Problems with the definition of renormalized Hamiltonians for momentum-space renormalization transformations

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Abstract

For classical lattice systems with finite (Ising) spins, we show that the implementation of momentum-space renormalization at the level of Hamiltonians runs into the same type of difficulties as found for real-space transformations: Renormalized Hamiltonians are ill-defined in certain regions of the phase diagram.

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1 Introduction

Despite the great success of renormalization-group (RG) ideas, both for computations and as a heuristic guide, many aspects of the theory still lack rigorous mathematical justification. The filling of this gap is more than just of academic interest. It has been repeatedly pointed out (e.g. [8, p. 82], [3, footnote in page 38], [13, p. 268]) that the method is not a black-box type of technique; its succesful application requires some understanding of the underlying physics or one may be led to incorrect conclusions. Studies on the foundations of real-space transformations [15, 16, 19, 29, 30] suggest that a similar remark applies to the underlying mathematics. Indeed, these studies show that in various occasions renormalized Hamiltonians are ill-defined. The finite-volume probabilities of the renormalized system exhibit a long-range dependence on boundary spins that is incompatible with the existence of a Hamiltonian, at least one defined in the usual (summable) sense. Such a "pathology" is usually referred to as *non-Gibbsianness*. This phenomenon, which appears after a *single* application of the transformation, was first detected in the vicinity of first-order phase transitions, but was later discoverd in other regions of phase diagrams, including at high magnetic fields [30, 28] and at high temperatures [28, 25]. It follows that the design of the renormalization transformation is crucial for the very *existence* of a renormalization flow in a suitable space.

Nevertheless, the lack of similar studies for momentum-space transformations left open the possibility that they could be free of this pathological behavior. That is, the question remained as to whether such transformations, possibly with a soft cutoff, would generally lead to a honest-to-God renormalized Hamiltonian [16, 9]. In this note we present a simple example showing that this is in general *not* the case, as already suspected by Griffiths [14]. There is no essential difference between real-space and momentum-space transformations at least in case the spins are bounded. The same mechanism —the existence of a phase transition in the system of original spins, constrained by a particular block-spin configuration— causes similar problems with the definition of renormalized Hamiltonians. For an earlier suggestion that momentum-space maps are not all that different from real-space maps, see also [17].

As we remark at the end of Section 4, these problems can be interpreted as a manifestation of the well-known "large-field problem". It might be hoped that the considerable experience accumulated in the treatment of this type of problems, could be of help to control the non-Gibbsianness "pathologies".

2 Momentum transformations

We consider Ising spins $\sigma_{\vec{x}} = -1, +1$ on a lattice, $x \in \mathbb{Z}^d$. For each finite periodic cube $V = [-N, N]^d$ in $\mathbb{Z}^d$ we define the Fourier-transformed variables

$$\widehat{\sigma}^V_{\vec{k}} \;:=\; \sum_{\vec{x}\in V} \sigma_{\vec{x}}\, e^{-i\vec{k}\vec{x}} \;, \tag{2.1}$$

where $\vec{k}\vec{x} := k_1x_1 + \cdots + k_dx_d$, and each k_i belongs to the Brillouin zone: $\mathcal{B}_N = \{-\pi, -\pi[1 - 1/(2N + 1)], \ldots, \pi[1 - 1/(2N + 1)], \pi\}$. The inverse of (2.1) is

$$\sigma_{\vec{x}} \;=\; \frac{1}{(2N+1)^d} \sum_{\vec{k}\in\mathcal{B}^d_N} \widehat{\sigma}^V_{\vec{k}}\, e^{i\vec{k}\vec{x}} \;, \tag{2.2}$$

for $\vec{x} \in V$.

A *momentum-space transformation* is defined in two steps:

1st. A cutoff is applied to the variables $\widehat{\sigma}^V_{\vec{k}}$:

$$\widehat{\sigma}'^V_{\vec{k}} \;:=\; \widehat{f}(\vec{k})\, \widehat{\sigma}^V_{\vec{k}} \;. \tag{2.3}$$

The *volume-independent* cutoff function $\widehat{f}$ is designed so as to keep only momenta smaller than a certain threshold k_0.

2nd. Momenta are rescaled by the factor k_0 so as to return to a Brillouin zone in $[-\pi, \pi]^d$:

$$\widehat{\sigma}'^V_{\vec{k}'} \;:=\; \widehat{f}(\vec{k}' k_0/\pi)\, \widehat{\sigma}^V_{\vec{k}' k_0/\pi} \;. \tag{2.4}$$

In addition, the renormalized variables $\widehat{\sigma}'^V_{\vec{k}'}$ are usually rescaled in applications. We will not do this, as we shall not apply the transformation more than once.

