Hölder Regularity and Dimension Bounds for Random Curves

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Dedicated to the memory of Roland L. Dobrushin*:*
Exemplary in science and in life.

Abstract Random systems of curves exhibiting fluctuating features on arbitrarily small scales (δ) are often encountered in critical models. For such systems it is shown that scale-invariant bounds on the probabilities of crossing events imply that typically all the realized curves admit Hölder continuous parametrizations with a common exponent and a common random prefactor, which in the scaling limit ($\delta \to 0$) remains stochastically bounded. The regularity is used for the construction of scaling limits, formulated in terms of probability measures on the space of closed sets of curves. Under the hypotheses presented here the limiting measures are supported on sets of curves which are Hölder continuous but not rectifiable, and have Hausdorff dimensions strictly greater than one. The hypotheses are known to be satisfied in certain two dimensional percolation models. Other potential applications are also mentioned.

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1. Introduction

1.a General framework

We consider here curves in R^d which are shaped on many scales, in a manner found in various critical models. The framework for the discussion are random systems, where the random object is expressed as a closed collection of polygonal curves of a small step size δ. Our main results are general criteria for establishing that, as the short distance cutoff is sent to zero, the curves retain a certain degree of regularity, yet at the same time are intrinsically rough. In some cases the object of study is a single random curve. In others, the random object contains many curves; in such situations, the regularity estimates are intended to cover the entire collection.

The criteria developed here can be applied to various stochastic geometric models. In the Appendix we mention as examples critical percolation, the minimal spanning trees in random geometry, the frontier of two-dimensional Brownian motion, and the level sets of a two-dimensional random field.

While our discussion does not require familiarity with any of these examples, let us comment that an important feature they share is the existence of two very different length scales: the *microscopic* scale on which the model's building variables reside, and the *macroscopic* scale on which the connected curves are tracked. In such situations it is natural to seek a meaningful formulation for the *scaling limit*, at which the microscopic scale (δ) is taken to zero. The regularity established here enables such a construction through compactness arguments.

To introduce the results let us start with some of the terminology.

i. We denote by $\mathcal{S}_\Lambda$ the space of curves in a closed subset $\Lambda \subset \mathbf{R}^d$, with the metric defined in Section 2.
The symbol $\mathcal{C}$ is reserved here for individual curves, and $\mathcal{F}$ for sets of curves. The space of closed sets of curves in Λ is denoted Ω_Λ.

ii. A *configuration of curves* in $\mathbf{R}^d$ with a short-distance cutoff $\delta \in (0,1]$ is a collection of polygonal paths of step size δ which forms a closed subset $\mathcal{F}_\delta (\subset \mathcal{S}_\Lambda)$.

iii. A *system of random curves* with varying short-distance cutoff is described by a collection of probability measures $\{\mu_\delta(d\mathcal{F}_\delta)\}_{0<\delta\leq\delta_{max}}$ on Ω_Λ, where each μ_δ describes random sets of curves in Λ consisting of polygonal paths of step size δ. (We shall often take $\delta_{max} = 1$).

To summarize: the individual realizations of the random systems are closed sets of curves denoted $\mathcal{F}_\delta(\omega)$. The entire system will occasionally also be referred to by the symbol $\mathcal{F}$.

Figure 1: Schematic depiction of some of the realized paths in a critical percolation model. The model is formulated (possibly on a lattice) on a scale which is much smaller than the length of the depicted region. Typical configurations exhibit large-scale connected clusters, but the connections are tenuous. In this work we discuss the regularity properties of the self-avoiding paths supported on such clusters.

We are particularly interested in statements concerning the probability measures $\{\mu_\delta(d\mathcal{F}_\delta)\}_{0<\delta\leq\delta_{max}}$ which hold uniformly in δ, and thus provide information about the scaling limit. The following notion facilitates the formulation of such statements.

Definition A random variable X associated with a system $\mathcal{F}$ is said to be *stochastically bounded*, for $\delta \to 0$, if:

i. a version X_δ is defined for each $0 < \delta(\leq \delta_{max})$, and

ii. for every $\varepsilon > 0$ there is $u < \infty$ such that

$$Prob_\delta\left(|X_\delta(\omega)| \geq u\right) \ \leq \ \varepsilon \tag{1.1}$$

uniformly in δ.

A random variable is said to be *stochastically bounded away from zero* if its inverse is stochastically bounded.

1.b Main results

The main theorems in this paper are derived under two hypotheses, which require scale-invariant bounds on certain crossing events. In order to apply these results, the

Figure 2: A 6-fold crossing of an annulus $D(x; r, R)$. Under the hypothesis **H1** the probability of a k-fold crossing of an annulus $D(x; r, R)$ is bounded by $K_k(r/R)^{\lambda(k)}$. The regularity estimates are based on the absence of k-crossings for k large enough and $0 < r << R = r^{1-\varepsilon} << 1$.

hypotheses need to be verified using specific information on the given system. The conditions are known to be satisfied in certain two dimensional critical percolation models, through the "Russo-Seymour-Welsh theory" [1, 2] and its recent extensions [3]. They are expected to be true for critical percolation also in higher dimensions, though not for $d > 6$ [4]. Another proven example is the minimal spanning tree in two dimensions [5], where the criticality is *self-induced* in the sense of [6]. These, and other systems, are presented in the Appendix.

The first condition concerns repeated crossings of the spherical shells

$$D(x; r, R) = \Big\{ y \in \mathbf{R}^d \mid r \leq |y - x| \leq R \Big\} \ . \tag{1.2}$$

The assumption is:

(H1) *Power-bound on the probability of multiple crossings.* For all $k < \infty$ and for all spherical shells with radii $\delta \leq r \leq R \leq 1$, the following bound holds uniformly in δ

$$Prob_\delta \left(\begin{array}{c} D(x; r, R) \text{ is traversed by } k \text{ separate} \\ \text{segments of a curve in } \mathcal{F}_\delta(\omega) \end{array} \right) \ \leq \ K_k \left(\frac{r}{R} \right)^{\lambda(k)} \tag{1.3}$$

with some $K_k \leq \infty$ and

$$\lambda(k) \ \to \infty \qquad \text{as } k \to \infty \ . \tag{1.4}$$

Based on this assumption, we derive the following result.

Theorem 1.1 (Regularity) *Let $\mathcal{F}$ be a system of random curves in a compact region $\Lambda \subset \mathbf{R}^d$, with variable short-distance cutoff $\delta > 0$, and assume the hypothesis* **H1** *is satisfied. Then for each $\varepsilon > 0$ all the curves $\mathcal{C} \in \mathcal{F}_\delta(\omega)$ can be simultaneously parametrized by continuous functions $\gamma(t)$ $t \in [0,1]$, such that for all $0 \le t_1 < t_2 \le 1$:*

$$|\gamma(t_1) - \gamma(t_2)| \ \le \kappa_{\varepsilon;\delta}(\omega) \ g(diam(\mathcal{C}))^{1+\varepsilon} \ \ |t_1 - t_2|^{\frac{1}{d-\lambda(1)+\varepsilon}} \ , \tag{1.5}$$

with a random variable $\kappa_{\varepsilon;\delta}(\omega)$ (common to all $\mathcal{C} \in \mathcal{F}_\delta(\omega)$) which stays stochastically bounded as $\delta \to 0$. The second factor depends on the curve's diameter through the function

$$g(r) \ = \ r^{-\frac{\lambda(1)}{d-\lambda(1)}} \ . \tag{1.6}$$

Remarks: *i.* The conclusion of Theorem 1.1 is stated in terms of the existence of Hölder continuous parametrizations, which offers a familiar criterion for regularity. We actually find it convenient to develop the results in terms of *tortuosity bounds*, that is, upper bounds on the function $M(\mathcal{C}, \ell)$ defined as the minimal number of segments produced if the curve $\mathcal{C}$ is partitioned into segments of diameter no greater that ℓ. The two notions are linked in Section 2.

ii. The dependence of the Hölder constant in eq. (1.5) on the diameter of the curve can be removed by lowering the Hölder exponent below $1/d$; indeed, interpolating between eq. (1.5) and the trivial relation $|\gamma(t_1) - \gamma(t_2)| \ \le \ \mathrm{diam}(\mathcal{C})$ gives

$$|\gamma(t_1) - \gamma(t_2)| \ \le (\kappa_{\varepsilon;\delta}(\omega))^{1+\tilde{\varepsilon}-d/\lambda} \ |t_1 - t_2|^{\frac{1}{d+\tilde{\varepsilon}}} \ , \tag{1.7}$$

where $\tilde{\varepsilon}$ is small when ε is small.

iii. For the main conclusion that the curves retain Hölder continuity at *some* $\alpha > 0$, it suffices to require instead of **H1** that eq. (1.3) holds for some k with $\lambda(k) > d$. While this is clearly a weaker assumption than eq. (1.4), so far it has been proven only in situations in which also (1.4) holds.

In addition to being of intrinsic interest, the above regularity property permits to construct the scaling (continuum) limit. The basic question concerning this limit is:

Q1. Is the collection of probability measures $\{\mu_\delta(d\mathcal{F}_\delta)\}_\delta$ tight?

Tightness means that, up to remainders that can be made arbitrarily small, the measures μ_δ share a common compact support. A positive answer to *Q1* implies the existence of limits $\lim \mu_{\delta,p(\delta)}$ at least along some sequences of $\delta_n \to 0$ ([7]). Without tightness, one cannot rule out the possibility that, as the cutoff is removed, curves give way to more general continua. The range of possibilities is rather vast: curves can converge (in a weaker sense than used here) to continua which do not support any continuous curve.

Theorem 1.1 yields a positive answer to *Q1*, as is explained here in Section 4. There are a number of dimension-like quantifiers for the description of the curves emerging in the scaling limit. Among them are (see Section 2)

- the Hausdorff dimension $dim_{\mathcal{H}}\mathcal{C}$,
- the upper box dimension (also known as the Minkowski dimension) $\overline{dim}_B\mathcal{C}$,
- and the reciprocal of the optimal Hölder regularity exponent

$$\alpha(\mathcal{C}) \;=\; \sup\left\{\alpha \middle| \begin{array}{c} \mathcal{C} \text{ can be parametrized as } \{\gamma(t)\}_{0\leq t\leq 1} \text{ with} \\ |\gamma(t)-\gamma(t')| \;\leq\; K_\alpha\,|t-t'|^\alpha \text{ , for all } 0\leq t\leq t'\leq 1 \end{array}\right\} . \tag{1.8}$$

The following result is derived in Section 4.

Theorem 1.2 (Scaling limit) *For any system $\mathcal{F}$ of random curves in a compact set $\Lambda \subset \mathbf{R}^d$, hypothesis* **H1** *implies that the limit*

$$\lim_{n\to\infty} \mu_{\delta_n}(d\mathcal{F}) \;=:\; \mu(d\mathcal{F}) \tag{1.9}$$

exists at least for some sequence $\delta_n \to 0$. The limiting probability measure (on Ω_Λ) is supported on configurations $\mathcal{F}$ containing only paths with

$$\overline{dim}_B\mathcal{C} \;=\; \alpha(\mathcal{C})^{-1} \;\leq\; d-\lambda(1) \tag{1.10}$$

and

$$dim_{\mathcal{H}}(\mathcal{C}) \;\leq d-\lambda(2)\;. \tag{1.11}$$

(The improvement in the dimension estimate of the last inequality over the preceding one is based on considerations of the *"backbone"*.)

