ell,m}} \tag{3.14}$$

where $\ell_1 = \ell - \ell^{1/9}$. The "variance" $V(G,m)$ of G is given by

$$V(G,m) = \limsup_{\ell\to\infty} V_\ell(G,m) \tag{3.15}$$

With an abuse of notation, we denote $V_\ell(G,m)$ by $V_\ell(G,r,\theta)$ when m is associated to the chemical potential $m = (r,0,0,0,\theta)$.

For an integer ℓ, we let $\overline{\ell} = \ell^5$ and $k = \ell\varepsilon^{-2/3}$. We assume that the box Λ_k is divided into cubes $\Lambda_{\ell,\sigma}$ of size $(2\overline{\ell}+1)$ with centers $\sigma \in (2\overline{\ell}+1)\mathbb{Z}^3$, $|\sigma| \le k$. We consider the sub-cubes $\Lambda_{\overline{\ell}_1,\sigma}$ of size $\overline{\ell}_1 = \overline{\ell} - \ell^{1/9}$ and their union is denoted by $\hat{\Lambda}_k = \cup_{|\sigma|\le k}\Lambda_{\ell_1,\sigma}$. The functions ω and $\hat{\omega}$ are the normalized indicator functions

$$\omega(x) = (2k+1)^{-3}\, \mathbb{1}\{x \in \Lambda_k\}, \quad \hat{\omega}(x) = |\hat{\Lambda}_k|^{-1}\, \mathbb{1}\{x \in \hat{\Lambda}_k\} \tag{3.16}$$

The following two theorems have been proven in [EMY3] (Sect. 6 and Sect. 4)

Theorem 3.2 *Let $d\mu_{L,n} = \Psi d\mu_{L,r,\theta}$ be a local Gibbs state with smooth chemical potentials $n_\beta(x) = \lambda_\beta(\varepsilon x)$ ($\beta = 0, \dots, 4$) and suppose G is a local function, J is a smooth function. Then for any density f with respect to $\mu_{L,r,\theta}$ and for any $\gamma > 0$*

$$\limsup_{\ell\to\infty}\limsup_{\varepsilon\to 0}\Big\{\varepsilon\int\sum_{x\in\Lambda_L} J(\varepsilon x)\Big((\hat{\omega} * \tau_y G)_x - E^{\mu_{L,\overline{r}_k^+}}[G]\Big) f\, d\mu_{L,r,\theta}$$
$$-\frac{c}{\gamma}V(G,r,\theta)\int\sum_{x\in\Lambda_L} J^2(z)\,dz - \gamma\varepsilon^{-1}D_{\Lambda_L}(\sqrt{f}) - \delta^{-1}\gamma\varepsilon^{-2}s(f|\Psi)\Big\} \le 0 \tag{3.17}$$

where $*$ is the convolution product on $\mathbb{Z}^3$, $k = \ell\varepsilon^{-2/3}$, c and δ are positive constants.

Define

$$H^\beta_\alpha = H^\beta_{0,\alpha}, \qquad H^\beta_{x,\alpha} = w^{(a),\beta}_{x,\alpha} - \sum_{\gamma=1}^{3}\sum_{\delta=0}^{4}\bar{D}^{\beta,\delta}_{\alpha,\gamma}\nabla_\gamma I_\delta(\eta_x) - \mathcal{L}^*\tau_x F^\beta_\alpha \tag{3.18}$$

Theorem 3.3 *There exists a positive diffusion matrix $\bar{D} = (\bar{D}^{\beta,\delta}_{\alpha,\gamma})$ such that*

$$\inf_{(F^\beta_\alpha)}\sum_{\alpha=1}^{3}\sum_{\beta=0}^{4} V(H^\beta_\alpha, r, \theta) = 0 \tag{3.19}$$

Remark. Consider the density $\Psi^\star_t$

$$\Psi^\star_t := Z^{-1}_{L,n}\exp\Big\{\varepsilon\sum_{x\in\Lambda_L}\sum_{\beta=0}^{4}\lambda^{(1)}_\beta(\varepsilon x, t)\Big\}$$

that differs from $\tilde{\Psi}_t$ by terms of order ε^2. Then Theorem 3.1 implies also

$$\lim_{\varepsilon\to 0}\varepsilon^{-2}s\big(f_t|\Psi^\star_t\big) = 0 \tag{3.20}$$

The proof of Theorem 3.1 is based on the following estimate

Theorem 3.4 *For any $\gamma > 0$,*

$$
\begin{aligned}
&\limsup_{\ell\to\infty}\limsup_{\varepsilon\to 0}\varepsilon^{-2}s(f_T,\widetilde{\Psi}_T) \le \\
&\qquad c\liminf_{\ell\to\infty}\liminf_{\varepsilon\to 0}\int_0^T \varepsilon^{-2}s(f_t,\widetilde{\Psi}_t)\,dt + \frac{CT}{\gamma}\sum_{\alpha=1}^{3}\sum_{\beta=0}^{4}V(H_\alpha^\beta,r,\theta)+\gamma c
\end{aligned}
\tag{3.21}
$$

where c is a positive constant.

In fact, Theorems 3.3, 3.4 and the Gronwall lemma imply Theorem 3.1.

Proof of Theorem 3.4. From Lemma 3.9 of [EMY3], the time derivative of the relative entropy $s(f_t|\widetilde{\Psi}_t)$ satisfies the following bound. There exists a constant c_t such that

$$
\lim_{\varepsilon\to 0}\varepsilon^{-2}\Big[\frac{d}{dt}s(f_t|\widetilde{\Psi}_t) - \varepsilon^3\int f_t\Big(\varepsilon^{-2}\mathcal{L}^* - \frac{\partial}{\partial t}\Big)\log\widetilde{\Psi}_t\,d\mu_{L,r,\theta}\Big] \le c_t \tag{3.22}
$$

Therefore, we have to estimate

$$
\mathcal{I} \equiv \varepsilon\int f_t\Big(\varepsilon^{-2}\mathcal{L}^* - \frac{\partial}{\partial t}\Big)\log\widetilde{\Psi}_t\,d\mu_{L,r,\theta} \tag{3.23}
$$

We start with the term $\varepsilon^{-1}\mathcal{L}^*\log\widetilde{\Psi}_t$.

$$
\begin{aligned}
\varepsilon^{-1}\mathcal{L}^*\log\widetilde{\Psi}_t = &\sum_x\sum_{\beta=0}^{4}\sum_{\alpha=1}^{3}(\lambda_\beta^{(1)}*\hat{\omega})(\varepsilon x,t)\nabla_\alpha^- w_{x,\alpha}^{*,\beta} \\
&+\varepsilon\sum_x\sum_{\beta=0}^{4}\sum_{\alpha=1}^{3}(\lambda_\beta^{(2)}*\hat{\omega})(\varepsilon x,t)\nabla_\alpha^- w_{x,\alpha}^{*,\beta} \\
&+\varepsilon\Phi(\eta)
\end{aligned}
\tag{3.24}
$$

which is then the sum of three terms $A_1 + A_2 + A_3$. For the second term A_2, we divide the currents into their symmetric and antisymmetric parts (see (2.42)). Then, performing summations by parts, we get $A_2 = B_1 + B_2$ with

$$
\begin{aligned}
B_1 =&\chi\varepsilon\sum_x\sum_{\beta=0}^{4}\sum_{\alpha=1}^{3}\nabla_\alpha^-\nabla_\alpha\lambda_\beta^{(2)}(\varepsilon x,t)(I_\beta(\eta_y)*\hat{\omega})_x \\
B_2 =&\varepsilon\sum_x\sum_{\beta=0}^{4}\sum_{\alpha=1}^{3}\nabla_\alpha\lambda_\beta^{(2)}(\varepsilon x,t)(w_{y,\alpha}^{(a),\beta}*\hat{\omega})_x
\end{aligned}
\tag{3.25}
$$

Using Taylor expansions, one can replace $\nabla_\alpha^- \nabla_\alpha \lambda_\beta^{(2)}(\varepsilon x, t)$ by $\varepsilon^2 \partial_\alpha^2 \lambda_\beta^{(2)}(\varepsilon x, t)$ with a negligible error. Moreover the measure $f_t \mu_{L,r,\theta}$ is very close to the equilibrium $\mu_{L,r,\theta}$, and since $E[B_1] = 0$, $E^{f_t}[B_1]$ is of order ε (see (iii), Lemma 3.8 in [EMY2]).

In order to use the non gradient estimates stated above, we introduce, for $\alpha = 1, \dots, 3$ and $\beta = 0, \dots, 4$, the quantities

$$g_{x,\alpha}^\beta = (w_{y,\alpha}^{(a),\beta} * \hat{\omega})_x + \xi_\alpha^\beta(\overline{I}_k^+(x)) \tag{3.26}$$

where, letting $Y = \overline{I}_k^+(x)$, we have $\xi_\alpha^\beta(Y) = -E^{\mu_{L,Y}}\left[(w_{y,\alpha}^{(a),\beta} * \hat{\omega})_0\right] = -E^{\mu_{L,Y}}\left[w_{0,\alpha}^{(a),\beta}\right]$, that is

$$\begin{aligned}
\xi_\alpha^0(Y) &= \sum_{v \in \mathcal{V}} (v \cdot e_\alpha)\left[f^2(v, n^{(Y)}) - f(v, n^{(Y)})\right] \\
\xi_\alpha^\beta(Y) &= \sum_{v \in \mathcal{V}} (v \cdot e_\alpha)(v \cdot e_\beta)\left[f^2(v, n^{(Y)}) - f(v, n^{(Y)})\right], \quad \text{for } \beta = 1, \dots, 3 \\
\xi_\alpha^4(Y) &= \sum_{v \in \mathcal{V}} (v \cdot e_\alpha)\frac{|v|^2}{2}\left[f^2(v, n^{(Y)}) - f(v, n^{(Y)})\right]
\end{aligned} \tag{3.27}$$

Here $f(v, n) = E^{\mu_{L,n}}[\eta(0, v)]$ and the parameters $n^{(Y)}$ are chosen such that $E^{\mu_{L,n^{(Y)}}}[I_\beta(\eta_0)] = Y_\beta$. Using Taylor expansions, we can rewrite B_2 as $C_1 + C_2$ where

$$\begin{aligned}
C_1 &= \varepsilon^2 \sum_x \sum_{\beta=0}^4 \sum_{\alpha=1}^3 \partial_\alpha \lambda_\beta^{(2)}(\varepsilon x, t) g_{x,\alpha}^\beta \\
C_2 &= \varepsilon \sum_x \sum_{\beta=0}^4 \sum_{\alpha=1}^3 \nabla_\alpha \lambda_\beta^{(2)}(\varepsilon x, t) \xi_\alpha^\beta(Y)
\end{aligned} \tag{3.28}$$

We now turn to the term A_1. Dividing again the currents into their symmetric and antisymmetric parts, we obtain

$$\begin{aligned}
A_1 = \chi \varepsilon^2 &\sum_x \sum_{\beta=0}^4 \sum_{\alpha=1}^3 \partial_\alpha^2 \lambda_\beta^{(1)}(\varepsilon x, t)(I_\beta(\eta_y) * \hat{\omega})_x \\
&+ \sum_x \sum_{\beta=0}^4 \sum_{\alpha=1}^3 \nabla_\alpha \lambda_\beta^{(1)}(\varepsilon x, t) g_{x,\alpha}^\beta \\
&+ \sum_x \sum_{\beta=0}^4 \sum_{\alpha=1}^3 \nabla_\alpha \lambda_\beta^{(1)}(\varepsilon x, t) \xi_\alpha^\beta(\overline{I}_k^+(x)) + o(1)
\end{aligned} \tag{3.29}$$