In Wilson's original approach [31, and references therein], the cutoff function $\widehat{f}(\vec{k})$ was chosen simply as the step function

$$\chi_{k_0}(\vec{k}) \;:=\; \begin{cases} 1 & \text{if } |k_i| \le k_0 \;\; i = 1, \ldots, d \\ 0 & \text{otherwise .} \end{cases} \tag{2.5}$$

It was quickly realized, however, that such a sharp cutoff leads to unwanted long-range terms in the renormalized Hamiltonian (see e.g. [31, p. 153], [21, Appendix 2]). To avoid such terms one usually takes smooth momentum-cutoffs, that is, functions $\widehat{f}$ which go to zero in a sufficiently differentiable fashion. Such functions are obtained, for instance, via a convolution

$$\widehat{f}(\vec{k}) \;=\; \sum_{\vec{\ell}\in\mathcal{B}^d_N} \delta_\Delta(\vec{k} - \vec{\ell})\, \chi_{k_0}(\vec{k}) \tag{2.6}$$

with a smooth delta-like function δ_Δ peaked at $k = 0$ of width Δ.

Rigorously speaking, one is interested in the limit $V \to \mathbb{Z}^d$ of this procedure. To make sense of this limit we return to real space, where the prescription (2.4) translates into the relation

$$\sigma'^V_{\vec{x}'} = \sum_{\vec{y} \in V} f^V(L\vec{x}' - \vec{y})\, \sigma_{\vec{y}} \quad , \quad x' \in V/L \;, \tag{2.7}$$

where f^V is the V-dependent (inverse) discrete Fourier transform of $\widehat{f}$ and

$$L := \frac{\pi}{k_0} \;. \tag{2.8}$$

The volume V is assumed to be a disjoint union of cubes of side L (i.e. N is a multiple of L). As $V \to \mathbb{Z}^d$, the function $f^V(\vec{x})$ tends to

$$f(\vec{x}) := \frac{1}{(2\pi)^d} \int_{-\pi}^{\pi} \widehat{f}(\vec{k})\, e^{-i\vec{k}\vec{x}}\, d\vec{k} \;. \tag{2.9}$$

A sharp cutoff in momentum space gives rise to a *non-summable* function f, i.e. $\sum_{\vec{x} \in \mathbb{Z}^d} |f(\vec{x})| = \infty$. [The inverse transform of χ_{k_0} is proportional to the function $\prod_{i=1}^d \sin(k_0 x_i)/(k_0 x_i)$]. Summability is restored if $\widehat{f}$ is smooth enough (for example once differentiable). In such a case, expression (2.7) remains valid in the thermodynamic limit:

$$\sigma'_{\vec{x}'} = \sum_{\vec{y} \in \mathbb{Z}^d} f(L\vec{x}' - \vec{y})\, \sigma_{\vec{y}} \;, \tag{2.10}$$

and the renormalized spins remain *bounded* in this limit. They may take a large number of values, but within some finite interval.

Expression (2.10) shows that a cutoff in momentum space, even a smooth one like (2.6), leads to *non-local* averages in real space, i.e. to functions f extending to infinity. This is the distinctive feature with respect to the real-space transformations analyzed, for instance, in [30]. Nevertheless, it is expected that "the physics behind integration over fluctuations having wavenumbers $[|k_i| > k_0]$ is the same as the physics behind the formation of blocks of spins having volume $[L^d]$ in real space" [23, Section 4.2]. To ensure this, the momentum-space cutoff should lead to an *almost local* average in real space. That is, the function f should decay rapidly outside of a region of size not much larger than L. We see that if f has a Fourier transform of the type (2.6), the contribution of spins outside the volume of size L is of order $\ln(\Delta/k_0)$. We conclude that the cutoff function $\widehat{f}$ must approach zero in a "gradual" manner, that is, with Δ of the order of k_0 in (2.6).

A *soft momentum-cutoff* is a function $\widehat{f}$ that is smooth and gradual in the above sense.