The sense of convergence in eq. (1.9) can be expressed by saying that there exists a family of *couplings* consisting of probability measures $\rho_n(d\mathcal{F}_{\delta_n}, d\mathcal{F})$ such that:

i) the marginal distributions satisfy

$$\rho_n(d\mathcal{F}_{\delta_n}, \Omega_\Lambda) \;=\; \mu_{\delta_n}(d\mathcal{F}_{\delta_n})\;, \qquad \rho_n(\Omega_\Lambda, d\mathcal{F}) \;=\; \mu(d\mathcal{F}) \tag{1.12}$$

Figure 3: A simultaneous crossing event for a family of cylinders of common aspect ratio. In hypothesis **H2** the probability of such an events is assumed to be less than $Const.\rho^k$, with $\rho < 1$ (here $k = 4$). The implication is a uniform lower bound on the Hausdorff dimensions of all the curves in the configuration.

and

ii) the two components are close, in the sense that

$$\int_{\Omega_\Lambda \times \Omega_\Lambda} dist(\mathcal{F}_{\delta_n}, \mathcal{F}) \;\; \rho_n(d\mathcal{F}_{\delta_n}, d\mathcal{F}) \underset{n\to\infty}{\longrightarrow} \; 0 \;, \tag{1.13}$$

with the distance between two configurations of curves defined by the Hausdorff metric:

$$dist(\mathcal{F}, \mathcal{F}') \leq \varepsilon \quad \Leftrightarrow \quad \left\{ \begin{array}{c} \text{for every } \gamma \in \mathcal{F} \text{ there is } \gamma' \in \mathcal{F}' \\ \text{with } \sup_t |\gamma(t) - \gamma'(t)| \leq \varepsilon \\ \text{and } \textit{vice-versa } (\mathcal{F} \leftrightarrow \mathcal{F}') \end{array} \right. \quad . \tag{1.14}$$

The positive answer to *Q1* invites a number of other questions, such as:

Q2. Is the limit independent of the sequence $\{\delta_n\}$, and is it shared by other models with different short-scale structure?

This question is beyond the scope of the present work. In some of the models of interest it is related to the purported universality of critical behavior.

To establish some minimal roughness for the realized curves, we require a hypothesis on the probability of simultaneous crossings of a family of cylinders. It suffices to restrict the assumption to spatially separated cylinders, with much latitude in the exact definition of the term.

Definition A collection of sets $\{A_j\}$ is *well separated* if the distance of each set A_j to the other sets $\{A_i\}_{i\neq j}$ is at least as large as the diameter of A_j.

The relevant hypothesis is:

(H2) *Power bound on the probability of simultaneous crossings.* There exist a cross section $\sigma > 0$ and some $\rho < 1$ with which for every collection of k well separated cylinders, $A_1, \ldots, A_k$, of aspect ratio σ and lengths $\ell_1, \ldots, \ell_k \geq \delta$

$$\mathrm{Prob}_\delta \left(\begin{array}{c} \text{each } A_j \text{ is traversed (in the} \\ \text{long direction) by a curve in } \mathcal{F}_\delta \end{array} \right) \leq K\rho^k . \tag{1.15}$$

An effective way to express curve roughness is in terms of *capacity* lower bounds. The capacity $\mathrm{Cap}_{s;\ell}(A)$ of a set $A \subset \mathbf{R}^d$ is defined in Section 5. For the purpose of this summary it suffices to note that the capacity of a fixed set A increases with the parameter ℓ, and that it provides the following lower bounds on coverings.

i. For every covering of A by sets $\{B_j\}$ of diameter at least ℓ,

$$\sum_j (\mathrm{diam} B_j)^s \geq \mathrm{Cap}_{s;\delta}(A) . \tag{1.16}$$

ii. The minimal number of elements for a covering of A by sets of diameter ℓ satisfies

$$N(\mathcal{C}, \ell) \geq \mathrm{Cap}_{s;\ell}(A) \cdot \ell^{-s} . \tag{1.17}$$

iii. The behavior of the capacity for small ℓ provides information on the *Hausdorff dimension*:

$$\inf_{0<\ell\leq 1} \mathrm{Cap}_{s;\ell}(A) > 0 \qquad \Rightarrow \qquad dim_{\mathcal{H}}(A) \geq s . \tag{1.18}$$

(Proof of *i.* is given in Section 5; *ii.* and *iii.* are direct consequences.)

Theorem 1.3 (Roughness) *If a system $\mathcal{F}$ of random curves in a compact subset $\Lambda \subset \mathbf{R}^d$ with a variable short distance cutoff satisfies the hypothesis* **H2**, *then there exists $d_{min} > 1$ such that for any fixed $r > 0$ and $s > d_{min}$ the random variable*

$$T_{s,r;\delta}(\omega) := \inf_{\mathcal{C}\in\mathcal{F}:\ diam(\mathcal{C})\geq r} \mathrm{Cap}_{s;\delta}(\mathcal{C}) \tag{1.19}$$

stays stochastically bounded away from zero, as $\delta \to 0$.

Furthermore, any scaling limit of the measures μ_δ, $\mu = \lim_{\delta_n \to 0} \mu_{\delta_n}$, is supported on configurations containing only curves with

$$dim_{\mathcal{H}}\mathcal{C} \geq d_{min} \ (> 1) . \tag{1.20}$$

In particular, the scaling limit contains no rectifiable curves.

Let us note that Theorem 1.3 complements Theorem 1.1, since by the monotonicity properties of the capacity, we can combine eq. (1.19) with eq. (1.17) to obtain

$$N(\mathcal{C},\ell) \;\geq\; \mathrm{Cap}_{s,\ell}(\mathcal{C}) \cdot \ell^{-s} \;\geq\; T_{s,r;\delta}(\omega)\; \ell^{-s} \tag{1.21}$$

for all $\ell \geq \delta$ whereas under the condition (1.5)

$$N(\mathcal{C},\ell) \;\leq\; \left[\frac{\kappa_{\varepsilon;\delta}(\omega)\; g(\mathrm{diam}\mathcal{C})^{1+\varepsilon}}{\ell}\right]^{d-\lambda(1)+\varepsilon} . \tag{1.22}$$

In particular, eq. (1.21) implies that the minimal number of steps of the lattice size (δ) needed in order to advance distance L exceeds $Const.\ (L/\delta)^{\tau}$. Some bounds of this form were previously obtained for the "lowest path" in 2D critical percolation models in the work of Kesten and Zhang [8], who refer to the optimal value of τ as the *tortuosity* exponent. We slightly modify their terminology, by requiring the power bounds to hold simultaneously on all scales.

The assumption in Theorem 1.3 can be weakened by restricting eq. (1.15) to collections of cylinders of comparable dimensions, but then the conclusion will be stated in terms of the box dimension.

In models where spatially separated events are independent, condition **H2** is implied by **H1**, provided the parameter $\lambda(1)$ of eq. (1.4) is positive. A similar observation applies to models without strict independence but with a correlation length of only microscopic size, such as the droplet percolation model.

There is a considerable disparity between the upper and the lower bounds derived here for the dimensions of curves in the scaling limit. Part of the reason is that our lower bounds are far from sharp. However, we also expect some of the systems considered here (e.g. the percolation models) to exhibit simultaneously curves of different dimensions.

The organization of the paper is as follows. In Section 2 we prepare for the discussion of random systems by clarifying some notions pertaining to single curves. Introduced there is the concept of *tortuosity*, which provides a measure of roughness manifestly independent of parametrization. The associated tortuosity exponent coincides with Richardson's exponent D. It is related here to the degree of Hölder regularity achievable through reparametrization (Theorem 2.3). Moreover, under the *tempered crossing* condition the tortuosity exponent coincides with the curve's upper box dimension (Theorem 2.5). In Section 3 we apply these relations to general systems of random curves, and prove the regularity result, Theorem 1.1. To tighten the regularity estimate we briefly discuss the concept of the *backbone*. Section 4 deals with the construction of scaling limits and the proof of Theorem 1.2, based on the afore-mentioned regularity properties. The proof of the roughness lower bounds is split into two parts. In Section 5 we derive a deterministic statement (Theorem 5.1) which presents a criterion for the roughness of a curve, based on the assumption that the *straight runs* of the curve are *sparse*. The analysis exploits the relation of the dimension with *capacity*, and involves suitable energy estimates. In Section 6, we apply this result to random systems, and prove Theorem 1.3 by establishing that **H2** implies the sparsity of straight runs. The Appendix includes examples of systems for which the general Theorems yield results of interest within the specific context.

2. Analysis of curves through tortuosity

In this section we introduce the space of curves, and the notion of tortuosity. The two basic results are Theorem 2.3, which relates the tortuosity exponent with the optimal Hölder continuity exponent, and Theorem 2.5 which provides useful conditions under which the tortuosity exponent agrees with the upper box dimension, and thus is finite.

2.a The space of curves

We regard curves as equivalence classes of continuous functions, modulo reparametrizations. More precisely, two continuous functions f_1 and f_2 from the unit interval into $\mathbf{R}^d$ describe the same curve if and only if there exist two monotone continuous bijections $\phi_i : [0,1] \to [0,1]$, $i = 1, 2$, so that $f_1 \circ \phi_1 = f_2 \circ \phi_2$.

Recall that the space of curves in a closed subset $\Lambda \subset \mathbf{R}^d$ is denoted here by S_Λ. The distance between two curves is measured by:

$$\mathrm{d}(\mathcal{C}_1, \mathcal{C}_2) \ := \ \inf_{\phi_1,\phi_2} \ \sup_{t\in[0,1]} |f_1(\phi_1(t)) - f_2(\phi_2(t))| \ , \tag{2.1}$$

where f_1 and f_2 is any pair of continuous functions representing $\mathcal{C}_1$ and $\mathcal{C}_2$, and the infimum is over the set of all strictly monotone continuous functions from the unit interval onto itself.

Lemma 2.1 *Equation (2.1) defines a metric on the space of curves.*

Proof: Clearly, $\mathrm{d}(\mathcal{C}_1, \mathcal{C}_2)$ is nonnegative, symmetric, satisfies the triangle inequality and $\mathrm{d}(\mathcal{C}, \mathcal{C}) = 0$. To prove strict positivity, assume $\mathrm{d}(\mathcal{C}_1, \mathcal{C}_2) = 0$, and choose parametrizations f_1 and f_2. We need to show that f_1 and f_2 describe the same curve, i.e., $\mathcal{C}_1 = \mathcal{C}_2$. We may choose f_1 and f_2 to be non-constant on any interval. Under these assumptions, there exist sequences of reparametrizations ϕ_1^i and ϕ_2^i such that

$$\sup_{t\in[0,1]} \left| f_1 \circ \phi_1^i \circ (\phi_2^i)^{-1}(t) - f_2(t) \right| \ = \ \sup_{t\in[0,1]} \left| f_1 \circ \phi_1^i(t) - f_2 \circ \phi_2^i(t) \right| \ \underset{i\to\infty}{\longrightarrow} \ 0 \ ; . \tag{2.2}$$

Monotonicity and uniform boundedness imply (Helly's theorem) that there are subsequences (again denoted ϕ_1^i and ϕ_2^i) so that $\phi_2^i \circ (\phi_1^i)^{-1}$ and their inverses $\phi_1^i \circ (\phi_2^i)^{-1}$ converge pointwise, at all but countably many points, to monotone limiting functions ϕ and $\tilde{\phi}$, with $f_1 = f_2 \circ \phi$ and $f_2 = f_1 \circ \tilde{\phi}$. To see that ϕ has no discontinuities, note that jumps of ϕ would correspond to intervals where $\tilde{\phi}$ is is constant. But $\tilde{\phi}$ cannot be constant on an interval, since, by our choice of parametrization, f_2 is not constant on any interval. ∎

With this metric, $\mathcal{S}_\Lambda$ is complete but, even for compact Λ it is not compact. This reflects the properties of the space of continuous functions $C([0,1],\Lambda)$.

2.b Measures of curve roughness

Let $M(\mathcal{C},\ell)$ be the minimal number of segments needed for a partition of a curve $\mathcal{C}$ into segments of diameter no greater than ℓ. Any bound on $M(\mathcal{C},\ell)$ will be called a *tortuosity bound*. In particular, we are interested in power bounds of the form

$$M(\mathcal{C},\ell) \;\leq\; K_s\,\ell^{-s}\,. \tag{2.3}$$

Optimization over the exponents yields the following dimension-like quantity.

