IMPROVED PEIERLS ARGUMENT FOR HIGH DIMENSIONAL ISING MODELS

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Abstract. We consider the low temperature expansion for the Ising model on $\mathbb{Z}^d$, $d \geq 2$, with ferromagnetic nearest neighbor interactions in terms of Peierls contours. We prove that the expansion converges for all temperatures smaller than $Cd(\log d)^{-1}$, which is the correct order in d.

Key Words. Ising Model, Peierls Contour, Low-Temperature Expansion, High Dimension

1. Introduction and Result

We consider the Ising model without an external magnetic field on the d-dimensional cubic lattice $\mathbb{Z}^d$. The model is defined by the formal Hamiltonian

$$H(\sigma) = -\sum_{(x,y)} \sigma_x \sigma_y, \tag{1.1}$$

where the configuration $\sigma \in \{-1,+1\}^{\mathbb{Z}^d}$ takes values σ_x and σ_y at sites x and y of $\mathbb{Z}^d$ and the sum is taken over all nearest neighbor bonds (x,y) of $\mathbb{Z}^d$. It is known [BKLS] that the critical inverse temperature β_{cr} of the Ising model behaves like $\frac{1}{4d}$ for large d. An estimate

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$\beta_{cr} > C_1/d$ for some $C_1 \leq \frac{1}{4}$ can be easily derived from the standard high temperature expansions [R] or by other arguments [G], [S]. Here and below we denote by C_1, $C_2, \ldots$ positive absolute constants.

At first sight it is surprising that the best currently available upper bound, $\beta_p < C_2$, given by the Peierls argument [R], is very far from the true value of β_{cr} when d is large. Naively one would like to have from the low temperature expansion an upper bound of the form $\beta_p < C_3/d$, i.e. at least of the same order as the lower bound. Unfortunately this is impossible. The Peierls argument, i.e. the convergence of the low temperature expansion, automatically implies the absence of percolation of the minority phase while it is known [ABL] that for the inverse temperature β higher than β_{cr} but lower than $C_4 \log d/d$ the minority phase does percolate. Thus for large d the upper estimate for β_{cr} given by the Peierls argument can not be close to β_{cr}. Nevertheless it is interesting to understand what is the radius of convergence for the low temperature expansion written in terms of Peierls contours. The answer, which is correct up to the constant factor, is given by

Theorem 1.1 *For the Ising model (1.1) the low temperature expansion, written in terms of Peierls contours, is convergent for* $\beta \geq 64(\log d)/d$.

The geometric problem closely related to this theorem is the upper bound for the number, $\sharp(n)$, of Peierls contours of size n. The best previous estimate was $\sharp(n) \leq 3^n$ [R]. Now we improve this estimate

Corollary 1.2 *The number,* $\sharp(n)$, *of different Peierls contours of size* n *is less than* $\exp[64n(\log d)/d]$.

On the other hand it is not hard to see that

Lemma 1.3 *The number* $\sharp(n)$ *is larger than* $\exp[(n-2d)(\log d)/(2d-2)]$.

Proof. Consider a chain of k lattice sites which starts at a given $x \in \mathbb{Z}^d$ and every next site is obtained from the previous one by the unit shift in one of the positive coordinate directions. Clearly one has d^{k-1} different chains of that type. Take the contour which is the boundary of the union of the unit d-dimensional cubes centered at the sites of the chain. The size of this contour is $(2d-2)k+2$ and different chains produce different contours. ■

Thus our estimate is correct in order though the constant 64 is certainly too large.

2. Proof of Theorem

We begin with some geometric notions which we need to define Peierls contours. A *plaquette* is a unit $(d-1)$-dimensional cube from the dual lattice centered at the middle of some bond of the initial lattice. Two plaquettes are called adjacent if they have a common $(d-2)$-dimensional face. Two lattice sites are called adjacent if they are endpoints of some lattice bond. A plaquette and a lattice site are adjacent if this plaquette intersects one of the lattice bonds incident on this site. A lattice site and a $(d-2)$-dimensional face are adjacent if the site is adjacent to one of four plaquettes incident to this face.

A set of plaquettes is connected if any two its plaquettes belong to a chain of pairwise adjacent plaquettes from this set. Similarly a lattice subset is connected if any two of its sites belong to a chain of pairwise adjacent sites from this set.