3 The phenomenon of non-Gibbsianness

A state (probability measure or distribution) is called Gibbsian if it can be written in terms of Boltzmann-Gibbs weights for an "acceptable Hamiltonian [which] ... must satisfy the additional requirement of locality ... [that is,] a quantity that is additive over distant lattice sites" [31, p. 145]. In other words, Hamiltonians must be such that the energy of disjoint volumes is additive except for boundary terms whose contribution is small in comparison with the volumes. For classical lattice systems, the appropriate requirement is that the flipping of one spin lead to a *finite* energy change whatever the configuration of the remaining spins is. This intuition has been precisely formalized in the form of a summability condition that we now present.

Hamiltonians are, in general, sums of many-body terms:

$$H(\sigma) \;=\; \sum_{B} \Phi_B(\sigma_B) \;, \tag{3.1}$$

where each Φ_B is a function only of the spins in the finite set $B \subset \mathbb{Z}^d$, i.e. of the variables $\sigma_B = \{\sigma_{\vec{x}}\}_{\vec{x}\in B}$. For Ising spins, these functions Φ_B are usually written in the form $J_B\sigma_B$; the general expression (3.1) is more suitable for spins larger than 1/2, where one would need powers of $\sigma_{\vec{x}}$, and also for some particular spin-1/2 interactions [27]. The finiteness of the single-flip energy change translates into the following summability requirement:

$$\sup_{\vec{x}} \sum_{B\ni\vec{x}} \|\Phi_B\| \;<\; \infty \;, \tag{3.2}$$

where $\|\Phi_B\| := \sup_\sigma |\Phi_B(\sigma_B)|$.

If this summability holds, then we can consider Hamiltonians with arbitrary boundary conditions. For each cube Λ in $\mathbb{Z}^d$, let

$$H_\Lambda(\sigma|\omega) \;=\; \sum_{B:B\cap\Lambda\neq\emptyset} \Phi_B((\sigma_\Lambda\omega)_B) \;, \tag{3.3}$$

where $\sigma_\Lambda\omega$ is the configuration

$$(\sigma_\Lambda\omega)_{\vec{x}} \;=\; \begin{cases} \sigma_{\vec{x}} & \text{if } \vec{x}\in\Lambda \\ \omega_{\vec{x}} & \text{if } \vec{x}\in\mathbb{Z}^d\setminus\Lambda \;. \end{cases} \tag{3.4}$$

For finite-range interactions, the dependence of the Hamiltonian (3.3) on the boundary condition ω is only through spins at sites not farther from Λ than the range of the interaction. More generally, even if the Hamiltonian involves terms with arbitrarily long range, the summability condition (3.2) implies that the dependence on the boundary spins decays with their distance to the volume Λ. This property, which is called *quasilocality*, is central for our argument. Lets us state it precisely. We take a sequence of cubes $U \supset \Lambda$ with larger and larger radius and *fix* a configuration ω in the intermediate ("buffer") region $U \setminus \Lambda$ (see Figure 1). It is not hard to see that

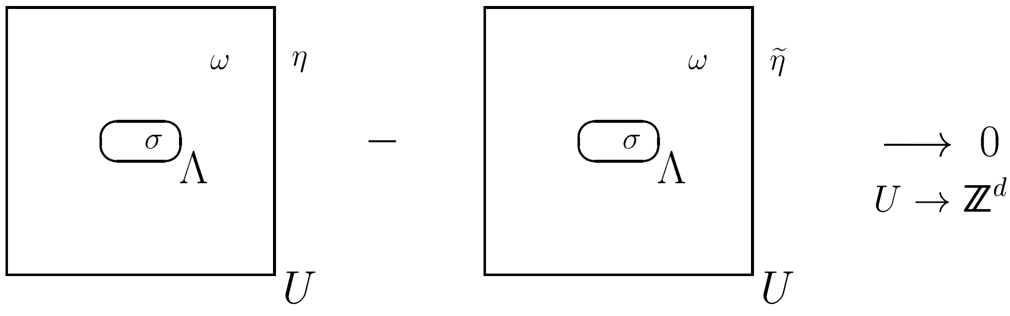

Figure 1: Test for quasilocality. As U tends to $\mathbb{Z}^d$ while keeping fixed the intermediate configuration ω, the energy inside Λ should asymptotically become independent of the configuration η or $\tilde{\eta}$ outside U

the summability condition implies (in fact, it is equivalent to) the following fact: As U tends to the whole $\mathbb{Z}^d$ the Hamiltonian becomes independent of what happens outside U:

$$\sup_{\eta\,\widehat{\eta}} \Bigl| H_\Lambda(\sigma|\omega_U\eta) - H_\Lambda(\sigma|\omega_U\widehat{\eta}) \Bigr| \underset{U\to\mathbb{Z}^d}{\longrightarrow} 0 \tag{3.5}$$

for all σ_Λ and all boundary conditions ω, for all cubes Λ. Hamiltonians H_Λ satisfying (3.5) are said *quasilocal.*

The condition (3.5) is appropriate for discrete finite spins. The renormalized spins considered in the next section, however, can take an infinite number of values. In such a situation the quasilocality condition must allow small variations in the spin configurations σ and ω [1].