Definition *For a given curve $\mathcal{C}$,*

$$\tau(\mathcal{C}) \;=\; \inf\{s>0|\; \ell^s\, M(\mathcal{C},\ell) \underset{\ell\to 0}{\longrightarrow} \; 0\} \tag{2.4}$$

is called the tortuosity exponent*.*

There are a number of other ways of dividing a curve to short segments which yield comparable results. Of particular interest to us is the observation that the tortuosity exponent can also be based on $\tilde{M}(\mathcal{C},\ell)$, which we define as the maximal number of points that can be placed on the curve so that successive points have distance at least ℓ. $M(\mathcal{C},\ell)$ and $\tilde{M}(\mathcal{C},\ell)$ are comparable, but have different continuity properties.

Lemma 2.2 *$M(\mathcal{C},\ell)$ and $\tilde{M}(\mathcal{C},\ell)$ are related by the inequalities*

$$M(\mathcal{C},3\ell) \;\leq\; \tilde{M}(\mathcal{C},\ell) \;\leq\; \inf_{\varepsilon} M(\mathcal{C},\ell-\varepsilon)\,. \tag{2.5}$$

Furthermore, $M(\mathcal{C},\ell)$ is lower semicontinuous, and $\tilde{M}(\mathcal{C},\ell)$ is upper semicontinuous on the space of curves.

Proof The first inequality holds because a segment of the curve of diameter at least 3ℓ certainly contains a point that has a distance of at least ℓ from both endpoints. The second inequality holds because no segment of diameter less than ℓ can contain two points of distance ℓ or more. The continuity properties follow easily from the fact that M was defined through minimization and $\tilde{M}$ through maximization of cut points. ∎

It follows from Lemma 2.2 that the tortuosity exponent coincides with Richardson's exponent D [9, 10], which in ref. [11] was termed the "divider dimension". It was pointed out there that D can take arbitrarily large values.

From a different perspective, the curve's regularity may be expressed through the degree of *Hölder continuity* achievable through reparametrization. One attempts to describe the curve by means of a continuous function $\mathcal{C} = \{\gamma(t)\}_{0\leq t\leq 1}$, satisfying:

$$|\gamma(t_1) - \gamma(t_2)| \;\leq\; K_\alpha\, |t_1 - t_2|^\alpha \quad \text{for all } 0 \leq t_1 \leq t_2 \leq 1, \tag{2.6}$$

with some exponent $\alpha > 0$. Greater values of the exponent correspond to higher degrees of regularity, and thus one is interested in

$$\alpha(\mathcal{C}) \;=\; \sup\{\alpha |\; \mathcal{C} \text{ admits a parametrization satisfying eq. (2.6) with exponent } \alpha\}\ . \tag{2.7}$$

The tortuosity exponent may remind one of the *upper box dimension*, which has a similar definition. Let $N(\mathcal{C},\ell)$ be the minimal number of sets of diameter ℓ needed to cover the curve. Then

$$\overline{dim}_B(\mathcal{C}) \;:=\; \inf\{s > 0|\; \ell^s\, N(\mathcal{C},\ell) \underset{\ell\to 0}{\longrightarrow} 0\}\ . \tag{2.8}$$

The two definitions are different, since a single set of diameter ℓ may contain a large number of segments of the curve. The box dimension can be calculated using only coverings with boxes taken from subdivisions of a fixed grid.

A trivial relation between the three parameters is

$$\overline{dim}_B(\mathcal{C}) \;\leq\; \tau(\mathcal{C}) \;\leq\; \alpha(\mathcal{C})^{-1}\ , \tag{2.9}$$

which follows immediately from

$$N(\mathcal{C},\ell) \;\leq M(\mathcal{C},\ell) \;\leq\; \left\lceil \frac{K_\alpha}{\ell} \right\rceil^{1/\alpha}\ , \tag{2.10}$$

where $\lceil x \rceil$ denotes the smallest integer at least as large as x.

2.c Tortuosity and Hölder continuity

It turns out that the tortuosity exponent and the optimal Hölder exponent are directly related:

Theorem 2.3 *For any curve $\mathcal{C}$ in $\mathcal{S}_{\mathbf{R}^d}$,*

$$\tau(\mathcal{C}) \;=\; \alpha(\mathcal{C})^{-1} \,. \tag{2.11}$$

More explicitly, uniform continuity is equivalent to a uniform upper tortuosity bound, as expressed in the following lemma.

Lemma 2.4 *If a curve $\mathcal{C}$ in $\mathbf{R}^d$ admits a parametrization as $\{\gamma(t)\}_{0\le t\le 1}$ so that for all t_1, t_2 in the unit interval*

$$\psi(|\gamma(t_1)-\gamma(t_2)|) \;\le\; |t_1-t_2| \;, \tag{2.12}$$

where $\psi : (0,1] \to (0,1]$ is a nondecreasing function, then, for all $\ell \le 1$,

$$M(\mathcal{C},\ell) \;\le\; \left\lceil \frac{1}{\psi(\ell)} \right\rceil \,. \tag{2.13}$$

Conversely, if

$$M(\mathcal{C},\ell) \;\le\; \frac{1}{\psi(\ell)} \tag{2.14}$$

for all $\ell \le 1$, then $\mathcal{C}$ can be parametrized as $\{\gamma(t)\}_0$ with a function satisfying

$$\tilde{\psi}(|\gamma(t_1)-\gamma(t_2)|) \;\le\; |t_1-t_2| \;, \tag{2.15}$$

for all $0 \le t_1 < t_2 \le 1$, with

$$\tilde{\psi}(\ell) \;=\; \frac{\psi(\ell/2)}{2(\log_2(4/\ell))^2} \,. \tag{2.16}$$

Proof: The tortuosity bound in eq. (2.13) follows from the uniform continuity condition in eq. (2.12) with the definition of $M(\mathcal{C},\ell)$, by partitioning the curve into segments corresponding to time intervals of length $\psi(\ell)$. To prove the reverse implication, we need to construct a parametrization satisfying the uniform continuity condition in eq. (2.15), given that eq. (2.14) holds for $0 < \ell \le 1$. Choose an auxiliary parametrization of the curve, $\mathcal{C} = \{\gamma(s)\}$, which is not constant on any interval. We associate with each curve segment, $\mathcal{C}_s = \gamma([0,s])$, the *"time of travel"*

$$t_\varepsilon(s) \;:=\; \frac{\sum_n (n+1)^{-2}\; \psi(\ell_n)\; M(\mathcal{C}_s,\ell_n)}{\sum_n (n+1)^{-2}\; \psi(\ell_n)\; M(\mathcal{C},\ell_n)} \,. \tag{2.17}$$

with $\ell_n = 2^{-n}$. Clearly, t is a strictly increasing continuous function of s, and hence defines a reparametrization of C. The denominator satisfies

$$\sum_n (n+1)^{-2} \; \psi(\ell_n) \, M(\mathcal{C}, \ell_n) \;\leq\; \sum_n (n+1)^{-2} \;<\; 2 \tag{2.18}$$

by the assumption (2.14). Consider two points $\gamma(s_1)$ and $\gamma(s_2)$ (with $s_1 < s_2$) that are at least Δq apart, and let Δt be the corresponding time difference. For large n we observe that

$$\ell_n \;<\; \Delta q \qquad \Longrightarrow \qquad M(\mathcal{C}_{s_2}, \ell_n) - M(\mathcal{C}_{s_1}, \ell_n) \geq 1 \; . \tag{2.19}$$

It follows that

$$\begin{aligned} \Delta t \;&\geq\; \frac{1}{2} \sum_{n:\ \ell_n < \Delta q} (n+1)^{-2} \; \psi(\ell_n) \\ &\geq\; \frac{\psi(\Delta q/2)}{2(\log_2(4/\Delta q))^2} \; , \end{aligned} \tag{2.20}$$

as claimed in eq. (2.15). ∎

2.d Tortuosity and box dimension

In view of Theorem 2.3 it is important for us to have conditions implying finiteness of the tortuosity exponent. It is also of interest to have efficient estimates of the exponent's value. Both goals are accomplished here through a criterion for the equality of $\tau(\mathcal{C})$ with the upper box dimension $\overline{dim}_B(\mathcal{C})$, which is relatively easier to estimate (and never exceeds d). Some criterion is needed since in general the tortuosity exponent may exceed the upper box dimension, and may even be infinite [11].

Definition

*i. We say that a curve $\mathcal{C}$ in R^d exhibits a k-*fold crossing of power ϵ, *at the scale $r \leq 1$ if it traverses k times some spherical shell $D(x; r^{1+\varepsilon}, r)$ (in the notation of eq. (1.2)).*

ii. A curve has the tempered-crossing property *if for every $0 < \epsilon < 1$ there are $k(\epsilon) < \infty$ and $0 < r_o(\epsilon) < 1$ such that on scales smaller than $r_o(\epsilon)$ it has no $k(\varepsilon)$-fold crossing of power ϵ.*

Note that the condition places restrictions on crossings at arbitrarily small scales; however, it is less restrictive at smaller scales since it rules out only crossings of spherical shells with increasingly large aspect ratio.

Theorem 2.5 *If a curve $\mathcal{C}$ has the tempered-crossing property, then*

$$\tau(\mathcal{C}) \ = \ \overline{dim}_B \mathcal{C} \ . \tag{2.21}$$

In particular, $\mathcal{C}$ admits Hölder continuous parametrizations with every exponent $\alpha < (\overline{dim}_B(\mathcal{C}))^{-1}$.

Proof: Since $M(\mathcal{C}, \ell) \leq N(\mathcal{C}, \ell)$, it is always true that $\tau(\mathcal{C}) \ \geq \ \overline{dim}_B \mathcal{C}$ (equation (2.9)). To establish the opposite inequality we first prove that if a curve $\mathcal{C}$ has no k-fold crossings of power ε at the scale ℓ then

$$M(\mathcal{C}, 2\ell) \ \leq \ k\, N(\mathcal{C}, \ell^{1+\varepsilon}) \ . \tag{2.22}$$

To prove eq. (2.22) we recursively partition the curve into segments of diameter at most 2ℓ. The segments are defined by a sequence of points x_i along the curve. We start with $x_1 = \gamma(0)$. After $x_1, \ldots, x_n$ are determined, the next point x_{n+1} is taken as the site of the first exit, after n_n, of γ from the ball of radius ℓ about x_n; if γ does not leave this ball, we terminate.

The number of stopping points produced by this algorithm is clearly an upper bound for $M(\mathcal{C}, 2\ell)$. In order to estimate this number, let us consider a covering of $\mathcal{C}$ by balls of diameter $\ell^{1+\varepsilon}$. Since there are no k-fold crossings of power ε at the scale ℓ, no such ball will contain more than k of the stopping sites. Hence eq. (2.22).

By the definition of the upper box dimension, for each $s > \overline{dim}_B \mathcal{C}$, the number $N(\mathcal{C}, \ell)$ of elements in a minimal covering satisfies

$$N(\mathcal{C}, \ell) \ \leq \ K_s \, \ell^{-s} \ . \tag{2.23}$$

for some constant K_s which depends on the curve. Therefore, for any $s > \overline{dim}_B(\mathcal{C})$

$$M(\mathcal{C}, 2\ell) \quad \leq \ k\, K_s \, \ell^{-s(1+\varepsilon)} \ , \tag{2.24}$$

with some $K_s(\mathcal{C}) < \infty$. Our assumptions imply that the exponent $s(1+\varepsilon)$ can be made arbitrarily close to $\overline{dim}_B \mathcal{C}$, and therefore $\tau(\mathcal{C}) \leq \overline{dim}_B \mathcal{C}$. That concludes the proof of eq. (2.21). The assertion about the Hölder regularity follows from Theorem 2.3. ■

Remark: The proof of Theorem 2.5 shows that the tortuosity exponent can be bounded by the box dimension under the weaker assumption that for some integer k and $\varepsilon > 0$, the curve has no k-fold crossings of power ε below some scale r_o. In this case inequality (2.24) implies the bound

$$\overline{dim}_B \mathcal{C} \ \leq \ \tau(\mathcal{C}) \ \leq \ (1+\varepsilon)\, \overline{dim}_B \mathcal{C} \ . \tag{2.25}$$

3. Regularity for curves in random systems

We now extend the discussion from a single curve to systems of random curves, in the terminology presented in the introduction. Our first goal is to prove Theorem 1.1. Following that we discuss the concept of the backbone, and thus improve the bounds on the dimension of curves.

