Scattering behaviour of quantum vector fields obtained from Euclidean covariant SPDEs

Sergio Albeverio[1,2,3],Hanno Gottschalk[1] and Jiang-Lun Wu [1,3]

[1] Fakultät und Institut für Mathematik der Ruhr-Universität Bochum, D-44780 Bochum, Germany and SFB237 Essen-Bochum-Düsseldorf, Germany
[2] Institut für angewandte Mathematik der Rheinischen Fridrich-Wilhelms- Universität Bonn, Wegelerstr. 6, D-53155 Bonn and BiBoS Research Centre, Bielefeld-Bochum-Bonn, Germany; CERFIM, Locarno and Acc. Arch., USI Mendrisio, Switzerland
[3]Institute of Applied Mathematics, Academia Sinica, Beijing 100080, P R China

Abstract

We discuss Euclidean covariant vector random fields as the solution of stochastic partial differential equations of the form $DA = \eta$, where D is a covariant (w.r.t. a representation τ of $SO(d)$) differential operator with "positive mass spectrum" and η is a non-Gaussian white noise. We obtain explicit formulae for the Fourier transformed truncated Wightman functions, using the analytic continuation of Schwinger functions discussed by Becker, Gielerak and Ługewicz.

Based on these formulae we give necessary and sufficient conditions on the mass spectrum of D which imply nontrivial scattering behaviour of relativistic quantum vector fields associated to the given sequence of Wightman functions. We compute the scattering amplitudes explicitly and we find that the masses of particles in the obtained theory are determined by the mass spectrum of D.

1 Introduction

Local and relativistic quantum fields can be obtained via analytic continuation from Euclidean random fields. ¿¿From the 80-ies on, this concept, which proved to be useful especially for scalar fields in the space-time dimension $d = 2$, was applied in a number of articles (see e.g. [4] and references therein) to random vector fields obtained as solutions of the stochastic partial differential equation (SPDE) $DA = \eta$, where D is a complex, quaternionic or octonionic Cauchy-Riemann differential operator and η a non Gaussian noise with values in the fields of complex numbers, quaternions or octonions, respectively. For the quaternionic case (space-time dimension $d = 4$) we have recently been able to prove that the associated quantum gauge fields have non-trivial scattering behaviour [3].

In the paper [5], the equation $DA = \eta$ was studied systematically in d dimensional space-time and spin dimension L (of η). Let $\tau : SO(d) \to Gl(L)$ be a representation s.t. η transforms covariantly under the representation of the Euclidean group induced by τ, i.e. $\tau(\Lambda)\eta(\Lambda^{-1}(x-y)) = \eta(x)$ in probability law $\forall \Lambda \in SO(d)$ and $y \in {}^d$. Let D be a differential operator covariant w.r.t. the representation τ, i.e. $\tau(\Lambda)D_x\tau(\Lambda^{-1}) = D_{\Lambda x} \forall \Lambda \in SO(d)$. Furthermore we assume that D is translation invariant, i.e. has constant coefficients. In this case, for the Fourier transformed Green's function $\hat{D}^{-1}(k)$ of D the following representation was obtained in [5]:

$$\hat{D}^{-1}(k) = \frac{Q_E(k)}{\prod_{l=1}^{N}(|k|^2 + m_l^2)^{\nu_l}} \tag{1}$$

with $m_l \in$,$m_j \neq m_l$ for $l \neq j$ and $\nu_l \in$. $Q_E(k)$ is an $L \times L$-matrix with polynomial entries of order $\leq \kappa = 2(\sum_{l=1}^{N} \nu_l - 1)$ which fulfills the Euclidean transformation law $\Lambda Q_E(k)\Lambda^{-1} = Q_E(\Lambda k) \forall \Lambda \in SO(d)$. Without loss of generality we assume that Q_E is prime w.r.t the factors $(|k|^2 - m_l^2)$, i.e. that none of them divides all of the polynomial matrix elements of Q_E. If one imposes a "positive mass spectrum" condition $m_l > 0$ for $l = 1, \ldots, N$ it follows immediately that D is invertible on the space of (^{L}-valued) distributions over ^{d} and we can therefore solve the above SPDE by setting $A = D^{-1} * \eta$.

