YANG-MILLS THEORY AND THE SEGAL-BARGMANN TRANSFORM

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ABSTRACT. We use a variant of the Segal-Bargmann transform to study canonically quantized Yang-Mills theory on a space-time cylinder with a compact structure group K. The non-existent Lebesgue measure on the space of connections is "approximated" by a Gaussian measure with large variance. The Segal-Bargmann transform is then a unitary map from the L^2 space over the space of connections to a *holomorphic* L^2 space over the space of complexified connections with a certain Gaussian measure. This transform is given roughly by $e^{t\Delta_{\mathcal{A}}/2}$ followed by analytic continuation. Here $\Delta_{\mathcal{A}}$ is the Laplacian on the space of connections and is the Hamiltonian for the quantized theory.

On the gauge-trivial subspace, consisting of functions of the holonomy around the spatial circle, the Segal-Bargmann transform becomes $e^{t\Delta_K/2}$ followed by analytic continuation, where Δ_K is the Laplacian for the structure group K. This result gives a rigorous meaning to the idea that $\Delta_{\mathcal{A}}$ reduces to Δ_K on functions of the holonomy. By letting the variance of the Gaussian measure tend to infinity we recover the standard realization of the quantized Yang-Mills theory on a space-time cylinder, namely, $-\frac{1}{2}\Delta_K$ is the Hamiltonian and $L^2(K)$ is the Hilbert space. As a byproduct of these considerations, we find a new one-parameter family of unitary transforms from $L^2(K)$ to certain holmorphic L^2-spaces over the complexification of K. This family of transformations interpolates between the two previously known unitary transformations.

Our work is motivated by results of Landsman and Wren and uses probabilistic techniques similar to those of Gross and Malliavin.

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†This research was partially supported by NSF Grant DMS 96-12651.
*Supported by an NSF Postdoctoral Fellowship.

1. Introduction

This paper uses techniques of stochastic analysis to address the problem of canonically quantizing Yang-Mills theory on a space-time cylinder. We outline our results briefly here, leaving a detailed description of the Yang-Mills interpretation to Section 2. Let K be a connected compact Lie group and $\mathfrak{k}$ be its Lie algebra endowed with a fixed Ad-K-invariant inner product. Let $\bar{\mathcal{A}}$ be a certain subspace of the $\mathfrak{k}$-valued distributions on $[0,1]$ and $\tilde{P}_s$ be a scaled white noise measure on $\bar{\mathcal{A}}$. See (4.2) and Definition 4.1 below. By taking the indefinite "integrals" of elements of $\bar{\mathcal{A}}$, the measure space $(\bar{\mathcal{A}}, \tilde{P}_s)$ may be identified with the space of $\mathfrak{k}$-valued paths on $[0,1]$ starting at 0 equipped with a Wiener measure of variance s. Elements of $\bar{\mathcal{A}}$ are to be interpreted as (generalized) connections on the spatial circle.

Our objective is to understand infinite-dimensional Laplacian operator $\Delta_{\mathcal{A}}$, where $\mathcal{A}$ is the Cameron-Martin subspace of $\bar{\mathcal{A}}$, namely, the Hilbert space of square-integrable $\mathfrak{k}$-valued functions. Since $\Delta_{\mathcal{A}}$ is poorly behaved (e.g., non-closable) as a operator on $L^2(\bar{\mathcal{A}}, \tilde{P}_s)$, we work with a variant $\tilde{S}_{s,t}$ of the *Segal-Bargmann transform*. This is defined to be $e^{t\Delta_{\mathcal{A}}/2}$ followed by analytic continuation. The transform is defined at first on cylinder functions but extends to a unitary map of $L^2(\bar{\mathcal{A}}, \tilde{P}_s)$ onto the *holomorphic* subspace of $L^2(\bar{\mathcal{A}}_{\mathbb{C}}, \tilde{M}_{s,t})$. Here $\bar{\mathcal{A}}_{\mathbb{C}}$ is a space of $\mathfrak{k}_{\mathbb{C}}$-valued distributions (where $\mathfrak{k}_{\mathbb{C}} = \mathfrak{k} + i\mathfrak{k}$) and $\tilde{M}_{s,t}$ is a certain Gaussian measure on $\bar{\mathcal{A}}_{\mathbb{C}}$.

We are particularly interested in the Itô map θ, which associates to almost every $A \in \bar{\mathcal{A}}$ a continuous K-valued path $\theta_{\cdot}(A)$. Geometrically, $\theta_\tau(A)$ represents the parallel transport of the connection A from 0 to τ, and $h(A) := \theta_1(A)$ represents the *holonomy* of A around the spatial circle. We similarly consider the Itô map $\theta^{\mathbb{C}}$ and the holonomy $h_{\mathbb{C}}$ for complex connections $C \in \bar{\mathcal{A}}_{\mathbb{C}}$, where $\theta^{\mathbb{C}}_\tau(C)$ and $h_{\mathbb{C}}(C)$ take values in the complexification $K_{\mathbb{C}}$ of the compact group K.

The main result is Theorem 5.2 of Section 5, which states:

> Suppose $f \in L^2(\bar{\mathcal{A}}, \tilde{P}_s)$ is of the form
> $$f(A) = \phi(h(A))$$
> where ϕ is a function on K. Then there exists a unique holomorphic function Φ on $K_{\mathbb{C}}$ such that
> $$\tilde{S}_{s,t} f(C) = \Phi(h_{\mathbb{C}}(C)).$$
> The function Φ is determined by the condition that
> $$\Phi|_K = e^{t\Delta_K/2}\phi.$$

Here Δ_K refers to the Laplacian for the compact group K. Recall that $\tilde{S}_{s,t}$ is defined to be $e^{t\Delta_{\mathcal{A}}/2}$ followed by analytic continuation. Theorem 5.2 says that on functions of the holonomy $\tilde{S}_{s,t}$ reduces to $e^{t\Delta_K/2}$ followed by analytic continuation. This is formally equivalent to the following imprecise principle.

> On functions of the holonomy $\Delta_{\mathcal{A}}$ reduces to Δ_K.

As a consequence of Theorem 5.2 and the "averaging lemma" in [H1] we obtain a Segal-Bargmann type transform for the compact group K (Theorem 5.3). To describe this theorem let ρ_s denote the distribution of $h(A)$ with respect to $\tilde{P}_s$ and $\mu_{s,t}$ denote the distribution of $h_{\mathbb{C}}(C)$ with respect to $\tilde{M}_{s,t}$. The measures ρ_s and $\mu_{s,t}$ are certain heat kernel measures on K and $K_{\mathbb{C}}$, respectively. Then Theorem 5.3 asserts:

> The map
>
> $$\phi \rightarrow \text{ analytic continuation of } e^{t\Delta_K/2}\phi$$
>
> is an isometric isomorphism of $L^2(K, \rho_s)$ onto the holomorphic subspace of $L^2(K_{\mathbb{C}}, \mu_{s,t})$.

This generalized Segal-Bargmann transform for K was known previously [H1] in the case $s = t$ and also in the limiting case $s \to \infty$. (See also [H2, H3, H4, D2, DG].) For general s and t this transform is new, and interpolates continuously between the two previously known cases. An analysis of this new transform, from purely finite-dimensional point of view, is given in [H5].

There is a simple explanation (not a proof) for Theorem 5.2. The Hilbert space $\mathcal{A}$ (the Cameron-Martin subspace) may be thought of as an infinite-dimensional flat Riemannian manifold. Let $H(K)$ be the infinite-dimensional group of finite-energy paths with values in K, starting at the identity. This has a natural right-invariant Riemannian metric. Theorem 8.1 in Section 8 asserts that the Itô map θ, restricted to the Cameron-Martin subspace $\mathcal{A}$, is an isometry of $\mathcal{A}$ onto $H(K)$. From this observation, one formally concludes that $\Delta_{\mathcal{A}}(f \circ \theta) = \left(\Delta_{H(K)} f\right) \circ \theta$ where $\Delta_{H(K)}$ is the "Laplace-Beltrami" operator associated to the right-invariant Riemannian structure on $H(K)$. Furthermore, if f depends only on the endpoint of the path (i.e., $f \circ \theta$ is a function of the holonomy) then an easy calculation shows that $\Delta_{H(K)}$ reduces to Δ_K (Theorem 8.9). However, even working on $\mathcal{A}$ there are serious domain issues to deal with, and of course $\mathcal{A}$ is a measure-zero subset of $\bar{\mathcal{A}}$. So the proof of Theorem 5.2 does not make direct use of this calculation. Nevertheless we present it as motivation, with a precise treatment of the domain issues, in Appendix A.

The main tool in the proof of Theorem 5.2 is the Hermite expansion, which for $L^2(\bar{\mathcal{A}}, \tilde{P}_s)$ takes the form of an expansion in terms of multiple Wiener integrals, the so-called Wiener chaos or homogeneous chaos expansion [Ka, Ito]. The Segal-Bargmann transform $\tilde{S}_{s,t}$ has a very simple action on this expansion, given in Theorem 4.7 of Section 4.2.

Although it is natural from the standpoint of Yang-Mills theory to consider functions of the holonomy $h(A) := \theta_1(A)$, it makes sense to apply $\tilde{S}_{s,t}$ to arbitrary functions of the parallel transport θ_τ, $\tau \in [0,1]$. Results similar to Theorem 5.2 hold, described in Section 6.

Finally, let us mention the paper [AHS], which considers a sort of Segal-Bargmann transform in the context of two-dimensional *Euclidean* Yang-Mills theory. That paper is not so much concerned with constructing the theory as with understanding the structure of the Euclidean Yang-Mills measure. Despite a superficial similarity, there is no overlap of results between [AHS] and the present paper.

Acknowledgments. Our work on this problem was motivated by the results of N.P. Landsman and K.K. Wren [LW, W1, W2]. (See also [L].) Although they take

a different approach to the quantization, their results strongly suggest a connection between Yang-Mills theory on a cylinder and the generalized Segal-Bargmann transform for K. See Section 2 for more on the relationship of our results to theirs. We thank both Landsman and Wren for valuable scientific discussions.

In the case $s = t$, a result very close to Theorem 5.2 was proved by L. Gross and P. Malliavin [GM]. We thank Gross for helpful comments on an earlier form of this paper. In the case $s = t$, Theorem 5.2 itself was proved in [Sa2] and in [HS]. We thank A. Sengupta for his insights into Theorem 5.2.

The first author is grateful to David Brydges for clarifying to him the role of the heat equation in Wick ordering and thanks l'Institut Henri Poincaré and l'École Normal Supériure for their hospitality.

We thank Nolan Wallach for valuable discussions concerning the energy representation, and we thank the referee for helpful comments.

2. The Yang-Mills interpretation

This section explains the motivation for, and the desired interpretation of, the results of the paper. It may be skipped without a loss of understanding of the statements.

The Segal-Bargmann transform was developed independently in the early 1960's by Segal [S1, S2, S3] in the infinite-dimensional context of scalar quantum field theories and by Bargmann [B] in the finite-dimensional context of quantum mechanics on $\mathbb{R}^n$. The paper [H1] introduced an analog of the Segal-Bargmann transform in the context of quantum mechanics on a compact Lie group. A natural next step is to attempt to combine the compact group with the field theory in order to obtain a transform in the context of quantum gauge theories. One such transform has already been obtained by Ashtekar, *et al.* [A], with application to quantum gravity.

This paper considers the canonical quantization of Yang-Mills theory in the simplest non-trivial case, namely that of a space-time cylinder. We consider first briefly the classical Yang-Mills theory. (See also [L, RR].) Let K be a connected compact Lie group (the structure group) together with an Ad-K-invariant inner product $\langle \cdot, \cdot \rangle$ on its Lie algebra $\mathfrak{k}$. We work in the temporal gauge, in which case the configuration space for the classical Yang-Mills theory is the space of $\mathfrak{k}$-valued 1-forms on the spatial circle. More precisely, let $\mathcal{A}$ denote the space of square-integrable $\mathfrak{k}$-valued 1-forms, which can be identified with $L^2([0,1];\mathfrak{k})$, where the circle is $[0,1]$ with ends identified. The phase space of the system is then $\mathcal{A}_{\mathbb{C}} = \mathcal{A} + i\mathcal{A}$. The dynamical part of the Yang-Mills equations (e.g., [Di, Eq. (2)]) may be expressed in Hamiltonian form, with the Hamiltonian function on $\mathcal{A}_{\mathbb{C}}$ given by

$$H(A + iP) = \frac{1}{2} \|P\|^2 = \frac{1}{2} \int_0^1 |P_\tau|^2 \, d\tau. \tag{2.1}$$

Note that since our spatial manifold is one-dimensional, the curvature term which usually appears in the Hamiltonian is zero.

There is also a constraint part to the Yang-Mills equation (e.g., [Di, Eq. (1)]), namely,

$$\frac{dP_\tau}{d\tau} + [A_\tau, P_\tau] = 0, \quad \tau \in [0,1]. \tag{2.2}$$

or equivalently,

$$J_h(A, P) := -\langle h' + [A, h], P \rangle_{\mathcal{A}} = 0$$

for all $h \in \mathcal{C}^1\left(S^1 \to \mathfrak{k}\right)$. The set of points (A, P) satisfying this constraint is preserved under the time evolution generated by (2.1).

Now let $\mathcal{G}$ be the **gauge group**, namely, the group of maps of the spatial circle into K. This acts on $\mathcal{A}$ by

$$(g \cdot A)_\tau = g_\tau A_\tau g_\tau^{-1} - \frac{dg}{d\tau} g_\tau^{-1}$$

and on $\mathcal{A}_{\mathbb{C}}$ by

$$(g \cdot (A, P))_\tau = \left((g \cdot A)_\tau, g_\tau P_\tau g_\tau^{-1}\right).$$

The gauge action preserves both the dynamics and the constraint. The function J_h on $\mathcal{A}_{\mathbb{C}}$ is the Hamiltonian generator of the action of the one-parameter subgroup e^{th} in $\mathcal{G}$, where $\left(e^{th}\right)_\tau = e^{th_\tau}$. That is, J is the moment mapping for the action of $\mathcal{G}$ [L, Sect. IV.3.6].