3.a Proof of the main regularity result

An essential step towards establishing regularity of random curves consists of showing that under the hypothesis **H1**, k-fold crossings of spherical shells are rare in a sense which provides a probabilistic version of the tempered crossing condition. For this purpose, let us define the random variables

$$r^{(\delta)}_{\varepsilon,k}(\omega) \ := \ \begin{cases} \inf\left\{ 0 < r \le 1 \ \middle|\ \begin{array}{c} \text{some shell } D(x; r^{1+\varepsilon}, r),\ x \in \Lambda, \text{ is traversed} \\ \text{by } k \text{ distinct segments of curves in } \mathcal{F}_\delta(\omega) \end{array} \right\} \\ \\ 1 \qquad\qquad \textit{if no such } k \textit{ crossing occurs} \end{cases} \tag{3.1}$$

Lemma 3.1 *Let $\mathcal{F}$ be a system of random curves with variable short distance cutoff, in a compact region $\Lambda \subset \mathbf{R}^d$. Let $\varepsilon > 0$, and assume that the condition (1.3) of* **H1** *holds for some $k < \infty$, and $\lambda(k)$ large enough so that*

$$\varepsilon\, \lambda(k) - d > 0\ . \tag{3.2}$$

Then the random variable $r_{\varepsilon,k}(\omega)$ is stochastically bounded away from zero, with

$$Prob_\delta\left(r^{(\delta)}_{\varepsilon,k}(\omega) \ \le u\right) \ \le \ Const.(\varepsilon, k)\ u^{\varepsilon\,\lambda(k)-d} \quad . \tag{3.3}$$

Proof: We need to estimate the probability that there is a k-fold crossing of power ε at some scale $r \le u$. Any such crossing gives rise to a crossing in a smaller spherical shell with discretized coordinates: $D(x; 3r_n^{1+\varepsilon}, r_n/2)$ with: $r_n = 2^{-n}$, $x \in (2r_n^{1+\varepsilon}/\sqrt{d})\ \mathbf{Z}^d$ (where $\mathbf{Z}^d$ is the integer lattice in $\mathbf{R}^d$), and n chosen so that $r_n < r \le r_{n+1}$. Using the assumption **H1** and adding the probabilities over the possible placements of the discretized shells, we find:

$$\begin{aligned} Prob_\delta\left(\begin{array}{c} \mathcal{F}_\delta(\omega) \text{ exhibits a } (k,\varepsilon) \text{ crossing} \\ \text{at some scale } r \in (r_n, r_{n+1}] \end{array} \right) \ &\le \ \left(\frac{\sqrt{d}}{2r_n^{1+\varepsilon}} \right)^d \ K_k \left(\frac{3r_n^{1+\varepsilon}}{r_n/2} \right)^{\lambda(k)} \\ &\le \ Const.\ r_n^{\varepsilon\,\lambda(k)-(1+\varepsilon)d}\ , \end{aligned} \tag{3.4}$$

where the constant depends only on k, $\lambda(k)$, and the dimension. This bound bound decays exponentially in n. Its sum over scales r_n ($\delta \leq r_n \leq u$) yields the claim. ■

Proof of Theorem 1.1 First let us note that the statement to be proven can be reformulated as follows.

*Let $\mathcal{F}$ be a system of random curves in a compact region $\Lambda \subset \mathbf{R}^d$, with variable short-distance cutoff $\delta > 0$, and assume the hypothesis **H1** is satisfied. Then for any $\varepsilon > 0$ there is a random variable $\tilde{\kappa}_{\varepsilon;\delta}(\omega)$, which stays stochastically bounded as $\delta \to 0$, with which the following tortuosity bound applies simultaneously to all the curves $\mathcal{C} \in \mathcal{F}_\delta(\omega)$:*

$$M(\mathcal{C}, \ell) \ \leq \tilde{\kappa}_{\varepsilon;\delta}(\omega) \ \frac{1}{(diam\mathcal{C})^{\lambda(1)+\varepsilon}} \ \ell^{-[d-\lambda(1)+\varepsilon]} \ . \tag{3.5}$$

In this formulation, the Hölder continuity estimate of eq. (1.5) is replaced by a tortuosity bound. The equivalence is based on Lemma 2.4, with the function $\psi(\ell) = Const.\ \ell^\alpha$ for which the inverse function is the power law with $s = 1/\alpha$. The logarithmic correction in eq. (2.16) is absorbed through the "infinitesimal slack" we have in in the power law.

Let now $\varepsilon > 0$. By the hypothesis **H1**, there exists k large enough so eq. (3.2) is satisfied. For such value of k, we learn from Lemma 3.1 and Theorem 2.5 (more specifically, eq. (2.22) there) that for ℓ small enough, i.e., $\ell < r_{\varepsilon,k}(\omega)$,

$$M(\mathcal{C}, 2\ell) \ \leq \ kN(\mathcal{C}, \ell^{1+\varepsilon}) \ . \tag{3.6}$$

In the complementary range, $\ell \geq r_{\varepsilon,k}(\omega)$, we use that

$$M(\mathcal{C}, 2\ell) \ \leq \ M\left(\mathcal{C}, 2r_{\varepsilon,k}(\omega)\right) \ \leq \ kN(\mathcal{C}, r_{\varepsilon,k}(\omega)^{1+\varepsilon}) \ . \tag{3.7}$$

It follows that

$$M(\mathcal{C}, 2\ell) \ \leq \ A_{\varepsilon,k;\delta}(\omega) \ N(\mathcal{C}, \ell^{1+\varepsilon}) \ , \tag{3.8}$$

where the random variable

$$A_{\varepsilon,k;\delta}(\omega) \ = \ \left(\frac{\ell}{r_{\varepsilon,k}(\omega)}\right)^{(1+\varepsilon)d} \tag{3.9}$$

remains stochastically bounded as $\delta \to 0$ by Lemma 3.1.

We shall now introduce some useful random variables which will permit to extract from eq. (3.8) bound s valid simultaneously for all curves $\mathcal{C} \in \mathcal{F}_\delta(\omega)$. Referring to the standard grid partition of Λ, let

$$\tilde{N}_\delta(r, \ell; \omega) \ := \ \left\{ \begin{array}{l} \text{the number of cubes } B \text{ of diameter } \ell \\ \text{which meet a curve } \mathcal{C} \in \mathcal{F}_\delta(\omega) \text{ with} \\ \text{diameter } diam(\mathcal{C}) \geq r \end{array} \right. \tag{3.10}$$

Its expectation value is $E(N_\delta(r,\ell))$. Summing over scales $r_n \geq \ell_m \geq \delta$, with $r_n = \ell_n = 2^{-n}$, we define

$$U_\delta(\omega) \ := \ \sum_{m \leq n} \frac{\tilde{N}_\delta(r_n, \ell_m; \omega)}{E(N_\delta(r_n, \ell_m))} \ (n+1)^{-2}(m+1)^{-2} \ . \tag{3.11}$$

This random variable stays stochastically bounded as $\delta \to 0$, by the Chebysheff inequality and the observation that the mean is independent of δ:

$$E(U_\delta) \ \leq \ \left[\sum_{n=1}^{\infty} 1/n^2\right]^2 \ . \tag{3.12}$$

For the mean value of $\tilde{N}_\delta(r,\ell;\omega)$ we find

$$\begin{aligned} E_\delta\left(\tilde{N}_\delta(r,\ell;\omega)\right) \ &= \ \sum_{B \subset \Lambda;\ diam(B)=\ell} Prob_\delta \begin{pmatrix} B \text{ meets a curve } \mathcal{C} \in \mathcal{F} \\ \text{with } diam(\mathcal{C}) \geq r \end{pmatrix} \\ &\leq \ \sum_{B \subset \Lambda;\ diam(B)=\ell} K \left(\frac{\ell/2}{r/2}\right)^{\lambda(1)} \\ &\leq \ \frac{K'|\Lambda|}{r^{\lambda(1)}} \left(\frac{1}{\ell}\right)^{d-\lambda(1)} \ . \end{aligned} \tag{3.13}$$

We now return to eq. (3.8). For curves with $diam(\mathcal{F}) \geq r$ we use

$$N(\mathcal{C},\ell) \ \leq \ \tilde{N}_\delta(r,\ell;\omega) \ \leq \ (\log_2 2/\ell))^2 \, (\log_2 (2/r))^2 \, U_\delta(\omega) \ E(\tilde{N}_\delta(r,\ell)) \ . \tag{3.14}$$

(the last inequality based on the definition (3.11). Combining the equations (3.8), (3.13), and (3.14), we learn:

$$\begin{aligned} M(\mathcal{C}, 2\ell) \quad \leq \quad & \Big[(1+\varepsilon)^2 \ K' \ |\Lambda| \ A_{\varepsilon;\delta}(\omega) \ U_\delta(\omega)\Big] \ \times \\ & \times \ \frac{(\log_2 (2/\ell))^2 \, (\log_2 (2/r))^2}{r^{\lambda(1)}} \ \left(\frac{1}{\ell}\right)^{(1+\varepsilon)[d-\lambda(1)]} \ . \end{aligned} \tag{3.15}$$

The product of stochastically bounded variables is stochastically bounded, and the logarithm can be absorbed by adjusting ε. Hence eq. (3.15) implies the claimed eq. (3.5). ∎

3.b Tortuosity of random systems and the backbone dimension

To summarize some of the results in a compact form, it may be useful to extend the notions of tortuosity and dimensions to systems of random curves with varying cutoff.

Definition *For a system $\mathcal{F}$ of curves in a compact set $\Lambda \subset \mathbf{R}^d$*

i. The upper tortuosity exponent $\overline{\tau}(\mathcal{F})$ *is the infimum of* $s > 0$ *for which the random variables*

$$\sup \{ M(\mathcal{C}, \ell) \, \ell^s \ | \mathcal{C} \in \mathcal{F}_\delta(\omega) \ , \, diam(\mathcal{C}) \geq r \ \} \tag{3.16}$$

remain stochastically bounded, as $\delta \to 0$ *at fixed* $0 < r \leq 1$.

ii. Similarly, the upper box dimension $\overline{dim}_B(\mathcal{F})$ *is defined through the boundedness of the variables given by*

$$\sup \{ N(\mathcal{C}, \ell) \, \ell^s \ | \mathcal{C} \in \mathcal{F}_\delta(\omega) \ , \, diam\mathcal{C} \geq r \ \} \tag{3.17}$$

with s, r *as above.*

The analysis carried above implies that if hypothesis **H1** holds then the upper tortuosity exponent $\overline{\tau}(\mathcal{F})$ is finite, and furthermore

$$\overline{\tau}(\mathcal{F}) \ = \ \overline{dim}_B(\mathcal{F}) \ \leq \ d - \lambda(1). \tag{3.18}$$

The dimension estimate eq. (3.18) reflects the fact that each point on a curve $\mathcal{C} \in \mathcal{F}$ is connected a macroscopic distance away. It might seem that most points on a curve are in fact at the end points of *two* line segments of macroscopic length. This suggests an improved upper bound, in which $d - \lambda(1)$ is replaced by the smaller $d - \lambda(2)$. However, one has to proceed here with caution.