The Schwinger functions (moments) associated to A can be calculated exlicitly as

$$\begin{aligned} S_{n,\alpha_1\cdots\alpha_n}(x_1, \ldots, x_n) &= \mathbf{E}\left[A^{\alpha_1}(x_1) \cdots A^{\alpha_n}(x_n)\right] \\ = \sum_{I \in \mathcal{P}^{(n)}} \prod_{\{j_1,\ldots,j_l\} \in I} & \underbrace{C^{\beta_{j_1}\cdots\beta_{j_l}} \int_{{}^d} \prod_{r=1}^{l} D^{-1}_{\beta_{j_r}\alpha_{j_r}}(x_{j_r} - x)\, dx}_{=S^T_{n,\alpha_{j_1}\cdots\alpha_{j_l}}(x_{j_1},\ldots,x_{j_l})} \end{aligned} \tag{2}$$

where $\mathcal{P}^{(n)}$ is the collection of all partitions of $\{1, \ldots, n\}$ into disjoint subsets and $C^{\beta_{j_1}\cdots\beta_{j_l}}$ are constants depending on the law of η. We have also used the Einstein convention of summation on repeated upper and lower greek indices w.r.t. an arbitrary invariant metric on the spin-space.

In this paper we study the analytic continuation of the truncated Schwinger functions S_n^T to truncated relativistic Wightman functions W_n^T and the scattering behaviour of these Wightman functions. The paper is organized as follows: In Section 2 we explicitly compute the Fourier transformed truncated Wightman functions of the model. This will be done combining techniques of [1] and [5]. In Section 3 we discuss the scattering behaviour of the truncated Wightman functions, showing that it is nontrivial, if and only if $\nu_l = 1$ for $l = 1, \ldots, N$. In this case we derive explicit formulae for the truncated scattering amplitudes.

2 Analytic continuaton of the Schwinger functions

In this section we obtain a representation of the truncated Schwinger functions S_n^T as Fourier-Laplace transforms, i.e.

$$S^T_{n,\alpha_1\cdots\alpha_n}(x_1, \ldots, x_n) = \mathcal{L}(\hat{W}^T_{n,\alpha_1\cdots\alpha_n})(x_1, \ldots, x_n) =$$

$$(2\pi)^{-dn/2}\int_{{}^{dn}} \exp\Big(\sum_{l=1}^{n} -k_l^0 x_l^0 + i\vec{k}_l\cdot\vec{x}_l\Big)\ \hat{W}^T_{n,\alpha_1\cdots\alpha_n}(k_1,\ldots,k_l)\ dk_1\cdots dk_n\ , \tag{3}$$

where $\hat{W}^T_{n\alpha_1\cdots\alpha_n}$ is a tempered distribution which fulfils the spectral property, i.e. it has support in the cone $\{(k_1,\ldots,k_n)\in {}^{dn} : q_j=\sum_{l=1}^{j} k_l \in \bar{V}_0^-, j=1,\ldots,n-1\}$, and $x_1^0<\ldots<x_n^0$. Here, $\bar{V}_0^-$ stands for the closed backward lightcone (that we do not use the forward lightcone for the formulation of the spectral condition is a matter of convention on the Fourier transform). Under this condition the above integral exists in the sense of tempered distributions. If such a representation exists, it follows from the general theory of quantum fields that S_n^T is the analytic continuation of the inverse Fourier transform $W_n^T=\mathcal{F}^{-1}(\hat{W}_n^T)$ of $\hat{W}_n^T$ from points with purely relativistically real time to the Euclidean points of purely imaginary time. Furthermore, it follows from the symmetry and Euclidean covariance of the S_n^T that W_n^T fulfills the requirements of Poincaré covariance and locality, see e.g. [10].

In order to obtain such a representation, we apply the formula for the expansion of an inverse polynomial into partial fractions on the inverse of the denominator of Eq. (1)

$$\frac{1}{\prod_{l=1}^{N}(|k|^2+m_l^2)^{\nu_l}}=\sum_{l=1}^{N}\sum_{j=1}^{\nu_l}\frac{b_{lj}}{(|k|^2+m_l^2)^j} \tag{4}$$

with $b_{lj}\in$ uniquely determined and $b_{l\nu_l}\neq 0$. Thus, $D^{-1}(x)$ can be represented as $Q_E(-i\nabla)\sum_{l=1}^{N}\sum_{j=1}^{\nu_l} b_{lj}(-\Delta+m_l^2)^{-j}(x)$ and