The parallel transport $\theta_\tau(A)$ of a connection $A \in \mathcal{A}$, is the solution to the K-valued differential equation

$$\frac{d\theta}{d\tau} = \theta_\tau A_\tau, \quad \theta_0 = e, \tag{2.3}$$

and θ transforms under gauge transformations as

$$\theta_\tau(g \cdot A) = g_0 \theta_\tau(A) g_\tau^{-1}, \quad g \in \mathcal{G}. \tag{2.4}$$

The **holonomy** of A is the parallel transport around the circle: $h(A) := \theta_1(A)$. (In the interests of consistency with [G, GM, HS] we have put θ_τ to the left of A_τ in (2.3). Although this is the reverse of the usual definition of parallel transport, it makes little difference. Theorem 5.2 would be unchanged with the other definition and Theorem 6.3 would require just the reversal of x and g in (6.2).)

If we *formally* apply the usual canonical quantization procedure to this classical Yang-Mills theory, we find that the quantum mechanical Hilbert space is $L^2(\mathcal{A}, \mathcal{D}A)$ and the Hamiltonian operator corresponding to the classical Hamiltonian (2.1) is $-\Delta_{\mathcal{A}}/2$. Here $\mathcal{D}A$ is the *non-existent* Lebesgue measure on $\mathcal{A}$ and $\Delta_{\mathcal{A}}$ is the Laplacian operator, that is, the sum of squares of derivatives in the directions of an orthonormal basis. The quantum operator corresponding to the function J_h is the vector field $\hat{J}_h$ given by

$$\begin{aligned} \hat{J}_h F(A) &= i \left.\frac{d}{dt}\right|_{t=0} F(A + t(h' + [A, h])) \\ &= -i \left.\frac{d}{dt}\right|_{t=0} F\left(e^{th} \cdot A\right). \end{aligned}$$

Thus the quantum analog of the constraint equation is to require that $F \in L^2(\mathcal{A}, \mathcal{D}A)$ be $\mathcal{G}$-invariant. (More precisely, this is true if $\mathcal{G}$ is connected, i.e., if K is simply connected. We will simply assume $\mathcal{G}$-invariance even if K is not simply connected, and will not address here the issue of "θ-angles." See [LW, W1, L].)

We consider at first the **based gauge group** $\mathcal{G}_0$

$$\mathcal{G}_0 = \{g \in \mathcal{G} \,|\, g_0 = g_1 = e\}.$$

It is not hard to verify using (2.4) (see [L]) that two connections are $\mathcal{G}_0$-equivalent if and only if they have the same holonomy. Thus, the $\mathcal{G}_0$-invariant functions are precisely the functions of the form $\phi(h(A))$, where ϕ is a function on K. For invariance under the full gauge group, ϕ would be required to be a class function.

Now let Δ_K denote the Laplacian (quadratic Casimir) operator on K associated to the chosen invariant inner product on $\mathfrak{k}$.

Claim 2.1 (Main Idea)**.** *Consider a function on $\mathcal{A}$ of the form $\phi(h(A))$, where ϕ is a function on K. Then*

$$\Delta_{\mathcal{A}}\phi(h(A)) = (\Delta_K\phi)(h(A)).$$

That is, on functions of the holonomy, the Laplacian for the space of connections should reduce to the Laplacian on the structure group. This idea is not new. It is stated without proof in [Wi, pp.166,169], and a rigorous result in this direction is given in [Di, Lem. 3.2]. (See the end of this section.) The challenge is not so much to prove the result but to give it a rigorous interpretation. (See also [Ra], where reduction is done before quantization.)

One approach is to approximate the non-existent Lebesgue measure by a Gaussian measure $\tilde{P}_s$ with large variance s, where "large" means that at the appropriate point in our calculations s will tend to infinity. The measure $\tilde{P}_s$ does not exist on $\mathcal{A}$ itself, but does exist on a certain space $\bar{\mathcal{A}}$ of generalized connections. We then take $\mathcal{G}_0$ to be the group of **finite-energy** maps $g : [0,1] \to K$ satisfying $g_0 = g_1 = e$, where finite energy means that g is absolutely continuous and $\int_0^1 \left|g_\tau^{-1}\, dg/d\tau\right|^2 d\tau < \infty$. The action of $\mathcal{G}_0$ on $\mathcal{A}$ may be extended to an action on $\bar{\mathcal{A}}$, and this action leaves $\tilde{P}_s$ quasi-invariant. We consider the Hilbert space $L^2(\bar{\mathcal{A}}, \tilde{P}_s)$ and define the gauge-trivial subspace to be:

$$L^2(\bar{\mathcal{A}}, \tilde{P}_s)^{\mathcal{G}_0} = \left\{ f \in L^2(\bar{\mathcal{A}}, \tilde{P}_s) \,|\forall g \in \mathcal{G}_0,\ f\left(g^{-1}\cdot A\right) = f(A) \text{ a.e.} \right\}. \tag{2.5}$$

Note that the map $f(A) \to f\left(g^{-1}\cdot A\right)$ is not unitary, since $\mathcal{G}_0$ leaves $\tilde{P}_s$ quasi-invariant but not invariant. We are deliberately not unitarizing the action of $\mathcal{G}_0$ as in [LW]; the point of letting $s \to \infty$ is to avoid having to do so. The following result shows clearly our motivation for not unitarizing.

Theorem 2.2. *Let $U(g)$ be the unitary gauge action, as for example in* [Di]*. If $f \in L^2(\bar{\mathcal{A}}, \tilde{P}_s)$ and $U(g)f = f$ for all $g \in \mathcal{G}_0$, then $f = 0$.*

The corresponding results in dimensions 3+1 and higher (and in certain (2+1)-dimensional cases) is a consequence of the irreducibility of the energy representation [Wa, AKT, GGV], at least for the case when K is semisimple. In the one-dimensional case considered here, the energy representation is reducible, so a different proof is needed, and is given in [DH]. Defining the gauge-trivial subspace in terms of the un-unitarized action as in (2.5) gives a non-zero Hilbert space, as we shall see momentarily. Unitarity is recovered, at least formally, in the $s \to \infty$ limit. (However, in cases where the energy representation is irreducible, the space defined in (2.5) contains only the constants. So our approach will not work without modification in high dimensions.)

The parallel transport map θ, and so also the holonomy, may be "extended" from $\mathcal{A}$ to $\bar{\mathcal{A}}$ by replacing the differential equation (2.3) with a *stochastic* differential equation, the Itô map (Section 5). A deep theorem of Gross asserts that the elements of $L^2(\bar{\mathcal{A}}, \tilde{P}_s)^{\mathcal{G}_0}$ are precisely functions of the holonomy. (See also [Sa1].) The reason this is not obvious is that although we have enlarged the space of connections by replacing $\mathcal{A}$ with $\bar{\mathcal{A}}$, we cannot unduly enlarge the group of gauge transformations without losing quasi-invariance, without which (2.5) does not make sense. As a result, two connections in $\bar{\mathcal{A}}$ with the same holonomy need not be gauge-equivalent.

(If K is simply connected, then the characterization of $L^2(\bar{\mathcal{A}},\tilde{P}_s)^{\mathcal{G}_0}$ is obtained by composing [G, Thm. 2.5] with the Itô map. The general case is easily reduced to the simply connected case. Note that our $\mathcal{G}_0$ corresponds to $\hat{K}_0$ (*not* K_0) in the notation of Gross.)

We are back, then, to the matter of computing the Laplacian on functions of the holonomy. Unfortunately, while $\Delta_{\mathcal{A}}$ is densely defined in $L^2(\bar{\mathcal{A}},\tilde{P}_s)$ (say on smooth, compactly supported cylinder functions), it is not closable. So it is not clear what it means to apply the Laplacian to a function of the holonomy. We consider, then, a variant of the Segal-Bargmann transform. The transform involves the heat operator $e^{t\Delta_{\mathcal{A}}/2}$, instead of the $\Delta_{\mathcal{A}}$ itself. More precisely, the Segal-Bargmann transform consists of $e^{t\Delta_{\mathcal{A}}/2}$ followed by analytic continuation. The transform maps from $L^2(\bar{\mathcal{A}},\tilde{P}_s)$ onto a certain L^2 space of holomorphic functions on a space $\bar{\mathcal{A}}_{\mathbb{C}}$ of complexified connections, rather than from $L^2(\bar{\mathcal{A}},\tilde{P}_s)$ to itself. Although the Segal-Bargmann transform is defined initially only on cylinder functions, it is an isometric map and so extends by continuity to all of $L^2(\bar{\mathcal{A}},\tilde{P}_s)$. In particular, it makes sense to apply the Segal-Bargmann transform to functions of the holonomy.

Our main result is Theorem 5.2, described already in the introduction. It asserts that for functions of the holonomy the Segal-Bargmann transform (roughly, the heat operator $e^{t\Delta_{\mathcal{A}}/2}$, followed by analytic continuation) becomes the heat operator $e^{t\Delta_K/2}$ for the structure group K, followed by analytic continuation. This holds for each fixed s, not just in the $s\to\infty$ limit. Thus Theorem 5.2 gives a rigorous meaning to the Main Idea in Claim 2.1.

Now, the gauge-trivial subspace, which consists of functions of the form $\phi\left(h\left(A\right)\right),$ may be identified with $L^2\left(K,\rho_s\right),$ where ρ_s is the distribution of $h\left(A\right)$ with respect to $\tilde{P}_s$. Similarly, the space of functions of the form $\Phi\left(h_{\mathbb{C}}\left(C\right)\right)$ may be identified with $L^2\left(K_{\mathbb{C}},\mu_{s,t}\right),$ where $\mu_{s,t}$ is the distribution of $h_{\mathbb{C}}\left(C\right)$ with respect to the relevant Gaussian measure on $\bar{\mathcal{A}}$. So restricting the Segal-Bargmann transform to the gauge-trivial subspace gives an isometric map from $L^2\left(K,\rho_s\right)$ into the holomorphic subspace of $L^2\left(K_{\mathbb{C}},\mu_{s,t}\right),$ given by $e^{t\Delta_K/2}$ followed by analytic continuation. A finite-dimensional argument shows that this transform maps *onto* the holomorphic subspace. This gives a unitary Segal-Bargmann-type transform (Theorem 5.3) for K, $B_{s,t}:L^2\left(K,\rho_s\right)\to\mathcal{H}L^2\left(K_{\mathbb{C}},\mu_{s,t}\right)$ given by

$$B_{s,t}f=\text{ analytic continuation of } e^{t\Delta_K/2}f.$$

Here $\mathcal{H}L^2$ denotes the space of square-integrable holomorphic functions. This unitary transform was previously obtained in [H1] for the case $s=t$ and the limiting case $s\to\infty$, which we now discuss.

Note that the formula for $B_{s,t}$ depends only on t; the s-dependence is only in the measures. Hence it makes sense to let s tend to infinity. In this limit ρ_s converges to normalized Haar measure on K and $\mu_{s,t}$ converges to a certain K-invariant measure ν_t on $K_{\mathbb{C}}$. So taking the Segal-Bargmann transform for $\bar{\mathcal{A}}$, restricting to the gauge-trivial subspace, and taking the large variance limit yields a unitary transform C_t, mapping $L^2\left(K,\text{ Haar}\right)$ onto the holomorphic subspace of $L^2\left(K_{\mathbb{C}},\nu_t\right).$ The transform is given, as always, by the time t heat operator followed by analytic continuation. Here t is an arbitrary positive parameter, which is to be interpreted physically as Planck's constant. See Theorem 5.5.

We arrive, then, at the expected conclusion: the physical Hilbert space for quantized Yang-Mills on a space-time cylinder is $L^2\left(K,\text{Haar}\right)$ and the Hamiltonian operator is $-\Delta_K/2$. As a bonus, we obtain a natural Segal-Bargmann transform C_t for the physical Hilbert space. Let us mention two other matters in passing. First, for invariance under the full gauge group $\mathcal{G}$ we would restrict attention to the Ad-invariant subspace of $L^2(K, \text{Haar})$. Second, the Wilson loop operators naturally act as multiplication operators in $L^2(\bar{\mathcal{A}}, \tilde{P}_s)$ and so also in $L^2(K, \text{Haar})$.

Let briefly compare our approach to others. Landsman and Wren [LW, W1, W2, L] use a method called Rieffel induction, in which gauge symmetry is implemented by means of a certain integral over the gauge group. Under this integration the classical coherent states for $\mathcal{A}$ map to the coherent states that are associated to the transform C_t described above (the $s \to \infty$ limit of $B_{s,t}$). However, it is not clear how to derive by this method the relevant measure ν_t on $K_{\mathbb{C}}$, and the computation of the reduced Hamiltonian [W1] is complicated.

Dimock [Di] adds a mass term to the Hamiltonian, which makes it self-adjoint in $L^2(\bar{\mathcal{A}}, \tilde{P}_s)$. However, because the mass term destroys gauge-invariance, Dimock obtains a result like the Main Idea only in the $s \to \infty$ limit [Di, Lem. 3.2].