There are two reasons for which $d - \lambda(2)$ may not provide a valid upper bound for the dimension:

i. The union $\bigcup_{\mathcal{C} \in \mathcal{F}_\delta(\omega); \ \mathrm{diam}\mathcal{C} > r} \mathcal{C}$ may be dominated by the collection of the *endpoints* of curves in $\mathcal{F}$ which are only *singly connected* a macroscopic distance away. For instance, that would occur if the connected clusters to which the curves of $\mathcal{F}_\delta(\omega)$ are restricted have many short branches (one could call this the *broccoli effect*).

ii. Certain curves $\mathcal{C} \in \mathcal{F}$ may be rougher at the their ends, where only one segment is accommodated in the available space, than in their interior. We expect this to be the case for some examples of self-avoiding paths. When that happens it will not be true that "most of the curve", as counted by covering boxes, consists of its interior.

Nevertheless, the proposed bound is obviously valid for the union of the interior parts of the curves, if that is defined as the collection of point on $\mathcal{C}$ whose distance to the endpoints is at least some $a > 0$, which remains fixed as $\delta \to 0$. The proof is by a direct adaptation of argument used in the proof of Theorem 1.1, making the suitable correction in eq. (3.13).

A situation like that has been addressed in the percolation context through the concept of the *backbone*. The term is used to distinguish between a *spanning cluster*, i.e., a cluster connecting two opposite faces of a macroscopic size cube which typically contains many *dangling ends*, and the smaller set of bonds which carry current between the faces ([12]).

A mathematically appealing formulation is possible in the continuum limit (at $\delta = 0$), for which we define the backbone $\mathcal{B}(\omega)$ of the system of curves $\mathcal{F}(\omega)$ as the union of all interior segments of curves $\mathcal{C} \in \mathcal{F}(\omega)$.

For the backbone, the Hausdorff and box dimensions need not coincide. Since the statement is closely related with the considerations of this section we present it here, even though it anticipates the construction which is better described in the next section.

Theorem 3.2 *In the scaling limit [defined in the next section]*

$$dim_{\mathcal{H}}\mathcal{B}(\omega) \;\leq\; d - \lambda(2) \quad (a.s.)\ , \tag{3.19}$$

whereas

$$\overline{dim}_B\mathcal{B}(\omega) \;=\; \overline{dim}_B\mathcal{F}(\omega) \;\leq\; d - \lambda(1) \quad (a.s.). \tag{3.20}$$

The last inequality can be saturated.

Proof: Equation (3.19) follows from the continuity of the Hausdorff dimension under countable unions, and the the previous observation on the dimension of the sets defined with fixed macroscopic cutoffs. Equation (3.20) holds since the box dimension of a set equals that of its closure, which for $\mathcal{B}(\mathcal{F}(\omega))$ is the union of all curves in $\mathcal{F}$. ∎

4. Compactness, tightness, and scaling limits

We now turn to the construction of scaling limits for a random system of curves. Such a system is described by a collection of probability measures μ_δ on the space of configurations of curves, Ω_Λ defined in the introduction. We shall see that the tortuosity bound 3.5 derived in Theorem 1.1 allows one to conclude the existence of limits for μ_δ.

The key to the proof of Theorem 1.2 is the relation of the space of curves with the space of continuous functions, $C([0,1],\Lambda)$, and the well developed theory of probability measures on the space of closed subsets of a complete separable metric space. We recall some of this theory below. The first step is the following counterpart to Arzelà-Ascoli theorem.

Lemma 4.1 (Compactness in $\mathcal{S}_\Lambda$) *A closed subset $\mathcal{K} \subset \mathcal{S}_\Lambda$, of the space of curves in a compact $\Lambda \subset \mathbf{R}^d$, is compact if and only if there exists a function $\psi : (0,1] \to (0,1]$ so that for all $\mathcal{C} \in \mathcal{K}$,*

$$M(\mathcal{C}, \ell) \;\le\; \frac{1}{\psi(\ell)} \quad \textit{for all } 0 < \ell \le 1 \, . \tag{4.1}$$

Proof: We first show that if a closed set $\mathcal{K} \subset \mathcal{S}_\Lambda$ consists of curves satisfying uniform tortuosity bounds then $\mathcal{K}$ is compact. It suffices here to show that each sequence of curves in $\mathcal{K}$ has an accumulation point in $\mathcal{S}_\Lambda$. The limit will be in $\mathcal{K}$ because $\mathcal{K}$ is closed.

By Lemma 2.4, we can parametrize each curve in the sequence by a continuous function satisfying the corresponding continuity condition eq. (2.15). That yields an equicontinuous family of functions in $C([0,1],\Lambda)$. Applying the Arzelà-Ascoli theorem we deduce the existence of a uniformly convergent subsequence. It is easy to see that the curves defined by these functions also converge, with respect to the metric on $\mathcal{S}$.

In the converse direction (which we do not use in this work), we need to show that if $\mathcal{K}$ is compact then $M(\mathcal{C},\ell)$ is uniformly bounded on it. That follows from Lemma 2.2, which shows that: i) $M(\mathcal{C},\ell)$ is bounded above by $\tilde{M}(\mathcal{C},\ell/3)$, ii) since $\tilde{M}(\mathcal{C},\ell)$ is upper semicontinuous by Lemma 2.2, it achieves its supremum on the compact set $\mathcal{K}$. ∎

Standard arguments, such as used for $C([0,1],\mathbf{R}^d)$, show that the space of curves $\mathcal{S}_{\mathbf{R}^d}$ is a complete and separable metric space. The completeness and separability of $\mathcal{S}_{\mathbf{R}^d}$ are passed on to Ω_Λ. For this space we get the following characterization of compactness.

Lemma 4.2 (Compactness in Ω_Λ) *A closed subset $\tilde{A}$ of Ω_Λ is compact, if and only if there exists some $\psi : (0,1] \to (0,1]$ for which each configuration $\mathcal{F} \in \tilde{A}$ consists exclusively of curves satisfying abound of the form eq. (4.1).*

Proof: The claim follows from the basic property of the Hausdorff metric, under which the closed subsets of a compact metric space form a compact space. ∎

The scaling limit we are interested in is taken in the space of probability measures on Ω_Λ, for compact $\Lambda \subset \mathbf{R}^d$. Our discussion will now make use of a number of useful general concepts and results. Let us just briefly list those. A thorough treatment can be found in ref. [7].

A family of probability measures $\{\mu_n\}$ is said to be *tight* if for every ϵ there exists a compact set A so that $\mu_n(A) \ge 1 - \varepsilon$.

The sequence μ_n is said to *converge to* μ if $\lim_{n\to\infty} \int f \; d\mu_n = \int f \; d\mu$ for every continuous function $f : \Omega \to \mathbf{R}$. If the distance function is uniformly bounded, as is the case for measures on Ω_Λ with compact Λ, this convergence statement is equivalent to the existence of a coupling as described in the introduction, below the statement of Theorem 1.2.

A collection of measures is said to be *relatively compact* if every sequence has a convergent subsequence. Tightness and compactness are equivalent in this general setting:

Theorem (Prohorov [13], see [7]) *A family of probability measures on a complete separable metric space is relatively compact if and only if it is tight.*

Thus, in order to prove Theorem 1.2, we need to show that for each $\varepsilon > 0$, up to remainders of probability $\leq \varepsilon$ the measures $\{\mu_\delta\}$ are supported on a common compact subset (of the space of configuration), which may depend on ε.

Proof of Theorem 1.2 By Theorem 1.1 and point *ii.* of the remark following it, there is for each $s > d$ and $\varepsilon > 0$ a choice of $K < \infty$, such that all curves in the random configuration $\mathcal{F}(\omega)$ drawn with the probability measure μ_δ can be parametrized Hölder continuously with exponent s and Hölder constant $\kappa_{s;\delta}$, as in eq. (1.5). By Lemma 2.4, this implies that

$$M(\mathcal{C}, \ell) \ \leq \ K_r \ell^{-s} \text{ for all curves } \mathcal{C} \in \mathcal{F}_\delta(\omega) \ , \tag{4.2}$$

except for a collection of configurations whose total probability is $\leq \varepsilon$. By Lemma 4.2, the set $A_\Lambda(K, s) \subset \Omega_\Lambda$ of all configurations consisting only of curves that satisfy eq. (4.2). is compact. In other words, finite upper tortuosity of $\mathcal{F}$ implies that upon truncation of small remainders the measures μ_δ are supported in the compact sets of the form $A_\Lambda(K, s)$. (Note that $K < \infty$ needs to be adjusted depending on ε and the choice of s.) This proves that the family μ_δ is tight. By Prohorov's theorem, that is equivalent to compactness.

To see that the limiting measure is supported on curves that can be parametrized Hölder continuously with any exponent less than $1/(d - \lambda(1))$, consider the collections $\mathcal{F}^{(r)}$ of curves of diameter a least r. The above argument shows that the measure restricted to this collection is almost supported on $A_\Lambda(K(r), s)$ for any $s > d - \lambda(1)$, and $K(r)$ large enough. By Prohorov's theorem, the limiting measure is supported on $\bigcup_{K>0} A_\Lambda(K, s)$, which proves the claim by Lemma 2.4. ■

Let us remark that the notion of convergence we use here (technically it is called *weak convergence* on the space of measures on Ω_Λ) is quite strong, due to our choice of topology on Ω_Λ. As eq. (1.13) makes it clear, for n large typical configurations of $\mathcal{F}_{\delta_n}$ are close to typical configurations of the scaling limit – close in the sense of the Hausdorff metric induced on the space of configurations, Ω_Λ, by the uniform metric in the space of curves S_Λ. This sense of convergence is stronger than that defined through the joint probability distributions of finite collections of macroscopic crossing events. In this respect, the notion of convergence used here is reminiscent of the sense in which Brownian motion is proven to approximate random walks, in Donsker's theorem [14].

5. Lower bounds for the Hausdorff dimension of curves

Our next goal (the third theme of this work) is to prove the statement of Theorem 1.3, that in a system satisfying the hypothesis **H2**, almost surely none of the curves which appear in the scaling limit are of Hausdorff dimension lower than some $d_{min} > 1$.

The proof is split into two parts. The first part, carried out in this section, consists of measure-theoretic analysis based on the assumption that a certain auxiliary deterministic condition is satisfied for a given curve. In the next section the proof is completed with a probabilistic argument showing that in a system of random curves satisfying the hypothesis **H2**, the auxiliary condition is met almost surely.

5.a Straight runs

Standard examples of curves of dimension greater than one are curves whose segments deviate from straight lines proportionally on all scales. However for random systems (and other setups) that criterion is too restrictive since one may expect exceptions to any rule to occur on many scales. The criterion which we develop here is the *sparsity of straight runs*, which is an abbreviated expression for the absence of sequences of nested straight runs occurring over an excessively dense collection of scales. The concept is defined with a macroscopic scale $L > 0$ and a shrinkage factor $\gamma > 1$, used to specify a sequence of length scales:

$$L_k \;=\; \gamma^{-k} L_o \, , \tag{5.1}$$

and an integer k_o, used to allow exceptions above a certain scale.

Definition *A curve in* $\mathbf{R}^d$ *is said to exhibit a* straight run *at scale* L *(*$= L_k$ *for some* k*), if it traverses some cylinder of length* L *and cross sectional diameter* $(9/\sqrt{\gamma})\, L$*, in the "length" direction, joining the centers of the corresponding faces. Two straight runs are* nested *if one of the defining cylinders contains the other.*

We say that straight runs are (γ, k_o)-sparse*, down to the scale* ℓ*, if* $\mathcal{C}$ *does not exhibit any nested collection of straight runs on a sequence of scales* $L_{k_1} > \ldots > L_{k_n}$*, with* $L_{k_n} \geq \ell$ *and*

$$n \geq \frac{1}{2} \max\{k_n, k_o\} \quad . \tag{5.2}$$

Following is the deterministic result, which is stated here only in the continuum ($\delta = 0$). For systems of random curves we will make use of the more detailed information which appears in the proof (see eq. (5.22)).

Theorem 5.1 *If the straight runs of a given curve $\mathcal{C}$ are (γ, k_o)-sparse , then $dim_{\mathcal{H}}\mathcal{C} \geq s$, with s given by*

$$\gamma^s = \sqrt{m(m+1)}\ , \tag{5.3}$$

and m an integer strictly smaller than γ.