$$S^T_{n,\alpha_1\cdots\alpha_n}=\mathbf{Q}_{E,n,\alpha_1\cdots\alpha_n}(-i\underline{\nabla})\sum_{l_1,\ldots,l_n=1}^{N}\ \sum_{j_1,\ldots,j_n=1}^{\nu_1,\ldots,\nu_n}\prod_{r=1}^{n} b_{l_r,j_r} S_{n,(m_{l_1},j_1),\ldots,(m_{l_n},j_n)} \tag{5}$$

with $\mathbf{Q}_{E,n,\alpha_1\cdots\alpha_n}(-i\underline{\nabla})=C^{\beta_1\cdots\beta_n}Q_E(-i\nabla)^{\otimes n}_{\beta_1\cdots,\beta_n\,\alpha_1\cdots\alpha_n}$ and

$$S_{n,(m_{l_1},j_1)\cdots(m_{l_n},j_n)}(x_1,\ldots,x_n)=\int_{{}^{d}}\prod_{r=1}^{n}(-\Delta+m_{l_r}^2)^{-j_r}(x_r-x)\ dx \tag{6}$$

In order to obtain a Laplace representation as in Eq. (3) for $S_{n,(m_{l_1},j_1),\ldots,(m_{l_n},j_n)}$ we introduce some notations. Let $\delta_m^{(j)\pm(k)}:=\theta(\pm k^0>0)\delta^{(j)}(k^2-m^2)$ where $\delta^{(j)}$ is the j-th derivative of the one-dimensional delta distribution, θ is the Heaviside function and $k^2={k^0}^2-|\vec{k}|^2$. Furthermore, let

$$\mathbf{Q}_{M,n}(\underline{k})=\mathbf{Q}_{M,n}((k_1^0,\vec{k}_1),\ldots,(k_n^0,\vec{k}_n))=\mathbf{Q}_{E,n}((ik_1^0,\vec{k}_1),\ldots,(ik_n^0,\vec{k}_n)). \tag{7}$$

We are now in the position to state the main theorem of this section:

Theorem 1 *(i) For $n \geq 3$ or $n = 2, m_{l_1} \neq m_{l_2}$ let $\hat{W}^T_{n,(m_{l_1},j_1),\ldots,(m_{l_n},j_n)}$ be defined as*

$$(2\pi)^{-\frac{d(n-2)+2}{2}} \left\{ \sum_{s=1}^{n} \prod_{v=1}^{s-1} \frac{(-1)^{j_v-1}}{(j_v-1)!} \delta^{-,(j_v-1)}_{m_{l_v}}(k_v) \frac{1}{(k_s^2 - m_{l_s}^2)^{j_s}} \right.$$
$$\left. \times \prod_{v=s+1}^{n} \frac{(-1)^{j_v-1}}{(j_v-1)!} \delta^{+,(j_v-1)}_{m_{l_v}}(k_v) \right\} \delta(\sum_{s=1}^{n} k_s) \qquad (8)$$

and for $n = 2, m_{l_1} = m_{l_2}$:

$$\frac{(-1)}{(j_1+j_2-1)!} \left\{ \sum_{l=0}^{j_1+j_2-1} \binom{j_1+j_2-1}{l} \frac{(-1)^l l! \delta^{-,(n-l)}_{m_{l_1}}(k_1)}{4(|\vec{k}_1|^2 + m_{l_1}^2)^l} \right.$$
$$\left. + (-1)^{j_1+j_2-1} \delta^{-,(j_1+j_2-1)}_{m_{l_1}}(k_1) \right\} \delta(k_1 + k_2)\,. \qquad (9)$$

Then $\hat{W}_{n,(m_{l_1},j_1),\ldots,(m_{l_n},j_n)}$ is a tempered distribution on ^{dn} which fulfills the spectral property and

$$S^T_{n,(m_{l_1},j_1),\ldots,(m_{l_n},j_n)} = \mathcal{L}(\hat{W}^T_{n,(m_{l_1},j_1),\ldots,(m_{l_n},j_n)}). \qquad (10)$$

