The period function of the nonholomorphic Eisenstein series for $PSL(2,\mathbb{Z})$

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Abstract

We calculate the period function of Lewis of the automorphic Eisenstein series $E(s,w) = \frac{1}{2}v^s \sum_{n,m\neq(0,0)}(mw+n)^{-2s}$ for the modular group $PSL(2,\mathbb{Z})$. This function turns out to be the function $B(\frac{1}{2},s+\frac{1}{2})\psi_s(z)$, where $B(x,y)$ denotes the beta function and ψ_s a function introduced some time ago by Zagier and given for $\Re s > 1$ by the series $\psi_s(z) = \sum_{n,m\geq 1}(mz+n)^{-2s} + \frac{1}{2}\zeta(2s)\,(1+z^{-2s})$. The analytic extension of ψ_s to negative integers s gives just the odd part of the period functions in the Eichler, Shimura, Manin theory for the holomorphic Eisenstein forms of weight $-2s+2$. We find this way an interesting connection between holomorphic and nonholomorphic Eisenstein series on the level of their respective period functions.

keywords: period function, Maass wave form, transfer operator, modular forms, dynamical approach, modular group, Selberg's zeta function.

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1 The transfer operator $\mathcal{L}_s$ and zeros of Selberg's zeta function

In the so called dynamical approach to Selberg's zeta function $Z_S(s)$ for the modular group $PSL(2,\mathbb{Z})$ this function is expressed through Fredholm determinants of the transfer operator $\mathcal{L}_s$ of the geodesic flow on the modular surface as [M]

$$Z_s(s) = det(1-\mathcal{L}_s)\,det(1+\mathcal{L}_s)\,. \tag{1}$$

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Thereby $\mathcal{L}_s$ denotes the transfer operator for $PSL(2,\mathbb{Z})$ which for $\Re s > 0$ has the following representation in the Banach space $B(D)$ of holomorphic functions on the disc $D = \{z \in \mathbb{C} : |z-1| < \frac{3}{2}\}$:

$$\mathcal{L}_s f(z) = \sum_{n=1}^{\infty} (\frac{1}{z+n})^{2s} \left[f(\frac{1}{z+n}) - f(0) \right] + f(0)\, \zeta(2s, z+1)\,, \tag{2}$$

with $\zeta(s,z)$ the Hurwitz zeta function. The zeros of $Z_S(s)$ hence can be related to those complex values of s for which the analytically continued operator $\mathcal{L}_s$ has eigenfunctions $f_s \in B(D)$ with eigenvalues $\lambda = 1$ or $\lambda = -1$.

In [CM] we discussed these eigenfunctions for s values corresponding to the so called trivial zeros of $Z_S(s)$, that is $s = s_n = -n\,,\ n \in \mathbb{N}$. It was shown in [CM], [LZ] and [Z2] that these eigenfunctions are the period polynomials $r_{\varphi_{-2s_n+2}}(z+1)$ of the holomorphic cusp forms φ_{-2s_n+2} of weight $-2s_n + 2$ for the group $PSL(2,\mathbb{Z})$ given in the Eichler, Shimura, Manin theory in terms of the cusp forms $\varphi_{-2s_n+2}(z)$ as [Z1]

$$r_{\varphi_{-2s_n+2}}(z) = \int_0^{i\infty} \varphi_{-2s_n+2}(z')(z-z')^{-2s_n} dz'\,, \tag{3}$$

respectively the period functions $p_{-2s_n+2}(z+1)$ of the holomorphic Eisenstein forms $G_{-2s_n+2}(z)$ of weight $-2s_n + 2$

$$G_{-2s_n+2}(z) = \sum_{m,n \neq (0,0)} (mz+n)^{2s_n-2}\,, \tag{4}$$

which are given in terms of these noncusp forms by a formula similar to the one in (3) which however has to be regularized at infinity since these Eisenstein forms do not vanish there [Z1].