Consider a configuration σ containing a finite number of sites x at which $\sigma_x = -1$. For every unit lattice bond (x,y) having $\sigma_x\sigma_y = -1$ draw a plaquette orthogonal to (x,y).

Such plaquettes form a closed surface consisting of several connected components which are known as Peierls contours. Clearly a *contour* γ is just a connected closed plaquette surface separating a finite set $\bar{\gamma} \subset \mathbb{Z}^d$ called the interior of γ from its complement $\bar{\gamma}^c = \mathbb{Z}^d \setminus \bar{\gamma}$ called the exterior of γ.

Two contours are called *compatible* if their union is not a connected set of plaquettes. A collection of contours is called compatible if any two contours from this collection are compatible. It is not hard to see that the correspondence between finite compatible collections of contours and configurations σ with a finite number of "$-$" spins is one-to-one.

The convergence of the low temperature expansion means the existence of an absolutely convergent polymer series for the logarithm of the partition function in any finite region with "+" (and by symmetry "$-$") boundary conditions. This series is the sum of statistical weights of so called polymers belonging to the region. Every contributing polymer is a finite family of contours which can be indexed in such a way that every next contour is incompatible with at least one of previous contours. The notion of polymer is dual to that of the compatible collection of contours. The statistical weight of the polymer is the product of the statistical weights of contributing contours times a combinatorial factor of more complicated structure. The details and precise definitions can be found in any reference on cluster (polymer) expansions. In particular [KP], [MS] or [D] contain general theorems which, applied to the Ising model, give us

Lemma 2.1 *The polymer expansion constructed for the Ising model in terms of Peierls contours is convergent at inverse temperature* β *if there exists a positive function* $a(\gamma)$ *such* *that for any contour* γ

$$\sum_{\gamma'} w(\gamma')e^{a(\gamma')} \le a(\gamma), \tag{2.1}$$

where the sum is taken over all contours γ' *incompatible with* γ, $w(\gamma') = \exp(-\beta|\gamma'|)$ *is the statistical weight of contour* γ' *and* $|\gamma'|$ *denotes the number of plaquettes in* γ'.

According to the Peierls argument [P] the probability that $\sigma_0 = -1$ in the Gibbs state $\langle \cdot \rangle^+$ with "+" boundary condition is equal to the probability that an odd number of contours surround the origin. This is less than the probability that $\bar{\gamma} \ni 0$ for at least one contour γ. A given contour γ has a probability not exceeding $w(\gamma)$. Hence the magnetization $\langle \sigma_0 \rangle^+$ is positive as soon as

$$\sum_{\gamma:\ \bar{\gamma}\ni 0} w(\gamma) < \frac{1}{2}. \tag{2.2}$$

Condition (2.1) is stronger than (2.2) but has a similar nature. To clarify the difference observe that γ' is incompatible with γ iff it contains at least one of the $(d-2)$-dimensional faces of γ. Therefore $\bar{\gamma}'$ contains one of the lattice sites adjacent to some $(d-2)$-dimensional face of γ. The number of $(d-2)$-dimensional faces in γ does not exceed $(d-1)|\gamma|$ as such a face is shared by two or four plaquettes from γ. Hence the number of adjacent lattice sites is less than $4d|\gamma|$ and (2.1) is satisfied for $a(\gamma) = \beta\exp[-d\beta/4]|\gamma|$ if

$$\sum_{\gamma:\ \bar{\gamma}\ni x} w(\gamma)\exp\left(\beta e^{-\frac{d\beta}{4}}|\gamma|\right) \le \frac{\beta}{4d}e^{-\frac{d\beta}{4}}, \tag{2.3}$$

where the sum is taken over all contours surrounding a site $x \in \mathbb{Z}^d$.

Estimate (2.3) is stronger than (2.2) because $\exp\left(\beta e^{-\frac{d\beta}{4}}|\gamma|\right) > 1$ and $\frac{\beta}{4d}e^{-\frac{d\beta}{4}} \le e^{-1}d^{-2} < 1/2$. In the proof of Lemma 2.1 the factor $\exp\left(\beta e^{-\frac{d\beta}{4}}|\gamma|\right)$ is used to dominate the combinatorial factor contributing to the statistical weight of the polymer. The control obtained is enough to see that the sum of the absolute values of the statistical weights of all polymers surrounding a site $x \in \mathbb{Z}^d$ is smaller than $\frac{\beta}{4d} e^{-\frac{d\beta}{4}}$. This gives the convergence of the polymer expansion implying various nice properties, e.g. an exponential decay of correlations in the state $\langle \cdot \rangle^+$.