(ii) Let $\hat{W}^T_{n,\alpha_1\cdots\alpha_n}$ be defined as

$$\hat{W}^T_{n,\alpha_1\cdots\alpha_n} = \mathbf{Q}_{M,n,\alpha_1\cdots,\alpha_n}(\underline{k}) \sum_{l_1,\ldots,l_n=1}^{N} \sum_{j_1,\ldots,j_n=1}^{\nu_1,\ldots,\nu_n} \prod_{r=1}^{n} b_{l_r j_r} \hat{W}_{n,(m_{l_1},j_1),\ldots,(m_{l_n},j_n)}, \qquad (11)$$

then $S^T_{n,\alpha_1\cdots,\alpha_n} = \mathcal{L}(\hat{W}^T_{n,\alpha_1\cdots,\alpha_n})$. Furthermore, $W^T_{n,\alpha_1\cdots,\alpha_n} = \mathcal{F}^{-1}(\hat{W}^T_{n,\alpha_1\cdots,\alpha_n})$ fulfills the requirements of temperedness, relativistic covariance w.r.t. the representation $\tilde{\tau} : L^{\uparrow}_{+}(^d) \to G(L)$, locality, spectral property and cluster property. Here $\tilde{\tau}$ is obtained by analytic continuation of τ to a representation of the proper complex Lorentz group over ^{d} (which contains $SO(d)$ as a real submanifold) and restriction of this representation to the real orthochronous proper Lorentz group.

Proof. The spectral property of $\hat{W}^T_{n,(m_{l_1},j_1),\ldots,(m_{l_n},j_n)}$ is a direct consequence of the definition Eq. (8), cf. the proof of Proposition 7.8 of [1].

In order to prove the Laplace representation formula for $S^T_{n,(m_{l_1},j_1),\ldots,(m_{l_n},j_n)}$ we proceed by induction over $\rho = \max\{j_l : l = 1,\ldots,n\}$. The statement for $\rho = 1$ is just the statement of Proposition 7.8 in [1]. Note that $(-\Delta+m^2)^{-j}(x) = \frac{(-1)}{(j-1)} \frac{d}{dm^2}(-\Delta+m^2)^{-(j-1)}(x)$ holds in the sense of tempered distributions. Let $j_{a_1} = \ldots = j_{a_u} = \rho$ and all other $j_l < \rho$. For $j_l < \rho$ let $j'_l = j_l$ and $j'_l = j_l - 1$ otherwhise. We then get by using Eq. (6), the induction hypothesis as well as continuity properties of the Laplace transform and approximation of derivations by differential quotients:

$$
\begin{aligned}
S^T_{n,(m_{l_1},j_1),\dots,(m_{l_n},j_n)} &= \prod_{w=1}^{u} \frac{(-1)}{(j_{a_w}-1)} \frac{d}{dm^2_{j_{a_w}}} S^T_{n,(m_{l_1},j'_1),\dots,(m_{l_n},j'_n)} \\
&= \prod_{w=1}^{u} \frac{(-1)}{(j_{a_w}-1)} \frac{d}{dm^2_{j_{a_w}}} \mathcal{L}(\hat{W}^T_{n,(m_{l_1},j'_1),\dots,(m_{l_n},j'_n)}) \\
&= \mathcal{L}(\prod_{w=1}^{u} \frac{(-1)}{(j_{a_w}-1)} \frac{d}{dm^2_{j_{a_w}}} \hat{W}^T_{n,(m_{l_1},j'_1),\dots,(m_{l_n},j'_n)}) \\
&= \mathcal{L}(\hat{W}^T_{n,(m_{l_1},j_1),\dots,(m_{l_n},j_n)}),
\end{aligned}
$$

where we have made use of the explicit formulae (8), (9) in the last step. The above way of deriving the $W^T_{n,(m_{l_1},j'_1),\dots,(m_{l_n},j'_n)}$ as distributions w.r.t. some mass parameter can also be used to prove the temperedness inductively, since we can differentiate terms like $\delta^{\pm,(j)}_m$ and $(|k|^2+m^2)$ w.r.t. ${k^0}^2$ instead of m^2 and use the fact that the change of variables ${k^0}^2 \leftrightarrow k^0$ is smooth and polynomially bounded for $k^2 > \min\{m_r : r = 1\dots,N\} - \epsilon > 0$.

(ii) The Laplace representation of $S^T_{n,\alpha_1\cdots\alpha_n}$ immediately follows from (5), (6) and the fact that $\mathbf{Q}_{E,n}(-i\underline{\nabla})\mathcal{L}(\hat{W}) = \mathcal{L}(\mathbf{Q}_{M,n}(\underline{k})\hat{W})$ for any tempered distribution $\hat{W}$ on ^{dn} with the spectral property. The rest of the theorem follows from [10] (for the cluster property, see the proof of Theorem 7.10 of [1]) $\blacksquare$

By considerations similar to those proving the temperedness of the $\hat{W}^T_n$ one can also show that the sequence of tempered distributions fulfills the sufficient Hilbert space structure condition on truncated Wightman functions introduced in [2]. Therefore, the sequence of Wightman functions are the vacuum expectation values of some quantum field theory in indefinite metric [2, 9].