The eigenfunctions of $\mathcal{L}_s$ corresponding to the spectral zeros of Z_S on $\Re s = \frac{1}{2}$ have also been called period functions by J. Lewis and D. Zagier in [LZ]. They were introduced by Lewis in [L] for the even Maass cusp forms φ_s which are automorphic eigenfunctions of the Laplace Beltrami operator $-\triangle_{LB} = -y^2\,(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$ on the modular surface with eigenvalue $\lambda = s(1-s)$, through the transformation $\mathcal{T}_s$ defined as

$$\mathcal{T}_s\, \varphi_s(z) = z \int_0^{\infty} \varphi_s(i\,v)\,(z^2+v^2)^{-s-1} v^s dv\,. \tag{5}$$

Then $\mathcal{T}_s \varphi_s(z+1)$ is an eigenfunction of $\mathcal{L}_s$ with eigenvalue $\lambda = 1$ [CM], [L]. An extension to odd Maass cusp forms has been given in [LZ] where also a formula analogous to the one in (5) can be found.

It was found in [CM] that formula (5) can also be applied to the constant Maass wave form $\varphi_s = c$ for the modular group. In this case one determines as its period function simply the function z^{-1} which results just in the eigenfunction $f(z) = \frac{1}{z+1}$ of $\mathcal{L}_s$ for $s = 1$ with eigenvalue $\lambda = 1$.

There still remain to be determined the eigenfunctions φ_s of $\mathcal{L}_s$ with eigenvalue $\lambda = 1$ corresponding to the spectral zeros s of $Z_S(s)$ related to the nontrivial zeros of Riemann's zeta function through the relation $\zeta(2s) = 0$.

In [LZ] and independently in [CM] it was shown that the function $\psi_s(z)$ defined for $\Re s > 1$ as [Z2]

$$\psi_s(z) = \sum_{m,n \geq 1} (mz+n)^{-2s} + \frac{1}{2}\zeta(2s)\,(1+z^{-2s}) \tag{6}$$

determines an eigenfunction of $\mathcal{L}_s$ with eigenvalue $\lambda = 1$ iff $\zeta(2s) = 0$. For s values corresponding to the trivial Riemann zero's, that is $s = -1, -2, -3, \cdots$, the analytic extension of ψ_s

leads to the period function p_{-2s+2} of the holomorphic Eisenstein series G_{-2s+2} [CM] discussed earlier.

In the same paper we conjectured that the analytic extension of ψ_s to s values corresponding to the nontrivial zeros $\zeta(2s)=0$ of Riemann's function should be related to the nonholomorphic Eisenstein series $E_s(z)$ for $PSL(2,\mathbb{Z})$, since the corresponding zeros of Selberg's zeta function are closely related to poles of the scattering matrix for this group.

2 The period function of the nonholomorphic Eisenstein series

The nonholomorphic Eisenstein series $E_s(w)$ is defined for $w \in \mathbb{H} = \{w \in \mathbb{C} : w = u + iv,\ v > 0\}$ and complex s with $\Re s > 1$ as

$$E_s(w) = \frac{1}{2} v^s \sum_{(m,n)\neq(0,0)} |mw+n|^{-2s} \,. \tag{7}$$

These functions are real analytic in w and define generalized eigenfunctions of the hyperbolic Laplace Beltrami operator $-\triangle_{LB}$ with eigenvalue $s(1-s)$ for $\Re s > 1$. $E_s(w)$ can be meromorphically continued into the entire complex s plane with only one simple pole at $s=1$ with residue $\frac{\pi}{2}$ independent of w [Z3].

Our main result then is

Theorem *The period function $\mathcal{T}_s E_s(z)$ as defined in (5) is given by*

$$\mathcal{T}_s E_s(z) = B(\frac{1}{2}, s+\frac{1}{2})\, \psi_s(z)\,, \tag{8}$$

where B denotes the beta function and ψ_s was defined in (6).