In the rest of the paper we check that (2.3) is true for $\beta \geq 64(\log d)/d$. A contour γ is called *primitive* if it can not be partitioned into two contours γ' and γ''. If γ is not primitive then the corresponding γ' and γ'' have no common plaquettes but have common $(d-2)$-dimensional faces. In particular $w(\gamma) = w(\gamma')w(\gamma'')$. For γ surrounding a given site x consider some decomposition of γ into primitive subcontours: note that such a decomposition may not be unique. For a fixed decomposition the set of corresponding primitive subcontours can be naturally provided with a tree-like structure. The root of the tree is the primitive subcontour γ_0 surrounding x. The fist-level subcontours γ_{1,i_1} are the subcontours having a common $(d-2)$-dimensional face with γ_0. The second-level subcontours γ_{2,i_2} are the subcontours (not included in the previous levels) which have a common $(d-2)$-dimensional face with at least one of the first-level subcontours. Generally the subcontours of the n-th level γ_{n,i_n} are the subcontours which have a common $(d-2)$-dimensional face with at least one of the subcontours from level $n-1$ and are not included in $\cup_{k=0}^{n-1} \cup_{i_k} \gamma_{k,i_k}$.

We will now show that (2.3) will be satisfied when the following inequality holds:

$$\sum_{\gamma:\, \bar{\gamma} \ni x,\ \gamma \text{ is primitive}} w(\gamma) \exp\left(2\beta e^{-\frac{d\beta}{4}} |\gamma|\right) \leq \frac{\beta}{4d} e^{-\frac{d\beta}{4}}, \tag{2.4}$$

with the sum over primitive contours only.

Denote by $n(\gamma)$ the number of levels in the decomposition of γ into primitive subcontours. By induction in $n(\gamma)$ we check that, for any n, (2.4) implies

$$\sum_{\gamma:\, \bar{\gamma}\ni x, n(\gamma)\le n} w(\gamma) \exp\left(\beta e^{-\frac{d\beta}{4}}|\gamma|\right) \le \frac{\beta}{4d} e^{-\frac{d\beta}{4}}, \tag{2.5}$$

and hence (2.3). For $n = 1$ (2.5) clearly follows from (2.4). Suppose now that (2.5) is true for $n = N-1$ and consider the case $n = N$. Without γ_0 the subcontours from $\cup_{k=1}^{N} \cup_{i_k} \gamma_{k,i_k}$ are decoupled into subtrees with γ_{1,i_1} serving as new roots. Clearly for each subtree the number of levels does not exceed $N-1$. By construction the subcontour γ_{1,i_1} surrounds a site y adjacent to some $(d-2)$-dimensional face of γ_0 and the set $A(\gamma_0)$ of all such sites has the cardinality $|A(\gamma_0)| \le 4d|\gamma_0|$. Using the induction hypothesis (in the second inequality below) one obtains

$$\begin{aligned}
&\sum_{\gamma:\, \bar{\gamma}\ni x, n(\gamma)\le N} w(\gamma) \exp\left(\beta e^{-\frac{d\beta}{4}}|\gamma|\right) \\
&\le \sum_{\gamma_0:\, \bar{\gamma}_0\ni x} w(\gamma_0) \exp\left(\beta e^{-\frac{d\beta}{4}}|\gamma_0|\right) \prod_{y\in A(\gamma_0)} \left(1 + \sum_{\gamma:\, \bar{\gamma}\ni y, n(\gamma)\le N-1} w(\gamma) \exp\left(\beta e^{-\frac{d\beta}{4}}|\gamma|\right)\right) \\
&\le \sum_{\gamma_0:\, \bar{\gamma}_0\ni x} w(\gamma_0) \exp\left(\beta e^{-\frac{d\beta}{4}}|\gamma_0|\right) \prod_{y\in A(\gamma_0)} \left(1 + \frac{\beta}{4d} e^{-\frac{d\beta}{4}}\right) \\
&\le \sum_{\gamma_0:\, \bar{\gamma}_0\ni x} w(\gamma_0) \exp\left(\beta e^{-\frac{d\beta}{4}}|\gamma_0|\right) \exp\left(4d|\gamma_0|\frac{\beta}{4d} e^{-\frac{d\beta}{4}}\right) \\
&= \sum_{\gamma_0:\, \bar{\gamma}_0\ni x} w(\gamma_0) \exp\left(2\beta e^{-\frac{d\beta}{4}}|\gamma_0|\right) \\
&\le \frac{\beta}{4d} e^{-\frac{d\beta}{4}},
\end{aligned} \tag{2.6}$$

which reproduces (2.5) for $n = N$.