3 Criteria for the existence of a scattering limit and calculation of the scattering amplitudes

In this section we replace the two point function $\hat{W}^T_2$ given in Theorem 1 by a two point function $\hat{W'}^T_2(k_1,k_2) = \mathbf{Q}_{M,2}(k_1,k_2)\sum_{s=1}^N \lambda_s \delta^-_{m_s}(k_1)\delta(k_1+k_2)$, $0 \neq \lambda_s \in$ which clearly has the same covariance properties as $\hat{W}^T_2$. The reason for this is that in any $\hat{W}^T_2$ there occur terms $\hat{W}^T_{2,(m_{l_1},j_1)(m_{l_2},j_2)}$ with $m_{l_1} = m_{l_2}$ and these terms lead to "exploding" scattering behaviour, as we shall explain below. Replacing $\hat{W}^T_2$ by $\hat{W'}^T_2$ simply means cancelling these terms in order to obtain stable one-particle states. In some special cases this can be motivated as a "renormalization procedure", see [4]. We will also drop the ' and write $\hat{W}^T_2$ for the "new" 2-point functions from now on.

We now briefly recall some basic notions of axiomatic scattering theory following [8].

For $l = 1, \ldots, n$ let $\varphi^t_{l,s_l}(x) = (2\pi)^{-dn/2} \int_{{}^d} e^{ik_l \cdot x} \hat{\varphi}_{l,s_l}(k) e^{i(k_l^0 - \omega_{l,s_l})t} \ dk_l$, where $\hat{\varphi}_{l,s_l}$ is a Schwartz function with values in ^{L} and support in $\{k_l^0 > 0, |k_l^2 - m_{s_l}^2| < \epsilon\}$ for ϵ small enough that non of these neighbourhoods intersect for $m_{s_l} \neq m_{s_l'}$ and $\omega_{l,s_l} = \sqrt{|k_l|^2 + m_{s_l}^2}$.

We say that a sequence of truncated Wightman functions W_n^T has non trivial scattering behaviour, if the limits

$$
\begin{aligned}
&\langle \varphi_{1,s_1} \cdots \varphi_{r,s_r}^{\mathrm{in}} | \varphi_{r+1,s_{r+1}} \cdots \varphi_{n,s_n}^{\mathrm{out}} \rangle^T = \\
&\lim_{t \to +\infty} \int_{{}^{dn}} W^T_{n,\alpha_1 \cdots \alpha_n}(x_1, \ldots, x_n) \varphi^{\alpha_1,-t,*}_{r,s_r}(x_1) \cdots \varphi^{\alpha_r,-t,*}_{1,s_1}(x_r) \\
&\qquad \times \varphi^{\alpha_{r+1},t}_{r+1,s_{r+1}}(x_{r+1}) \cdots \varphi^{\alpha_n,t}_{n,s_n}(x_n) dx_1 \cdots dx_n; \\
&\langle \varphi_1 \cdots \varphi_r^{\mathrm{in/out}} | \varphi_{r+1} \cdots \varphi_n^{\mathrm{in/out}} \rangle^T = \\
&\lim_{t \to \pm\infty} \int_{{}^{dn}} W^T_{n,\alpha_1 \cdots \alpha_n}(x_1, \ldots, x_n) \varphi^{\alpha_1,-t,*}_{r,s_r}(x_1) \cdots \varphi^{\alpha_r,-t,*}_{1,s_1}(x_r) \\
&\qquad \times \varphi^{\alpha_{r+1},-t}_{r+1,s_{r+1}}(x_{r+1}) \cdots \varphi^{\alpha_n,-t}_{n,s_n}(x_n) dx_1 \cdots dx_n
\end{aligned} \tag{12}
$$

exist and are not identically equal to zero. Here $*$ stands for complex conjugation. By the following theorem we give necessary and sufficient conditions, in terms of the mass spectrum of D, for nontrivial scattering behaviour.