The Euclidean method for Yang-Mills on a cylinder constructs a probability measure directly on connections modulo gauge transformations. Let h^t denote the holonomy around the spatial circle at time t. Then it can be shown that: 1) for each t, h^t is distributed as Haar measure on K, and 2) h^t is a K-valued Brownian motion. Thinking in terms of the temporal gauge, it is then reasonable to take as the time-zero Hilbert space $L^2(K, \text{Haar})$, and (since the infinitesimal generator of Brownian motion on K is $\Delta_K/2$) to take as the Hamiltonian $-\Delta_K/2$. Since the Euclidean Yang-Mills measure does not exist on the space of connections, but only on connections modulo gauge transformations, the Euclidean method does not directly address the relationship between $\Delta_{\mathcal{A}}$ and Δ_K.

Finally in Appendix A, $\Delta_{\mathcal{A}}$ is considered as an operator acting on functions on $\mathcal{A}$ rather than on $\bar{\mathcal{A}}$. While functions of the form $\phi(h(A))$, with ϕ smooth on K, are differentiable on $\mathcal{A}$, the Hessian of such a function is not in general trace-class. Hence, it is not possible to define $\Delta_{\mathcal{A}}\phi(h(A))$ as the trace of the Hessian of $\phi(h(A))$. This problem is circumvented by computing the trace by a two-step procedure–see Definition 8.5. With this definition, we prove a rigorous version of the Main Idea in Claim 2.1 (Theorem 8.9). Since $\mathcal{A}$ is a set of $\tilde{P}_s$-measure zero, these results do not bear directly on Theorem 5.2.

3. Segal-Bargmann for $\mathbb{R}^d$

We consider a variant of the classical Segal-Bargmann transform that depends on two parameters, one of which we wish to let tend to infinity. See Section 2 for motivation. This is in contrast to the conventional version of the transform, which has only one parameter (or none, depending on the author). However, in the $\mathbb{R}^d$ case, this two-parameter transform is not truly new, but can be reduced to the classical one-parameter version by elementary changes of variable. This reduction is described in [H5]. In Section 3.1 we describe the transform itself. In Section 3.2 we describe Hermite expansions on both the domain and range of the transform, and we describe the action of the transform on these expansions. Hermite expansions play a key role in the proof of our main result, Theorem 5.2 in Section 5.

3.1. **The Transform for $\mathbb{R}^d$.** Let Δ be the standard Laplacian on $\mathbb{R}^d$ and P_s be the associated Gaussian measure. Explicitly for $s>0$, $dP_s(x) = P_s(x)\, dx$, where

$$P_s(x) = (2\pi s)^{-d/2} e^{-x^2/2s}.$$

Here $x = (x_1, \cdots, x_d)$, $x^2 = x_1^2 + \cdots + x_d^2$, and dx is standard Lebesgue measure on $\mathbb{R}^d$. Note that the function $P_s(x)$ admits an analytic continuation to $\mathbb{C}^d$, denoted $P_s(z)$. Now for any number t with $t < 2s$ (i.e., $s > t/2$) define a map

$$S_{s,t} : L^2\left(\mathbb{R}^d, P_s\right) \to \mathcal{H}\left(\mathbb{C}^d\right)$$

by

$$S_{s,t} f(z) = \int_{\mathbb{R}^d} P_t(z-x)\, f(x)\, dx, \quad z \in \mathbb{C}^d, \tag{3.1}$$

where $\mathcal{H}\left(\mathbb{C}^d\right)$ denotes the space of holomorphic functions on $\mathbb{C}^d$. The integral is well defined since for $t < 2s, P_t(z-x)/P_s(x)$ is in $L^2(\mathbb{R}^d, P_s(x))$. Using Morera's theorem one may show that $S_{s,t} f$ is indeed holomorphic.

Since P_t is just the fundamental solution at the identity of the heat equation $du/dt = \frac{1}{2}\Delta u$, $S_{s,t} f$ may be expressed as

$$S_{s,t} f = \text{ analytic continuation of } e^{t\Delta/2} f. \tag{3.2}$$

Here $e^{t\Delta/2}$ is to be interpreted as the usual contraction semigroup on $L^2\left(\mathbb{R}^d, dx\right)$, extended by continuity to $L^2\left(\mathbb{R}^d, P_s\right)$.

Definition 3.1. *For $s > t/2$, let $A_{s,t}$ be the constant-coefficient elliptic differential operator on $\mathbb{C}^d$ given by*

$$A_{s,t} = \left(s - \frac{t}{2}\right) \sum_{k=1}^{d} \frac{\partial^2}{\partial x_k^2} + \frac{t}{2} \sum_{k=1}^{d} \frac{\partial^2}{\partial y_k^2}.$$

Let $M_{s,t}$ denote the Gaussian measure given by $e^{A_{s,t}/2}(\delta_0)$. Explicitly $dM_{s,t} = M_{s,t}(z)\, dz$, where dz is the standard Lebesgue measure on $\mathbb{C}^d$ and

$$M_{s,t}(z) = (\pi r)^{-d/2} (\pi t)^{-d/2} e^{-x^2/r} e^{-y^2/t}$$

Here $r = 2(s - t/2)$.

The Gaussian measures P_s and $M_{s,t}$ may also be described by their Fourier transforms:

$$\begin{aligned} \int_{\mathbb{R}^d} \exp(i\lambda \cdot x)\, dP_s(x) &= \exp\left(-\frac{s}{2}\lambda^2\right) \\ \int_{\mathbb{C}^d} \exp(i\lambda \cdot x + i\alpha \cdot y)\, dM_{s,t}(z) &= \exp(-\frac{1}{4}(r\lambda^2 + t\alpha^2)) \end{aligned} \tag{3.3}$$

for all λ and α in $\mathbb{R}^d$.

Let $\mathcal{H}L^2\left(\mathbb{C}^d, M_{s,t}\right)$ denote the Hilbert space of holomorphic functions on $\mathbb{C}^d$ which are square-integrable with respect to $M_{s,t}$.

Theorem 3.2 (Extended Segal–Bargmann Transform)**.** *For all s and t with $s > t/2 > 0$, the map $S_{s,t}$ defined in (3.1) is an isometric isomorphism of $L^2\left(\mathbb{R}^d, P_s\right)$ onto $\mathcal{H}L^2\left(\mathbb{C}^d, M_{s,t}\right)$. The standard case is $s = t$.*

We will prove this in Section 3.2, using Hermite expansions. The surjectivity of $S_{s,t}$ is proved by showing that the holomorphic polynomials are dense in $\mathcal{H}L^2\left(\mathbb{C}^d, M_{s,t}\right)$, which is item 4 of Theorem 3.6 below.

3.2. **Action on the Hermite Expansion.** The classical Segal-Bargmann transform for $\mathbb{R}^d$ takes the Hermite expansion of a function on $\mathbb{R}^d$ to the Taylor expansion of the corresponding holomorphic function on $\mathbb{R}^d$, and this property determines the transform [B]. Our variant of the Segal-Bargmann transform also has a simple action on the Hermite expansion, which reduces to the above result when $s = t$. Since we require Hermite expansions on both $\mathbb{R}^d$ and $\mathbb{C}^d$, we will prove abstract results which cover both cases simultaneously.

Let V be a real finite-dimensional vector space and L be a constant coefficient pure second order elliptic operator on V, i.e., $L = \sum_{i,j=1}^N g_{ij}\partial^2/\partial x_i \partial x_j$ where $N = \dim(V)$, $\{x_i\}_{i=1}^N$ are linear coordinates on V, and $\{g_{ij}\}$ is a positive-definite symmetric matrix. For $v, w \in V$, let $g(v,w) = \sum_{i,j=1}^N g^{ij} x_i(v) x_j(w)$, where $\{g^{ij}\}$ denotes the matrix inverse of $\{g_{ij}\}$. Then g is an inner product on V which is naturally induced by L. Indeed, if g^* denotes the dual inner product of V^* and $\alpha, \beta \in V^*$, then $g^*(\alpha,\beta) = \frac{1}{2}L(\alpha\beta)$. If $\{e_i\}_{i=1}^N$ is an orthonormal basis for (V, g), then $L = \sum_{i=1}^N \partial_i^2$ where $\partial_i = \partial_{e_i}$. This follows from the observation that

$$\frac{1}{2}\sum_{i=1}^N \partial_i^2(\alpha\beta) = \sum_{i=1}^N \alpha(e_i)\beta(e_i) = g^*(\alpha,\beta) = \frac{1}{2}L(\alpha\beta).$$

Definition 3.3 (Heat Kernel Measure). *For a pair V and L be as above, we associate the Gaussian measures*

$$dQ_t(v) = \left(\frac{1}{2\pi t}\right)^{N/2} \exp\left(-\frac{1}{2t}g(v,v)\right) dv \quad \forall t > 0,$$

where dv denotes Lebesgue measure on V normalized so that unit cube in V relative to g has unit volume. We will abbreviate Q_1 by Q. For any measurable function f on V and $v \in V$, let

$$e^{tL/2}f(v) = \int_V f(v-w)\, dQ_t(w) \tag{3.4}$$

whenever the integral exists.

The measures Q_t may also be described by their Fourier transforms, namely Q_t is the unique measure on V such that

$$\int_V e^{i\lambda(w)}\, dQ_t(w) = \exp\left(-\frac{tg^*(\lambda,\lambda)}{2}\right) = \exp\left(-t\frac{L(\lambda^2)}{4}\right).$$

for all $\lambda \in V^*$.

Given a reasonable function f on V (say continuous and exponentially bounded), it is well known and easily checked that $u(t,v) := e^{tL/2}f(v)$ is a solution to the heat equation $\partial u(t,v)/\partial t = \frac{1}{2}Lu(t,v)$ such that $\lim_{t\searrow 0} u(t,v) = f(v)$. It is also easily checked that if f is a polynomial function of v, then $e^{tL/2}f$ may be computed by the finite Taylor series expansion:

$$e^{tL/2}f = \sum_{k=0}^{\infty}\left(\frac{tL}{2}\right)^k f. \tag{3.5}$$

The above sum in finite since $L^k f = 0$ whenever $2k$ is greater than the degree of f. On polynomials, (3.5) defines $e^{tL/2}f$ for all $t \in \mathbb{R}$ in such a way that $e^{-tL/2}$ is the inverse of $e^{tL/2}$.

Definition 3.4. *The* n^{th} ***level Hermite subspace of*** $L^2(V,Q)$ *is the space* $\mathcal{F}_n(L) = e^{-L/2}\mathcal{P}_n(V)$*, where* $\mathcal{P}_n(V)$ *denotes the space of homogeneous polynomials of degree* n *on* V*.*

The following result is well-known. We include a proof for completeness and so that some calculations will be available for later use.

Proposition 3.5. *Let* V *and* L *be as above. Then*

1. $L^2(V,Q)$ *is the orthogonal Hilbert space direct sum of the subspaces* $\mathcal{F}_n(L)$ *for* $n = 0, 1, 2 \ldots$ *.*
2. $\mathcal{F}_n(L)$ *is the set of all polynomials on* V *of degree* n *which are orthogonal to all polynomials of degree at most* $n-1$*.*
3. *For every* $f \in L^2(V,Q)$*,* $e^{L/2}f$ *is a well defined, real-analytic function on* V*. Moreover, if the "Hermite" expansion of* $f \in L^2(V,Q)$ *is* $f = \sum_{n=0}^{\infty} f_n$ *with* $f_n \in \mathcal{F}_n(L)$*, then* $f_n = e^{-L/2}p_n$*, where* $p_n(v) = \frac{1}{n!}(\partial_v^n e^{L/2} f)(0)$ *and* $(\partial_v^n f)(0) = \frac{d^n}{dt^n} f(tv)|_{t=0}$*. We will write this succinctly as*

$$f(v) = \sum_{n=0}^{\infty} \frac{1}{n!} e^{-L/2}(\partial_v^n e^{L/2} f)(0). \tag{3.6}$$

Proof. Let $\{e_i\}_{i=1}^N$ be an orthonormal basis for (V,g) so that $L = \sum_{i=1}^N \partial_i^2$, where $\partial_i = \partial_{e_i}$. For functions p, q on V let $(p,q) = \int_V \bar{p}(v)q(v)\,dQ(v)$ be the L^2 inner product. Taking p and q to be polynomials on V and $v = 0$ in (3.4), we find, using the fact that Q is even, that

$$(e^{-L/2}p, e^{-L/2}q) = e^{L/2}(\overline{e^{-L/2}p} \cdot e^{-L/2}q)|_0.$$

Since $e^{tL/2}(\overline{e^{-tL/2}p}\, e^{-tL/2}q)$ is a polynomial in (t,v), it follows by Taylor's theorem that

$$e^{L/2}\left(\overline{e^{-L/2}p} \cdot e^{-L/2}q\right) = \sum_{n=0}^{\infty} \frac{1}{n!} \left.\frac{d^n}{dt^n}\right|_{t=0} e^{tL/2}(e^{-tL/2}\bar{p} \cdot e^{-tL/2}q).$$

Using the product rule repeatedly shows that

$$\begin{aligned}
&\frac{d}{dt} e^{tL/2}(e^{-tL/2}\bar{p} \cdot e^{-tL/2}q) \\
&= e^{tL/2}\left(\frac{L}{2}(e^{-tL/2}\bar{p} \cdot e^{-tL/2}q) - (\frac{L}{2}e^{-tL/2}\bar{p} \cdot e^{-tL/2}q) - (e^{-tL/2}\bar{p} \cdot \frac{L}{2}e^{-tL/2}q)\right) \\
&= e^{tL/2}\left(\sum_{i=1}^N \partial_i e^{-tL/2}\bar{p} \cdot \partial_i e^{-tL/2}q)\right).
\end{aligned}$$

This equation may now be used inductively to show

$$\left.\frac{d^n}{dt^n}\right|_{t=0} e^{tL/2}(e^{-tL/2}\bar{p} \cdot e^{-tL/2}q). = \sum_{i_1, i_2, \ldots i_n = 1}^N \partial_{i_1}\partial_{i_2}\cdots\partial_{i_n}\bar{p} \cdot \partial_{i_1}\partial_{i_2}\cdots\partial_{i_n} q.$$

Combining the previous four displayed equations shows that

$$(e^{-L/2}p, e^{-L/2}q) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{i_1, i_2, \ldots i_n = 1}^N (\partial_{i_1}\partial_{i_2}\cdots\partial_{i_n}\bar{p}(v) \cdot \partial_{i_1}\partial_{i_2}\cdots\partial_{i_n} q(v))\,|_{v=0}, \tag{3.7}$$

for all polynomials p and q on V.