From now on we discuss primitive contours only and we call them contours. Denote by γ_i, $i = 1, \dots, d$ the set of plaquettes of γ orthogonal to the coordinate axis number

i. Let $|\gamma_{i^*}| = \min_i |\gamma_i|$. The direction i^* is called γ-vertical and all plaquettes of γ are separated into horizontal ones, i.e. those belonging to γ_i, γ_{i^*}, and vertical ones, i.e. those from $\gamma \setminus \gamma_{i^*}$. From now on we denote them γ^{hor} and γ^{ver} respectively.

Consider a site $x \in \mathbb{Z}^d$ and a contour γ surrounding x. Draw a γ-vertical line through x. This line intersects some plaquette $p \in \gamma^{hor}$ and the distance from x to p is less than $\frac{|\gamma^{ver}|}{2d-2}$. Now it is clear that the sum in (2.4) does not exceed

$$\sum_{\gamma:\, \gamma^{hor} \ni p} d \frac{|\gamma^{ver}|}{2d-2} w(\gamma) \exp\left(2\beta e^{-\frac{d\beta}{4}} |\gamma|\right), \tag{2.7}$$

where p is fixed and the factor d counts the number of choices for the vertical direction.

Our main observation is the following simple estimate

$$|\gamma| \geq \frac{d}{2}|\gamma^{hor}| + \frac{1}{2}|\gamma^{ver}|, \tag{2.8}$$

which is an immediate consequence of the definition of γ^{hor}. It reduces (2.7) to

$$\sum_{\gamma:\, \gamma^{hor} \ni p} |\gamma^{ver}| \exp\left(2\beta e^{-\frac{d\beta}{4}} |\gamma|\right) \exp\left(-\frac{d\beta}{2}|\gamma^{hor}| - \frac{\beta}{2}|\gamma^{ver}|\right) \leq \frac{\beta}{4d} e^{-\frac{d\beta}{4}}. \tag{2.9}$$

The elementary inequalities:

$$2\beta e^{-\frac{d\beta}{4}} \leq \frac{\beta}{16} \quad \text{for} \quad \beta \geq \frac{20 \log 2}{d}, \tag{2.10}$$

$$|\gamma^{ver}| e^{-\frac{7\beta}{16}|\gamma^{ver}|} \leq e^{-\frac{6\beta}{16}|\gamma^{ver}|} \quad \text{for} \quad \beta \geq \frac{16 \log |\gamma^{ver}|}{|\gamma^{ver}|}, \tag{2.11}$$

$$\frac{16 \log d}{d} \geq \frac{16 \log (2d-2)}{2d-2} \geq \frac{16 \log |\gamma^{ver}|}{|\gamma^{ver}|} \quad \text{for} \quad d \geq 3 \tag{2.12}$$

reduce (2.9) to the bound

$$\sum_{\gamma:\, \gamma^{hor} \ni p} \exp\left(-\frac{3d\beta}{8}|\gamma^{hor}| - \frac{3\beta}{8}|\gamma^{ver}|\right) \leq \frac{\beta}{4d} e^{-\frac{d\beta}{4}} \tag{2.13}$$

Let a *floor* of a contour γ be a connected component of γ^{hor}.

Lemma 2.2 *A contour γ is uniquely defined by the family of its floors.*

Proof. To show this we reconstruct all plaquettes of γ from the family of its floors. We start with the floors of γ situated at the minimal vertical level which we denote by m. Their boundary consists of $(d-2)$-dimensional faces. Each of these faces has a unique vertical plaquette growing from it in the positive vertical direction. We add all these vertical plaquettes to the set of already reconstructed plaquettes. All floors of γ situated at vertical level $m+1$ are added next. The new boundary set of already reconstructed plaquettes consists of $(d-2)$-dimensional faces situated at level $m+1$. Again each of these faces has a unique vertical plaquette growing from it in the positive vertical direction. Adding all these vertical plaquettes one moves to the level $m+2$. Then we add all floors situated at the level $m+2$ and so on. The procedure terminates after a finite number of steps when the boundary of the set of reconstructed plaquettes becomes empty. ∎