From the coefficients $D_{\alpha,\gamma}^{\beta,\delta}$, $\beta, \delta = 0, \dots, 4$ and $\alpha, \gamma = 1, \dots, 3$ of the diffusion matrix in Theorem 3.3, we define

$$D_{\alpha,\gamma}^{\beta,\delta} = \chi \delta_{\alpha,\gamma} \delta_{\beta,\delta} + \bar{D}_{\alpha,\gamma}^{\beta,\delta} \tag{3.30}$$

and we introduce these terms in A_1

$$A_1 = \varepsilon^2 \sum_x \sum_{\beta,\delta=0}^{4} \sum_{\alpha,\gamma=1}^{3} D^{\beta,\delta}_{\alpha,\gamma} (\partial_\alpha \partial_\gamma \lambda^{(1)}_\beta)(\varepsilon x, t)(I_\delta(\eta_y) * \hat{\omega})_x$$

$$+ \varepsilon \sum_x \sum_{\beta=0}^{4} \sum_{\alpha=1}^{3} \partial_\alpha \lambda^{(1)}_\beta(\varepsilon x, t) \Big[g^\beta_{x,\alpha} - \sum_{\delta=0}^{4} \sum_{\gamma=1}^{3} \bar{D}^{\beta,\delta}_{\alpha,\gamma} \big(\nabla_\gamma I_\delta(\eta_y) * \hat{\omega}\big)_x \Big] \tag{3.31}$$

$$- \sum_x \sum_{\beta=0}^{4} \sum_{\alpha=1}^{3} \nabla_\alpha \lambda^{(1)}_\beta(\varepsilon x, t) \xi^\beta_\alpha(\overline{I}^+_k(x)) + o(1)$$

Putting together equations (3.25), (3.28) and (3.31), we have

$$A_1 + A_2 + A_3 = C_1 + C_2 + C_3 + C_4 + o(1) \tag{3.32}$$

where

$$C_3 = \varepsilon \sum_x \sum_{\beta=0}^{4} \sum_{\alpha=1}^{3} \partial_\alpha \lambda^{(1)}_\beta(\varepsilon x, t) \Big[g^\beta_{x,\alpha} - \sum_{\delta=0}^{4} \bar{D}^{\beta,\delta}_{\alpha,\gamma} \big(\nabla_\gamma I_\delta(\eta_y) * \hat{\omega}\big)_x - \big(\mathcal{L}^*(\tau_y F^\beta_\alpha) * \hat{\omega}\big)_x \Big]$$

$$C_4 = \varepsilon^2 \sum_x \sum_{\beta=0}^{4} \sum_{\alpha=1}^{3} \Big[\sum_{\delta=0}^{4} \sum_{\gamma=1}^{3} D^{\beta,\delta}_{\alpha,\gamma} (\partial^2_{\alpha,\gamma} \lambda^{(1)}_\beta)(\varepsilon x, t)(I_\delta(\eta_y) * \hat{\omega})_x$$

$$- \varepsilon^{-2} \nabla_\alpha \lambda^{(1)}_\beta(\varepsilon x, t) \xi^\beta_\alpha(\overline{I}^+_k(x)) \Big] \tag{3.33}$$

We now compute the second term in the definition of $\mathcal{I}$ (see (3.23))

$$- \varepsilon \int f_t \frac{\partial}{\partial t} \log \widetilde{\Psi}_t \, d\mu_{L,r,\theta}$$

$$= - \int f_t \Big[\varepsilon^2 \sum_x \sum_{\beta=0}^{4} \frac{\partial}{\partial t} \lambda^{(1)}_\beta(\varepsilon x, t) \overline{I}_{\beta,k}(x) + \varepsilon^3 \sum_x \sum_{\beta=0}^{4} \frac{\partial}{\partial t} \lambda^{(2)}_\beta(\varepsilon x, t) \overline{I}_{\beta,k}(x) \Big] + \text{const} \tag{3.34}$$

With the same argument as the one we used to bound B_1, the second term in the previous formula is negligible up to a constant. Therefore, $\mathcal{I}$ can be written as

$$\mathcal{I} = \int \big[C_1 + C_3 + C_5 \big] f_t \, d\mu_{L,r,\theta} \tag{3.35}$$

where

$$C_5 = \varepsilon \sum_x \Gamma_x\big(\overline{I}^+_k(x)\big) \tag{3.36}$$

and

$$\Gamma_x(Y) = \varepsilon \sum_{\beta=0}^{4} \Bigg\{ -\frac{\partial}{\partial t}\lambda_\beta^{(1)}(\varepsilon x, t) Y_\beta + \sum_{\delta=0}^{4}\sum_{\gamma=1}^{3} D_{\alpha,\gamma}^{\beta;\delta}(\partial^2_{\alpha,\gamma}\lambda_\beta^{(1)})(\varepsilon x, t) Y_\delta$$
$$-\varepsilon^{-2}\sum_{\alpha=1}^{3}\nabla_\alpha \lambda_\beta^{(1)}(\varepsilon x, t)\xi_\alpha^\beta(Y) - \varepsilon^{-1}\sum_{\alpha=1}^{3}\nabla_\alpha\lambda_\beta^{(2)}(\varepsilon x, t)\xi_\alpha^\beta(Y)\Bigg\} \tag{3.37}$$

Applying Theorem 3.2 to C_1 and C_3, we obtain the following bound for $\mathcal{I}$.

$$\mathcal{I} \leq E^{f_t}[C_5] + \frac{c}{\gamma}\sum_{\alpha=1}^{3}\sum_{\beta=0}^{4} V(H_\alpha^\beta, r, \theta) + \delta^{-1}\varepsilon^{-2} s(f_t, \tilde{\Psi}_t) + \gamma\varepsilon^{-1} D_{\Lambda_L}(\sqrt{f_t}) + \text{const} + o(1) \tag{3.38}$$

where $\lim_{\ell\to\infty}\lim_{\varepsilon\to 0} o(1) = 0$. Using the same arguments as in [EMY3] (see formulas above (3.32)), the constant "const" satisfies

$$\varepsilon \sum_x \Gamma_x(m_x) + \text{const} = o(1) \tag{3.39}$$

with $m_x = (m_x^0, \ldots, m_x^4)$ and $m_x^\beta = E^{\Psi_t}[I_\beta(\eta_x)]$. Therefore, the entropy inequality gives

$$\mathcal{I} \leq \frac{c}{\gamma}\sum_{\alpha=1}^{3}\sum_{\beta=0}^{4} V(H_\alpha^\beta, r, \theta) + (\delta^{-1}+1)\varepsilon^{-2} s(f_t, \tilde{\Psi}_t) + \gamma\varepsilon^{-1} D_{\Lambda_L}(\sqrt{f_t})$$
$$+ \varepsilon \log E^{\Psi_t}\Big[\exp\Big\{\sum_x \big(\Gamma_x(\overline{I}_k^+(x)) - \Gamma_x(m_x)\big)\Big\}\Big] + o(1) \tag{3.40}$$

It results from the a priori estimates on entropy (3.8) and (3.9) that, when integrated in time, the term involving the Dirichlet form in the previous expression is bounded above by $c\gamma$. Therefore, keeping in mind Theorem 3.3, Theorem 3.4 will follow from the Gronwall lemma if we are able to prove that the first term of the second line of (3.40) vanishes with ε. From the large deviation lemma (stated as Lemma 3.7 in [EMY3]), it suffices to show that Γ_x satisfies the conditions

$$\frac{\partial \Gamma_x}{\partial Y_\beta}\Big|_{Y_\beta = m_x^\beta} = o(\varepsilon), \quad \beta = 0, \ldots, 4 \tag{3.41}$$

To this end, we compute the first order expansion of the variation of the antisymmetric part of the current $\xi_\alpha^\beta(Y+\delta Y) - \xi_\alpha^\beta(Y)$ in $\delta Y = (\delta Y_0, \ldots, \delta Y_4)$. Denoting $h(v,n) = f(v,n) - f^2(v,n)$ and letting $\delta n = n^{(Y+\delta Y)} - n^{(Y)} = (\delta n_0, \delta \underline{n}, \delta n_4)$, we have

$$h(v, n+\delta n) - h(v,n) = \big(\delta n_0 + \delta\underline{n}\cdot v + \delta n_4 \frac{|v|^2}{2}\big) h^*(v,n) + o(\delta n) \tag{3.42}$$

with $h^*(v,n) = f(v,n)(1-f(v,n))(1-2f(v,n))$. From Taylor expansions,

$$h^*(v,n) = h_0^* + \varepsilon h_1^*\big(\lambda_0^{(1)} + \underline{\lambda}^{(1)}\cdot v + \lambda_4^{(1)}\frac{|v|^2}{2}\big) + o(\varepsilon) \tag{3.43}$$

where $h_0^* = h_1$, and $h_1^* = \overline{h}_2$ (see (2.63)). Then, using the properties (IR) and (IP) of the velocity set $\mathcal{V}$, we have

$$\begin{aligned}
&\xi_\alpha^0(Y+\delta Y) - \xi_\alpha^0(Y) = \sum_{v\in V}(v\cdot e_\alpha)\big(h(v,n+\delta n) - h(v,n)\big)\\
&= \varepsilon\frac{\langle\overline{h}_2|v|^2\rangle}{3}\lambda_\alpha^{(1)}\delta n_0 + \sum_{\gamma=1}^{3}\Big[\frac{1}{3}\langle h_1|v|^2\rangle\delta_{\alpha,\gamma} + \varepsilon\frac{1}{3}\langle\overline{h}_2|v|^2\rangle\lambda_0^{(1)} + \varepsilon\frac{1}{6}\langle\overline{h}_2|v|^4\rangle\lambda_4^{(1)}\Big]\delta n_\gamma\\
&+\varepsilon\frac{1}{6}\langle\overline{h}_2|v|^4\rangle\lambda_\alpha^{(1)}\delta n_4 + O(\varepsilon^2)
\end{aligned} \tag{3.44}$$

$$\begin{aligned}
&\xi_\alpha^\beta(Y+\delta Y) - \xi_\alpha^\beta(Y) = \sum_{v\in V}(v\cdot e_\alpha)(v\cdot e_\beta)\big(h(v,n+\delta n) - h(v,n)\big)\\
&= \Big[\frac{1}{3}\langle h_1|v|^2\rangle\delta_{\alpha,\beta} + \varepsilon\frac{1}{3}\langle\overline{h}_2|v|^2\rangle\lambda_0^{(1)}\delta_{\alpha,\beta} + \varepsilon\frac{1}{3}\langle\overline{h}_2|v|^4\rangle\lambda_4^{(1)}\delta_{\alpha,\beta}\Big]\delta n_0\\
&\quad+\sum_{\gamma=1}^{3}\varepsilon\langle\overline{h}_2 v_\alpha v_\beta v_\gamma(\underline{\lambda}^{(1)}\cdot v)\rangle\delta n_\gamma\\
&\quad+\Big[\frac{1}{6}\langle h_1|v|^4\rangle\delta_{\alpha,\beta} + \varepsilon\frac{1}{6}\langle\overline{h}_2|v|^4\rangle\lambda_0^{(1)}\delta_{\alpha,\beta} + \varepsilon\frac{1}{12}\langle\overline{h}_2|v|^6\rangle\lambda_4^{(1)}\delta_{\alpha,\beta}\Big]\delta n_4 + O(\varepsilon^2)
\end{aligned} \tag{3.45}$$

$$\begin{aligned}
&\xi_\alpha^4(Y+\delta Y) - \xi_\alpha^4(Y) = \sum_{v\in V}(v\cdot e_\alpha)\frac{|v|^2}{2}\big(h(v,n+\delta n) - h(v,n)\big)\\
&= \varepsilon\frac{1}{6}\langle\overline{h}_2|v|^4\rangle\lambda_\alpha^{(1)}\delta n_0 + \sum_{\gamma=1}^{3}\Big[\frac{1}{6}\langle h|v|^4\rangle\delta_{\alpha,\gamma} + \varepsilon\frac{1}{6}\langle\overline{h}_2|v|^4\rangle\lambda_0^{(1)}\delta_{\alpha,\gamma} + \varepsilon\frac{1}{12}\langle\overline{h}_2|v|^6\rangle\lambda_4^{(1)}\delta_{\alpha,\gamma}\Big]\delta n_\gamma\\
&+\varepsilon\frac{1}{12}\langle\overline{h}_2|v|^6\rangle\lambda_\alpha^{(1)}\delta n_4 + O(\varepsilon^2)
\end{aligned} \tag{3.46}$$