Using (3.7) we may prove items 1 and 2 as follows. Notice that $\mathcal{F}_n(L)$ consists of polynomials of degree n and that $\oplus_{k=0}^{n-1}\mathcal{F}_k(L)$ consists of *all* polynomials on V of degree $n-1$ or less. By (3.7), it is easily seen that if $p \in \mathcal{P}_n(V)$ and $q \in \mathcal{P}_m(V)$ with $m \neq n$, then $(e^{-L/2}p, e^{-L/2}q) = 0$. Hence $\mathcal{F}_n(L)$ is orthogonal to $\oplus_{k=0}^{n-1}\mathcal{F}_k(L)$. Since polynomials are dense in $L^2(V,Q)$, these observations immediately imply the first two items of the theorem.

For item 3, suppose for the moment that f is a polynomial on V. By Taylor's theorem applied to $e^{L/2}f$,

$$e^{L/2}f = \sum_{n=0}^{\infty} \frac{1}{n!}(\partial_v^n e^{L/2}f)(0).$$

Applying $e^{-L/2}$ to both sides of this equation then proves (3.6) when f is a polynomial.

For general $f \in L^2(V,Q)$, we must first show that $e^{L/2}f$ is defined and smooth. Letting $q(v) = dQ(v)/dv$, we may write

$$e^{L/2}f(v) = \int_V f(v-w)q(w)\,dw = \int_V f(w)\frac{q(v-w)}{q(w)}q(w)\,dw. \tag{3.8}$$

Since $q(v-w)/q(w) = \exp(-\frac{1}{2}g(v,v) + g(v,w)) \in L^2(V, Q(dw))$, it follows that $f(v-w)q(w)$ is integrable and hence $e^{L/2}f$ is defined. More generally one may show $\sup_{v\in K} |\partial^\alpha q(v-w)/q(w)| \in L^2(V, Q(dw))$ for all compact sets $K \subset V$. Hence $e^{L/2}f(v)$ is smooth and

$$\partial^\alpha e^{L/2}f(v) = \int_V f(w)\frac{\partial^\alpha q(v-w)}{q(w)}q(w)dw. \tag{3.9}$$

So for each integer $n \geq 0$, let $(P_n f)(v) = \frac{1}{n!}e^{-L/2}(\partial_v^n e^{L/2}f)(0) \in \mathcal{F}_n(L)$. Because of (3.9), $P_n : L^2(V,Q) \to \mathcal{F}_n(L)$ is a well defined continuous linear map. Moreover, $P_n f$ is the same as orthogonal projection onto $\mathcal{F}_n(L)$ when f is a polynomial. Hence it follows by density of polynomials in $L^2(V,Q)$ that P_n is orthogonal projection onto $\mathcal{F}_n(L)$. This proves (3.6) for general $f \in L^2(V,Q)$. □

We will need the following holomorphic version of Proposition 3.5.

Theorem 3.6. *Suppose that V is a real vector space and L be a pure second order constant coefficient differential operator on $C^\infty(V)$. Also assume that V is equipped with a complex structure J, i.e., $J : V \to V$ is a linear map such that $J^2 = -I$. Using J, V may considered to be a complex vectors space by defining $iv = Jv$ for $v \in V$. Let $\mathcal{H}(V)$ denote the space of holomorphic functions on V and let $\mathcal{H}L^2(V,Q) = \mathcal{H}(V) \cap L^2(V,Q)$ be the space of L^2 holomorphic functions. Let $\mathcal{HF}_n(L) = \mathcal{H}(V) \cap \mathcal{F}_n(L)$. Then*

1. *$\mathcal{H}L^2(V,Q)$ is the orthogonal Hilbert space direct sum of the subspaces $\mathcal{HF}_n(L)$ for $n = 0,1,2\ldots$.*
2. *$\mathcal{HF}_n(L)$ is the set of all holomorphic polynomials on V of degree n or less which are orthogonal to all holomorphic polynomials on V of degree $n-1$ or less.*
3. *Let $f = \sum_{n=0}^{\infty} f_n$ be the Hermite expansion of $f \in \mathcal{H}L^2(V,Q)$. Then $f_n \in \mathcal{HF}_n(L)$ for $n = 0,1,2,\ldots$.*
4. *The holomorphic polynomials on V are dense in $\mathcal{H}L^2(V,Q)$.*

Proof. Since $\mathcal{F}_m(L)$ and $\mathcal{F}_n(L)$ are orthogonal for $m \neq n$, $\mathcal{HF}_n(L)$ and $\mathcal{HF}_n(L)$ are also clearly orthogonal for $m \neq n$.

Now for $f \in L^2(V,Q)$ we have already seen that $e^{L/2}f$ is a smooth function. If f is also holomorphic, then $e^{L/2}f$ is holomorphic. To see this it suffices to show, for each $u,v \in V$, that $e^{L/2}f(u+zv)$ is holomorphic as a function of $z \in \mathbb{C}$. This is easily done using Morera's Theorem and the fact that f is holomorphic. We omit the details.

Hence if $f \in \mathcal{H}L^2(V,Q)$, then $p_n(v) := \frac{1}{n!}(\partial_v^n e^{L/2} f)(0)$ is a holomorphic polynomial that is homogeneous of degree n. Since L preserves the space of holomorphic functions, it follows that $f_n = e^{-L/2}p_n$ is both holomorphic and in $\mathcal{F}_n(L)$, i.e., $f_n \in \mathcal{HF}_n(L)$. Hence we have proved items 1, 3, and 4 of the Theorem. Finally, for item 2, if p is a holomorphic polynomial of degree less than or equal to n, then

$$p(v) = \sum_{k=0}^{n} \frac{1}{k!} e^{-L/2}(\partial_v^n e^{L/2} p)(0).$$

Since $\oplus_{k=0}^{n-1}\mathcal{HF}_k(L)$ is the collection $\mathcal{H}_{n-1}$ of holomorphic polynomials of degree less than or equal to $n-1$, it follows p is orthogonal to $\mathcal{H}_{n-1}$ if and only if $p(v) = \frac{1}{n!}e^{-L/2}(\partial_v^n e^{L/2}p)(0)$ which is equivalent to p being in $\mathcal{HF}_n(L)$. □

We now apply our results in two cases: $V = \mathbb{R}^d$ and $L = s\Delta$, and $V = \mathbb{C}^d$ and $L = A_{s,t}$.

Definition 3.7. *Let $\mathcal{F}_{n,s}(\mathbb{R}^d) = \mathcal{F}_n(s\Delta) \subset L^2(\mathbb{R}^d, P_s)$ and $\mathcal{F}_{n,s,t}(\mathbb{C}^d) = \mathcal{F}_n(A_{s,t}) \subset L^2(\mathbb{C}^d, M_{s,t})$. Let $\mathcal{HF}_{n,s,t}(\mathbb{C}^d)$ denote the holomorphic polynomials in $\mathcal{F}_{n,s,t}(\mathbb{C}^d)$.*

Theorem 3.8. *The transform $S_{s,t}$ in (3.1) takes $\mathcal{F}_{n,s}(\mathbb{R}^d)$ onto $\mathcal{HF}_{n,s,t}(\mathbb{C}^d)$. Specifically, let p be a homogeneous polynomial of degree n on $\mathbb{R}^d$ and let $p_{\mathbb{C}}$ be its analytic continuation to $\mathbb{C}^d$. Then*

$$S_{s,t}\left(e^{-s\Delta/2}p\right) = e^{-A_{s,t}/2}(p_{\mathbb{C}}). \tag{3.10}$$

Note that if $s=t$, then the operator $A_{s,t}$ is zero on all holomorphic functions. So when $s=t$ the holomorphic subspace of $\mathcal{F}_{n,s,t}(\mathbb{C}^d)$ is precisely the space of holomorphic polynomials which are homogeneous of degree n. In that case, the transform $S_{t,t}$ takes the Hermite expansion of $f \in L^2(\mathbb{R}^d, P_t)$ to the Taylor expansion of $S_{t,t}f$. If $s \neq t$, then by Theorem 3.6 and Theorem 3.8, $S_{s,t}$ takes the Hermite expansion of $f \in L^2(\mathbb{R}^d, P_s)$ to an L^2-convergent expansion of $S_{s,t}f$ in terms of non-homogeneous holomorphic polynomials. For $s \neq t$ it is not clear (to us) whether the Taylor series of a function in $\mathcal{H}L^2(\mathbb{C}^d, M_{s,t})$ is always L^2-convergent.

Proof. By the definition of $S_{s,t}$,

$$S_{s,t}\left(e^{-s\Delta/2}p\right) = (e^{t\Delta/2}e^{-s\Delta/2}p)_{\mathbb{C}} = (e^{(t-s)\Delta/2}p)_{\mathbb{C}}.$$

On the other hand since $p_{\mathbb{C}}$ is holomorphic, $\partial p_{\mathbb{C}}/\partial y_k = i\partial p_{\mathbb{C}}/\partial x_k$ and hence

$$A_{s,t}p_{\mathbb{C}} = \left(s - \frac{t}{2} - \frac{t}{2}\right)\sum_{k=1}^{d} \frac{\partial^2}{\partial x_k^2} p_{\mathbb{C}} = (s-t)\sum_{k=1}^{d}\frac{\partial^2}{\partial x_k^2}p_{\mathbb{C}} = -(t-s)(\Delta p)_{\mathbb{C}}.$$

Therefore, $e^{-A_{s,t}/2}(p_{\mathbb{C}}) = (e^{(t-s)\Delta/2}p)_{\mathbb{C}} = S_{s,t}\left(e^{-s\Delta/2}p\right)$. □

Proof of Theorem 3.2. If p and q are polynomials on $\mathbb{R}^d$, then

$$\begin{aligned}
A_{s,t}(\overline{p_{\mathbb{C}}}q_{\mathbb{C}}) - (A_{s,t}\overline{p_{\mathbb{C}}})q_{\mathbb{C}} - \overline{p_{\mathbb{C}}}(A_{s,t}q_{\mathbb{C}}) &= 2\left(s-\frac{t}{2}\right)\sum_{k=1}^{d}\overline{\frac{\partial p_{\mathbb{C}}}{\partial x_k}}\frac{\partial q_{\mathbb{C}}}{\partial x_k} + \frac{t}{2}\sum_{k=1}^{d}\overline{\frac{\partial p_{\mathbb{C}}}{\partial y_k}}\frac{\partial q_{\mathbb{C}}}{\partial y_k} \\
&= 2\left(s-\frac{t}{2}\right)\sum_{k=1}^{d}\overline{\frac{\partial p_{\mathbb{C}}}{\partial x_k}}\frac{\partial q_{\mathbb{C}}}{\partial x_k} + \frac{t}{2}\sum_{k=1}^{d}\left(-i\overline{\frac{\partial p_{\mathbb{C}}}{\partial x_k}}\right)\left(i\frac{\partial q_{\mathbb{C}}}{\partial x_k}\right) \\
&= 2s\sum_{k=1}^{d}\overline{\frac{\partial p_{\mathbb{C}}}{\partial x_k}}\frac{\partial q_{\mathbb{C}}}{\partial x_k}.
\end{aligned}$$

This formula and computations similar to those used to prove (3.7) show that

$$\begin{aligned}
&(e^{-A_{s,t}/2}p_{\mathbb{C}}, e^{-A_{s,t}/2}q_{\mathbb{C}})_{L^2(M_{s,t})} \\
&= e^{A_{s,t}/2}(\overline{e^{-A_{s,t}/2}p_{\mathbb{C}}}\cdot e^{-A_{s,t}/2}q_{\mathbb{C}})|_{z=0} \\
&= \sum_{n=0}^{\infty}\frac{s^n}{n!}\sum_{k_1,k_2,\ldots k_{,n}=1}^{d}\overline{\partial_{x_{k_1}}\partial_{x_{k_2}}\cdots\partial_{x_{k_n}}p_{\mathbb{C}}}\cdot\partial_{x_{k_1}}\partial_{x_{k_2}}\cdots\partial_{x_{k_n}}q_{\mathbb{C}}\Big|_{z=0} \qquad (3.11)\\
&= \sum_{n=0}^{\infty}\frac{s^n}{n!}\sum_{k_1,k_2,\ldots k_{,n}=1}^{d}\overline{\partial_{x_{k_1}}\partial_{x_{k_2}}\cdots\partial_{x_{k_n}}p}\cdot\partial_{x_{k_1}}\partial_{x_{k_2}}\cdots\partial_{x_{k_n}}q\Big|_{x=0} \qquad (3.12)\\
&= e^{s\Delta/2}(\overline{e^{-s\Delta/2}p}\cdot e^{-s\Delta/2}q)\,(0) = (e^{-s\Delta/2}p, e^{-s\Delta/2}q)_{L^2(P_s)}.
\end{aligned}$$

Here $\partial_{x_i} = \partial/\partial x_i$. In light of (3.10), which holds by linearity for all polynomials, this shows that $S_{s,t}$ is isometric on polynomials.