Clearly, if for some integer m the above condition is met for all $\gamma > m$ then the bound becomes

$$dim_{\mathcal{H}}\mathcal{C} \ \geq\ 1 + \frac{\ln(1+1/m)}{2\,\ln m} \quad . \tag{5.4}$$

We will prove Theorem 5.1 by cutting the given curve $\mathcal{C}$ into a hierarchical family of subsegments at different scales, with segments at the same scale separated by a certain minimal distance. This family defines a Cantor-like (i.e., closed, perfect, and totally disconnected) subset $\tilde{C}$ of $\mathcal{C}$. If $\mathcal{C}$ contains no straight runs at all, a scaling argument easily shows that the dimension of $\tilde{C}$, and hence the dimension of $\mathcal{C}$, exceeds one. We use capacity arguments to show that this holds also under the weaker condition that straight runs are sparse. For the construction of the family of subsegments which defines the fractal subset $\tilde{\mathcal{C}}$, we modify the exit-point algorithm from the proof of Theorem 2.5.

5.b Construction of fractal subsets

Let γ be a positive number, m an integer in $[\gamma/2, \gamma]$, and k_{max} a positive integer. By an iterative procedure we shall construct for a given curve $\mathcal{C}$ a nested sequence $\Gamma_o, \ldots, \Gamma_{k_{max}}$ of collections of segments of $\mathcal{C}$, at scales

$$L_k \ = \ \gamma^{-k}\, L_o\ , \quad k = 0, \ldots, k_{max}\ , \tag{5.5}$$

with $L_o = \mathrm{diam}(\mathcal{C})$, having the following properties.

i. Each Γ_k is a collection of segments of diameter at least L_k.

ii. In each generation (as defined by k), distinct segments are at distances at least εL_k with $\varepsilon = \gamma/m - 1$.

Figure 4: The algorithm used for marking the points x_j in the construction of Γ_k.

iii. Each segment of Γ_k ($k > 1$) is contained in one of the segments of Γ_{k-1}. The number of immediate *descendants* thus contained in a given element of Γ_{k-1} is at least m, and very frequently at least $(m+1)$.

To define $\tilde{\mathcal{C}}$ let $\mathcal{C}_k$ be the union of the segments in Γ_k. Then $\tilde{\mathcal{C}} = \cap_{k \leq k_{max}} \Gamma_k$. In the construction, we find it convenient to use the *span* of a curve, which we define to be the distance between the curve's end points, in place of the diameter.

Lemma 5.2 (Construction of $\tilde{\mathcal{C}}$) *There is an algorithmic construction which for each curve $\mathcal{C}$, yields a sequence of collections of segments with the above properties* i. - iii.*, and with the further property that unless a segment $\eta \in \Gamma_k$ exhibits a straight run of scale L_k the number of its descendants is at least $m+1$.*

Proof: We may assume (by trimming) that the span of $\mathcal{C}$ equals L_o. Let Γ_o consist of only one element: a segment which starts at one end of the curve and stops upon the first exit from a ball of radius $\text{diam}(\mathcal{C})$. Once Γ_k has been constructed, we form Γ_{k+1} by selecting for each element $\eta \in \Gamma_k$ a collection of descendants $\eta_1, \ldots \eta_N$, which are subsegments of η cut by two sequences of points x_j and y_j, strung along it in the order: $y_1 < x_1 < y_2 < x_2 < \ldots$. The cutting points are selected by the following procedure.

We let y_1 be the starting point, and x_1 the first exit of η from the ball of radius $L_{k+1} = L_k/\gamma$ centered at y_1. Then, recursively, we chose x_n as the first point on η having distance at least L_k/m from the already constructed subsegments $\eta_1, \ldots, \eta_{n-1}$; terminating if no such point can be found. The point y_n is selected as the last entrance, prior to x_n into the ball of radius L_{k+1} centered at x_n.

It can be verified that the sequence of subsegments η_j, $j = 1, \ldots, N$, with the endpoints $\{x_j, y_j\}$ have the properties:

Figure 5: The subdivision of an element of Γ_k into Γ_{k+1}. Unless there is a straight run in a cylinder positioned as indicated, the number of elements increases at a higher rate than the factor by which the radius shrinks (γ). Under the hypothesis **H2** straight runs are sparse in a sense which permits to derive a lower bound on the Hausdorff dimension based on this picture.

i. each η_N spans distance L_{k+1},

ii. the distance of each point on η_n to the union of the segments $\eta_1 \dots \eta_{n-1}$ lies between εL_{k+1} and L_k/m, while

iii. the distance from y_n to the starting point x is at most nL_{k-1}/m.

We need to estimate the number of segments generated by the above procedure. It is easy to construct from the collection of segments a polygon with step size at most L_{k-1}/m, connecting the endpoints of η. Choose as the vertex before the last the point x_N, and for any given vertex in η_n $(n > 1)$, select the preceding one from some η_i with $i < n$ so that the resulting leg has length at most L_k/m; if $n = 1$, terminate and use y_1 as the initial point. Clearly, the polygon has at least m interior vertices, and hence the number N of subsegments is at least m.

Assume now that η does not exhibit a straight run at scale L_k. We claim that the number N of descendants is at least $m+1$. By construction, at least one of the segments η_n has a distance of less than L_k/m from the lateral boundary of the cylinder of width $9/\sqrt{\gamma}$ defining a straight run from$x$ to y. If that segment contributes a vertex to the polygon, then this vertex must lie outside the cylinder of width $(9/\sqrt{\gamma} - 4/m)\, L_k \geq (2/\sqrt{m})\, L_k$. Then the polygon has length at least $\sqrt{1 + 4/m}\, L_k \geq (1 + 1/m)\, L_k$, and hence contains at least $m + 1$ interior vertices, coming from distinct subsegments. On the other hand, if some subsegment does not contribute a vertex to the polygon, we also have $N > m$. This completes the proof of the lemma. ∎

One may think of the elements of $\cup_k \Gamma_k$ as vertices of a tree, with the root in Γ_o, and edges joining each segment to its immediate descendants. For any two points $x, y \in \tilde{\mathcal{C}}$

which are not in the same element of $\Gamma_{k_{max}}$

$$|x - y| \ \geq \ \varepsilon L_{k(x,y)} \ , \tag{5.6}$$

where $k(x, y)$ is the index of the first generation at which the two points are separated. Following are two general results which we shall use to estimate the dimension of $\tilde{\mathcal{C}}$.

5.c Energy estimates

For a metric space A and $\ell \geq 0$, let $\mathrm{Cov}_\ell(A)$ denote the collection of coverings of A by sets of diameter not smaller than ℓ. By the definition of the Hausdorff dimension, a lower bound on $dim_{\mathcal{H}} A$ means that for some $s > 0$ the quantity

$$\inf_{\{B_j\} \in \mathrm{Cov}_\ell(A)} \sum_j (diam B_j)^s \tag{5.7}$$

does not tend to 0 as $\ell \to 0$ (in which case $dim_{\mathcal{H}} A \geq s$). It is difficult to use this definition directly to find lower bounds on the Hausdorff dimension. We shall therefore make use of the relation of Hausdorff measures with capacities and deduce a lower bound on dimension from an upper bound on the *energy* of a judiciously chosen probability measure (*charge distribution*) supported on the set A.

Lemma 5.3 *For $s > 0$ and $\ell \geq 0$, let the capacity* $\mathrm{Cap}_{s;\ell}(A)$ *of a subset of* $\mathbf{R}^d$ *be defined by:*

$$\frac{1}{\mathrm{Cap}_{s;\ell}(A)} \ = \ \inf_{\mu \geq 0, \int_A d\mu = 1} \int\int_{A \times A} \frac{\mu(dx)\mu(dy)}{\max\{|x-y|, \ell\}^s} \tag{5.8}$$

Then, for every collection of sets $\{B_j\}$ covering A, with $\min_j$ *$diam(B_j) \geq \ell$:*

$$\sum_j (diam B_j)^s \ \geq \ \mathrm{Cap}_{s;\ell}(A) \ . \tag{5.9}$$

(The case $\ell = 0$ can be found in Falconer [15]. The statement is related to the theorem of Erdős and Gillis [16] that the s-dimensional Hausdorff measure of A is infinite whenever $\mathrm{Cap}_{s;0}$ is positive.)

Proof: By monotonicity, it clearly suffices to prove eq. (5.9) for any covering by *disjoint* sets. Let $\{B_j\}$ be such a collection, and μ a probability measure supported on A. Then

$$\begin{aligned} \int\int_{A \times A} \frac{\mu(dx)\mu(dy)}{\max\{|x-y|, \ell\}^s} \ &\geq \ \sum_j \int\int_{x,y \in B_j} \frac{\mu(dx)\mu(dy)}{\max\{\mathrm{diam}(B_j), \ell\}^s} \\ &= \ \sum_j \frac{\mu(B_j)^2}{\mathrm{diam}(B_j)^s} \ . \end{aligned} \tag{5.10}$$

We also have

$$1 \;=\; \left(\sum_j \mu(B_j)\right)^2 \;\leq\; \left(\sum_j \frac{\mu(B_j)^2}{\text{diam}(B_j)^s}\right)\;\left(\sum_j \text{diam}(B_j)^s\right) \tag{5.11}$$

(by the Schwarz inequality). Combining the last two relations we learn that

$$\sum_j \text{diam}(B_j)^s \;\cdot\; \left(\int\int_{A\times A} \frac{\mu(dx)\mu(dy)}{\max\{|x-y|,\ell\}^s}\right) \;\geq\; 1 \tag{5.12}$$

for any probability measure supported on A, and any covering of A by sets with diameters $\geq \ell$. Minimizing over μ one obtains the relation claimed in eq. (5.9). ■

Lemma 5.4 *Let A be a compact subset of $\mathbf{R}^d$. Assume there is a sequence $\Gamma_o, \ldots, \Gamma_{k_{max}}$ of (nonempty) collections of closed disjoint subsets of $\mathbf{R}^d$, such that for each $k = 0, \ldots, k_{max} (\leq \infty)$:*

i. Each element of is contained in some element of Γ_{k-1}, and each element of Γ_k contains at least one such "descendant".

ii. Any two distinct sets in Γ_k are a distance at least εL_k apart, where $L_k = \gamma^{-k}, L_o$ with some $L_o > 0$, $\gamma > 1$ and $0 < \varepsilon \leq \gamma$.

iii. For each element $\eta \subset \Gamma_k$: $\eta \text{Cap} A \neq \emptyset$.

For points $x \in \cup_{\eta\in\Gamma_k}\eta$, let $n_k(x)$ be the number of immediate descendants of the set containing x within Γ_{k-1}. Assume, furthermore:

iv. there is some $\beta > 1$, such that

$$\prod_{j=1}^{k} n_j(x) \geq \beta^k \;, \qquad \textit{for all } k = k_o, \ldots, k_{max} \tag{5.13}$$

with some common k_o, whenever $x \in \cup_{\eta\in\Gamma_k}\eta$.

Then, for $s > 0$ such that $\gamma^s < \beta$, and $\ell = \gamma^{-k_{max}} L_o$:

$$\text{Cap}_{s;\ell}(A) \;\geq\; (\varepsilon\, L_o)^s \left[\gamma^{s\,k_o} \;+\; \frac{\beta}{1-\gamma^s/\beta}\right]^{-1} \;. \tag{5.14}$$

Remark: It should be appreciated that ℓ and k_{max} do not appear on the right side in eq. (5.14). If straight runs are sparse on all scales (that is, $k_{max} = \infty$), then the limit $\ell \to 0$ of eq. (5.9) yields a bound on the s-dimensional Hausdorff measure of A.