Proof The nonholomorphic Eisenstein series

$$E_s(w) = \frac{1}{2}\, v^s \sum_{m,n\in\mathbb{Z},\, m^2+n^2\neq 0} |m\,w+n|^{-2s}\,, \qquad (w = u + i\,v)\,. \tag{9}$$

can be written for $w = iv$ and $\Re s > 1$ as follows

$$\begin{aligned} E_s(0+i\,v) &= \frac{1}{2}\, v^s \sum_{m,n\in\mathbb{Z},\, m^2+n^2\neq 0} \left(|mvi+n|^2\right)^{-s} \\ &= \frac{1}{2}\, v^s \sum_{m,n\in\mathbb{Z},\, m^2+n^2\neq 0} \left(m^2v^2+n^2\right)^{-s} \\ &= \frac{1}{2}\, v^s \Big[\, 2\sum_{m\geq 1, n=0} + 2\sum_{m=0, n\geq 1} + 4\sum_{m,n\geq 1}\, \Big] \left(m^2v^2+n^2\right)^{-s} \\ &= v^{-s}\,\zeta(2s) + v^s\,\zeta(2s) + 2\,v^s \sum_{m,n\geq 1} \left(m^2v^2+n^2\right)^{-s}\,. \end{aligned} \tag{10}$$

Inserting this expression into definition (5) gives

$$\mathcal{T}_s E_s(z) \;=\; z\int_0^\infty \Big[\, v^{-s}\,\zeta(2s) + v^s\,\zeta(2s) + 2\,v^s \sum_{m,n\geq 1} \left(m^2v^2+n^2\right)^{-s} \Big]\, (\,z^2+v^2)^{-s-1}\, v^s\, d\,v$$

$$
\begin{aligned}
&= \zeta(2s)\, z \int_0^\infty \frac{d\,v}{(\,z^2+v^2)^{\,s+1}} + \zeta(2s)\, z \int_0^\infty \frac{v^{2s}\,d\,v}{(\,z^2+v^2)^{\,s+1}} \\
&\quad + 2\,z \sum_{m,n\geq 1} \int_0^\infty v^{2s}\,\frac{(m^2v^2+n^2)^{-s}}{(\,z^2+v^2)^{\,s+1}}\; d\,v\,, \qquad (11)
\end{aligned}
$$

where we used uniform convergence of the infinite series to get the last equality. With the substitution $\xi = (\frac{v}{z})^2$ the first two intergrals above can be rewritten as

$$
\frac{1}{2\,z^{2s+1}} \int_0^\infty \frac{\xi^{-1/2}\,d\,\xi}{(1+\xi)^{s+1}} \qquad \textit{respectively} \qquad \frac{1}{2\,z^{2s+1}} \int_0^\infty \frac{\xi^{s-1/2}\,d\,\xi}{(1+\xi)^{s+1}}\,. \qquad (12)
$$

However, for $|arg\,\beta| < \pi\,, \Re\nu > \Re\mu > 0$ the following formula holds [GR] 3.194.3

$$
\int_0^\infty \frac{\xi^{\mu-1}\,d\xi}{(1+\beta\xi)^\nu} = \beta^{-\mu}\,B(\mu,\nu-\mu)\,, \qquad (13)
$$

where $B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$ is the beta function. For the special values $\beta = 1,\ \nu = s+1$ and $\mu = 1/2$ respectively $\mu = s+1/2$ and for $\Re s > -1/2$ one then gets for the integrals in (12)

$$
B(1/2, s+1/2) \qquad \textit{respectively} \qquad B(s+1/2, 1/2)\,. \qquad (14)
$$

Obviously these two beta functions are the same.

For the third integral we use the following formula [GR] 3.197.5 which holds for $|arg\;\alpha| < \pi$ and $-\Re(\mu+\nu) > \Re\lambda > 0$

$$
\int_0^\infty x^{\lambda-1}\,(1+x)^\nu\,(1+\alpha x)^\mu\,d\,x = B(\lambda, -\mu-\nu-\lambda)\,F(-\mu,\lambda;-\mu-\nu;1-\alpha)\,,
$$

with $F(-\mu,\lambda;-\mu-\nu;1-\alpha)$ the hypergeometric function. Substituting $\mu = -s$, $\nu = -s-1$, $\lambda = s+1/2$, $\alpha = (\frac{m}{n}z)^2$ and $x = (\frac{v}{z})^2$ we get for $\Re s > -1/2$

$$
\int_0^\infty v^{2s}\,\frac{(m^2v^2+n^2)^{-s}}{(\,z^2+v^2)^{s+1}}\;d\,v = \frac{B(s+1/2,s+1/2)}{2\,n^{2s}\,z}\,F(s,s+\frac{1}{2};2s+1;1-(\frac{m}{n}z)^2)\,. \qquad (15)
$$