The Hamiltonians (3.3) are used to construct the Boltzmann-Gibbs weights

$$\rho_\Lambda(\sigma|\omega) \;:=\; \frac{\exp[-\beta H_\Lambda(\sigma|\omega)]}{Z_\Lambda(\omega)} \tag{3.6}$$

$[Z_\Lambda(\omega)$ is the obvious normalization factor], which do inherit the quasilocality property (3.5):

$$\sup_{\eta\,\widetilde{\eta}} \Bigl| \rho_\Lambda(\sigma|\omega_U\eta) - \rho_\Lambda(\sigma|\omega_U\widetilde{\eta}) \Bigr| \underset{U\to\mathbb{Z}^d}{\longrightarrow} 0\,, \tag{3.7}$$

for all cubes Λ, configurations σ_Λ and boundary conditions ω. For spins taking infinitely many values, the condition should be slightly modified as described in note [1].

A momentum-space renormalization amounts to passing to *renormalized Boltzmann-Gibbs weights*

$$\rho'_{\Lambda'}(\sigma'|\omega') \;=\; \frac{1}{Z_{\Lambda'}(\omega')} \;\text{``}\!\!\sum_{\sigma_\Lambda\omega\,\in\,\mathcal{C}(\sigma'_{\Lambda'}\omega')}\!\!\text{''}\; \exp[-\beta H_\Lambda(\sigma|\omega)]\,, \tag{3.8}$$

where $\mathcal{C}(\sigma'_{\Lambda'}\omega')$ denotes the set of (original) spin configurations compatible with the "block-spin" configuration $\sigma'_{\Lambda'}\omega'$, that is the set of η such that

$$\sum_{\vec{y}\in\mathbb{Z}^d} f(L\vec{x}' - \vec{y})\,\eta_{\vec{y}} \;=\; \begin{cases} \sigma'_{\vec{x}'} & \text{if } \vec{x}' \in \Lambda' \\ \omega'_{\vec{x}'} & \text{if } \vec{x}' \in \mathbb{Z}^d \setminus \Lambda' \,. \end{cases} \tag{3.9}$$

The symbol "$\sum$" in (3.8) reminds us that the operation involved is not a standard sum because we are "summing" over uncountably many configurations. The operation is, therefore, a sum combined with a suitable limit procedure or, in mathematical terms, an integral with respect to the countable product measure $\prod_{\vec{x}\in\mathbb{Z}^d}[(1/2)\sum_{\eta_{\vec{x}}}]$. (The reader interested in the rigorous construction of (3.8) is referred to the discussion in [30, pp. 987–90].)

Most of the studies based on renormalization ideas assume that renormalized weights (3.9) can also be written as Boltzmann-Gibbs weights for an *acceptable* —in Wilson's and Kogut's sense— renormalized Hamiltonian. That is, it is assumed that the family of identities

$$\exp[-\beta' H'_{\Lambda'}(\sigma'|\omega')] \;:=\; \text{“}\!\!\sum_{\sigma_\Lambda\omega\,\in\,\mathcal{C}(\sigma'_{\Lambda'}\omega')}\!\!\text{”}\; \exp[-\beta H_\Lambda(\sigma|\omega)]\;, \tag{3.10}$$

gives rise to Hamiltonians H' that can be written in the form (3.3) for a suitable interaction that *satisfies the summability condition* (3.2). The purpose of this paper is to show that this assumption can be *false* at least at low enough temperatures. This is the phenomenon of non-Gibbsianness referred to in the title of this section.