From these equalities, we get

$$\begin{aligned}
&\sum_{\beta=0}^{4}\sum_{\alpha=1}^{3}\nabla_\alpha\lambda_\beta^{(2)}(\varepsilon x,t)\big(\xi_\alpha^\beta(Y+\delta Y) - \xi_\alpha^\beta(Y)\big)\\
&= \frac{1}{3}\langle h_1|v|^2\rangle\,\mathrm{div}\underline{\lambda}^{(2)}\delta n_0 + \sum_{\gamma=1}^{3}\frac{1}{3}\partial_\gamma\Big[\langle h_1|v|^2\rangle\lambda_0^{(2)} + \frac{1}{2}\langle h_1|v|^4\rangle\lambda_4^{(2)}\Big]\delta n_\gamma\\
&\quad+\frac{1}{6}\langle h_1|v|^4\rangle\,\mathrm{div}\underline{\lambda}^{(2)}\delta n_4
\end{aligned} \tag{3.47}$$

and

$$
\begin{aligned}
&\varepsilon^{-1}\sum_{\beta=0}^{4}\sum_{\alpha=1}^{3}\nabla_\alpha\lambda_\beta^{(1)}(\varepsilon x,t)\big(\xi_\alpha^\beta(Y+\delta Y)-\xi_\alpha^\beta(Y)\big)\\
&=\varepsilon\Big[\varepsilon^{-1}\frac{1}{3}\langle h_1\rangle\,\mathrm{div}\underline{\lambda}^{(1)}+\frac{1}{3}\nabla\Big(\langle\overline{h}_2|v|^2\rangle\lambda_0^{(1)}+\frac{1}{2}\langle\overline{h}_2|v|^4\rangle\lambda_4^{(1)}\Big)\cdot\underline{\lambda}^{(1)}\Big]\delta n_0\\
&\quad+\varepsilon\sum_{\gamma=1}^{3}\Big[\varepsilon^{-1}\frac{1}{3}\partial_\gamma\Big(\langle h_1|v|^2\rangle\lambda_0^{(1)}+\frac{1}{2}\langle h_1|v|^4\rangle\lambda_4^{(1)}\Big)\\
&\qquad+\frac{1}{3}\partial_\gamma\lambda_0^{(1)}\Big(\langle\overline{h}_2|v|^2\rangle\lambda_0^{(1)}+\frac{1}{2}\langle\overline{h}_2|v|^4\rangle\lambda_4^{(1)}\Big)\\
&\qquad+\langle\overline{h}_2(v_1^4-3v_1^2v_2^2\rangle\partial_\gamma\lambda_\gamma^{(1)}\lambda_\gamma^{(1)}+\langle\overline{h}_2v_1^2v_2^2\rangle\Big(\underline{\lambda}^{(1)}\cdot\nabla\lambda_\gamma^{(1)}+\partial_\gamma(|\underline{\lambda}^{(1)}|^2)\Big)\\
&\qquad+\frac{1}{6}\Big(\langle\overline{h}_2|v|^4\rangle\lambda_0^{(1)}+\frac{1}{2}\langle\overline{h}_2|v|^6\rangle\lambda_4^{(1)}\Big)\partial_\gamma\lambda_4^{(1)}\Big]\delta n_\gamma\\
&\quad+\varepsilon\Big[\frac{1}{6}\nabla\Big(\langle\overline{h}_2|v|^4\rangle\lambda_0^{(1)}+\frac{1}{2}\langle\overline{h}_2|v|^6\rangle\lambda_4^{(1)}\Big)\cdot\underline{\lambda}^{(1)}\\
&\qquad+\frac{1}{6}\Big(\langle\overline{h}_2|v|^4\rangle\lambda_0^{(1)}+\frac{1}{2}\langle\overline{h}_2|v|^6\rangle\lambda_4^{(1)}\Big)\,\mathrm{div}\underline{\lambda}^{(1)}\Big]\delta n_4
\end{aligned}
\tag{3.48}
$$

From our assumptions on the chemical potentials (see (2.73)), we have

$$
\mathrm{div}\underline{\lambda}^{(1)}=0 \tag{3.49}
$$

Moreover it results from the Boussinesq condition (2.74) that

$$
\langle h_1|v|^2\rangle\lambda_0^{(1)}+\frac{1}{2}\langle h_1|v|^4\rangle\lambda_4^{(1)}=\mathrm{const} \tag{3.50}
$$

Therefore, the terms in (3.48) with a factor ε^{-1} vanish. Notice that we also assumed $\mathcal{V}$ to be such that

$$
\langle\overline{h}_2(v_1^4-3v_1^2v_2^2)\rangle=0 \tag{3.51}
$$

We now compute the variations δn as a function of δY. Recalling that

$$
\begin{aligned}
\delta Y_0&=\sum_{v\in\mathcal{V}}\big(f(v,n+\delta n)-f(v,n)\big)\\
\delta Y_\alpha&=\sum_{v\in\mathcal{V}}(v\cdot e_\alpha)\big(f(v,n+\delta n)-f(v,n)\big)\\
\delta Y_4&=\sum_{v\in\mathcal{V}}\frac{|v|^2}{2}\big(f(v,n+\delta n)-f(v,n)\big)
\end{aligned}
\tag{3.52}
$$

the variations δY and δn satisfy the following linear equations (up to a negligible error term)

$$\begin{aligned} \delta Y_0 =& \langle h_0 \rangle \delta n_0 + \frac{h_0}{2} \langle |v|^2 \rangle \delta n_4 \\ \delta Y_\alpha =& \frac{1}{3} \langle h_0 |v|^2 \rangle \delta n_\alpha \\ \delta Y_4 =& \frac{1}{2} \langle h_0 |v|^2 \rangle \delta n_0 + \frac{1}{4} \langle h_0 |v|^4 \rangle \delta n_4 \end{aligned} \tag{3.53}$$

Thus

$$\begin{aligned} \delta n_0 =& \frac{1}{\langle h_0 \rangle \phi} \Big(\frac{\langle h_0 |v|^4 \rangle}{\langle h_0 \rangle} \delta Y_0 - 2 \frac{\langle h_0 |v|^2 \rangle}{\langle h_0 \rangle} \delta Y_4 \Big) \\ \delta n_\alpha =& \frac{3}{\langle h_0 |v|^2 \rangle} \delta n_\alpha \\ \delta n_4 =& \frac{2}{\langle h_0 \rangle \Phi} \Big(2 \delta Y_4 - \frac{\langle h_0 |v|^2 \rangle}{\langle h_0 \rangle} \delta Y_0 \Big) \end{aligned} \tag{3.54}$$

with

$$\Phi = \frac{\langle h_0 |v|^4 \rangle}{\langle h_0 \rangle} - \frac{\langle h_0 |v|^2 \rangle^2}{\langle h_0 \rangle^2}.$$

We then obtain the following partial derivatives for Γ_x (see (3.37)) with respect to δY_β. First, for $\beta = 1, \ldots, 3$,

$$\begin{aligned} \frac{\partial \Gamma_x}{\partial Y_\beta} &= -\varepsilon \frac{\partial}{\partial t} \lambda_\beta^{(1)} + \sum_{\delta=0}^{4} \sum_{\gamma=0}^{3} D_{\alpha,\gamma}^{\beta,\delta} (\partial_\alpha \partial_\gamma \lambda_\delta^{(1)}) \\ &\quad - \varepsilon \frac{3}{\langle h_0 |v|^2 \rangle} \Big[\frac{1}{6} \langle \overline{h}_2 |v|^2 \rangle \partial_\beta \big(\lambda_0^{(1)} \big)^2 + \frac{1}{6} \langle \overline{h}_2 |v|^4 \rangle \partial_\beta \big(\lambda_0^{(1)} \lambda_4^{(1)} \big) + \frac{1}{24} \langle \overline{h}_2 |v|^2 \rangle \partial_\beta \big(\lambda_4^{(1)} \big)^2 \\ &\qquad + \langle \overline{h}_2 v_1^2 v_2^2 \rangle \big(\underline{\lambda}^{(1)} \cdot \nabla \lambda_\beta^{(1)} + \frac{1}{2} \partial_\beta |\underline{\lambda}^{(1)}|^2 \big) \Big] \\ &\quad - \varepsilon \frac{3}{\langle h_0 |v|^2 \rangle} \Big[\frac{1}{3} \langle h_1 |v|^2 \rangle \partial_\beta \lambda_0^{(2)} + \frac{1}{6} \langle h_1 |v|^4 \rangle \partial_\beta \lambda_4^{(2)} \Big] + o(\varepsilon) \\ &= -\varepsilon \frac{\partial}{\partial t} \lambda_\beta^{(1)} + \varepsilon D_\alpha (\partial_\alpha^2 \lambda_\delta^{(1)}) - \varepsilon \partial_\beta p' - \varepsilon K' \underline{\lambda}^{(1)} \cdot \nabla \lambda_\beta^{(1)} + o(\varepsilon) \end{aligned} \tag{3.55}$$

where we used the properties of the matrix D in Theorem 4.5 and $\mathrm{div} \underline{\lambda} = 0$, K' is defined in (2.89), $p' = 3(h_0 \langle |v|^2 \rangle)^{-1} p$ and p is given by (2.70). Since $\lambda_\beta^{(1)}$ satisfies (2.88), we obtain

$$\frac{\partial \Gamma_x}{\partial Y_\beta} = o(\varepsilon) \tag{3.56}$$

We now turn to the case $\beta = 4$.