Proof: For a bound on the capacity it suffices to produce a single probability measure supported on A with a correspondingly small "energy integral" (see eq. (5.8)). We construct the measure μ so that for each $\eta \in \Gamma_k$ the total measure of $A \cap \eta$ is distributed evenly among its immediate descendants. This means that for each $k = 0, \ldots, k_{max}$ and each $\eta \in \Gamma_k$

$$\mu(\eta) \ = \ \prod_{j=1}^{k} n_j(\eta)^{-1} \ , \tag{5.15}$$

where (fpr $j \leq k$), the number $n_j(\eta)$ is the constant value which $n_j(x)$ takes for $x \in \eta$. To specify the measure uniquely, we designate as its support $\{x_{min}(\eta) \mid \eta \in \Gamma_{k_{max}}\}$, where for each $\eta \in \Gamma_{k_{max}}$, the point $x_{min}(\eta)$ is the earliest point in η, with respect to the lexicographic order of $\mathbf{R}^d$.

For $x, y \in A$, if the two points are in separate elements of $\Gamma_{k_{max}}$ we let $k(x, y)$ denote the index of the level at which they separated. In estimating the energy integral we shall use the bound:

$$|x - y| \ \geq \ \varepsilon \ L_{k(x,y)} \ . \tag{5.16}$$

for points which are separated in $\Gamma_{k_{max}}$. Otherwise, we use $\max\{|x - y|, \ell\} \geq \ell$. Thus

$$\begin{aligned} \mathcal{E}(\mu) \ &\equiv \ \int\int_{A\times A} \frac{\mu(dx)\mu(dy)}{\max\{|x-y|, \ell\}^s} \\ &\leq \ \int\int_{k(x,y)\leq k_{max}} (\varepsilon L_{k(x,y)})^{-s} \ \mu(dx)\mu(dy) \ + \\ &\qquad\qquad + \sum_{\eta\in\Gamma_{k_{max}}} L_{k_{max}}^{-s} \int\int_{\eta\times\eta} \mu(dx)\mu(dy) \ . \end{aligned} \tag{5.17}$$

Splitting the first integral on the right according to the value of $k(x, y)$ (separating out the case $k(x, y) \leq k_o$) and replacing L_k by $\gamma^{-k} L_o$ throughout, we obtain

$$\mathcal{E}(\mu) \ \leq \ (\varepsilon\, L_o)^{-s} \ \gamma^{s\, k_o} \ + \ \sum_{k=k_o+1}^{k_{max}} (\varepsilon\, L_o)^{-s} \ \gamma^{s\, k} \sum_{\eta\in\Gamma_{k-1}} \mu(\eta)^2 \ + \ L_o^{-s}\, \gamma^{s\, k_{max}} \sum_{\eta\in\Gamma_{k_{max}}} \mu(\eta)^2 \ . \tag{5.18}$$

Since $\varepsilon \leq \gamma$, the last term on the right hand side of (5.18) can be replaced there by adding the term $k = k_{max+1}$ to the preceding sum. Finally, we use the assumption eq. (5.13) together with the definition of the measure in eq. (5.15) and

$$\sum_{\eta\in\Gamma_k} \mu(\eta) \ = \ 1 \tag{5.19}$$

to see that

$$\sum_{\eta\in\Gamma^k} \mu(\eta)^2 \;=\; \prod_{j=0}^{k} n_j(\eta)^{-1} \sum_{\eta\in\Gamma_k} \mu(\eta) \;\leq\; \beta^{-k}\;. \tag{5.20}$$

This yields a geometric series bound for the sum over k in eq. (5.18), which results in the bound stated in eq. (5.14). ■

Proof of Theorem 5.1: Let $\mathcal{C}$ be a curve where straight runs are (γ, k_o) sparse down to scale $\ell = \gamma^{-k_{max}}$. The hierarchical construction of Lemma 5.2 results in a fractal subset $\tilde{\mathcal{C}}$ of $\mathcal{C}$. Since straight runs are sparse by assumption, $\tilde{C}$ satisfies the branching condition (5.13) of Lemma 5.4, with the the value of β defined by the relation:

$$\beta \;=\; \sqrt{m(m+1)}\;. \tag{5.21}$$

Thus, Lemma 5.4 implies that for any s such that $\gamma^s < \beta$:

$$\mathrm{Cap}_{s;\ell}(\mathcal{C}) \;\geq\; (\varepsilon\, L_o)^s \left[\gamma^{s\,k_o} \;+\; \frac{\beta}{1-\gamma^s/\beta}\right]^{-1} \;; \tag{5.22}$$

this inequality holds for all $\ell \in (\gamma^{-k_{max}}, 1]$. By Lemma 5.3, the same lower bound holds for $\inf_{\{B_j\}\in \mathrm{Cov}_\ell(\tilde{\mathcal{C}})} \sum_j (diam B_j)^s$. Since we may choose k_{max} as large as we please, the s-Hausdorff measure of $\mathcal{C}$ is positive, and hence, the Hausdorff dimension is at least s. ■

6. Lower bounds on curve dimensions in random systems

We shall now combine the previous deterministic results with a probabilistic estimate, and prove Theorem 1.3. The proof consists of showing that, with high probability, straight runs are sparse, and then applying the results of the previous section.

Lemma 6.1 *(Sparsity of straight runs.) Assume a system of random curves in a compact set $\Lambda \in \mathbf{R}^d$ satisfies the hypothesis* **H2**. *For $\gamma > 4d$, define a sequence of length scales $L_k = \gamma^k$. Then there are constants $K_\Lambda, K_1 < \infty, K_2 > 0$, with which for any fixed sequence $k_1 < k_2 < \ldots < k_n$*

$$Prob_\delta \left(\begin{array}{c} \textit{there is a nested sequence} \\ \textit{of straight runs at scales } L_{k_1}, \ldots, L_{k_n} \end{array} \right) \;\leq\; K_\Lambda\; \gamma^{2d\,k_n}\; e^{(K_1 - K_2\sqrt{\gamma})\,n}\;. \tag{6.1}$$

provided $\gamma^{-k_n} > \delta$.

Proof: If a curve traverses a cylinder of length L, width $(9/(\sqrt{\gamma})\, L$, then it also traverses a cylinder of width $(10/\sqrt{\gamma})\, L$ and length $L/2$ centered at a line segment joining discretized points in $L'\mathbf{Z}^d$, provided $L' \le L/\gamma$. The number of possible positions of such a cylinder in a set of diameter ℓ is bounded above by $(\ell/L')^{2d}$. The number of positions of n nested cylinders at scales $L_{k_1}, \ldots, L_{k_n}$ is thus bounded by

$$K_\Lambda \gamma^{2d\,k_1} \gamma^{2d\,(k_2-k_1)} \ldots \gamma^{2d\,(k_n-k_{n-1})} \le K_\Lambda \gamma^{2d\,k_n} \tag{6.2}$$

Fix now a sequence A_i, $i = 1, \ldots, n$ of nested cylinders of length $L_{k_i}/2$ and width $(10/\sqrt{\gamma})\,L_{k_i}$. Let σ be aspect ratio for which **H2** holds with some $\rho < 1$. Cut each of the cylinders into $\sqrt{\gamma}/(10\sigma)$ shorter cylinders of aspect ratio σ, and pick a maximal number of well separated cylinders from this collection. Since A_{i+1} intersects at most two of the shorter cylinders obtained by subdividing A_i, the number of cylinders in a maximal collection is at least $n\left(\sqrt{\gamma}/(20\sigma) - 2\right)$. The probability of a curve traversing all of the A_i is bounded above by the probability of crossing the shorter cylinders. Applying **H2** gives

$$Prob_\delta \begin{pmatrix} A'_1, \ldots, A'_n \text{ are crossed} \\ \text{by a curve in } \mathcal{F}_\delta(\omega) \end{pmatrix} \le K_1 e^{(K_2 - K_3\sqrt{\gamma})\,n} \tag{6.3}$$

Summing over the possible positions and adjusting the constants completes the proof.

Proof of Theorem 1.3 We first show that for each system of random curves in a compact set $\Lambda \subset \mathbf{R}^d$ satisfying the hypothesis **H2**, there exist $m < \infty$ and $q < 1$ such that for every $\gamma > m$

$$Prob_\delta \begin{pmatrix} \text{straight runs are } (\gamma, k_o)\text{-sparse} \\ \text{in } \Lambda\text{, down to scale } \delta \end{pmatrix} \ge 1 - \frac{q^{k_o}}{1-q}\ ; \tag{6.4}$$

in other words, the random variable given by

$$k_{o;\delta}(\omega) = \inf\{k \ge 0 \mid \text{straight runs are } (\gamma, k_o)\text{-sparse down to scale } \delta\} \tag{6.5}$$

is stochastically bounded as $\delta \to 0$. To see this, note that for specified k,

$$\begin{aligned} Prob \begin{pmatrix} \text{there exist a nested sequence} \\ \text{of straight runs on scales} \\ k_1 < \ldots < k_n = k \text{ with } k \ge n \ge k/2 \end{pmatrix} &\le \sum_{n=k/2}^{k} \binom{k}{n} K_1\, m^{2d\,k}\, e^{-K_2\sqrt{m}\,n} \\ &\le K_1\,(2m)^{2d\,k}\, e^{-(K_2\sqrt{m})\,k/2}\ . \end{aligned} \tag{6.6}$$

Choosing m large enough so that

$$q \equiv m^{2d}\, e^{-(K_2\sqrt{m}/2)} < 1 \tag{6.7}$$

and summing the geometric series over k we obtain eq. (6.4).

As in the proof of Theorem 5.1, it follows with Lemmas 5.2 and 5.4 from eq. (6.5) that all curves in a given configuration satisfy the bound

$$\mathrm{Cap}_{s;\delta}(\mathcal{C}) \;\geq\; (\varepsilon\, diam(\mathcal{C}))^s \left[\gamma^{s\, k_{o;\delta}(\omega)} \;+\; \frac{\beta}{1-\gamma^s/\beta}\right]^{-1} , \tag{6.8}$$

with m and γ as above, $\beta = \sqrt{m(m+1)}$, and s small enough so that $\gamma^s < \beta$. Choosing γ sufficiently close to m we may take $s > 1$, which proves the claim. ■

APPENDIX

A. Models with random curves

In order to provide some context for the discussion of systems of random curves, we present here a number of guiding examples. Familiarity with this material is not necessary for reading the work, however it does offer a better perspective both on the motivation and on the choice of criteria employed here. We start with some systems exhibiting the percolation transition.

A.a Percolation models

Among the simplest examples to present (for a review see [17, 18]) is the independent bond percolation model on the cubic d dimensional lattice, which we scale down to $\delta\mathbf{Z}^d$, $\delta << 1$. "Bonds" are pairs $b = \{x, y\}$ of neighboring lattice sites. Associated with them are independent and identically distributed random variables $n_b(\omega)$, with values in $\{0, 1\}$. The one-parameter family of probability measures is parametrized by:

$$p \;=\; Prob\,(n_b = 1)\ . \tag{A.1}$$

For a given realization, the bonds with $n_b(\omega) = 1$ are referred to as occupied. The lattice decomposes into clusters of connected sites, with two sites regarded as connected if there is a path of occupied bonds linking them.

For an intuitive grasp of the terminology one may think of the example in which the occupied bonds represent electrical conductors (of size $\delta << 1$) embedded randomly in an insulating medium. If a macroscopic piece of material with such characteristics is placed between two conducting plates which are maintained at different potentials, the resulting current will be restricted to the macroscopic-scale clusters connecting the two plates (the "spanning clusters").

The model exhibits a phase transition. Its simplest manifestation is that the probability of there being an *infinite cluster* changes from 0 for $p < p_c$, to 1 for $p > p_c$. The transition is also noticeable in finite volumes of macroscopic size: for $p < p_c$ the probability of observing a *spanning cluster* in $[0,1]^d$ is vanishingly small, whereas for $p > p_c$ this probability is extremely close to 1. In both cases the probabilities of the unlikely events decay as $\exp(-const.\ /\delta)$, when $\delta \to 0$ at fixed p ($\neq p_c$).