This lack of Gibbsianness will be proven in the next section by showing that the renormalized weights fail to satisfy the quasilocality property (3.7), or, more precisely, its generalized version for spins with infinitely many values [1]. We shall show that there exists one configuration ω' for which the Boltzmann-Gibbs weight $\rho(\sigma'|\omega')$ violate the quasilocality condition for some volume Λ' (formed by only two sites!) and some configuration σ'. We emphasize that the existence of *one* such a configuration ω' is enough to disprove the existence of an acceptable (i.e. summable) renormalized Hamiltonian. The summability property (3.2) implies the quasilocality (3.7) (or its generalization for spins with infinitely many vaules) for *all* ω'. Nevertheless, it is natural to wonder how relevant the phenomenon is from the physical point of view, specially if the violation of quasilocality involves very atypical configurations ω'. We shall comment on this point in Section 4.

The lack of quasilocality can be interpreted as exhibiting some sort of "action at a distance": Infinitely far away spin flips produce a sizeable change close to the origin, *even when the intermediate renormalized spins are frozen in the configuration* ω'. This is in contrast with the usual behavior in equilibrium statistical mechanical (Gibbsian behavior) where changes at infinity can propagate only through fluctuations of intermediate spins. It is not hard to imagine the explanation: The fixing of a renormalized configuration still leaves some fluctuation possible in the system of original spins compatible with it. These fluctuations act as "hidden degrees of freedom" that in some instances can bring information from infinity. This happens when the constrained system of original spins develops long-range correlations, i.e. when it undergoes a *phase transition*. The argument of next section consists, precisely, in showing conditions and discussing examples under which such phase transitions do take place.

4 Non-Gibbsianness due to momentum transformations

We consider the nearest-neighbor ferromagnetic Ising model in $\mathcal{L} = \mathbb{Z}^2$

$$H \;=\; -\sum_{\langle\vec{x},\vec{y}\rangle} \sigma_{\vec{x}}\, \sigma_{\vec{y}} + h \sum_{\vec{y}} \sigma_{\vec{y}} \;, \tag{4.1}$$

at low temperatures, that is large β. It has been shown that the low-temperature states for this model under a (local) block-average transformation with even block-sizes are mapped onto non-Gibbsian states [30, Theorem 4.6]. A very simple example of this phenomenon for 1 by 2 blocks was presented in [26]. We shall now prove a similar result for a momentum transformation of the type introduced above.

Let us first sketch some intuition behind our argument. In real space, a momentum transformation looks approximately like an average:

$$\sigma'_{\vec{x}'} \;=\; \sum_{\vec{y}\in\mathbb{Z}^d} f(L\vec{x}' - \vec{y})\, \sigma_{\vec{y}} \;, \tag{4.2}$$

where f is the Fourier (anti)transform (2.9) of the cutoff function $\widehat{f}(\vec{k})$. Expressions like (4.2) have been studied for example in [11, 12] (see also [21, Appendix 2] for a stochastic version). Even when (4.2) involves a sum over all spins $\vec{y}$ of the lattice, one would expect that each $\sigma'_{\vec{x}'}$ is essentially determined only by the spins $\sigma_{\vec{y}}$ inside the block of side L centered at $L\vec{x}'$. Therefore, the mechanism causing non-Gibbsianness for average transformations [30, Section 4.3.5] should apply to the present case with minor adaptations.

We will take for our example the identity in one direction and in the other direction the soft cutoff function:

$$\widehat{f}(k) \;=\; \begin{cases} cos^2(k) & \text{for} |k| \le \frac{\pi}{2}, \\ 0 & \text{otherwise} \,. \end{cases} \tag{4.3}$$

This function integrates out half of the momenta degrees of freedom in this direction, which corresponds to taking a (not strictly local) average over blocks of size 1 by 2, centered at sites with even coordinates in the direction in which we renormalize.

Its Fourier transform is easily computable. Indeed, $f(0) = \frac{1}{4}$, $f(2) = f(-2) = \frac{1}{8}$, and for all other n

$$f(n) = -\frac{2}{\pi}\, sin\Bigl(n\frac{\pi}{2}\Bigr) \times \frac{1}{(n-2)\, n\, (n+2)} \;. \tag{4.4}$$

(In particular $f(n) = 0$ for all even $n \neq 0, \pm 2$).

The initial (and crucial) part of the argument consists in exhibiting a transformed configuration ω' such that the corresponding constrained system of original spins has a phase transition *at zero temperature.* The configuration in question is $\omega'_{\vec{x}} = 0$ for all $\vec{x} \in \mathbb{Z}^2$. The corresponding original configurations must, therefore, satisfy the constraint

$$\sum_l f(l)\, \omega_{2n+l} = 0 \tag{4.5}$$

for each n in the direction under consideration. We claim that the only four groundstates are the 4-periodic configuration (strip state)

$$++--++--++ \tag{4.6}$$

and its translates over distances 1,2 or 3 (while in the other direction they are of course translation invariant). It is immediate to check that these configurations are compatible with the constraint. Moreover, it is not difficult to check that under the constraint (4.5) they are groundstates.