$$\begin{aligned}
\frac{\partial \Gamma_x}{\partial Y_4} &= -\varepsilon \frac{\partial}{\partial t}\lambda_4^{(1)} + \varepsilon \sum_{\delta=0,4} D^{4,\delta}_{\alpha,\gamma}(\partial_\alpha \partial_\gamma \lambda_\delta^{(1)}) \\
&- \varepsilon \frac{2}{3\langle h_0\rangle \Phi}\Bigg[\Big(\langle h_1|v|^4\rangle - \langle h_1|v|^2\rangle \frac{\langle h_1|v|^2\rangle}{\langle h_0\rangle}\Big)\,\mathrm{div}\underline{\lambda}^{(2)} \\
&+ \nabla\Big(\langle \overline{h}_2|v|^4\rangle \lambda_0^{(1)} + \frac{1}{2}\langle \overline{h}_2|v|^6\rangle \lambda_4^{(1)}\Big)\cdot \underline{\lambda}^{(1)} \\
&- \frac{\langle h_0|v|^2\rangle}{\langle h_0\rangle}\nabla\Big(\langle \overline{h}_2|v|^2\rangle \lambda_0^{(1)} + \frac{1}{2}\langle \overline{h}_2|v|^4\rangle \lambda_4^{(1)}\Big)\cdot \underline{\lambda}^{(1)}\Bigg] + o(\varepsilon)
\end{aligned} \tag{3.57}$$

Finally, for $\beta = 0$, we have

$$\begin{aligned}
\frac{\partial \Gamma_x}{\partial Y_0} &= -\varepsilon \frac{\partial}{\partial t}\lambda_0^{(1)} + \varepsilon \sum_{\delta=0,4} D^{0,\delta}_{\alpha,\gamma}(\partial_\alpha \partial_\gamma \lambda_\delta^{(1)}) \\
&- \varepsilon \frac{1}{3\langle h_0\rangle \Phi}\Bigg[\frac{1}{\langle h_0\rangle}\Big(\langle h_0|v|^4\rangle\langle h_1 v^2\rangle - \langle h_1|v|^4\rangle\langle h_0|v|^2\rangle\Big)\,\mathrm{div}\underline{\lambda}^{(2)} \\
&- \frac{\langle h_0|v|^2\rangle}{\langle h_0\rangle}\nabla\Big(\langle \overline{h}_2|v|^4\rangle \lambda_0^{(1)} + \frac{1}{2}\langle \overline{h}_2|v|^6\rangle \lambda_4^{(1)}\Big)\cdot \underline{\lambda}^{(1)} \\
&+ \frac{\langle h_0|v|^4\rangle}{\langle h_0\rangle}\nabla\Big(\langle \overline{h}_2|v|^2\rangle \lambda_0^{(1)} + \frac{1}{2}\langle \overline{h}_2|v|^4\rangle \lambda_4^{(1)}\Big)\cdot \underline{\lambda}^{(1)}\Bigg] + o(\varepsilon)
\end{aligned} \tag{3.58}$$

To get

$$\frac{\partial \Gamma_x}{\partial Y_4} = o(\varepsilon), \qquad \frac{\partial \Gamma_x}{\partial Y_0} = o(\varepsilon) \tag{3.59}$$

we have to show that the equations below are satisfied

$$\begin{aligned}
&\frac{\partial}{\partial t}\lambda_4^{(1)} + \frac{2}{3\langle h_0\rangle \Phi}\Bigg[\Big(\langle h_1|v|^4\rangle - \langle h_1|v|^2\rangle \frac{\langle h_0|v|^2\rangle}{\langle h_0\rangle}\Big)\,\mathrm{div}\underline{\lambda}^{(2)} \\
&\quad + \nabla\Big(\langle \overline{h}_2|v|^4\rangle \lambda_0^{(1)} + \frac{1}{2}\langle \overline{h}_2|v|^6\rangle \lambda_4^{(1)}\Big)\cdot \underline{\lambda}^{(1)} \\
&\quad - \frac{\langle h_0|v|^2\rangle}{\langle h_0\rangle}\nabla\Big(\langle \overline{h}_2|v|^2\rangle \lambda_0^{(1)} + \frac{1}{2}\langle \overline{h}_2|v|^4\rangle \lambda_4^{(1)}\Big)\cdot \underline{\lambda}^{(1)}\Bigg] \\
&= \sum_{\alpha=1}^{3}\sum_{\delta=0,4} D^{4,\delta}_{\alpha,\alpha}(\partial_\alpha^2 \lambda_\delta^{(1)})
\end{aligned} \tag{3.60}$$

$$
\begin{aligned}
&\frac{\partial}{\partial t}\lambda_0^{(1)} + \frac{1}{3\langle h_0\rangle\Phi}\Bigg[\Big(\langle h_0|v|^4\rangle\langle h_1 v^2\rangle - \frac{\langle h_1|v|^4\rangle\langle h_0|v|^2\rangle}{\langle h_0\rangle}\Big)\,\mathrm{div}\underline{\lambda}^{(2)} \\
&\qquad - \frac{\langle h_0|v|^2\rangle}{\langle h_0\rangle}\nabla\Big(\langle\overline{h}_2|v|^4\rangle\lambda_0^{(1)} + \frac{1}{2}\langle\overline{h}_2|v|^6\rangle\lambda_4^{(1)}\Big)\cdot\underline{\lambda}^{(1)} \\
&\qquad + \frac{\langle h_0|v|^4\rangle}{\langle h_0\rangle}\nabla\Big(\langle\overline{h}_2|v|^2\rangle\lambda_0^{(1)} + \frac{1}{2}\langle\overline{h}_2|v|^4\rangle\lambda_4^{(1)}\Big)\cdot\underline{\lambda}^{(1)}\Bigg] \\
&\qquad = \sum_{\alpha=1}^{3}\sum_{\delta=0,4} D_{\alpha,\alpha}^{0,\delta}(\partial_\alpha^2\lambda_\delta^{(1)})
\end{aligned}
\tag{3.61}
$$

Equations (3.60) and (3.61) are not closed, because of the term involving $\mathrm{div}\underline{\lambda}^{(2)}$. To get the final equations multiply (3.60) by Φ_2 and (3.61) by $-2\Phi_1$ (see (2.86)) and sum the two equations to get

$$
\begin{aligned}
&\Phi_2\frac{\partial}{\partial t}\lambda_4^{(1)} - 2\Phi_1\frac{\partial}{\partial t}\lambda_0^{(1)} + \frac{1}{3\langle h_0\rangle\Phi}\langle h_1|v|^2\rangle\Bigg[\Big(\Phi_2 + \Phi_1\frac{\langle h_0|v|^2\rangle}{\langle h_0\rangle}\Big)\Psi_1 \\
&\quad - \Big(\Phi_2\frac{\langle h_0|v|^2\rangle}{\langle h_0\rangle} + \Phi_1\frac{\langle h_0|v|^4\rangle}{\langle h_0\rangle}\Big)\Psi_2\Bigg] \\
&\quad = \sum_{\alpha=1}^{3}\Bigg[\big(\Phi_2 D_{\alpha,\alpha}^{4,4} - 2\Phi_1 D_{\alpha,\alpha}^{4,0}\big)(\partial_\alpha^2\lambda_4^{(1)}) + \big(\Phi_2 D_{\alpha,\alpha}^{4,0} - 2\Phi_1 D_{\alpha,\alpha}^{0,0}\big)(\partial_\alpha^2\lambda_0^{(1)})\Bigg]
\end{aligned}
$$

Using the condition (3.50) we get the third equation in (2.88) which is satisfied by $\lambda_4^{(1)}$ by assumption.

Moreover, by summing the two equations, we get the condition on (2.87) $\mathrm{div}\underline{\lambda}^{(2)}$ which is satisfied by assumption. □

4. Transport coefficients and Green-Kubo formulas.

Fix a point $x \in \mathbb{Z}^d$. For any f and g functions of η_x define the scalar product

$$
(f,g) := E^{\mu_r}[f(\eta_x); g(\eta_x)] = E^{\mu_r}[f(\eta_x)g(\eta_x)] - E^{\mu_r}[f(\eta_x)]E^{\mu_r}[g(\eta_x)]. \tag{4.1}
$$

We have the orthogonality relations:

$$
(\eta(x,v), \eta(x,v')) = \delta_{v,v'}h_0
$$

The set $I_\beta, \beta = 0,\dots,4$ is not orthogonal because $(I_0, I_4) \neq 0$. Nevertheless, we will use this set in the following. Note that the set I'_β, which differs from the previous one only for the presence of I_4' instead of I_4, is orthogonal. We have that

$$
\begin{aligned}
(I_0, I_0) = \langle h_0 \rangle, \quad (I_\alpha, I_\alpha) = \frac{1}{3}\langle |v|^2 h_0 \rangle, \quad \alpha = 1,2,3 \\
(I_4, I_4) = \langle h_0 \frac{|v|^4}{4} \rangle, \quad (I_0, I_4) = (I_4, I_0) = \langle h_0 \frac{|v|^2}{2} \rangle
\end{aligned}
\tag{4.2}
$$

We introduce a function space $\mathcal{G}$ similar to the one in [EMY3]

Definition 4.1 Let $\mathcal{G}$ be the space of local functions of η satisfying

$$
E^{\mu_r}[g] = 0, \quad \sum_x E^{\mu_r}[g; I_\alpha(\eta_x)] = 0, \quad \alpha = 0, \ldots, 4. \tag{4.3}
$$

Let ν_m be the product measure such that $E^{\nu_m}[I_\alpha] = m_\alpha$ for $\alpha \geq 0$ and $\hat{g}(m) = E^{\nu_m}[g]$. Then the second condition in (4.3) is equivalent to

$$
\frac{\partial \hat{g}(m)}{\partial m_\alpha}\Big|_{m=\bar{m}} = 0, \quad \alpha = 0, \ldots, d. \tag{4.4}
$$

where m is the vector $\{m_\alpha, \alpha = 0, \ldots, 4\}$ and $\bar{m}_\alpha = E^{\mu_r}[I_\alpha]$, $\alpha = 0, \ldots, 4$, are the values corresponding to the equilibrium measure μ_r.

Let

$$
\mathcal{G}^0 = \{\sum_{\alpha=0}^{4} \sum_{j=1}^{3} a_{\alpha,j} \nabla_j I_\alpha(\eta_0), a_{\alpha,j} \in \mathbb{R}\}
$$

be the space of the gradients of the conserved quantities. Here

$$
\nabla_j g(\eta) = g(\tau_{e_j}\eta) - g(\eta),
$$

with $(\tau_x \eta)(y, v) = \eta(y + x, v)$ and $(\tau_x f)(\eta) = f(\tau_x \eta)$. Define the semi-norm

$$
\| \cdot \|_{-1}^2 = V(\,\cdot\,)
$$

where $V(\,\cdot\,)$ is the infinite volume variance defined by (3.11). We introduce the following equivalence relation: two elements of $\mathcal{G}$ are equivalent if they differ by an element of $\mathcal{G}^0$. The quotient of $\mathcal{G}$ w.r.t. this equivalence relation will be denoted by $\mathcal{G} \backslash \mathcal{G}^0$.

Structure Theorems.

The following Theorems are proved as in [EMY3], Sect. 4 and 6.