The contour γ is uniquely decomposed into horizontal floors γ_i^{hor} and vertical plaquette stacks γ_j^{ver}. A *vertical plaquette stack* is a chain of pairwise adjacent vertical plaquettes extending between two floors. None but the first and the last plaquettes in the stack have a common $(d-2)$-dimensional face with the floors and every next plaquette in the stack can be obtained from the previous one by the unit vertical shift. One may associate with γ an abstract connected graph with floors being the vertices and vertical stacks being the links of the graph. Note that in this graph two vertices may have more than one link joining them. Let $T = \{\gamma_0^{hor}, \gamma_{m,i_m}^{hor}, \gamma_{m,i_m}^{ver}\}$ be a spanning tree of this graph. We assume that its root, γ_0^{hor}, passes through the given plaquette p. The links of the first level, γ_{1,i_1}^{ver}, join the root with the corresponding vertices of the first level, γ_{1,i_1}^{hor}, and so on. We associate with the tree T the statistical weight

$$w(T) = \exp\left(-\frac{3d\beta}{8}|\gamma_0^{hor}| - \sum_m \sum_{i_m}\left(\frac{3d\beta}{8}|\gamma_{m,i_m}^{hor}| + \frac{3\beta}{8}|\gamma_{m,i_m}^{ver}|\right)\right). \tag{2.14}$$

Then in view of Lemma 2.2 bound (2.13) follows from

$$\sum_{T:\ \gamma_0^{hor}(T)\ni p} w(T) \le \frac{\beta}{4d}e^{-\frac{d\beta}{4}}, \tag{2.15}$$

where the sum is taken over all abstract trees of the type described above. In the same way as (2.3) follows from (2.4) one can see that (2.15) is a consequence of

$$\sum_{(\gamma^{ver},\gamma^{hor}):\ \gamma^{ver}\ni p} \exp\left(-\frac{3d\beta}{8}|\gamma^{hor}| - \frac{3\beta}{8}|\gamma^{ver}|\right)\exp\left(\beta e^{-\frac{d\beta}{4}}|\gamma^{hor}|\right) \le \frac{\beta}{4d}e^{-\frac{d\beta}{4}}. \tag{2.16}$$

Here the sum is taken over all pairs $(\gamma^{ver},\gamma^{hor})$ consisting of the vertical stack, γ^{ver}, starting at the fixed plaquette p and the floor, γ^{hor}, connected to the end of the stack γ^{ver}. The analogue of estimate (2.6) is applicable as the number of the starting plaquettes for the first-level stacks growing from γ_0^{hor} does not exceed $(2d-2)|\gamma_0^{hor}| \le 4d|\gamma_0^{hor}|$.

In view of (2.10) the bound (2.16), for $\beta > \frac{20\log 2}{d}$, is reduced to

$$\sum_{(\gamma^{ver},\gamma^{hor}):\ \gamma^{ver}\ni p} \exp\left(-\frac{5d\beta}{16}|\gamma^{hor}| - \frac{5\beta}{16}|\gamma^{ver}|\right) \le \frac{\beta}{4d}e^{-\frac{d\beta}{4}}. \tag{2.17}$$

Now we have

$$\begin{aligned}
&\sum_{(\gamma^{ver},\gamma^{hor}):\ \gamma^{ver}\ni p} \exp\left(-\frac{5d\beta}{16}|\gamma^{hor}| - \frac{5\beta}{16}|\gamma^{ver}|\right) \\
&\le \sum_{|\gamma^{ver}|=1}^{\infty} \exp\left(-\frac{5\beta}{16}|\gamma^{ver}|\right) \sum_{|\gamma^{hor}|=1}^{\infty} \exp\left(-\frac{5d\beta}{16}|\gamma^{hor}|\right)(2d)^{|\gamma^{hor}|} \\
&= \frac{2de^{-\frac{5d\beta}{16}-\frac{5\beta}{16}}}{\left(1-2de^{-\frac{5d\beta}{16}}\right)\left(1-e^{-\frac{5\beta}{16}}\right)}
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{32d}{5\beta} \frac{e^{-\frac{5d\beta}{16}}}{1-e^{-\frac{5d\beta}{16}}} \\
&\leq \frac{\beta}{4d} e^{-\frac{d\beta}{4}},
\end{aligned}
\tag{2.18}
$$

where the last inequality is true for $\beta \geq 64\frac{\log d}{d}$. ■

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