Indeed, assume that we would have a row of 3 identical spins, say plus, next to each other, then we claim that the above constraint could not be satisfied. If the middle plus would be on an even site, say at zero, this would require that

$$\sum_{|l|\geq 3} f(l) \;\geq\; 2\, f(1) \quad \Big(= \frac{4}{3 \times \pi}\Big) \tag{4.7}$$

which a simple calculation shows to be impossible. If, on the other hand, the middle plus would be on an odd site, the interval of 3 plus spins would need to have a minus spin both to its left and to its right (because of the argument above), and the constraint centered at either the left or the right site could only be fulfilled if

$$\sum_{|l|\geq 3} f(l) \;\geq\; f(0) \quad (= \frac{1}{4}) \tag{4.8}$$

which again is impossible. This shows that the constrained system has multiple, namely four, ground states.

The remaining part of the argument follows closely the presentation in [30, Section 4.3.5]. There are three additional steps:

(1) Existence of a phase transition at nonzero temperature for the constrained system. This follows from a well-known theory (Pirogov-Sinai theory [24, Chapter 2], [30, Appendix B]; note that as remarked in [18], the theory also applies to systems with constraints.). There is one extremal phase associated to each of the four ground states.

(2) Selection of the phases of the constrained system via block-spin boundary conditions. This can be done by choosing a profile η' such that if it is imposed in a sufficiently large (but finite) volume, the configuration deep inside this volume

has to be close to the prescribed ground state. This is straightforward, though a little cumbersome to write. The idea is as follows: Pick first the ground-state of the constrained system that corresponds to the phase one wants to select. Then pick a (very) large volume Δ, and outside it set the original spins equal to all "+" or all"$-$". Now compute the resulting block-spin configuration according to (4.2). It will have the property that, for regions U sufficiently inside Δ, the block spins will be very close to zero. The profile we talked about corresponds, then, to the block spins in some ring around Δ and in $\Delta \setminus U$. One sees that by playing with the parity of the boundary of Δ and changing "+" to "$-$" outside Δ, one can select alternatively the four phases of the constrained system [2].

(3) "Unfixing" of the spins close to the origin. This is an uncomplicated and inessential step. See the discussion in [30, Section 4.2].)

We see that the argument is insensitive to the presence of a magnetic field (because the constrained system is asked to have small magnetization), thus we are proving non-Gibbsianness for low temperatures but *arbitrary magnetic field*.

Moreover, we remark that in case the blocks are large, having small magnetisation in a block at low temperatures represents a large fluctuation from the typical behavior, in which the magnetisation is either positive or negative of order $O(L^d)$. Renormalized effective interactions are known not to be adequate to describe such large values of the fluctuation field [12]. This large (fluctuation) field region can in good cases be treated by polymer expansions.

5 Comments and conclusions

The present example of non-Gibbsianness as a consequence of momentum-space transformations confirms the suspicion of Griffiths [14] that "no peculiarities of this sort have been found..., which may merely reflect the fact that no one has looked for them!". Nevertheless, one should not draw too radical conclusions from this occurrence. On the practical side, the main implication of non-Gibbsianness is that one has to be very careful in designing renormalization group transformations. This is in complete agreement with what the founders and various practitioners of Renormalization-Group methods have been saying all along.

Indeed, already Wilson and Kogut in their classic review emphasized

> Otherwise, [that is non-perturbatively], the locality of [the renormalized interactions] is a non-trivial problem, which will not be discussed further [31, p. 145].

And more explicitly, M. E. Fisher in his "Renormalization Group Desiderata", listed the conditions needed for a succesful renormalization scheme in Hamiltonian space:

> A Renormalization Group for a space of Hamiltonians should satisfy the following: A) Existence *in* the thermodynamic limit,... C) Spatial locality ..., one should be able to identify the same regions of space and associated local variables before and after the transformation [8, Section 5.4.2]

Our example adds to the numerous instances showing that perversely or sloppily designed transformations can lead people into trouble. As N. Goldenfeld points out in his book *Lectures on Phase Transitions and the Renormalization Group* [13, p. 268]: "It is dangerous to proceed without thinking about the physics". The moral is, then, that renormalization transformations must be carefully crafted and case-tailored. Already Wilson, as quoted in [22, p. 492], warned: "One can not write a renormalization cookbook".