Theorem 4.2 *For all $g \in \mathcal{G}$, one has $\| g \|_{-1} < \infty$. Furthermore, let $\overline{\mathcal{G}}$ denote the closure of $\mathcal{G}$ under the semi-norm $\| \cdot \|_{-1}$ and define*

$$\ll g, h \gg = \frac{1}{4} [\, \| g+h \|_{-1} + \| g-h \|_{-1} \,]. \tag{4.5}$$

Then $\ll \cdot, \cdot \gg$ is an inner product and $\overline{\mathcal{G}}$ equipped with this inner product is a Hilbert space. Moreover $\overline{\mathcal{G}}$ can be decomposed as

$$\overline{\mathcal{G}} = \overline{\mathcal{L}_s[\mathcal{G} \backslash \mathcal{G}^0] \oplus \mathcal{G}^0} := \mathcal{H}. \tag{4.6}$$

We introduce the currents σ_α^β

$$\sigma_\beta^\alpha = w_\beta^\alpha - \sum_{\gamma \geq 0} A_\beta^{\alpha,\gamma} \tilde{I}_\gamma(\eta_0), \quad \sigma_\beta^{*\alpha} = w_\beta^{*\alpha} - \sum_{\gamma \geq 0} A_\beta^{*\alpha,\gamma} \tilde{I}_\gamma(\eta_0) \tag{4.7}$$

where $\tilde{I}_\gamma(\eta_0) = I_\gamma(\eta_0) - E^{\mu_r}[I_\gamma(\eta_0)]$. The constants $A_\beta^{\alpha,\gamma}$ $(A_\beta^{*\alpha,\gamma})$ are fixed by the condition that $\sigma_\beta^\alpha(\sigma_\beta^{*\alpha}) \in \mathcal{G}$.

Theorem 4.3 .

(i) $\overline{\mathcal{L}\mathcal{G} + \mathcal{G}_0} = \overline{\mathcal{G}} = \overline{\mathcal{L}^*\mathcal{G} + \mathcal{G}_0}$

(ii) Let $\mathcal{G}_w = \left\{ \sum_{\alpha \geq 0, j \geq 1} b_j^\alpha \sigma_j^\alpha \right\}$. *Then*

$$\overline{\mathcal{G}_w + \mathcal{L}_s\mathcal{G}} = \overline{\mathcal{G}} = \overline{\mathcal{G}_w + \mathcal{L}\mathcal{G}} \ ; \qquad \overline{\mathcal{G}_{w^*} + \mathcal{L}^*\mathcal{G}} = \overline{\mathcal{G}}. \tag{4.8}$$

The proof of these Theorems is based essentially on the properties of the symmetric simple exclusion (see [LY] and [EMY3]) and on the following properties of the collision operator: the collisions are local, binary and conserve only the quantities I_α. Since these properties are true also for the model we are considering in this paper, the proof is easily extended to it.

The main step in proving the previous Theorem is solving the resolvent equation for $\mathcal{L}$. We summarize some of the results on the resolvent equation, that we will need below, in next Theorem.

If f and g are local functions, we define the inner product

$$\langle f, g \rangle_0 = (f, \sum_x \tau_x g) = (\sum_x \tau_x f, g)$$

Theorem 4.4 . *Let H_0 and H_1 be the spaces*

$$H_0 = \{f \ local : ||f||_0^2 := \langle f, f\rangle_0 < \infty\}, \quad H_1 = \{f \ local : ||f||_1^2 := \langle f, (-\mathcal{L}_s)f\rangle_0 < \infty\}$$

Let $h \in \mathcal{G}$. Then

(i) *For any $\varepsilon > 0$ there exists a local function $u_\varepsilon \in \mathcal{G}$ such that*

$$||\mathcal{L}u_\varepsilon - h||_{-1} \leq \varepsilon$$

(ii) *u_ε converges strongly in H_1 to some $g \in H_1$*

(iii) *The solution u_λ of the resolvent equation $\lambda u_\lambda - \mathcal{L}u_\lambda = h$ converges strongly in H_1 to the same g*

(iv) $\lim_{\lambda\to 0} ||\lambda u_\lambda||_0 = 0$

The proof of these statements can be found in [EMY3], [V], [LY].

Transport coefficients.

The diffusion matrix D used in the entropy method in Section 3 represents the "components" of the currents in the directions of the gradients of the conserved quantities referred to the non-orthogonal base $\{I_\alpha\}$, namely the matrix $D_{i,j}^{\alpha,\beta}$ is characterized by

$$\sigma_i^\alpha - \sum_{j=1}^{3}\sum_{\beta=0}^{4} D_{i,j}^{\alpha,\beta}\nabla_j I_\alpha \in \overline{\mathcal{L}\mathcal{G}}$$

The following theorem is taken by [EMY3], the only difference in the present case being that the energy I_4 is an independent conserved quantity and the fact that the base $\{I_\alpha\}$ is not orthogonal. To take into account the latter, we introduce the 4×4 compressibility matrix $\bar\Theta$ and the matrix Θ

$$\bar\Theta^{\alpha,\beta} = (I_\alpha, I_\beta); \qquad \Theta_{i,j}^{\alpha,\beta} = \delta_{i,j}\bar\Theta^{\alpha,\beta} \tag{4.9}$$

Theorem 4.5 .

(i) Let $u_j^\beta = \chi\nabla_j I_\beta$. Put $a\cdot d = \sum_{j\geq 1,\beta\geq 0} a_j^\beta d_j^\beta$. Define T to be the linear transformation from $\mathcal{G}$ to $\mathcal{G}$ s.t.

$$T(b\cdot\sigma + \mathcal{L}g) = b\cdot u + \mathcal{L}_s g. \tag{4.10}$$

Then T is bounded above by 1, hence can be extended by continuity to $\overline{\mathcal{G}}$. Moreover

$$\begin{aligned} &(a) \quad T\nabla_j I_\alpha(\eta) \perp \mathcal{L}^*\mathcal{G} \quad \text{for } j\geq 1, \alpha\geq 0 \\ &(b) \quad \ll T\nabla_i I_\alpha(\eta)\,,\, \sigma_j^{*\ \beta} \gg = \Theta_{i,j}^{\alpha,\beta} \end{aligned}$$

(ii) Let M be the matrix (with double indices (α, i) and (β, j))

$$M^{\alpha,\beta}_{i,j} = \ll \nabla_i I_\alpha(\eta), T\nabla_j I_\beta(\eta) \gg .$$

Then the elements $D^{\alpha,\beta}_{i,j}$ of the diffusion matrix D are given by

$$D^{\alpha,\beta}_{i,j} = (\Theta M^{-1})^{\alpha,\beta}_{i,j} \tag{4.11}$$

As a quadratic form, $D \geq D_s$ and $D \neq D_s$, where $(D_s)^{\alpha,\beta}_{i,j} = \chi\delta_{\alpha,\beta}\delta_{i,j}$. Furthermore, there exist constants $B_k, k = 1, \cdots, 6$ such that

$$D^{\alpha,\beta}_{j,k} = \delta_{j,k}\delta_{\alpha,\beta}[B_1 + B_2\delta_{\alpha,j})] + B_3\delta_{\alpha,j}\delta_{\beta,k} + B_6\delta_{j,\beta}\delta_{k,\alpha}. \tag{4.12}$$

for $\alpha, \beta = 1, 2, 3$,

$$D^{0,0}_{i,j} = \delta_{i,j}B_0, \quad D^{4,4}_{i,j} = \delta_{i,j}B_4 \qquad D^{\alpha,\beta}_{i,j} = 0 \begin{cases} \alpha = 0, 4 \text{ and } \beta = 1, 2, 3 \\ \alpha = 1, 2, 3 \text{ and } \beta = 0, 4 \end{cases} \tag{4.13}$$

and

$$(\Theta^{-1}D)^{0,4}_{i,j} = (\Theta^{-1}D)^{4,0}_{i,j} = \delta_{i,j}B_5 \tag{4.14}$$

Due to the property of T, M^{-1} is bounded below by M_s^{-1} as a quadratic form, where $M_s = \ll \nabla_i I_\alpha(\eta), \nabla_j I_\beta(\eta) \gg = \frac{\Theta}{\chi}$. Therefore, we have $D \geq D_s$ as a quadratic form.

Eq. (4.12) and (4.13) are a consequence of the symmetry properties by spatial rotations and reflection of the dynamics of the matrix M, as shown in [EMY3]. Eq. (4.14) instead is true because of the time-reversal symmetry. In fact, let us introduce T^*, the adjoint of T w.r.t. the scalar product $\ll \cdot, \cdot \gg$,

$$T^*\Big(\sum_{j\geq 1, \beta\geq 0} b^\beta_j (\sigma^*)^\beta_j + \mathcal{L}^* g\Big) = \sum_{j\geq 1, \beta\geq 0} b^\beta_j u^\beta_j + \mathcal{L}_s g. \tag{4.15}$$

It is easy to see that the time reversal invariance (2.43) implies

$$T^*S = ST$$

Hence

$$\ll \nabla_i I_0(\eta), T\nabla_j I_4(\eta) \gg = \ll S\nabla_i I_0(\eta), ST\nabla_j I_4(\eta) \gg = \ll S\nabla_i I_0(\eta), T^*S\nabla_j I_4(\eta) \gg$$
$$= \ll TS\nabla_i I_0(\eta), S\nabla_j I_4(\eta) \gg = \ll T\nabla_i I_0(\eta), \nabla_j I_4(\eta) \gg$$

We have used that $S^* = S, S^2 = I$ and $SI_0 = I_0$ $SI_4 = I_4$. That proves (4.14).

Variational formulas for the diffusion coefficients D and its inverse D^{-1} can be obtained, which are very useful in proving for example positivity and boundedness properties. We refer to [EMY3] and [LOY2], where similar formulas are proven and discussed for a slightly simpler model, because the extension to the present one is straightforward. Here instead we give a rigorous argument to express the transport coefficients in terms of the Green-Kubo formulas.

Green-Kubo formulas.

We start with a Lemma that express D in terms of the strong limit in H_1 of the sequence u_ε introduced in Theorem 4.4. The proof is essentially the one given in [EMY3] (at the end of Sect.5) for showing the formal equivalence between (4.11) and the Green-Kubo formula.

Lemma 4.6 . *The diffusion matrix* $\bar{D} = D - \chi \mathbb{I}$ *satisfies*

$$a \cdot (\bar{D}\Theta)a = \lim_{\varepsilon \to 0} ||a \cdot u_\varepsilon||_1^2 \tag{4.16}$$

Proof.