Since polynomials are dense in $L^2(\mathbb{R}^d, P_s)$ and the linear functionals $f \in L^2(\mathbb{R}^d, P_s) \to (S_{s,t}f)(z) \in \mathbb{C}$ are continuous for each $z \in \mathbb{C}^d$, it follows that $S_{s,t}$ is isometric on all of $L^2(\mathbb{R}^d, P_s)$. Note that $e^{-s\Delta/2}$ is isometric and hence injective and hence invertible on the space of polynomials of degree at most n. This plus the fact that every holomorphic polynomial on $\mathbb{C}^d$ is the analytic continuation of its restriction to $\mathbb{R}^d$ shows that every holomorphic polynomial is in the image of $S_{s,t}$. But Item 4 of Theorem 3.6 asserts that the holomorphic polynomials are dense in $\mathcal{H}L^2(\mathbb{C}^d, M_{s,t})$ and therefore $S_{s,t}$ is surjective. □

4. Segal-Bargmann for the Wiener space

Because it is formulated in terms of Gaussian measures, our variant of the Segal-Bargmann transform admits an infinite-dimensional ($d \to \infty$) limit. While this could be formulated in terms of an arbitrary abstract Wiener space, we will for concreteness consider only the classical Wiener space case relevant to this paper. Thus $\mathbb{R}^d$ will be replaced by an infinite-dimensional space of Lie algebra-valued generalized functions on $[0,1]$, with a white noise measure. By integrating once, this space may be identified with the space of continuous Lie algebra-valued functions on $[0,1]$ with a Wiener measure. Similarly, $\mathbb{R}^d$ will be replaced by a space of generalized functions with values in the complex Lie algebra (with a white noise measure), which may be identified with the space of continuous functions with values in the complex Lie algebra (with a Wiener measure).

4.1. **The Transform for the Wiener Space.** Let K be a compact connected Lie group. Fix once and for all an Ad-K-invariant inner product $\langle \cdot, \cdot \rangle$ on the Lie algebra $\mathfrak{k}$ of K. Let $K_{\mathbb{C}}$ be the complexification of K in the sense of [Ho, H1], and let $\mathfrak{k}_{\mathbb{C}} = \mathfrak{k} + i\mathfrak{k}$ be the Lie algebra of $K_{\mathbb{C}}$.

We then consider the space of connections on the spatial circle. These are Lie algebra-valued 1-forms, which can be identified with Lie algebra-valued functions on the interval $[0,1]$, where the circle is this interval with ends identified. Specifically, let

$$\mathcal{A} = L^2\left([0,1]\,;\mathfrak{k}\right), \tag{4.1}$$

where the norm is computed using Lebesgue measure on $[0,1]$ and the inner product on $\mathfrak{k}$. We need also a larger space $\bar{\mathcal{A}}$, which may be taken to be

$$\bar{\mathcal{A}} = \{A = \frac{da_\tau}{d\tau}\,|a \in W(\mathfrak{k})\,\}, \tag{4.2}$$

a subspace of $\mathfrak{k}$-valued distributions. Here $W(\mathfrak{k})$ denotes the set of continuous paths a from $[0,1]$ to $\mathfrak{k}$ such that $a_0 = 0$, and $\frac{da_\tau}{d\tau}$ denotes the distributional derivative of a. For each $A \in \bar{\mathcal{A}}$, the function $a \in W(\mathfrak{k})$ is unique, and so may be thought of as a function of A. We will write, suggestively,

$$a_\tau(A) = \int_0^\tau A_\sigma \, d\sigma. \tag{4.3}$$

Note that we are reversing convention by using the lowercase letter a for the anti-derivative of A. We can make $\bar{\mathcal{A}}$ into a Banach space whose norm is the supremum norm on a.

We similarly define

$$\mathcal{A}_{\mathbb{C}} = L^2\left([0,1]\,;\mathfrak{k}_{\mathbb{C}}\right)$$

using Lebesgue measure on $[0,1]$ and the sesquilinear extension of the inner product from $\mathfrak{k}$ to $\mathfrak{k}_{\mathbb{C}}$; and

$$\bar{\mathcal{A}}_{\mathbb{C}} = \{C = \frac{dc_\tau}{d\tau}\,|c \in W(\mathfrak{k}_{\mathbb{C}})\,\},$$

where $W(\mathfrak{k}_{\mathbb{C}})$ is defined analogously to $W(\mathfrak{k})$. As in the real case, c is unique and we will write

$$c_\tau(C) = \int_0^\tau C_\sigma \, d\sigma. \tag{4.4}$$

Definition 4.1. *Let $\tilde{P}_s$ denote the unique Gaussian measure on $\bar{\mathcal{A}}$ such that for all continuous linear functionals ϕ on $\bar{\mathcal{A}}$*

$$\int_{\bar{\mathcal{A}}} e^{i\phi(A)}\, d\tilde{P}_s(A) = \exp\left(-\frac{s}{2}\left\|\phi\right\|^2\right),$$

where $\|\phi\|$ denotes the norm of ϕ as a linear functional on $\mathcal{A}$.

Let $\tilde{M}_{s,t}$ denote the unique Gaussian measure on $\bar{\mathcal{A}}_{\mathbb{C}}$ such that for all continuous linear functionals ϕ and ψ on $\bar{\mathcal{A}}$

$$\int_{\bar{\mathcal{A}}_{\mathbb{C}}} e^{i\phi(A)+i\psi(B)}\, d\tilde{M}_{s,t}(A+iB) = \exp\left(-\frac{1}{4}(r\left\|\phi\right\|^2 + t\left\|\psi\right\|^2)\right),$$

where $\|\phi\|$ and $\|\psi\|$ denote norms as linear functionals on $\mathcal{A}$ and $r = 2\,(s - t/2)$.

These measures have the formal expressions:

$$\begin{aligned} d\tilde{P}_s(A) &= \frac{1}{Z_1} e^{-\|A\|^2/2s}\,\mathcal{D}A \quad \text{and} \\ d\tilde{M}_{s,t}(A+iB) &= \frac{1}{Z_2} e^{-\|A\|^2/r-\|B\|^2/t}\,\mathcal{D}A\,\mathcal{D}B. \end{aligned}$$

Here $\|\cdot\|$ is the L^2 – norm for $\mathcal{A}$, $\mathcal{D}A$ and $\mathcal{D}B$ refer to the (non-existent) Lebesgue measure on $\mathcal{A}$, and Z_1, Z_2 are "normalization constants." Note that the measure $\tilde{P}_s$ may be thought of as the heat kernel measure at the origin, that is, the fundamental solution at the origin of the equation $du/dt = \frac{1}{2}\Delta_{\mathcal{A}}u$, where $\Delta_{\mathcal{A}}$ is the sum of squares of derivatives in the directions of an orthonormal basis for $\mathcal{A}$.

The space $\mathcal{A}$ is the Cameron-Martin subspace for the Gaussian measure space $(\bar{\mathcal{A}}, \tilde{P}_s)$ and is a set of $\tilde{P}_s$-measure zero. Similarly, $\mathcal{A}_{\mathbb{C}}$ is the Cameron-Martin subspace for $(\bar{\mathcal{A}}_{\mathbb{C}}, \tilde{M}_{s,t})$ and is a set of $\tilde{M}_{s,t}$-measure zero.

The measure $\tilde{P}_s$ is the law of a scaled $\mathfrak{k}$-valued white noise on $[0,1]$. This is equivalent to saying that if A is distributed as $\tilde{P}_s$ then $a_\tau(A)$ (defined in (4.3)) is a scaled $\mathfrak{k}$-valued Brownian motion. Specifically, if $\{X^k\}$ is an orthonormal basis for $\mathfrak{k}$, then $a_\tau^k := \langle X^k, a_\tau\rangle$ are real-valued Brownian motions satisfying

$$E\left\{a_\sigma^k a_\tau^l\right\} = s \min\{\sigma,\tau\}\,\delta_{kl}. \tag{4.5}$$

Similarly, the measure $\tilde{M}_{s,t}$ is the law of a scaled $\mathfrak{k}_{\mathbb{C}}$-valued white noise on $[0,1]$. Let c be as in (4.4), and decompose c as $c_\tau = \mathrm{Re}c_\tau + i\mathrm{Im}c_\tau$, with $\mathrm{Re}c_\tau$ and $\mathrm{Im}c_\tau$ taking values in $\mathfrak{k}$. Then $\langle X^k, \mathrm{Re}c_\tau\rangle$ and $\langle X^k, \mathrm{Im}c_\tau\rangle$ are independent real-valued Brownian motions satisfying:

$$\begin{aligned} E\left\{\langle X^k, \mathrm{Re}c_\sigma\rangle \langle X^l, \mathrm{Re}c_\tau\rangle\right\} &= \left(s - \frac{t}{2}\right)\min\{\sigma,\tau\}\,\delta_{kl} \\ E\left\{\langle X^k, \mathrm{Im}c_\sigma\rangle \langle X^l, \mathrm{Im}c_\tau\rangle\right\} &= \frac{t}{2}\min\{\sigma,\tau\}\,\delta_{kl}. \end{aligned} \tag{4.6}$$

We now define the two-parameter version of the Segal-Bargmann transform for $L^2(\bar{\mathcal{A}}, \tilde{P}_s)$. The reader should keep in mind that we are "trying" to work on the space $\mathcal{A}$, with the larger space $\bar{\mathcal{A}}$ introduced as a technical necessity.

Definition 4.2. *Let $\{e_1, \cdots, e_d\}$ be a finite orthonormal set in the real Hilbert space $\mathcal{A}$ with the property that each linear functional $\langle e_j, \cdot\rangle$ extends continuously to $\bar{\mathcal{A}}$. Each linear functional $\langle e_j, \cdot\rangle$ then has a unique complex-linear extension from $\bar{\mathcal{A}}$ to $\bar{\mathcal{A}}_{\mathbb{C}}$. A* ***cylinder function*** *on $\bar{\mathcal{A}}$ is a function that can be expressed in the form*

$$f(A) = \phi(\langle e_1, A\rangle, \cdots \langle e_d, A\rangle), \tag{4.7}$$

where ϕ is a measurable function on $\mathbb{R}^d$ and $\{e_1, \cdots, e_d\}$ is a orthonormal basis as above. A ***holomorphic cylinder function*** *on $\bar{\mathcal{A}}_{\mathbb{C}}$ is a function of the form*

$$F(C) = \Phi(\langle e_1, C\rangle, \cdots \langle e_d, C\rangle),$$

where Φ is a holomorphic function on $\mathbb{C}^d$. The ***holomorphic subspace*** *of $L^2(\bar{\mathcal{A}}_{\mathbb{C}}, \tilde{M}_{s,t})$, denoted $\mathcal{H}L^2(\bar{\mathcal{A}}_{\mathbb{C}}, \tilde{M}_{s,t})$, is the L^2 closure of the L^2 holomorphic cylinder functions.*

The transform $\tilde{S}_{s,t}$ will be defined in Theorem 4.3 below so as to coincide with the finite dimensional transform $S_{s,t}$ acting on cylinder functions, and then extending by continuity to all of $L^2(\bar{\mathcal{A}}, \tilde{P}_s)$. In the standard case $(s = t)$ one can and often does define the transform differently (e.g., [BSZ, GM]), with the range Hilbert space being a certain space of holomorphic functions on $\mathcal{A}_{\mathbb{C}}$ rather than on $\bar{\mathcal{A}}_{\mathbb{C}}$. Since the necessary dimension-independent pointwise bounds hold only when $s=t$, this approach does not work when $s\neq t$. Formally, $\tilde{S}_{s,t}f$ is the analytic continuation of $e^{t\Delta_{\mathcal{A}}/2}f$; this description may be taken fairly literally when f is a cylinder function.

Theorem 4.3. *Fix s and t with $s>t/2>0$. There exists a unique isometric map $\tilde{S}_{s,t}$ of $L^2(\bar{A},\tilde{P}_s)$ onto $\mathcal{H}L^2(\bar{A}_{\mathbb{C}},\tilde{M}_{s,t})$ such that for all $f\in L^2(\bar{A},\tilde{P}_s)$ of the form*

$$f(A)=\phi\left(\langle e_1,A\rangle,\cdots,\langle e_d,A\rangle\right)$$

with $\{e_1,\cdots,e_d\}$ as in Definition 4.2, $\tilde{S}_{s,t}f$ is given by

$$\tilde{S}_{s,t}f(C)=\left(S_{s,t}\phi\right)\left(\langle e_1,C\rangle,\cdots,\langle e_d,C\rangle\right).$$

Proof. We want to define $\tilde{S}_{s,t}$ to coincide with $S_{s,t}$ on cylinder functions. The fact that $\tilde{S}_{s,t}f$ is well defined independent of how f is represented as a cylinder function is a consequence of the two observations: 1) the measure P_t on $\mathbb{R}^d$ is rotationally-invariant, and 2) the $(d+k)$-dimensional measure P_t factors as the product of the corresponding d-dimensional and k-dimensional measures.

Now isometricity on cylinder functions follows immediately from Theorem 3.2. Since cylinder functions are dense, $\tilde{S}_{s,t}$ has a unique isometric extension to $L^2(\bar{A},\tilde{P}_s)$. The surjectivity in Theorem 3.2 shows that every L^2 holomorphic cylinder function is in the image of $\tilde{S}_{s,t}$. Since by definition the L^2 holomorphic cylinder functions are dense in the holomorphic subspace, we conclude that $\tilde{S}_{s,t}$ maps onto $\mathcal{H}L^2(\bar{A}_{\mathbb{C}},\tilde{M}_{s,t})$. □

Definition 4.4. *Let $\Delta_{\mathcal{A}}$ be the unique operator on cylinder functions such that*

$$\Delta_{\mathcal{A}}f(A)=(\Delta\phi)\left(\langle e_1,A\rangle,\cdots,\langle e_d,A\rangle\right)$$

when f is a cylinder function as in (4.7) of Definition 4.2. The domain of $\Delta_{\mathcal{A}}$ is taking to be the set of those cylinder functions for which ϕ is smooth and both ϕ and $\Delta\phi$ are in $L^2\left(\mathbb{R}^d,P_s\right)$.