Theorem 2 *(i) The sequence W_n^T has nontrivial scattering behaviour, if and only if $\nu_l = 1$ for $l = 1, \ldots, N$. If at least one of the ν_l fulfills $\nu_l > 1$, then the first limit in Eq. (12) diverges polynomially in t for $n \geq 3$.*
(ii) If $\nu_1 = \cdots, \nu_N = 1$, then

$$
\begin{aligned}
\langle \varphi^{\mathrm{ex}}_{1,s_1} | \varphi^{\mathrm{ex}}_{2,s_2} \rangle^T &= (2\pi)^{-d} \lambda_{s_1} \int_{{}^d \times {}^d} \delta^+_{m_{s_1}}(k_1) \delta(k_1 - k_2) Q_{M,2,\alpha\beta}(k_1, k_2) \\
&\quad \times \hat{\varphi}^{\alpha,*}_{1,s_1}(k_1) \hat{\varphi}^{\beta}_{2,s_2}(k_2) \ dk_1 dk_2 \ ,
\end{aligned} \tag{13}
$$

if $s_1 = s_2$ and $\langle \varphi^{\mathrm{ex}}_{1,s_1} | \varphi^{\mathrm{ex}}_{2,s_2} \rangle^T = 0$ otherwise. Here, the two ex *stand for all combinations of* in *and* out*. For $n \geq 3$,*

$$
\begin{aligned}
&\langle \varphi_{1,s_1} \cdots \varphi_{r,s_r}^{\mathrm{in}} | \varphi_{r+1,s_{r+1}} \cdots \varphi_{n,s_n}^{\mathrm{out}} \rangle^T = \\
&2\pi i \prod_{l=1}^N b_{s_l,1} \int_{{}^{4n}} \mathbf{Q}_{M,n}(-k_1, \ldots, -k_r, k_{r+1}, \ldots, k_n)_{\alpha_1 \cdots \alpha_n} \\
&\times \prod_{l=1}^r \delta^+_{m_{s_{r-l+1}}}(k_l) \prod_{l=r+1}^n \delta^+_{m_{s_l}}(k_l) \delta(\sum_{l=1}^r k_l - \sum_{l=r+1}^n k_l) \\
&\times \hat{\varphi}^{\alpha_1,*}_{r,s_r}(k_1) \cdots \hat{\varphi}^{\alpha_r,*}_{1,s_1}(k_r) \hat{\varphi}^{\alpha_{r+1}}_{r+1,s_{r+1}}(k_{r+1}) \cdots \hat{\varphi}^{\alpha_n}_{n,s_n}(k_n) \ dk_1 \cdots dk_n
\end{aligned} \tag{14}
$$

and

$$
\langle \varphi_{1,s_1} \cdots \varphi_{r,s_r}^{\mathrm{in}} | \varphi_{r+1,s_{r+1}} \cdots \varphi_{n,s_n}^{\mathrm{in}} \rangle^T = \langle \varphi_{1,s_1} \cdots \varphi_{r,s_r}^{\mathrm{out}} | \varphi_{r+1,s_{r+1}} \cdots \varphi_{n,s_n}^{\mathrm{out}} \rangle^T = 0. \tag{15}
$$

Proof. We first consider point (ii). This is a rather straight forward generalization of the Theorems 1 and 2 of [3] and it can be obtained using the techniques developed there, see [7] for the details of the proof and further generalizations. Note that the condition that $Q_E(k)$ is prime w.r.t the factors $(|k|^2+m_s^2)$ implies that $\mathbf{Q}_{M,n}$ does not vanish identically on the mass-shells and therefore the scattering amplitudes are not identically equal to zero.

It remains to prove point (i). Note that $(\frac{\partial}{\partial {k_l^0}^2}) = (\pm\frac{1}{2k_l^0}\frac{\partial}{\partial k_l^0})$ on the support of $\hat\varphi_{l,s_l}(\pm k)$. Therefore we get $\delta_{m_{s_l}}^{\pm,(j_l-1)}(k_l) = (\pm\frac{1}{2k_l^0}\frac{\partial}{\partial k_l^0})^{j_l-1}\delta^{\pm}_{m_{s_l}}(k_l)$ and $\frac{1}{(k_l^2-m_{s_l}^2)^{j_l}} = \frac{(-1)^{j_l-1}}{(j_l-1)!}(\pm\frac{1}{2k_l^0}\frac{\partial}{\partial k_l^0})^{j_l-1}\frac{1}{(k_l^2-m_{s_l}^2)}$ on these support sets. Using Parseval's theorem and integration by parts we get for the right hand side of Eq. (9)