The currents σ_β^α and $\nabla_\alpha I^\beta$ belong to the space $\overline{\mathcal{G}}$. The diffusion coefficient is found as the matrix such that

$$\sigma - D\nabla I \in \mathcal{L}\mathcal{G}$$

Hence there exists some $g \in \overline{\mathcal{L}\mathcal{G}}$ such that

$$||\sigma - D\nabla I - g||_{-1} = 0$$

The function g is non-local but can be approximated by local functions $u_\varepsilon \in \mathcal{L}\mathcal{G}$ such that, by the second structure Theorem,

$$\sigma - D\nabla I - \mathcal{L}u_\varepsilon = h_\varepsilon, \quad \lim_{\varepsilon \to 0} ||h_\varepsilon||_{-1} = 0 \tag{4.17}$$

D is given by the expression (4.11)

$$a \cdot (D^{-1}\Theta)a = \ll a \cdot \nabla I, Ta \cdot \nabla I \gg .$$

By (4.17) we have $\nabla I = D^{-1}[\sigma - \mathcal{L}u_\varepsilon - h_\varepsilon]$. By the definition of the map T we have that $T(\sigma - \mathcal{L}u_\varepsilon) = \chi\nabla I - \mathcal{L}_s u_\varepsilon$. Hence

$$a \cdot (D^{-1}\Theta)a = \ll a \cdot \ D^{-1}[\sigma - \mathcal{L}u_\varepsilon - h_\varepsilon], a \cdot D^{-1}[\chi\nabla I - \mathcal{L}_s u_\varepsilon - Th_\varepsilon] \gg$$

which implies, using the symmetry of the matrices $D^{-1}\Theta$ and Θ,

$$a \cdot (D\Theta)a = \chi \ll a \cdot (\sigma - \mathcal{L}u_\varepsilon), a \cdot (\chi \nabla I + \mathcal{L}_s u_\varepsilon) \gg + R_\varepsilon$$

where $R_\varepsilon = R_1 + R_2 + R_3$ and

$$R_1 = \ll a \cdot h_\varepsilon, a \cdot (\bar{\chi} \nabla I - \mathcal{L}_s u_\varepsilon) \gg \leq \text{const} \, ||h_\varepsilon||_{-1} \, ||\chi \nabla I - \mathcal{L}_s u_\varepsilon||_{-1} = o(\varepsilon)$$
$$R_2 = \ll a \cdot (\sigma - \mathcal{L}u_\varepsilon), a \cdot Th_\varepsilon, \gg \leq \text{const} \, ||Th_\varepsilon||_{-1} \, ||\sigma - \mathcal{L}u_\varepsilon||_{-1} = o(\varepsilon)$$
$$R_3 = \ll a \cdot h_\varepsilon, a \cdot Th_\varepsilon \gg \leq \text{const} \, ||h_\varepsilon||_{-1}^2 = o(\varepsilon)$$

It is easy to see by using the properties of the scalar product that ([EMY3])

$$\ll a \cdot (\sigma - \mathcal{L}u_\varepsilon), a \cdot (\chi \nabla I + \mathcal{L}_s u_\varepsilon) \gg = \chi a \cdot \Theta a + \ll a \cdot \mathcal{L}u_\varepsilon, a \cdot \mathcal{L}_s u_\varepsilon \gg$$

Hence

$$a \cdot (\Theta D)a = \chi a \cdot \Theta a + \ll a \cdot \mathcal{L}u_\varepsilon, a \cdot \mathcal{L}_s u_\varepsilon \gg + R_\varepsilon$$

Since the scalar product has the property that, for f, h local functions in $\mathcal{G}$,

$$\ll f, \mathcal{L}_s h \gg = - < f, h >_0$$

we have

$$\ll a \cdot \mathcal{L}u_\varepsilon, a \cdot -\mathcal{L}_s u_\varepsilon \gg = - < a \cdot \mathcal{L}u_\varepsilon, a \cdot u_\varepsilon >_0 = < a \cdot u_\varepsilon, -\mathcal{L}_s a \cdot u_\varepsilon >_0 = ||a \cdot u_\varepsilon||_1^2$$

Since R_ε vanishes when ε goes to zero we have proved the Lemma.

Theorem 4.7 *The following Green-Kubo representation for the transport matrix D holds:*

$$a \cdot D\Theta a = \chi a \cdot \Theta a + < \sigma \cdot a, (-\mathcal{L})^{-1} \sigma \cdot a >_0 \tag{4.18}$$

Proof. By iii) of Theorem 4.4, we have that $\lim_{\lambda \to 0} ||u_\lambda||_1 = \lim_{\varepsilon \to 0} ||u_\varepsilon||_1$. On the other hand, u_λ is the solution of the resolvent equation $\lambda u_\lambda - \mathcal{L}u_\lambda = \sigma$ so that

$$\lambda ||a \cdot u_\lambda||_0^2 - ||a \cdot u_\lambda||_1^2 = < a \cdot \sigma, a \cdot u_\lambda >_0 = < a \cdot \sigma, (\lambda - \mathcal{L})^{-1} a \cdot \sigma >_0$$

By iii) and iv) of Theorem 4.4 $\lambda ||u_\lambda||_0 \to 0$ and $||u_\lambda||_1 \to ||g||_1$ for some g in H_1. Hence

$$\lim_{\lambda \to 0} ||a \cdot u_\lambda||_1^2 = < a \cdot \sigma, (-\mathcal{L})^{-1} a \cdot \sigma >_0 = ||a \cdot g||_1^2 < \infty$$

Finally by ii) of Theorem 4.4 we have that

$$a \cdot (\bar{D}\Theta)a = \lim_{\varepsilon \to 0} ||a \cdot u_\varepsilon||_1^2 = \lim_{\lambda \to 0} ||a \cdot u_\lambda||_1^2 = < a \cdot \sigma, (-\mathcal{L})^{-1}, a \cdot \sigma >_0$$

For any fixed $\lambda > 0$ we can write

$$< f, (\lambda - \mathcal{L})^{-1} f >_0 = \int_0^\infty dt < f, P_t(\lambda) f >_0$$

where $P_t(\lambda)$ is the semigroup generated by $\mathcal{L} - \lambda$. By using the theorem of dominated convergence we get in the limit $\lambda \to 0$ that

$$< f, (-\mathcal{L})^{-1} f >_0 = \int_0^\infty dt < f, P_t f >_0$$

In conclusion we get the Green-Kubo formula for $\bar{D}$

$$a \cdot (\bar{D}\Theta) a = \int_0^\infty dt < a \cdot \sigma, P_t a \cdot \sigma >_0 \tag{4.19}$$

The diffusion coefficient has been defined in different ways depending on the point of view adopted. Spohn [S] proposed a definition based on the long time behavior of the structure function. Adapting its definition to our model we introduce the coefficient $D^{(1)}$ as

$$(D^{(1)}\Theta)_{i,j}^{\alpha,\beta} = \lim_{t\to\infty} \frac{1}{t} \left\{ \sum_x x_i x_j E\Big[I_\alpha(x,t); I_\beta(0,0)\Big] - \sum_{\gamma,\delta=0}^{4} (v_i^{\alpha,\gamma} t) \bar{\Theta}^{\gamma\delta} (v_j^{\delta,\beta} t) \right\}$$

where v_i is the tensor $v_i^{\alpha,\beta}$ defined as

$$(\bar{\Theta} v_i)^{\alpha,\beta} = \sum_x x_i E\Big[I_\alpha(x,t); I_\beta(0,0)\Big]$$

and $\bar{\Theta}$ is the compressibility matrix.

On the other hand the linear response theory gives the transport coefficients expressed in terms of the Green-Kubo formulas. Call $D^{(2)}$ the transport matrix as given by the Green-Kubo formula

$$(D^{(2)}\Theta)_{i,j}^{\alpha,\beta} = \chi \Theta_{ij}^{\alpha,\beta} + \int_0^\infty dt < \sigma_i^\alpha, P_t a \cdot \sigma_j^\beta >_0$$

Finally, there is the transport matrix D that appears in the Navier-Stokes equations given by (4.11). One expects that all these definitions be related. In [LOY2] it is discussed this problem for the asymmetric simple exclusion process and it is shown that $(D)^{(1)}$ can be written also as

$$(D)_{i,j}^{(1)} = \delta_{i,j} + \lim_{t\to\infty} \frac{1}{2t\ \Theta} \int_0^t ds \int_0^s ds' \Big[< \sigma_i^\alpha, P_{s'} \sigma_j^\beta >_0 + < \sigma_j^\alpha, P_{s'} \sigma_i^\beta >_0 \Big]$$

with the compressibility Θ a scalar in that case, which is the natural form for the diffusion coefficient in the equation for the equilibrium fluctuation of the density field. For our model one can prove, following the argument in [LOY2] and [S], that

$$(D^{(1)}\Theta)^{\alpha,\beta}_{i,j} = \chi\Theta^{\alpha,\beta}_{i,j} + \lim_{t\to\infty}\frac{1}{t}\int_0^t ds\int_0^t ds' < \sigma_i^\alpha, P_t\sigma_j^\beta >_0$$

Note that in our case $D\Theta$ is symmetric separately in the indices α, β and i, j. Since we have shown above that the time integral in (4.18) is finite, we can conclude that $D^{(1)} = D^{(2)}$. Moreover, we have already shown that $D = D^{(2)}$, so that we can conclude the equivalence of all the definitions.

Remark 1. In [LY] it is proven, by using martingale methods, in the case of the simple exclusion process, that $D^{(1)} = \left(D^{(2)}\right)^s$ where $D^{(s)}$ is the symmetrization of D (and that $D^{(1)} = \lim_\varepsilon ||u_\varepsilon||_1^2$). By using the analogous of Lemma 4.6 for the simple exclusion process it is possible to conclude also that $D = \left(D^{(2)}\right)^s$.

Remark 2. In [CY], it is proved for the asymmetric simple exclusion process that the scaling limit of the equilibrium fluctuation of the density is Gaussian with diffusion coefficient $D^{(1)}$ which is as explained above equal to D, so proving what in physics literature is called fluctuation-dissipation theorem. An analogous result holds for the model we are considering here and we will discuss it in a forthcoming paper.

5. Derivation of the Navier-Stokes equations with a force term.

Given a smooth function F on $\mathbb{R}^3$ representing the macroscopic force, we consider an infinitesimal generator $\mathcal{L}^F$ which consists in flipping the velocities of particles in such a way that the asymmetric part of the incoming momentum current is equal to the force. More precisely, for a local function f, it is given by

$$\mathcal{L}^F f(\eta) = \sum_x \sum_{v,w\in\mathcal{V}} \eta(x,v) p^{F,\varepsilon}_{v,w}(x)\left[f\left(\eta^{x,v,w}\right) - f(\eta)\right] \tag{5.1}$$

with $\eta^{x,v,w}$ the configuration obtained from η by exchanging the values of $\eta(x,v)$ and $\eta(x,w)$. The rate $p^{F,\varepsilon}_{v,w}(x)$ is chosen as

$$p^{F,\varepsilon}_{v,w}(x) = \sum_{\alpha=1}^{3}\left(\kappa - \frac{F_\alpha(\varepsilon x)}{2\langle h_0\rangle v_\alpha}\right)\delta_{v^{(\alpha)}}(w) \tag{5.2}$$

where the velocity $v^{(\alpha)}$ is obtained by flipping the sign of the i-th component of v, and h_0 is defined by (2.63). Moreover, $\kappa > 0$ is chosen large enough to make the jump rate positive for all v, w. Remark that with this choice, the flip only affects the velocities in the same species, therefore the local mass and kinetic energy are still conserved by this generator. However, the momentum (local or global) is no more conserved and the reference measure $\mu_{L,r,\theta}$ is not invariant under $\mathcal{L}^F$. Indeed a simple computation shows that its adjoint $(\mathcal{L}^F)^*$ in $L^2(\mu_{L,r,\theta})$ is given by

$$((\mathcal{L}^F)^* f)(\eta) = \sum_x \sum_{v,w\in\mathcal{V}} \eta(x,v) \left[p^{F,\varepsilon}_{w,v}(x) f(\eta^{x,v,w}) - p^{F,\varepsilon}_{v,w}(x) f(\eta)\right] \tag{5.3}$$

and that $(\mathcal{L}^F)^* 1 \neq 0$. Note that $(\mathcal{L}^F)^*$ can be written as

$$((\mathcal{L}^F)^* f)(\eta) = (\widetilde{\mathcal{L}}^F f)(\eta) + \Big(\sum_x \sum_{v\in\mathcal{V}} \sum_{\alpha=1}^{3} \frac{F_\alpha(\varepsilon x)}{\langle h_0\rangle\, v_\alpha} \eta(x,v)\Big) f(\eta) \tag{5.4}$$

where $\widetilde{\mathcal{L}}^F$ is the Markov generator defined by

$$(\widetilde{\mathcal{L}}^F f)(\eta) = \sum_x \sum_{v,w\in\mathcal{V}} \eta(x,v) p^{F,\varepsilon}_{w,v}(x) \left[f(\eta^{x,v,w}) - f(\eta)\right] \tag{5.5}$$

The velocity flip generator is slowed down by the factor ε^3, so that the generator of the full dynamics is given by

$$\bar{\mathcal{L}} = \mathcal{L} + \varepsilon^3 \mathcal{L}^F = \mathcal{L}^{\mathrm{ex}} + \mathcal{L}^{\mathrm{c}} + \varepsilon^3 \mathcal{L}^F \tag{5.6}$$

The main result of this section is the next theorem.