One checks as in the proof of Theorem 4.3 that $\Delta_{\mathcal{A}}$ is well defined, independent of how f is represented as a cylinder function. Although it is densely defined, $\Delta_{\mathcal{A}}$ is a non-closable operator.

4.2. **Action on the Hermite expansion.**

We now turn to the infinite-dimensional version of the Hermite expansion, and the action of $\tilde{S}_{s,t}$ on it.

Definition 4.5. *The* ***n^{th} level Hermite subspace of*** *$L^2(\bar{A},\tilde{P}_s)$, denoted $\mathcal{F}_{n,s}\left(\bar{\mathcal{A}}\right)$, is the L^2 closure of the space of functions of the form*

$$f(A)=\phi\left(\langle e_1,A\rangle,\cdots,\langle e_d,A\rangle\right)$$

where $\{e_1,\cdots,e_d\}$ is as in Definition 4.2 and where $\phi\in\mathcal{F}_{n,s}\left(\mathbb{R}^d\right)$. The ***$n^{th}$ level holomorphic Hermite subspace of*** *$L^2(\bar{A}_{\mathbb{C}},\tilde{M}_{s,t})$, denoted $\mathcal{HF}_{n,s,t}\left(\bar{\mathcal{A}}_{\mathbb{C}}\right)$, is the L^2 closure of the space of functions of the form*

$$F(C)=\Phi\left(\langle e_1,C\rangle,\cdots,\langle e_d,C\rangle\right)$$

where Φ is in $\mathcal{HF}_{n,s,t}\left(\mathbb{C}^d\right)$ as defined in Definition 3.7.

Recall that $a_\tau = \int_0^\tau A_\sigma \, d\sigma$ is a scaled $\mathfrak{k}$-valued Brownian motion whose components with respect to an orthonormal basis $\{X^k\}$ for $\mathfrak{k}$ are denoted a_τ^k. Now consider the n-simplex

$$\Delta_n = \{(\tau_1, \cdots, \tau_n) \in \mathbb{R}^n \, | 0 \le \tau_1 \le \tau_2 \le \cdots \le \tau_n \le 1\}.$$

Let $\mathbf{H}$=$\{H_{k_1,\cdots,k_n} | k_i = 1, \ldots, \dim \mathfrak{k}\}$ be a collection of square-integrable complex-valued functions on Δ_n and let

$$\sum_{k_1,\cdots,k_n=1}^{\dim \mathfrak{k}} \int_{\Delta_n} H_{k_1,\cdots,k_n}(\tau_1, \cdots, \tau_n) \, da_{\tau_1}^{k_1} \cdots da_{\tau_n}^{k_n}$$

denote the multiple Wiener integral of $\mathbf{H}$ relative to a, see [Ito] or Definitions 4.12 and 4.12 below.

Similarly, $c_\tau = \int_0^\tau C_\sigma \, d\sigma$ for $C \in (\bar{\mathcal{A}}_\mathbb{C}, \tilde{M}_{s,t})$ is a $\mathfrak{k}_\mathbb{C}$-valued Brownian motion. Regard the orthonormal basis $\{X_k\}$ for $\mathfrak{k}$ as a basis of $\mathfrak{k}_\mathbb{C}$ as a complex vector space and let c_τ^k be the corresponding complex-valued components of c_τ. If $\mathbf{H}$=$\{H_{k_1,\cdots,k_n}\}$ as above let

$$\sum_{k_1,\cdots,k_n=1}^{\dim \mathfrak{k}} \int_{\Delta_n} H_{k_1,\cdots,k_n}(\tau_1, \cdots, \tau_n) \, dc_{\tau_1}^{k_1} \cdots dc_{\tau_n}^{k_n}$$

denote the multiple Wiener integral of $\mathbf{H}$ with respect to c. By expanding in terms of the real and imaginary parts of c_τ^k, we could express this as an integral in terms of independent real-valued Brownian motions. Note that this integral is a formally holomorphic function of c (and hence of C) since it depends only on the complex increments of c.

Proposition 4.6. *The Hilbert space $L^2(\bar{\mathcal{A}}, \tilde{P}_s)$ is the orthogonal direct sum of the subspaces $\mathcal{F}_{n,s}(\bar{\mathcal{A}})$. The Hilbert space $\mathcal{H}L^2(\bar{\mathcal{A}}_\mathbb{C}, \tilde{M}_{s,t})$ is the orthogonal direct sum of the subspaces $\mathcal{HF}_{n,s,t}(\bar{\mathcal{A}}_\mathbb{C})$.*

A function $f \in L^2(\bar{\mathcal{A}}, \tilde{P}_s)$ is in $\mathcal{F}_{n,s}(\bar{\mathcal{A}})$ if and only if there exist square-integrable complex-valued functions $H_{k_1,\cdots,k_n}$ on Δ_n such that

$$f(A) = \sum_{k_1,\cdots,k_n=1}^{\dim \mathfrak{k}} \int_{\Delta_n} H_{k_1,\cdots,k_n}(\tau_1, \cdots, \tau_n) \, da_{\tau_1}^{k_1} \cdots da_{\tau_n}^{k_n}.$$

A function $F \in \mathcal{H}L^2(\bar{\mathcal{A}}_\mathbb{C}, \tilde{M}_{s,t})$ is in $\mathcal{HF}_{n,s,t}(\bar{\mathcal{A}}_\mathbb{C})$ if and only if there exist square-integrable complex-valued functions $H_{k_1,\cdots,k_n}$ on Δ_n such that

$$F(C) = \sum_{k_1,\cdots,k_n=1}^{\dim \mathfrak{k}} \int_{\Delta_n} H_{k_1,\cdots,k_n}(\tau_1, \cdots, \tau_n) \, dc_{\tau_1}^{k_1} \cdots dc_{\tau_n}^{k_n}.$$

Here as usual $a_\tau = \int_0^\tau A_\sigma \, d\sigma$ and $c_\tau = \int_0^\tau C_\sigma \, d\sigma$, and in either case the H's are unique up to a set of measure zero in Δ_n.

The expansion of a function $f \in L^2(\bar{\mathcal{A}}, \tilde{P}_s)$ into a sum over n of such stochastic integrals is called the Wiener chaos expansion and goes back to Kakutani [Ka] and Itô [Ito]. Nevertheless, we will give a proof of this result (after Lemma 4.11 below) to emphasize the relation of this result to those in Section 3. The transform $\tilde{S}_{s,t}$ has the following simple action on the Wiener chaos expansion, which will be used in the proof of Theorem 5.2 in Section 5.

Theorem 4.7. *The transform $\tilde{S}_{s,t}$ takes $\mathcal{F}_{n,s}\left(\bar{\mathcal{A}}\right)$ onto $\mathcal{HF}_{n,s,t}\left(\bar{\mathcal{A}}_{\mathbb{C}}\right)$. Specifically, if $f\in L^2(\bar{\mathcal{A}},\tilde{P}_s)$ is of the form*

$$f=\sum_{k_1,\cdots,k_n=1}^{\dim K}\int_{\Delta_n}H_{k_1,\cdots,k_n}\left(\tau_1,\cdots,\tau_n\right)da_{\tau_1}^{k_1}\cdots da_{\tau_n}^{k_n}$$

then $\tilde{S}_{s,t}f$ is given by

$$\tilde{S}_{s,t}f=\sum_{k_1,\cdots,k_n=1}^{\dim K}\int_{\Delta_n}H_{k_1,\cdots,k_n}\left(\tau_1,\cdots,\tau_n\right)dc_{\tau_1}^{k_1}\cdots dc_{\tau_n}^{k_n}.$$

This result is the infinite-dimensional analog of Theorem 4.3, with $e^{-s\Delta/2}$ and $e^{-A_{s,t}/2}$ hidden in the definition of the stochastic integrals. The proof will be given at the end of this section.

We use a slightly unorthodox definition of the multiple Wiener integral, which emphasizes the role of the heat equation. Equation (4.12) below shows that our definition agrees with the usual one. The symmetric group S_n acts on the complex Hilbert space $L^2\left([0,1]^n;\mathfrak{k}_{\mathbb{C}}^{\otimes n}\right)$ by $(\sigma\cdot f)\left(x_1,x_2,\ldots,x_n\right):=\sigma f\left(x_{\sigma 1},x_{\sigma 2},\ldots,x_{\sigma n}\right)$, where σf denotes action of S_n on $\mathfrak{k}_{\mathbb{C}}^{\otimes n}$ determined by $\sigma(\xi_1\otimes\cdots\otimes\xi_n)=\xi_{\sigma^{-1}1}\otimes\cdots\otimes\xi_{\sigma^{-1}n}$. In particular, if $f=f_1\otimes\cdots\otimes f_n$ with $f_i\in L^2\left([0,1];\mathfrak{k}_{\mathbb{C}}\right)$, then $\sigma\cdot f=f_{\sigma^{-1}1}\otimes\cdots\otimes f_{\sigma^{-1}n}$. The symmetric subspace, denoted $\mathcal{S}L^2\left([0,1]^n;\mathfrak{k}_{\mathbb{C}}^{\otimes n}\right)$, is the space of those $f\in L^2\left([0,1]^n;\mathfrak{k}_{\mathbb{C}}^{\otimes n}\right)$ for which $\sigma\cdot f=f$ for all $\sigma\in S_n$. A function $f\in\mathcal{S}L^2\left([0,1]^n;\mathfrak{k}_{\mathbb{C}}^{\otimes n}\right)$ is determined by its restriction to Δ_n. The restriction is in $L^2\left(\Delta_n;\mathfrak{k}_{\mathbb{C}}^{\otimes n}\right)$ and every element of $L^2\left(\Delta_n;\mathfrak{k}_{\mathbb{C}}^{\otimes n}\right)$ arises as such a restriction. For a symmetric f, its norm-squared over $[0,1]^n$ is $n!$ times its norm-squared over Δ_n. Finally, note that given functions $H_{k_1,\cdots,k_n}\in L^2\left(\Delta_n\right)$ there is a unique $f\in\mathcal{S}L^2\left([0,1]^n;\mathfrak{k}_{\mathbb{C}}^{\otimes n}\right)$ such that

$$\left(f\left(\tau_1,\cdots,\tau_n\right),X_{k_1}\otimes\cdots\otimes X_{k_n}\right)=H_{k_1,\cdots,k_n}\left(\tau_1,\cdots,\tau_n\right)\text{ for }\left(\tau_1,\cdots,\tau_n\right)\in\Delta_n,$$

i.e. $f|_{\Delta_n}=\sum H_{k_1,\cdots,k_n}X_{k_1}\otimes\cdots\otimes X_{k_n}$. Here $(\cdot,\cdot)$ refers to the bilinear extension of the inner product from $\mathfrak{k}$ to $\mathfrak{k}_{\mathbb{C}}$.

Definition 4.8. *Let $\mathcal{E}$ denote the subspace of $L^2\left([0,1]^n;\mathfrak{k}_{\mathbb{C}}^{\otimes n}\right)$ consisting of finite linear combinations of functions of the form*

$$f\left(\tau_1,\cdots,\tau_n\right)=f_1\left(\tau_1\right)\otimes\cdots\otimes f_n\left(\tau_n\right)\tag{4.8}$$

*where $f_i\in L^2\left([0,1];\mathfrak{k}_{\mathbb{C}}\right)$ is of finite variation. As above, f will be denoted by $f_1\otimes\cdots\otimes f_n$. .The **multiple Stratonovich integral** is the linear map* $\mathrm{Strat}_{n,s}:\mathcal{E}\to L^2(\bar{\mathcal{A}},\tilde{P}_s)$ *determined by*

$$\mathrm{Strat}_{n,s}\left(f_1\otimes\cdots\otimes f_n\right)=\frac{1}{n!}\prod_{i=1}^{n}\int_0^1\left(f_i\left(\tau\right),da_\tau\right).\tag{4.9}$$

Here $(\cdot,\cdot)$ refers to the bilinear extension of the inner product from $\mathfrak{k}$ to $\mathfrak{k}_{\mathbb{C}}$.

Remark 4.9. Since f_i is assumed to be of finite variation, the integrals in (4.9) make sense for all continuous functions a as Stiltjies integrals. Moreover each $\left(f_i\left(\tau\right),da_\tau\right)$

is a continuous linear function of a. Using these remarks $\mathrm{Strat}_{n,s}(f)$ for $f \in \mathcal{E}$ is completely determined by its values on $\mathcal{A}$ which are:

$$\mathrm{Strat}_{n,s}(f)(A) = \frac{1}{n!}(f, A \otimes \cdots \otimes A)_{L^2\left([0,1]^n;\mathfrak{k}_{\mathbb{C}}^{\otimes n}\right)} \quad \text{for all} \quad A \in \mathcal{A}. \tag{4.10}$$

Thus the right side of (4.10) is a cylinder function, where the ϕ in Definition 4.2 is a homogeneous polynomial of degree n, and every such cylinder function arises in this way.

We will be interested in this map just on the symmetric subspace of $\mathcal{E}$, denoted $\mathcal{E}_{\mathcal{S}}$.

Definition 4.10. *The* ***multiple Itô integral*** *is the bounded linear map $I_{n,s}$ from $\mathcal{S}L^2\left([0,1]^n;\mathfrak{k}_{\mathbb{C}}^{\otimes n}\right)$ to $L^2(\bar{\mathcal{A}}, \tilde{P}_s)$ which is determined uniquely by*

$$I_{n,s}(f) = e^{-s\Delta_{\mathcal{A}}/2}\mathrm{Strat}_{n,s}(f) \text{ for all } f \in \mathcal{E}_{\mathcal{S}}. \tag{4.11}$$

The fact that there exists such a bounded linear operator satisfying this equation is a consequence of Lemma 4.11 below.