$$\int_{d^n} \hat W^T_{n,(m_r,1)\cdots(m_1,1)(m_{r+1},1)\cdots(m_n,1)}(k_1,\ldots,k_n)\psi(t,k_1,\ldots,k_n)\ dk_1\cdots dk_n,$$

where $\psi(t,k_1,\ldots,k_n)$ is a sum over functions of the type

$$\prod_{l=1}^n(\pm\frac{1}{2k_l^0}\frac{\partial}{\partial k_l^0})^{j_l-1}e^{\pm i(k_l^0\pm\omega_{l,s_l})t}\hat h_l^{(*)}(\pm k_l),$$

where j_l takes values from 1 to ν_l and the $\hat h_l$ are scalar functions obtained by contraction of the $\hat\varphi^{\alpha_l}_{l,s_l}$ with the tensor-valued polynomial $\mathbf{Q}_{M,n,\alpha_1,\ldots,\alpha_n}(\underline{k})$ and multiplication with factors $1/k_l^0$ and thus having the same support properties as the $\hat\varphi^{\alpha_l}_{l,s_l}$. If $\nu_l>1$ for some l, such functions can be written as a sum of functions of the type $t^r\times\prod_{l=1}^n e^{\pm i(k_l^0\pm\omega_{l,s_l})t}\hat g_l^{(*)}(\pm k_l)$ with $0\le r\le\sum_{l=1}^n(\nu_l-1)$. If we consider the unique term with maximal $r=\sum_{l=1}^n(\nu_l-1)\ge 1$, then, by (ii) of this theorem, the corresponding term in the scattering amplitude consists of an expression which converges to a constant $\neq 0$ (provided the $\hat\varphi_{l,s_l}$ are chosen adequately) multiplied by a factor t^r. Thus, the scattering amplitude in this case diverges polynomially as $t\to\infty$. ■

Theorem 2 shows that the scattering amplitudes of some of the quantum field models diverge polynomially. This is a remarkable fact, since it demonstrates that a generalization the standard axiomatic scattering theory (Haag-Ruelle theory) for quantum fields in positive metric to the case of quantum fields in indefinite metric is possible only under additional conditions (e.g. conditions on the infrared singularities of the theory).

Aknowledgements. We would like to thank C. Becker and R. Gielerak for interesting discussions. The financial support of D.F.G. via SFB 237 is gratefully acknowledged.

References

[1] S. Albeverio, H. Gottschalk, J.-L. Wu, Convoluted generalized white noise, Schwinger functions and their analytic continuation to Wightman functions. Rev. Math. Phys. **8**(1996), 763–817.

[2] S. Albeverio, H. Gottschalk, J.-L. Wu, Models of local relativistic quantum fields with indefinite metric (in all dimensions). Commun. Math. Phys. **184**(1997), 309–351.

[3] S. Albeverio, H. Gottschalk, J.-L. Wu, Nontrivial scattering amplitudes for some local relativistic quantum field models with indefinite metric, Phys. Lett.**B 405** (1997) 243-248.

[4] S. Albeverio, K. Iwata, T. Kolsrud, Random fields as solutions of the inhomogeneous quaternionic Cauchy–Riemann equation.I.Invariance and analytic continuation. Commun. Math. Phys. **132**(1990), 555–580.

[5] C. Becker, R. Gielerak, P. Ługiewicz, Covariant SPDEs and quantum field structures, J. Phys. **A 31** (1998) 231-258.

[6] F. Contantinescu, Distributionen und ihre Anwendung in der Physik(Teubner, Stuttgart, 1973).

[7] H. Gottschalk, Greens functions for scattering in local relativistic quantum field theory, thesis in preparation, Bochum 1998.

[8] K. Hepp, On the connection between the LSZ and Wightman quantum field theory. Commun. Math. Phys. **1**(1965), 95–111.

[9] G. Morchio, F. Strocchi, Infrared singularities, vacuum structure and pure phases in local quantum field theory. Ann. Inst. H. Poincaré **A33**(1980), 251–282.

[10] K. Osterwalder, R. Schrader, Axioms for Euclidean Green's functions. I. Commun. Math. Phys. **31**, 83–112 (1973); II. Commum. Math. Phys. **42**, 281–305 (1975).