Theorem 5.1 *We make the same assumptions of Theorem 2.2, with equations (2.75) replaced by the Navier-Stokes equation with the force,*

$$\begin{aligned} &\mathrm{div} u^{(1)} = 0 \\ &\partial_t u^{(1)}_\beta + \partial_\beta p + K\, u^{(1)} \cdot \nabla u^{(1)}_\beta - F_\beta = \sum_{\alpha=1}^{3} D_{\alpha,\beta} \partial^2_\alpha u^{(1)}_\beta \end{aligned} \tag{5.7}$$

and (2.83), where the coefficients K, $D_{\alpha.\beta}$ and $\mathcal{K}_\alpha$ are those in Theorem 2.2.

We start the process $\eta_t(x,v)$ with generator $\varepsilon^{-2}\bar{\mathcal{L}}$ from the measure $\mu_{L,n}$ defined in (2.28), with chemical potentials n_α of the form (2.55), satisfying (2.73) and (2.74).

Then the empirical fields $\nu_0(z,t)$, $(\nu_1(z,t),\dots,\nu_3(z,t))$ and $\nu_4(z,t)$ converge, for $t \le t_0$, weakly in probability to $\rho^{(1)}(z,t)dz$, $u^{(1)}(z,t)dz$ and $\mathcal{E}^{(1)}(z,t)dz$. Moreover, the linear combination (2.80) of $\rho^{(1)}$ and $\mathcal{E}^{(1)}$ is the smooth solution of the equation for the temperature (2.83).

We define the chemical potentials $\lambda^{(1)}$ and $\lambda^{(2)}$ in the following way. We suppose that $\underline{\lambda}^{(1)} = (\lambda_1^{(1)}, \ldots, \lambda_3^{(1)})$, $\lambda_4^{(1)}$ and p' are solutions of

$$
\begin{aligned}
&\mathrm{div}\underline{\lambda}^{(1)} = 0 \\
&\partial_t \lambda_\beta^{(1)} + \partial_\beta p' + K' \underline{\lambda}^{(1)} \cdot \nabla \lambda_\beta^{(1)} - \frac{3}{\langle |v|^2 h_0 \rangle} F_\beta = \sum_{\alpha=1}^{3} D_{\alpha,\beta} \partial_\alpha^2 \lambda_\beta^{(1)}, \quad \beta = 1, \ldots, 3 \\
&\frac{\partial}{\partial t} \lambda_4^{(1)} + H' \underline{\lambda}^{(1)} \cdot \nabla \lambda_4^{(1)} = \sum_{\alpha=1}^{3} \mathcal{K}_\alpha \partial_\alpha^2 \lambda_4^{(1)}
\end{aligned}
\tag{5.8}
$$

where K', H' are the same coefficients as those in (2.88). Moreover $\lambda_0^{(1)}$ is chosen such that the Boussinesq condition (2.74) holds, $\underline{\lambda}^{(2)}$ is such that (2.87) is valid and $\lambda_0^{(2)}$, $\lambda_4^{(2)}$ are taken such that the pressure p defined by (2.70) satisfies $3p = \langle |v|^2 h_0 \rangle\, p'$.

Now let n_β, $\beta = 0, \ldots, 4$, be the chemical potentials defined by (3.1) with $\lambda^{(1)}$ and $\lambda^{(2)}$ as above. We also consider the densities Ψ_t and $\widetilde{\Psi}_t$ w.r.t. $\mu_{L,r,\theta}$ given by formulas (3.2) and (3.4). As in Sect. 3, Theorem 5.1 will follow if we can prove the following estimate for the non equilibrium density $\bar{f}_t$ of the dynamics generated by $\bar{\mathcal{L}}$.

Theorem 5.2 *There are functions F_j^i such that the specific relative entropy $s(f_t|\widetilde{\Psi}_t)$ satisfies*

$$
\lim_{\varepsilon \to 0} \varepsilon^{-2} s\big(\bar{f}_t | \widetilde{\Psi}_t\big) = 0 \tag{5.9}
$$

One of the ingredients needed in the derivation of this entropy estimate is to get an a priori bound on the entropy production, namely something like inequalities (3.8) and (3.9). But since the reference measure is no longer invariant, such estimates are not trivial. That is the aim of the next proposition.

Proposition 5.3 *There are positive constants c_1 and c_2 such that for any $t \geq 0$,*

$$
\frac{d}{dt} s(\bar{f}_t) \leq -\varepsilon c_1 D_{\Lambda_L}\big(\sqrt{\bar{f}_t}\big) + s(\bar{f}_t) + c_2 \varepsilon^2 \tag{5.10}
$$

where D_{Λ_L} is the Dirichlet form defined after (3.15). In particular, there is a positive constant c such that for any $t \geq 0$

$$
s(\bar{f}_t) \leq c\varepsilon^2 (1 + e^t) \tag{5.11}
$$

Proof. We start computing the time derivative of the relative entropy $s(\bar{f}_t)$.

$$
\begin{aligned}
\frac{d}{dt} s(\bar{f}_t) &= \varepsilon \int \log \bar{f}_t (\bar{\mathcal{L}}^* \bar{f}_t) \, d\mu_{L,r,\theta} + \varepsilon \int \bar{\mathcal{L}}^* \bar{f}_t \, d\mu_{L,r,\theta} \\
&= \varepsilon \int \log \bar{f}_t (\mathcal{L}^* \bar{f}_t) \, d\mu_{L,r,\theta} + \varepsilon^4 \int \log \bar{f}_t (\mathcal{L}^F)^* \bar{f}_t \, d\mu_{L,r,\theta}
\end{aligned}
\tag{5.12}
$$

Using the basic inequality

$$
a(\log b - \log a) \leq -(\sqrt{b} - \sqrt{a})^2 + (b - a) \tag{5.13}
$$

for $a > 0$ and $b > 0$, we get the usual bound

$$
\varepsilon \int \log \bar{f}_t \mathcal{L}^* \bar{f}_t \, d\mu_{L,r,\theta} \leq -\varepsilon c_1 D_{\Lambda_L}(\sqrt{\bar{f}_t}) \tag{5.14}
$$

The second term in (5.12) can be written as

$$
\varepsilon^4 \int \bar{f}_t \mathcal{L}^F \log \bar{f}_t \, d\mu_{L,r,\theta} \tag{5.15}
$$

then, using again (5.13), it is bounded above by

$$
-\varepsilon^4 D^F_{\Lambda_L}(\sqrt{\bar{f}_t}) + \varepsilon^4 \int \mathcal{L}^F \bar{f}_t \, d\mu_{L,r,\theta}, \tag{5.16}
$$

where $D^F_{\Lambda_L}$ is the Dirichlet form related to the symmetric part of the force generator, that is

$$
D^F_{\Lambda_L}(f) = \frac{\kappa}{2} \sum_x \sum_{v,w} \int \left(f(\eta^{x,v,w}) - f(\eta) \right)^2 \, d\mu_{L,r,\theta} \geq 0 \tag{5.17}
$$

We now turn to the second term in (5.16). We first rewrite it as

$$
\varepsilon^4 \int (\mathcal{L}^F)^* 1 \, \bar{f}_t d\mu_{L,r,\theta} = \varepsilon^4 E^{\bar{f}_t} \Big[\sum_x \sum_{v,i} \frac{F_i(\varepsilon x)}{v_i} \eta(x,v) \Big] \tag{5.18}
$$

Following the proof of Lemma 2.2 of [EMY1], namely applying the entropy inequality and performing a Taylor expansion, it is easy to see that

$$
\varepsilon^2 E^{\bar{f}_t} \Big[\varepsilon^2 \sum_x \sum_{v,i} \frac{F_i(\varepsilon x)}{v_i} \eta(x,v) \Big] \leq \varepsilon^2 E \Big[\varepsilon^2 \sum_x \sum_{v,i} \frac{F_i(\varepsilon x)}{v_i} \eta(x,v) \Big] + s(\bar{f}_t) + c\varepsilon^2 \tag{5.19}
$$

where c is a positive constant. The first term on the r.h.s of the previous inequality vanishes since the velocity set $\mathcal{V}$ is symmetric and since the velocities having the same kinetic energy are uniformly distributed under $\mu_{L,r,\theta}$.

Putting together all these inequalities, we get

$$\frac{d}{dt}s(\bar{f}_t) \leq -\varepsilon c_1 D_{\Lambda_L}(\sqrt{\bar{f}_t}) + s(\bar{f}_t) + c\varepsilon^2 \tag{5.20}$$

□

Proof of Theorem 5.2. As in Sect. 3, we start computing the time derivative of the specific relative entropy. But since $(\mathcal{L}^F)^*$ is not a Markov generator, we can not quote directly [EMY1] and we get a formula slightly different from (3.22).

An usual computation (see [Y1]) yields to the bound

$$\frac{d}{dt}\varepsilon^{-2}s(\bar{f}_t|\widetilde{\Psi}_t) \leq \varepsilon \int \bar{f}_t \widetilde{\Psi}_t^{-1}\Big(\varepsilon^{-2}\bar{\mathcal{L}}^* - \frac{\partial}{\partial t}\Big)\widetilde{\Psi}_t \, d\mu_{L,r,\theta} \tag{5.21}$$

Using expressions (5.6) and (5.4) to rewrite the adjoint operator of $\bar{\mathcal{L}}^*$, the r.h.s of the previous inequality is also given by

$$\varepsilon \int \bar{f}_t \widetilde{\Psi}_t^{-1}\Big(\varepsilon^{-2}\widetilde{\mathcal{L}}^* - \frac{\partial}{\partial t}\Big)\widetilde{\Psi}_t \, d\mu_{L,r,\theta} + \varepsilon^2 \int \bar{f}_t \sum_x \sum_v \sum_{\alpha=1}^{3} \frac{F_\alpha(\varepsilon x)}{\langle h_0 \rangle \, v_\alpha}\eta(x,v)\, d\mu_{L,r,\theta} \tag{5.22}$$

where $\widetilde{\mathcal{L}}^*$ is the operator $\widetilde{\mathcal{L}}^* = \mathcal{L}^* + \varepsilon^3 \widetilde{\mathcal{L}}^F$, $\widetilde{\mathcal{L}}^F$ was defined in (5.5).