As an example, for $i = 1, 2, , \ldots, n$, let $f_i = X_{k_i} 1_{[l_i, m_i]} \in L^2([0,1];\mathfrak{k}_{\mathbb{C}})$ where $l_1, \cdots, l_n$ and $m_1, \cdots, m_n$ are real numbers such that $0 \leq l_i < m_i \leq l_{i+1} \leq 1$. Because of the conditions on the l's and m's, $f := \sum_{\sigma \in S_n} f_{\sigma 1} \otimes \cdots \otimes f_{\sigma n}$ is the unique functions in $\mathcal{E}_{\mathcal{S}}$ such that

$$f|_{\Delta_n}(\tau_1, \cdots, \tau_n) = \left(X_{k_1} 1_{[l_1, m_1]}(\tau_1)\right) \otimes \cdots \otimes \left(X_{k_n} 1_{[l_n, m_n]}(\tau_n)\right).$$

By Definition 4.8,

$$\mathrm{Strat}_{n,s}(f) = \prod_{i=1}^{n} \int_0^1 (f_i(\tau), da_\tau) = \prod_{i=1}^{n} \left(a_{m_i}^{k_i} - a_{l_i}^{k_i}\right)$$

Again the assumptions on the l's and m's imply that $\{f_1, f_2, \ldots, f_n\} \subset L^2([0,1];\mathfrak{k}_{\mathbb{C}})$ is an orthonormal set and hence that $\Delta_{\mathcal{A}}\mathrm{Strat}_{n,s}(f) = 0$. Therefore the multiple Wiener integral coincides with the multiple Stratonovich integral, i.e.,

$$I_{n,s}(f) = \prod_{i=1}^{n} \left(a_{m_i}^{k_i} - a_{l_i}^{k_i}\right). \tag{4.12}$$

This expression agrees with any other reasonable definition of the multiple Wiener integral. For more on multiple Stratonovich integrals and there relationship to multiple Itô integrals, see [HM, JK] and the references therein.

Lemma 4.11. *For $f \in \mathcal{E}_{\mathcal{S}}$,*

$$\|I_{n,s}(f)\|^2_{L^2(\bar{\mathcal{A}}, \tilde{P}_s)} = \frac{s^n}{n!}\|f\|^2_{L^2\left([0,1]^n;\mathfrak{k}_{\mathbb{C}}^{\otimes n}\right)} = s^n \|f\|^2_{L^2\left(\Delta_n;\mathfrak{k}_{\mathbb{C}}^{\otimes n}\right)}.$$

Proof of Lemma 4.11. Choose an orthonormal set $\{e_1, \cdots, e_d\} \subset \mathcal{A}$ with each e_i of finite variation and such that

$$f \in \mathrm{span}\{e_{i_1} \otimes \cdots \otimes e_{i_n}\}_{i_j=1}^{d}. \tag{4.13}$$

Then there is a homogeneous polynomial p of degree n on $\mathbb{R}^d$ such that $\mathrm{Strat}_{n,s}(f)(A) = p(\langle e_1, A\rangle, \ldots, \langle e_d, A\rangle)$. Since the distribution (under $\tilde{P}_s$) of $(\langle e_1, A\rangle, \ldots, \langle e_d, A\rangle)$ is the Gaussian measure $e^{s\Delta_{\mathbb{R}^d}/2}\delta_0$, (3.7) with $L = s\Delta_{\mathbb{R}^d}$ implies that

$$\|I_{n,s}(f)\|^2_{L^2(\bar{\mathcal{A}},\tilde{P}_s)} = \frac{s^n}{n!} \sum_{i_1,i_2,\ldots i_n=1}^{d} |(\partial_{i_1}\partial_{i_2}\cdots\partial_{i_n}p)(0)|^2 \,, . \tag{4.14}$$

where $\partial_i = \partial/\partial x_i$ and the factor s^n results from the fact that $L = s\Delta_{\mathbb{R}^d}$ rather than $\Delta_{\mathbb{R}^d}$. By (4.10) of Remark 4.9,

$$p(\langle e_1, A\rangle, \ldots, \langle e_d, A\rangle) = \frac{1}{n!}\langle f, A\otimes\cdots\otimes A\rangle_{L^2\left([0,1]^n;\mathfrak{k}_{\mathbb{C}}^{\otimes n}\right)}.$$

This equation and the chain rule gives,

$$\begin{aligned}
(\partial_{i_1}\partial_{i_2}\cdots\partial_{i_n}p)(0) &= \frac{1}{n!}\partial_{e_{i_1}}\partial_{e_{i_2}}\cdots\partial_{e_{i_n}}\langle f, A\otimes\cdots\otimes A\rangle_{L^2\left([0,1]^n;\mathfrak{k}_{\mathbb{C}}^{\otimes n}\right)}|_{A=0} \\
&= \frac{1}{n!}\sum_{\sigma\in S_n}\langle f, e_{i_{\sigma 1}}\otimes\cdots\otimes e_{i_{\sigma n}}\rangle_{L^2\left([0,1]^n;\mathfrak{k}_{\mathbb{C}}^{\otimes n}\right)} \\
&= \langle f, e_{i_1}\otimes\cdots\otimes e_{i_n}\rangle_{L^2\left([0,1]^n;\mathfrak{k}_{\mathbb{C}}^{\otimes n}\right)},
\end{aligned} \tag{4.15}$$

where in the last equality we have used $f\in\mathcal{E}_{\mathcal{S}}$. Combining (4.13)–(4.15) proves the Lemma. □

We may now define the integrals that appear in Proposition 4.6.

Definition 4.12. *For $f \in L^2(\Delta_n)$, let*

$$\int_{\Delta_n} f\, da_{\tau_1}^{k_1}\cdots da_{\tau_n}^{k_n} = I_{n,s}(g),$$

where g is the symmetric extension to $[0,1]^n$ of the function $f\cdot X_{k_1}\otimes\cdots\otimes X_{k_n}$ in $L^2\left(\Delta_n;\mathfrak{k}_{\mathbb{C}}^{\otimes n}\right)$.

Proof of Proposition 4.6. Identifying $\mathcal{S}L^2\left([0,1]^n;\mathfrak{k}_{\mathbb{C}}^{\otimes n}\right)$ with $L^2\left(\Delta_n;\mathfrak{k}_{\mathbb{C}}^{\otimes n}\right)$ by restriction, the Itô integral is an isometric map of $L^2\left(\Delta_n;\mathfrak{k}_{\mathbb{C}}^{\otimes n}\right)$ into $L^2(\bar{\mathcal{A}},\tilde{P}_s)$. From the proof of Lemma 4.11, the image of $L^2\left(\Delta_n;\mathfrak{k}_{\mathbb{C}}^{\otimes n}\right)$ in $L^2(\bar{\mathcal{A}},\tilde{P}_s)$ is precisely $\mathcal{F}_{n,s}(\bar{\mathcal{A}})$ in Definition 4.5. This gives the characterization of $\mathcal{F}_{n,s}(\bar{\mathcal{A}})$ given in the proposition. That the spaces $\mathcal{F}_{n,s}(\bar{\mathcal{A}})$ are orthogonal and that their sum is all of $L^2(\bar{\mathcal{A}},\tilde{P}_s)$ follow from the corresponding results (Proposition 3.5) for the spaces $\mathcal{F}_{n,s}(\mathbb{R}^d)$ in $L^2(\mathbb{R}^d, P_s)$ and the density of cylinder functions.

We now turn to the complex case. As in the real case we think of the integrand as an element of $L^2\left(\Delta_n;\mathfrak{k}_{\mathbb{C}}^{\otimes n}\right)$, which we then identify with the symmetric subspace of $L^2\left([0,1]^n;\mathfrak{k}_{\mathbb{C}}^{\otimes n}\right)$. Continuing the notation of Definition 4.4, let $\widetilde{A}_{s,t}$ be the unique operator on cylinder functions such that $\widetilde{A}_{s,t}f(A) = (A_{s,t}\phi)(\langle e_1, A\rangle, \cdots, \langle e_d, A\rangle)$. Define the complex Stratonovich integral by analogy to (4.9) to be

$$\mathrm{Strat}_{n,s,t}(f_1\otimes\cdots\otimes f_n) = \frac{1}{n!}\prod_{i=1}^{n}\int_0^1 (f_i(\tau), dc_\tau) \tag{4.16}$$

and the complex Itô integral to be

$$I_{n,s,t}(f) = e^{-\widetilde{A}_{s,t}/2}\mathrm{Strat}_{n,s,t}(f). \tag{4.17}$$

The analog of Lemma 4.11 in the complex case is:

$$\|I_{n,s,t}(f)\|^2_{L^2(\bar{\mathcal{A}}_{\mathbb{C}},\tilde{M}_{s,t})} = \frac{s^n}{n!}\|f\|^2_{L^2\left([0,1]^n;\mathfrak{k}_{\mathbb{C}}^{\otimes n}\right)} = s^n\|f\|^2_{L^2(\Delta_n;\mathfrak{k}_{\mathbb{C}}^{\otimes n})}.$$

The proof is the same provided that (3.12) is used in place of (3.7). Alternatively, this equation is a consequence of Theorem 3.8 and the isometricity of $S_{s,t}$ in Theorem 3.2. The rest of the proof is the same as the real case. □

Proof of Theorem 4.7. Identify the integrand as above with an element of $\mathcal{S}L^2\left([0,1]^n;\mathfrak{k}_{\mathbb{C}}^{\otimes n}\right)$. By the continuity of the transform and the integrals is suffices to prove the result on the dense subspace $\mathcal{E}_{\mathcal{S}}$. But comparing (4.9) and (4.11) to (4.16) and (4.17) and using Theorem 3.8 gives the result on $\mathcal{E}_{\mathcal{S}}$. □

Remark 4.13. In Section 5 we will need to know that, at least in certain cases, the *multiple* Wiener integral coincides with the *iterated* Itô integral. It is easily seen that the two coincide for nice integrands, as in (4.12). Moreover, using repeatedly the isometry property of the one-dimensional Itô integral shows that the iterated Itô integral has the same isometry property (Lemma 4.11) as the multiple Wiener integral. It follows that the multiple Wiener and iterated Itô integrals coincide, *provided* that the iterated Itô integral makes sense. For us it is enough to have this for integrands which are constant on a set of the form $0 \le u \le \tau_1 \le \cdots \le \tau_n \le v$ and zero elsewhere (Lemmas 5.7 and 5.8), in which case there is no difficulty.

5. Functions of the holonomy

5.1. Statements. Recall that $a_\tau(A) := \int_0^\tau A_\sigma\,d\sigma$ and $c_\tau(A) := \int_0^\tau C_\sigma\,d\sigma$ are Brownian motions with values in $\mathfrak{k}$ and $\mathfrak{k}_{\mathbb{C}}$ respectively.

Definition 5.1 (Itô Maps). *Let θ_τ and $\theta_\tau^{\mathbb{C}}$ denote the solutions to the Stratonovich stochastic differential equations*

$$d\theta_\tau = \theta_\tau \circ da_\tau \quad \text{with } \theta_0 = e \in K, \tag{5.1}$$

$$d\theta_\tau^{\mathbb{C}} = \theta_\tau^{\mathbb{C}} \circ dc_\tau \quad \text{with } \theta_0^{\mathbb{C}} = e \in K_{\mathbb{C}}. \tag{5.2}$$

We define the holonomies $h(A)$ and $h_{\mathbb{C}}(C)$ by

$$h(A) = \theta_1(A)$$
$$h_{\mathbb{C}}(C) = \theta_1^{\mathbb{C}}(C).$$

Notice that θ and h are defined on $(\bar{\mathcal{A}}, \tilde{P}_s)$, and that $\theta^{\mathbb{C}}$ and $h_{\mathbb{C}}$ are defined on $(\bar{\mathcal{A}}_{\mathbb{C}}, \tilde{M}_{s,t})$.

Recall that $\theta_\tau(A)$ and $\theta_\tau^{\mathbb{C}}(C)$ are to be interpreted as the parallel transport from 0 to τ of the generalized connections A and C, respectively. The meaning of the stochastic differential equations (5.1) and (5.2) is described in detail in Section 5.3. We are now ready to state our main result.

Theorem 5.2. *Fix s and t with $s > t/2 > 0$. Suppose $f \in L^2(\bar{\mathcal{A}}, \tilde{P}_s)$ is of the form*

$$f(A) = \phi(h(A)),$$

where ϕ is a function on K. Then there exists a unique holomorphic function Φ on $K_{\mathbb{C}}$ such that

$$\tilde{S}_{s,t} f(C) = \Phi(h_{\mathbb{C}}(C)).$$

The function Φ is determined by the condition that

$$\Phi|_K = e^{t\Delta_K/2}\phi.$$

Here Δ_K is the Laplace-Beltrami operator on K associated to the bi-invariant Riemannian metric that agrees at the identity with the chosen inner product on $\mathfrak{k}$. The meaning of $e^{t\Delta_K/2}$ is discussed following Theorem 5.3.