On the foundational side, examples like the present one confirm the view expressed by Benfatto and Gallavotti [3] in the opening sentence of their book, *Renormalisation Group*: "The notion of Renormalisation Group is not well-defined". It is clear that the mathematical formalization of the method requires much more than a naive approach in terms of Hamiltonians and flows of coupling constants. We mention two directions of work which can potentially lead to a better mathematical understanding of the renormalization group framework.

On the one hand, the connection with the large-field problem suggests to combine renormalization group ideas with geometrical expansions —cluster or polymer expansions— to circumvent the ill-definedness of the renormalized Hamiltonian. These expansions have indeed been succesfully applied in the rigorous control of renormalization group transformations of unbounded-spin systems [11, 12, and references therein]. A related approach, for bounded-spin systems, resorted to the renormalization of Peierls-like contours [10].

On the other hand, the more recent program started by Dobrushin [5, 6, 20, 4, 7] studies non-Gibbsianness with techniques borrowed from the treatment of Griffiths singularities. He proposes a more general class of allowable Hamiltonians, leading to the notion of "weak Gibbsianness" as the right framework for a unified treatment.

We think our result illustrates and clarifies to some extent the reason why finding a good renormalization group scheme is such a non-trivial task, not only for strictly local but also for only approximately local transformations. We produced an example in the low-temperature regime, but the fact that the mechanisms of non-Gibbsianness are so similar for real-space and momentum-space transformations, leads us to the conjecture that, as in real space, also in momentum-space one cannot trust that in general the critical region is free of problems.

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References

[1] Formally, quasilocality for spins with infinitely many values is defined as follows. Let us denote $\mathcal{O}_{\delta\,,\Lambda}(\sigma)$ the set of configurations differing less than δ at sites in Λ:

$$\mathcal{O}_{\delta\,,\Lambda}(\sigma) \;=\; \Big\{\widetilde{\sigma} \,:\, |\widetilde{\sigma}_{\vec{x}} - \sigma_{\vec{x}}| \le \delta \;,\; \vec{x} \in \Lambda\Big\} \;.$$

Then $H_\Lambda(\sigma|\cdot)$ is quasilocal at ω if for any $\epsilon > 0$ and $\mathcal{O}_{\delta\,,\Lambda}(\sigma)$ there exists a $\lambda > 0$ and $\Gamma \subset \mathbb{Z}^d$ such that

$$|H_\Lambda(\widetilde{\sigma}|\widetilde{\omega}) - H_\Lambda(\sigma|\omega)| \;<\; \epsilon$$

for every $\widetilde{\sigma} \in \mathcal{O}_{\delta\,,\Lambda}(\sigma)$ and $\widetilde{\omega} \in \mathcal{O}_{\lambda\,,\Gamma}(\omega)$.

[2] Formally, the selection argument consists in showing that for any given set $U \subset \mathbb{Z}^d$ there exists a block-spin configuration η' defined on a ring $\Gamma \setminus U$ such that the Pirogov-Sinai theory implies that boundary conditions close to η' on the ring, select inside U original configurations very close to the prescribed ground state. This selection must hold whatever the block-spin configuration outside Γ. This last arbitrariness is needed to show that the lack of quasilocality is *essential* in probabilistic terms. The configuration η', referred to as a "profile" makes a smooth transition from the all-(almost) "0" to the all-(almost) "+" or "−" configuration.

[3] G. Benfatto and G. Gallavotti. *Renormalization Group.* Physics Notes, Vol. 1, Princeton University Press, Princeton, 1995.

[4] J. Bricmont, A. Kupiainen, and R. Lefevere. Renormalization group pathologies and the definition of Gibbs states. *Commun. Math. Phys.*, 194:359–388, 1998.

[5] R. L. Dobrushin. A Gibbsian representation for non-Gibbs fields. Lecture given at the workshop "Probability and Physics", Renkum, The Netherlands, 1995.

[6] R. L. Dobrushin and S. B. Shlosman. Gibbsian description of "non-Gibbsian" fields. *Russian Math. Surveys*, 52:285–97, 1997.

[7] R. L. Dobrushin and S. B. Shlosman. To appear in *Commun. Math. Phys.*, 1998.