The generally believed picture in dimensions $2 \leq d < 6$ is that for p in the vicinity of the critical point ($|p - p_c| = O(\delta^{1/\nu})$, macroscopic clusters do occur but are tenuous. Much of this is proven for $2D$ ([1, 2, 19]) though gaps in proof remain for $d > 2$ ([20, 4]). Typical configurations exhibit many *choke points*, where the change of the occupation status of a single bond will force a large scale shift in the available connecting routes ([1]), and possibly even break a connected cluster into two large components, as indicated in Figure 1. The clusters are "fractal" in the sense that they exhibit fluctuating structure on many scales [10]. This is the situation addressed in this work.

For a given configuration of the model, we let $\mathcal{F}_\delta(\omega)$ stand for the collection of all the self-avoiding paths along the occupied bonds (possibly restricted to a specified subset $\Lambda \subset \mathbf{R}^d$). This random configuration of paths provides an explicit way of keeping track of the possible connecting routes within a given bond configuration.

One of the goals of this work was to establish that the description of the model in terms of a system of random curves ([21]) remains meaningful even in the scaling limit ($\delta \to 0$). It may be noted that the alternate (and more common) description of the random configuration in terms of the collection of *connected clusters*, is problematic in that limit. Clusters are naturally viewed as elements of the space of closed subsets of $\mathbf{R}^d$, with the distance provided by the Hausdorff metric. As long as $\delta \neq 0$ the two formulation of the model, as a system of random clusters or a system of random curves, are equivalent. However the ubiquity of choke points renders the random cluster description insufficient for the scaling limit. (The Hausdorff metric is not sensitive enough to pick up small differences, such as flips of individual bonds, which may have a drastic effect on the available routes.)

It is expected that in the scaling limit the configurations of the connected paths in the critical bond percolation model are hard to distinguish from those arising from a number of other systems of different microscopic structure, e.g., percolation models where the conducting objects are randomly occupied sites of the lattice $\delta\mathbf{Z}^d$ (viewed as a subset of $\mathbf{R}^d$), or droplets of radius δ randomly distributed in $\mathbf{R}^d$. The definition of $\mathcal{F}_\delta(\omega)$ for the such models may require minor adjustments, one in the notion of self-avoidance and the other in the selection of the polygonal approximation. For the droplet model both are taken care of by restricting the attention to the polygonal paths joining centers of intersecting droplets *which do not re-enter* any of the droplets. We form the set $\mathcal{F}_\delta(\omega)$ as the collection of all such paths.

In two dimensions, our hypotheses **H1** and **H2** are satisfied by the independent bond, site, and droplet percolation models. The $k = 1$ case of the bounds (1.3) (with $\lambda(1) > 0$) and (1.15), is a particular implications of the Russo-Seymour-Welsh theory [1, 2, 3]. The statement that $\lambda(k) \to \infty$ follows by the van den Berg – Kesten inequality [22], which implies that for independent systems the probability of multiple crossings is dominated by the corresponding product of the probabilities of single events. (More detailed analysis implies that $\lambda(k)$ actually grows quadratically in k [4, 23, 24].) The conditions **H1** and **H2** are expected to hold also for other dimensions $d < 6$, but not for $d > 6$ ([4]).

Thus, our general results imply the following statement, which was outlined in ref. [21].

Theorem A.1 *In two dimensions, in each of the above mentioned percolation models, based on random bonds, random sites, or random droplets, at the critical point all the non-repeating paths supported on the connected clusters within the compact region* $[0,1]^2$ *can be simultaneously parametrized by functions* $\gamma(t)$, $0 \le t \le 1$, *satisfying the Hölder continuity condition eq. (1.5). The continuity constants* $\kappa_{\varepsilon;\delta}(\omega)$, *which cover simultaneously all curves in* $[0,1]^2$ *remain stochastically bounded as* $\delta \to 0$. *(This holds for any* $\varepsilon > 0$ *as explained in Theorem 1.1).*

Furthermore, for each of these critical models the probability distribution of the random collection of curves $\mathcal{F}_\delta(\omega)$ *has a limit (in the sense of Theorem 1.2), at least for some sequence of* $\delta_n \to 0$. *The limiting measure is supported on collections of curves whose Hausdorff dimensions satisfy*

$$d_{min} \ \le \ dim_{\mathcal{H}}(\mathcal{C}) \ \le d - \lambda(2) \ , \tag{A.2}$$

with some non-random $d_{min} > 1$.

In fact, by similar reasoning we can also deduce the existence of a one-parameter family of such limits, corresponding to values of p which deviate from p_c by an amount which is scaled down to zero as $\delta \to 0$ (in essense: $p(\delta; t) = p_c + t\delta^{1/\nu}$).

The apparent universality of critical behavior leads one to expect that the scaling limits constructed here are common to the models listed above. If so, then the limiting measures will have the full rotation and reflection symmetry of $\mathbf{R}^n$ (and in two dimensions exhibit also *self-duality*). Remarkably, there is evidence for an even higher symmetry: conformal invariance (see [25, 26, 21, 27]), at the special point $p = p_c$, i.e.,

$t = 0$ for the one-parameter family alluded to above. The mathematical derivation of such universality of the scaling limits, and of the conformal invariance at the critical point, form outstanding open problems.

A.b Random spanning trees

The regularity criteria presented here can also be verified for a number of random spanning trees in two dimensions [5]. Each is a translation-invariant process describing a tree graph spanning a set of sites in $\mathbf{R}^d$ with neighboring sites spaced distances of order $\delta \ll 1$ apart.

MST (*Minimal Spanning Tree*)
: The underlying graph is the regular lattice $\delta\mathbf{Z}^d \subset \mathbf{R}^d$, with edges connecting nearest neighbors. Associated with the edges $b = \{x, y\}$ is a collection of independent random *call numbers* (or *edge lengths*) $u(b)$, with the uniform probability distribution in $[0, 1]$. For a bounded region, $\Lambda \subset \mathbf{R}^d$, the minimal spanning tree $\Gamma_{\delta;\Lambda}(\omega)$ is the tree spanning the set $\Lambda \cap \delta\mathbf{Z}^d$ minimizing the *total edge length* (i.e. the sum of the call numbers).

EST (*Euclidean (Minimal) Spanning Tree*)
: The vertices of the graph are generated as a random collection of points, with the Poisson distribution of density δ^{-d} on Λ. $\Gamma_{\delta;\Lambda}$ is the covering tree graph which minimizes the total (Euclidean) edge length.

UST (*Uniformly Random Spanning Tree*)
: The spanning tree $\Gamma_{\delta;\Lambda}$ is drawn uniformly at random from the set of trees spanning the vertices in $\Lambda \cap \delta\mathbf{Z}^d$ using the nearest neighbor edges.

In each of the above cases there is a well-defined limit

$$\Gamma_\delta(\omega) \;=\; \lim_{\Lambda \nearrow \mathbf{R}^d} \Gamma_{\delta,\Lambda}(\omega) \tag{A.3}$$

where λ is increased through a sequence which exhausts $\mathbf{R}^d$. (The restrictions of $\Gamma_{\delta,\Lambda}$ to compact subsets $\tilde{\Lambda} \subset \mathbf{R}^d$ are monotone decreasing in Λ once $\Lambda \supset \tilde{\Lambda}$.) The limiting spanning tree is independent of the sequence of volumes and is translational invariant, in the stochastic sense.

In general, the limit $\Gamma_\delta(\omega)$ may be either a single tree or a collection of trees. For two dimensions it is known that each of MST, EST and UST almost surely consists of a single tree, with a single topological end, i.e. a single route to infinity ([29, 30, 31, 32, 28, 3]). The structure of UST changes from a tree to a forest in dimensions $d > 4$[29], while MST and EST are expected to change similarly for $d > 8$ [33, 34] (the transition may appear differently from the scaling limit perspective, [5]).

For each n-tuple of points $x_1, \ldots, x_n \in \mathbf{R}^d$, let $T^{(n)}_{x_1,\ldots,x_n}(\omega)$ be the tree subgraph of $\Gamma_\delta(\omega)$ with vertices corresponding to the closest n sites in $\Gamma_\delta(\omega)$. Our methods can be applied to the question, analogous to $Q1$ in the introduction:

Q. Is there a limiting distribution for these graphs, as $\delta \to 0$?

To control the limit for $T^{(n)}_{x_1,\ldots,x_n}(\omega)$, one needs information on the curves supported on $\Gamma_\delta(\omega)$. This collection of curves forms the set $\mathcal{F}^{(2)}_\delta(\omega)$ to which the analysis of this work may be applied.

Random spanning trees provide striking examples of the phenomenon we encountered in critical percolation, that the formulation of the model in terms of random clusters is inadequate for the description of the scaling limit. Here, the Hausdorff distance between any two different realizations (as subsets of $\mathbf{R}^d$) is δ, and hence the space of configurations seems to collapse to a single point. That can be resolved by looking at the curves, as is done here. Let us add that the more complete description of the spenning trees requires the consideration of all the embedded finite trees, and that defines the object $\mathcal{F}_\delta(\omega)$ for those systems. However, their study can be based on the analysis of the curves which provide the tree branches.

In contrast with independent percolation, spatially separated events are not independent for stochastic trees. Moreover, $\lambda(1) = 0$, since any two vertices are connected with probability one. Nevertheless, the hypotheses **H1** and **H2** are valid (with $\lambda(2) > 0$) for the three spanning tree processes processes in $d = 2$ dimensions [5]. Instrumental in the derivation are the relations of MST and EST with invasion percolation (studied in ref. [28]), and of UST with the loop-erased random walk via the *Wilson algorithm* [35], The latter relation permits to draw also non-trivial conclusions about the scaling limit of the loop-erased random walk (LERW) in d=2 dimensions.

A.c The frontier of Brownian motion.

Yet another example of a random curve is provided by the *frontier* of the two dimensional *Brownian motion* ($\{\ b(t) \mid t \in [0,1], b(0) = 0\}$), abbreviated here as FBM. The frontier of a sample path is defined as the boundary of the unbounded connected component of the complement of the path in $\mathbf{R}^2$.

For the FBM, $\mathcal{F}(\omega)$ consists of a single curve. Its dimension has been considered in the literature: it is conjectured that $dim(\text{FBM}) = 4/3$ (almost surely) [36, 10], the best rigorous bounds are $1.015 \leq dim(\text{FBM}) \leq 1.475$ [37, 38].

Our general results do apply to this example. We shall not derive here the hypotheses **H1/H2**. Let us note, however, that **H2** is easy to establish by making use of the observation that the event depicted in Figure 3 requires that the Brownian path should hit each of the boxes but not traverse it in the width direction. Thus, the mechanism behind our lower bound is similar in spirit to the earlier work of Bishop et.al. [39], in the reliance on the fact that Brownian paths move erratically. The resulting upper bound, while not as tight an estimate of the dimension as that of Burdzy and Lawler [38], is expressed as a bound on the tortuosity, and hence can be used to establish that FBM is parametrizable as a Hölder continuous curve.

A.d The trail of three dimensional Brownian motion

The trail of Brownian motion is the set of sites it visits in times $0 \leq t < \infty$. In the transient case, $d > 2$, the trail almost surely forms a closed random set of Hausdorff dimension 2. Can it support curves of dimension arbitrarily close to 1? In a recent work of Lawler [40] this question was answered negatively for the interesting case $d = 3$, through analysis involving a number of results concerning the Brownian motion intersection exponent. Let us note that a negative answer can also be deduced from the general Theorem 1.3, since **H2** is rather easy to establish (for $d > 2$) within the setup relevant for this problem.

A.e Contour lines of random functions

As the last example of a system of random lines let us mention contour lines of a random function. Kondev and Henley [41] have considered the distribution of the level sets of a family of random functions defined on a lattice, $\phi_\delta(\omega) : \delta\mathbf{Z}^2 \to \mathbf{R}$. They present an interesting conjecture concerning the scale invariance for the distribution of the loops bounding the connected regions with $\phi(x) > \phi(0)$. It would be of interest to see an extension of our analysis to such systems.

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