Repeating the argument at the beginning of the proof of Theorem 3.4, it results from the entropy bound (5.11) by straightforward Taylor expansions that there exists a positive constant c_t such that the following inequality holds

$$\lim_{\varepsilon \to 0} \varepsilon^{-1} \int f_t \Big(\widetilde{\mathcal{L}}^* \log \widetilde{\Psi}_t - \widetilde{\Psi}_t^{-1} \widetilde{\mathcal{L}}^* \widetilde{\Psi}_t\Big)\, d\mu_{L,r,\theta} \leq c_t \tag{5.23}$$

As a consequence, the time derivative of the relative entropy satisfies

$$\frac{d}{dt}\varepsilon^{-2}s(\bar{f}_t|\widetilde{\Psi}_t) \leq \mathcal{J} + c_t \tag{5.24}$$

where

$$\begin{aligned}\mathcal{J} &= \varepsilon \int \bar{f}_t \Big(\varepsilon^{-2}\widetilde{\mathcal{L}}^* - \frac{\partial}{\partial t}\Big)\log \widetilde{\Psi}_t \, d\mu_{L,r,\theta} + \varepsilon^2 \int \bar{f}_t \sum_x \sum_v \sum_{\alpha=1}^{3} \frac{F_\alpha(\varepsilon x)}{\langle h_0 \rangle \, v_\alpha}\eta(x,v)\, d\mu_{L,r,\theta} \\ &= \bar{\mathcal{I}} + \varepsilon^2 \int \bar{f}_t \widetilde{\mathcal{L}}^F \log \widetilde{\Psi}_t \, d\mu_{L,r,\theta} + \varepsilon^2 \int \bar{f}_t \sum_x \varphi(\eta_x)\, d\mu_{L,r,\theta}\end{aligned} \tag{5.25}$$

with

$$\varphi(\eta_x) = \sum_v \sum_{\alpha=1}^{3} \frac{F_\alpha(\varepsilon x)}{\langle h_0 \rangle\, v_\alpha} \eta(x,v) \tag{5.26}$$

and $\bar{\mathcal{I}}$ is the same expression as the one introduced in (3.23) but with $\bar{f}_t$ instead of f_t.

We first deal with the third term in the previous equality. We denote by P_x the orthogonal projector of functions of η_x on the subspace generated by the invariants $I_\beta(\eta_x)$, $\beta = 0, \dots, 4$ with respect to the scalar product $(f(\eta_x), g(\eta_x))_x = E^{\Psi_t}[f(\eta_x), g(\eta_x)] - E^{\Psi_t}[f(\eta_x)]E^{\Psi_t}[g(\eta_x)]$. Since the density Ψ_t is close to 1 when ε is small, there exist a function $h(v)$ and a smooth function $G(\varepsilon x)$ such that

$$P_x \varphi(\eta_x) = \sum_{\beta=0}^{4} (\varphi_x, I_\beta) I_\beta(\eta_x) + \varepsilon \sum_x G(\varepsilon x) \sum_v h(v)\eta(x,v) + o(\varepsilon) \tag{5.27}$$

where $(\cdot,\cdot)$ is the scalar product (4.1).

The projection coefficients (φ, I_β) can be easily computed and we obtain the following expressions

$$(\varphi, I_0) = (\varphi, I_4) = 0, \qquad (\varphi, I_\beta) = 1, \;\; \beta = 1, \dots, 3$$

From the entropy bound (5.11) and Lemma 2.2 of [EMY1], we know that

$$E^{f_t}\Big[\varepsilon^3 \sum_x G(\varepsilon x) \sum_v h(v)\eta(x,v)\Big] = E\Big[\varepsilon^3 \sum_x G(\varepsilon x) \sum_v h(v)\eta(x,v)\Big] + o(1) = \text{const} + o(1) \tag{5.28}$$

Therefore

$$\begin{aligned}\varepsilon^2 \int \bar{f}_t \sum_x \varphi(\eta_x)\, d\mu_{L,r,\theta} =& \frac{3\varepsilon^2}{\langle |v|^2 h_0 \rangle} \int \bar{f}_t \sum_x \sum_{\beta=1}^{3} F_\beta(\varepsilon x) I_\beta(\eta_x)\, d\mu_{L,r,\theta} \\ &+ \varepsilon^2 \int \bar{f}_t \sum_x (I - P_x)\varphi(\eta_x)\, d\mu_{L,r,\theta} + \text{const} + o(1)\end{aligned} \tag{5.29}$$

We now turn to the second term in the sum (5.25). First, we have to determine the currents related to the force generator $\widetilde{\mathcal{L}}^F$. Since the mass and the energy are conserved by a flip of velocity on a given site, it is clear that $\widetilde{\mathcal{L}}^F I_\beta(\eta_x) = 0$ for both $\beta = 0$ and $\beta = 4$. Moreover, for $\beta = 1, \dots, 3$, we have

$$\widetilde{\mathcal{L}}^F I_\beta(\eta_x) = -2\kappa I_\beta(\eta_x) - \frac{1}{\langle h_0 \rangle} \sum_v \eta(x,v)(1 - \eta(x, v^{(\beta)}))F_\beta(\varepsilon x) \tag{5.30}$$

Then

$$
\begin{aligned}
&\varepsilon^2 \int f_t \widetilde{\mathcal{L}}^F \log \widetilde{\Psi}_t \, d\mu_{L,r,\theta} \\
&= -\varepsilon^3 E^{\bar f_t} \Big[\sum_x \sum_{\beta=1}^3 \lambda_\beta^{(1)} * \hat{w}(\varepsilon x, t) \frac{F_\beta(\varepsilon x)}{\langle h_0 \rangle} \sum_v \eta(x,v)(1-\eta(x,v^{(\beta)})) \Big] \\
&\quad - 2\kappa \varepsilon^3 E^{\bar f_t} \Big[\sum_x \sum_{\beta=1}^3 \lambda_\beta^{(1)} * \hat{w}(\varepsilon x, t) I_\beta(\eta_x) \Big] + o(1)
\end{aligned}
\tag{5.31}
$$

Because of the entropy bound (5.11), it results from Lemma 2.2 of [EMY1] that we may replace the expectations w.r.t. $\bar f_t d\mu_{L,r,\theta}$ by the expectations w.r.t. $\mu_{L,r,\theta}$ and that such a replacement produces a error term of order ε.

Since $E[I_\beta(\eta_x)] = 0$ and since

$$
E[\eta(x,v)(1-\eta(x,v^{(\beta)})] = \langle h_0 \rangle \tag{5.32}
$$

we obtain

$$
\varepsilon^2 \int \bar f_t \widetilde{\mathcal{L}}^F \log \widetilde{\Psi}_t \, d\mu_{L,r,\theta} = -\varepsilon^3 \sum_x \sum_{\beta=1}^3 \lambda_\beta^{(1)}(\varepsilon x, t) F_\beta(\varepsilon x) + o(1) = \text{const} + o(1) \tag{5.33}
$$

Finally, following step by step Sect. 3, the first term $\bar{\mathcal{I}}$ in the sum (5.25) satisfies inequality (3.38) when f_t is replaced by $\bar f_t$. So putting all together, we have shown that $\frac{d}{dt}\varepsilon^{-2} s(\bar f_t | \widetilde{\Psi}_t)$ is bounded above by

$$
\begin{aligned}
&E^{\bar f_t} \Big[\varepsilon \sum_x \bar\Gamma_x(\bar I_k^+(x)) \Big] + \frac{c}{\gamma} \sum_{\alpha=1}^3 \sum_{\beta=0}^4 V(H_\alpha^\beta, r, \theta) + \delta^{-1} \varepsilon^{-2} s(\bar f_t, \tilde\Psi_t) \\
&+ \gamma \varepsilon^{-1} D_{\Lambda_L}(\sqrt{\bar f_t}) + \text{const} + o(1)
\end{aligned}
\tag{5.34}
$$

where

$$
\begin{aligned}
\bar\Gamma_x(Y) =& \Gamma_x^1(Y) + \Gamma_x^2(Y) \\
\Gamma_x^1(Y) =& \Gamma_x(Y) + \varepsilon \sum_{\beta=1}^3 \frac{3F_\beta(\varepsilon x)}{\langle |v|^2 h_0 \rangle} Y_\beta \\
\Gamma_x^2(Y) =& \varepsilon E^{\mu_{L,n(Y)}} \big[(I-P_x)\varphi(\eta_x) \big]
\end{aligned}
\tag{5.35}
$$

with Γ_x defined in (3.37).

As in Sect. 3, the constant "const" is given by the relation

$$
\varepsilon \sum_x \bar\Gamma_x(\bar m_x) + \text{const} = o(1) \tag{5.36}
$$

with $\bar{m}_x = (\bar{m}_x^0, \dots, \bar{m}_x^4)$ and $\bar{m}_x^\beta = E^{\Psi_t}[I_\beta(\eta_x)]$. Therefore, it follows from the entropy inequality that

$$\frac{d}{dt}\varepsilon^{-2}s(\bar{f}_t|\widetilde{\Psi}_t) \leq \frac{c}{\gamma}\sum_{\alpha=1}^{3}\sum_{\beta=0}^{4} V(H_\alpha^\beta, r, \theta) + (\delta^{-1}+1)\varepsilon^{-2}s(\bar{f}_t, \tilde{\Psi}_t) + \gamma\varepsilon^{-1}D_{\Lambda_L}(\sqrt{\bar{f}_t}) + o(1)$$
$$+ \varepsilon \log E^{\Psi_t}\Big[\exp\Big\{\varepsilon \sum_x \big(\bar{\Gamma}(\bar{I}_k^+(x)) - \bar{\Gamma}(\bar{m}_x)\big)\Big\}\Big] \tag{5.37}$$

In the case where there is a force term, we have entropy estimates (cf. Proposition 5.3) similar to (3.8) and (3.9), therefore the large deviation lemma of [EMY3] can be still applied and Theorem 5.2 will be proved if $\bar{\Gamma}_x$ satisfies

$$\frac{\partial \bar{\Gamma}_x}{\partial Y_\beta}\Big|_{Y_\beta = \bar{m}_x^\beta} = 0, \quad \beta = 0, \dots, 4 \tag{5.38}$$

Since by construction $(I - P_x)\varphi(\eta_x)$ is orthogonal to the invariants $I_\beta(\eta_x)$ w.r.t. the scalar product $(\cdot,\cdot)_x$, it is clear that

$$\frac{\partial \Gamma_x^2}{\partial Y_\beta}\Big|_{Y_\beta = \bar{m}_x^\beta} = 0, \quad \beta = 0, \dots, 4 \tag{5.39}$$

so that it remains to check that

$$\frac{\partial \Gamma_x}{\partial Y_\beta}\Big|_{Y_\beta = \bar{m}_x^\beta} + \varepsilon\delta_{i,\beta}\sum_x\sum_v \frac{3f_\beta(\varepsilon x)}{\langle |v|^2 h_0\rangle} = 0, \quad \beta = 0, \dots, 4, \quad i = 1, 2, 3 \tag{5.40}$$

Since the only change with respect to the computations that have been done in the last part of Sect. 3 consists in the extra term $\varepsilon\sum_x\sum_v \frac{3F_\beta(\varepsilon x)}{\langle |v|^2 h_0\rangle}$ in the expression of $\frac{\partial \Gamma_x}{\partial Y_\beta}$ for $\beta = 1, \dots, 3$, we get (5.40) because the chemical potentials $\underline{\lambda}^{(1)}$ now satisfy (5.8). □

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