Applying relation $\Gamma(2\eta) = \pi^{-1/2}\,2^{2\eta-1}\,\Gamma(\eta)\,\Gamma(1/2+\eta)$ [MOS] p.3 for $\eta = s+1/2$ the function $B(s+1/2,s+1/2)$ can be also expressed as

$$
\begin{aligned}
B(s+1/2,s+1/2) &= \frac{\Gamma(s+1/2)\Gamma(s+1/2)}{\Gamma(2s+1)} = \frac{\Gamma(s+1/2)\Gamma(1/2)}{\pi^{1/2}}\frac{\Gamma(s+1/2)}{\Gamma(2s+1)} \\
&= B(1/2,s+1/2)\,2^{-2s} \qquad (16)
\end{aligned}
$$

Using next the identity [MOS] p.38, valid for all complex η with $\eta \notin [1,\infty)$

$$
F(a,a+1/2;2a+1;\eta^2) = 2^{2a}\,(1+\sqrt{1-\eta^2})^{-2a} \qquad (17)
$$

for $a = s$ and $\eta^2 = 1-(\frac{m}{n}z)^2$ we get the relation

$$
\frac{F(s,s+1/2;2s+1;1-(\frac{m}{n}z)^2)}{(2n)^{2s}} = (\frac{1}{mz+n})^{2s}\,. \qquad (18)
$$

Combining now (15), (16), (18) and (12), (14) with (11) shows that

$$\begin{aligned} \mathcal{T}_s E_s(z) &= B(1/2, s+1/2)\, \Big[\sum_{m,n\geq 1} \big(\frac{1}{mz+n}\big)^{2s} + \frac{1}{2}\zeta(2s)\big(1+(\frac{1}{z})^{2s}\big) \Big] \\ &= B(1/2, s+1/2)\, \psi_s(z)\,. \end{aligned} \tag{19}$$

Notice that (18) holds for all z with $z \not\in \mathbb{R}$. Hence also relation (19) holds true first only in that region. Analytic continuation however then gives the final result for all z in the entire half plane $\Re z > 0$. Known analyticity properties of the functions B and ψ_s [Z2] then tell us the result to be true in the entire complex s plane, also. □

An immediate consequence of this Theorem is

Corollary 1 *The eigenfunctions of the transfer operator $\mathcal{L}_s$ with eigenvalue $\lambda = 1$ for s-values related to the nontrivial zeros of Riemann's zeta function through $\zeta(2s) = 0$ are just the period functions $\mathcal{T}_s\, E_s(z+1)$ of the nonholomorphic Eisenstein series for the same s-values.*

It is interesting that the period functions of the Eisenstein series $E(s,z)$ for s-values with $\zeta(2s) = 0$ and not those on the critical line $\Re s = \frac{1}{2}$, which are known to determine the continuous spectrum of the Laplace Beltrami operator for $PSL(2,\mathbb{Z})$ are responsible for the spectral zeros of the Selberg function related to the scattering theory on the modular surface.

Combining next the result of our Theorem with what is known about the analytic extension of the function ψ_s to negative integer values of s we get

Corollary 2 *The analytic extension of the function $\frac{\mathcal{T}_s\, E_s(z)}{B(\frac{1}{2}, s+\frac{1}{2})}$ to negative integer s values is identical to the odd part of the period function $p_{-2s+2}(z)$ of the holomorphic Eisenstein form for $PSL(2,\mathbb{Z})$ of weight $-2s+2$.*

This shows that there exists a remarkable connection between nonholomorphic automorphic Eisenstein series $E(s,z)$ and all the holomorphic modular Eisenstein forms of even weight through the new period functions of Lewis and Zagier for Maass wave forms.

Our results also show that the transfer operator $\mathcal{L}_s$ combines in a surprising way the theory of holomorphic forms and nonholomorphic Maass wave forms through its eigenfunctions to the eigenvalues $\lambda = 1$ and $\lambda = -1$. These eigenfunctions are either period polynomials or period functions associated to these forms for the group $PSL(2,\mathbb{Z})$.

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