[8] M. E. Fisher. Scaling, universality and renormalization group theory. In F. J. W. Hahne, editor, *Critical Phenomena (Stellenbosch 1982)*, pages 1–139. Springer-Verlag (Lecture Notes in Physics #186), Berlin–Heidelberg–New York, 1983.

[9] M. E. Fisher, 1992. Private communication.

[10] K. Gawędzki, R. Kotecký, and A. Kupiainen. Coarse graining approach to first order phase transitions. *J. Stat. Phys.*, 47:701–724, 1987.

[11] K. Gawędzki and A. Kupiainen. A rigorous block spin approach to massless lattice theories. *Commun. Math. Phys.*, 77:31–64, 1980.

[12] K. Gawędzki and A. Kupiainen. Asymptotic freedom beyond perturbation theory. In K. Osterwalder and R. Stora, editors, *Critical Phenomena, Random Systems, Gauge Theories [Les Houches 1984], Part I*, pages 185–293, Amsterdam, 1986. North-Holland.

[13] N. Goldenfeld. *Lectures on Phase Transitions and the Renormalization Group.* Frontiers in Physics Vol. 85, Addison Wesley, Reading, Massachusetts, 1992.

[14] R. B. Griffiths. Mathematical properties of renormalization-group transformations. *Physica*, 106A:59–69, 1981.

[15] R. B. Griffiths and P. A. Pearce. Position-space renormalization-group transformations: Some proofs and some problems. *Phys. Rev. Lett.*, 41:917–920, 1978.

[16] R. B. Griffiths and P. A. Pearce. Mathematical properties of position-space renormalization-group transformations. *J. Stat. Phys.*, 20:499–545, 1979.

[17] A. Hasenfratz and P. Hasenfratz. Singular renormalization group transformations and first order phase transitions (I). *Nucl. Phys.*, B295 [FS21]:1–20, 1988.

[18] W. Holsztynski and J. Slawny. Peierls condition and the number of ground states. *Commun. Math. Phys.*, 61:177–190, 1978.

[19] R. B. Israel. Banach algebras and Kadanoff transformations. In J. Fritz, J. L. Lebowitz, and D. Szász, editors, *Random Fields (Esztergom, 1979), Vol. II*, pages 593–608. North-Holland, Amsterdam, 1981.

[20] J. Lörinczi and C. Maes. Weakly Gibbsian measures for lattice spin systems. *J. Stat. Phys.*, 89:561–79, 1997.

[21] S. K. Ma. *Modern Theory of Critical Phenomena.* Benjamin, Reading, Mass., 1976.

[22] T. Niemeijer and J. M. J. van Leeuwen. Renormalization theory for Ising-like spin systems. In C. Domb and M. S. Green, editors, *Phase Transitions and Critical Phenomena, Vol. 6*. Academic Press, London–New York–San Francisco, 1976.

[23] P. Pfeuty and G. Toulouse. *Introduction to the Renormalization Group and to Critical Phenomena*. Wiley, Chichester–New York, 1977.

[24] Ya. G. Sinai. *Theory of Phase Transitions: Rigorous Results*. Pergamon Press, Oxford–New York, 1982.

[25] A. C. D. van Enter. Ill-defined block-spin transformations at arbitrarily high temperatures. *J. Stat. Phys.*, 83:761–5, 1996.

[26] A. C. D. van Enter. On the possible failure of the Gibbs property for measures on lattice systems. *Markov Processes Rel. Fields*, 2:209–24, 1996.

[27] A. C. D. van Enter and R. Fernández. A remark on different norms and analyticity for many-particle interactions. *J. Stat. Phys.*, 56:965–972, 1989.

[28] A. C. D. van Enter, R. Fernández, and R. Kotecký. Pathological behavior of renormalization-group maps at high fields and above the transition temperature. *J. Stat. Phys.*, 79:969–992, 1995.

[29] A. C. D. van Enter, R. Fernández, and A. D. Sokal. Renormalization transformations in the vicinity of first-order phase transitions: What can and cannot go wrong. *Phys. Rev. Lett.*, 66:3253–3256, 1991.

[30] A. C. D. van Enter, R. Fernández, and A. D. Sokal. Regularity properties and pathologies of position-space renormalization-group transformations: Scope and limitations of Gibbsian theory. *J. Stat. Phys.*, 72:879–1167, 1993.

[31] K. G. Wilson and J. Kogut. The renormalization group and the ϵ-expansion. *Phys. Reports*, 12C:75–200, 1974.