Observe that the space of functions $f \in L^2(\bar{\mathcal{A}}, \tilde{P}_s)$ of the form $\phi \circ h$ may be identified with $L^2(K, \rho_s)$, where ρ_s is the distribution of h with respect to $\tilde{P}_s$. It is known that ρ_s coincides with the heat kernel measure on K; thus the Hilbert space $L^2(K, \rho_s)$ coincides with the one considered in [H1]. Similarly, the space of functions $F \in L^2(\bar{\mathcal{A}}_{\mathbb{C}}, \tilde{M}_{s,t})$ of the form $F = \Phi \circ \theta_{\mathbb{C}}$, with Φ a not-necessarily-holomorphic function on $K_{\mathbb{C}}$, may be identified with $L^2(K_{\mathbb{C}}, \mu_{s,t})$, where $\mu_{s,t}$ is the distribution of $h_{\mathbb{C}}$ with respect to $\tilde{M}_{s,t}$, which is a certain heat kernel measure on $K_{\mathbb{C}}$. So if we apply the isometric transform $\tilde{S}_{s,t}$ to functions of the form $\phi \circ h$, then we obtain an isometric map of $L^2(K, \rho_s)$ into the holomorphic subspace of $L^2(K_{\mathbb{C}}, \mu_{s,t})$. The proof of surjectivity in Theorem 2 of [H1] applies essentially without change to show that this map is onto the holomorphic subspace. The following theorem summarizes these observations.

Theorem 5.3. *For all s and t with $s > t/2 > 0$, the map*

$$\phi \rightarrow \text{ analytic continuation of } e^{t\Delta_K/2}\phi$$

is an isometric isomorphism of $L^2(K, \rho_s)$ onto the space of holomorphic functions in $L^2(K_{\mathbb{C}}, \mu_{s,t})$.

This result was proved in [H1] for the case $s = t$. Part of the theorem is that $e^{t\Delta_K/2}\phi$ always has a unique analytic continuation to $K_{\mathbb{C}}$. Note that the measure ρ_s has a density with respect to Haar measure which is strictly positive and continuous, and therefore bounded and bounded away from zero by compactness. This means that $L^2(K, \rho_s)$ is the same space of functions as $L^2(K, \text{Haar})$. So $e^{t\Delta_K/2}$ is to be interpreted as the standard contraction semigroup on $L^2(K, \text{Haar})$.

If we restrict the transform $\tilde{S}_{s,t}$ to functions of the holonomy, then it makes sense to allow s to tend to infinity. See Section 2 for a discussion of why this limit is natural from the point of view of Yang-Mills theory.

Theorem 5.4. *Normalize Haar measure dx on the compact group K to have mass one. Then $f \in L^2(K, \rho_s)$ if and only if $f \in L^2(K, dx)$, and*

$$\|f\|_{L^2(K,dx)} = \lim_{s\rightarrow\infty} \|f\|_{L^2(K,\rho_s)}.$$

The measure

$$d\nu_{s,t}(g) = \int_K d\mu_{s,t}(gx)\, dx, \quad g \in K_{\mathbb{C}}$$

is independent of s and will be denoted $\nu_t(g)$. For all $F \in \mathcal{H}(K_{\mathbb{C}})$, $F \in L^2(K_{\mathbb{C}}, \mu_{s,t})$ if and only if $F \in L^2(K_{\mathbb{C}}, \nu_t)$, and

$$\|F\|_{L^2(K_{\mathbb{C}},\nu_t)} = \lim_{s\rightarrow\infty} \|F\|_{L^2(K_{\mathbb{C}},\mu_{s,t})}.$$

Thus the map

$$f \rightarrow \text{ analytic continuation } \left(e^{t\Delta_K/2} f\right)$$

is an isometric isomorphism of $L^2(K, dx)$ onto $\mathcal{H}L^2(K_{\mathbb{C}}, \nu_t)$.

The last isometric isomorphism, with domain $L^2(K, dx)$, was obtained in [H1, Thm. 2], and was denoted C_t.

Recall that the transform $\tilde{S}_{s,t}$ maps into the holomorphic subspace of $L^2(\bar{\mathcal{A}}_{\mathbb{C}}, \tilde{M}_{s,t})$. Theorem 5.2 together with the "onto" part of Theorem 5.3 gives the following.

Theorem 5.5. *Suppose Φ is a holomorphic function on $K_{\mathbb{C}}$ such that $\Phi \circ h_{\mathbb{C}}$ is in $L^2(\bar{\mathcal{A}}_{\mathbb{C}}, \tilde{M}_{s,t})$. Then $\Phi \circ h_{\mathbb{C}}$ is in $\mathcal{H}L^2(\bar{\mathcal{A}}_{\mathbb{C}}, \tilde{M}_{s,t})$.*

It would be desirable to have a direct proof of this result. See the discussion in Section 2.5 of [HS], which contains the $s = t$ case of Theorem 5.5. (Note that there is a gap in one of the two proofs of this result in [HS]. The paper [DHu] will close this gap. See the discussion at the end of Section 7.) A more general version of Theorem 5.5 is given in Theorem 6.3 of Section 6.

5.2. **Heuristics.** We now give a simple heuristic argument for Theorem 5.2 based on the following proposition. The proof will be given in the next subsection. We are grateful to Ambar Sengupta for showing us the significance of Proposition 5.6. See also Appendix A for another "explanation" of Theorem 5.2.

Proposition 5.6. *If $A \in \bar{\mathcal{A}}$ is distributed according to the measure $\tilde{P}_s$ and B is a fixed element of $\mathcal{A}$, then $\theta(A + B)$ has the same distribution as $\theta(A)\theta(B)$.*

Proof. Direct calculation shows that for A and B smooth,

$$\theta_\tau(A)\,\theta_\tau(B) = \theta\Big(\mathrm{Ad}\theta_\tau(B)^{-1}(A_\tau) + B_\tau\Big).$$

Standard stochastic techniques show that this remains true almost surely if A is a white noise and B is in $\mathcal{A}$. But the white noise measure $\tilde{P}_s$ is invariant under the pointwise adjoint action, which is just a "rotation" of $\mathcal{A}$. ☐

Using Proposition 5.6, we may formally calculate $e^{t\Delta_{\mathcal{A}}/2}f$. The measure $\tilde{P}_t$ is the fundamental solution at the origin of the heat equation, which means that $e^{t\Delta_{\mathcal{A}}/2}$ should be given by convolution with $\tilde{P}_t$. Similarly, ρ_t is the fundamental solution at the identity of the heat equation on K. So if $f(A) = \phi(\theta_1(A))$ then

$$\begin{aligned} e^{t\Delta_{\mathcal{A}}/2}f(B) &= \int_{\bar{\mathcal{A}}} \phi(\theta_1(A+B))\, d\tilde{P}_t(A) \\ &= \int_{\bar{\mathcal{A}}} \phi(\theta_1(A)\,\theta_1(B))\, d\tilde{P}_t(A) \\ &= \int_K \phi(x\theta_1(B))\, d\rho_t(x) \\ &= e^{t\Delta_K/2}\phi(\theta_1(B)). \end{aligned}$$

We have used the proposition between the first and second lines. Assuming this is valid for $B \in \bar{\mathcal{A}}$ and then analytically continuing formally to $\bar{\mathcal{A}}_{\mathbb{C}}$ we obtain Theorem 5.2.

Some rigorous variant of this argument is used in [GM, Sa2, HS, AHS], all of which consider only the $s = t$ case. In that case it is possible to work with holomorphic functions on $\mathcal{A}_{\mathbb{C}}$ instead of $\bar{\mathcal{A}}_{\mathbb{C}}$, so that the above argument is essentially rigorous. The results of [GM] are stated only on $\mathcal{A}_{\mathbb{C}}$, while the other papers work first on $\mathcal{A}_{\mathbb{C}}$ and then extend to $\bar{\mathcal{A}}_{\mathbb{C}}$. However, this approach does not work when $s \neq t$, since the pointwise bounds needed to obtain (everywhere-defined) holomorphic functions on $\mathcal{A}_{\mathbb{C}}$ do not hold when $s \neq t$.

5.3. **Proofs.** Let us explain more precisely Definition 5.1 of the Itô maps. A continuous K–valued semi-martingale θ_τ satisfies (5.1) if and only if for all $f \in C^\infty([0,1]\times K)$,

$$f(\tau,\theta_\tau) = f(0,e) + \int_0^\tau \frac{\partial f}{\partial \sigma}(\sigma,\theta_\sigma)\, d\sigma + \sum_{k=1}^{\dim \mathfrak{k}} \int_0^\tau X_k f(\sigma,\theta_\sigma) \circ da_\sigma^k, \tag{5.3}$$

where $a_\tau = \sum_{k=1}^{\dim \mathfrak{k}} a_\tau^k X_k$. Here $\{X_k\}_{k=1}^{\dim \mathfrak{k}}$ is an orthonormal basis for $\mathfrak{k}$, with each X_k viewed as a left-invariant vector field on K.

Noting that $\{a_\cdot^k\}_{k=1}^{\dim \mathfrak{k}}$ are independent Brownian motions with variance s, (5.3) may be written in Itô form as

$$\begin{aligned} f(\tau,\theta_\tau) &= f(0,e) + \int_0^\tau \left(\frac{\partial f}{\partial \sigma}(\sigma,\theta_\sigma) + \frac{s}{2}\Delta_K f(\sigma,\theta_\sigma)\right) d\sigma \\ &\quad + \sum_{k=1}^{\dim \mathfrak{k}} \int_0^\tau X_k f(\sigma,\theta_\sigma)\, da_\sigma^k. \end{aligned} \tag{5.4}$$

where $\Delta_K = \sum X_k^2$.

Write $c_\tau = \sum_{k=1}^{\dim \mathfrak{k}} a_\tau^k X_k + \sum_{k=1}^{\dim \mathfrak{k}} b_\tau^k Y_k$, where $Y_k = JX_k$. **Warning:** we are using the same letter a for both the process in $\mathfrak{k}$ and the real part of the process in $\mathfrak{k}_{\mathbb{C}}$, which do not even have the same distribution. The context should make it clear whether we are in the real or the complex setting. A continuous $K_{\mathbb{C}}$–valued semi-martingale $\theta_\tau^{\mathbb{C}}$ is said to solve (5.2) provided that for all $u \in C^\infty([0,1]\times K_{\mathbb{C}})$,

$$\begin{aligned} u(\tau,\theta_\tau^{\mathbb{C}}) &= u(0,e) + \int_0^\tau \frac{\partial u}{\partial \sigma}(\sigma,\theta_\sigma^{\mathbb{C}})\, d\sigma \\ &\quad + \sum_{k=1}^{\dim \mathfrak{k}} \int_0^\tau X_k u(\sigma,\theta_\sigma^{\mathbb{C}}) \circ da_\sigma^k + \sum_{k=1}^{\dim \mathfrak{k}} \int_0^\tau Y_k u(\sigma,\theta_\sigma^{\mathbb{C}}) \circ db_\sigma^k. \end{aligned} \tag{5.5}$$

Noting that $\{a_\cdot^k\}_{k=1}^{\dim \mathfrak{k}}$ and $\{b_\cdot^k\}_{k=1}^{\dim \mathfrak{k}}$ are independent Brownian motions with variances $(s-t/2)$ and $t/2$, respectively, (5.5) may be written in Itô form as

$$\begin{aligned} u\left(\tau,\theta_\tau^{\mathbb{C}}\right) &= u(0,e) + \int_0^\tau \left(\frac{\partial u}{\partial \sigma}\left(\sigma,\theta_\sigma^{\mathbb{C}}\right) + \frac{1}{2}A_{s,t}^{K_{\mathbb{C}}} u\left(\sigma,\theta_\sigma^{\mathbb{C}}\right)\right) d\sigma \\ &\quad + \sum_{k=1}^{\dim \mathfrak{k}} \int_0^\tau X_k u\left(\theta_\sigma^{\mathbb{C}}\right) da_\sigma^k + \sum_{k=1}^{\dim \mathfrak{k}} \int_0^\tau Y_k u\left(\theta_\sigma^{\mathbb{C}}\right) db_\sigma^k. \end{aligned} \tag{5.6}$$

where $A_{s,t}^{K_{\mathbb{C}}}$ is defined, by analogy to $A_{s,t}$, to be $(s-t/2)\,\Sigma X_k^2 + \frac{t}{2}\Sigma Y_k^2$. For existence and uniqueness of solutions to (5.5) and (5.6) see, for example, Elworthy [El], Emery [Em], Ikeda and Watanabe [IW], or Kunita [Ku, Theorem 4.8.7].

We now begin working toward the proof of Theorem 5.2. For use in Section 6, we will actually compute $\tilde{S}_{s,t}$ on functions of the form $\phi(\theta_u^{-1}\theta_v)$, for two fixed times u and v. The transformed function is then $\Phi\left(\left(\theta_u^{\mathbb{C}}\right)^{-1}\theta_v^{\mathbb{C}}\right)$, where Φ is holomorphic on $K_{\mathbb{C}}$ and where $\Phi|_K = e^{(v-u)t\Delta_K/2}\phi$. Theorem 5.2 is the special case $u=0$, $v=1$.

The following Lemma is essentially a special case of the results of Veretennikov and Krylov[VK].

Lemma 5.7. *Suppose that $0 \le u < v \le 1$, and ϕ is a measurable function on K such that $\phi(\theta_u^{-1}\theta_v) \in L^2(\bar{\mathcal{A}}, \tilde{P}_s)$. Then the Wiener Chaos expansion of $\phi(\theta_u^{-1}\theta_v)$ is*

$$\phi(\theta_u^{-1}\theta_v) = \sum_{n=0}^{\infty} \sum_{k_1,\cdots,k_n=1}^{\dim \mathfrak{k}} \int_{\Delta_n(u,v)} \alpha_{k_1,\cdots,k_n}\, da_{\tau_1}^{k_1} \cdots da_{\tau_n}^{k_n}, \tag{5.7}$$

where

$$\alpha_{k_1,\cdots,k_n} = \left(X_{k_1} \cdots X_{k_n} e^{(v-u)s\Delta_K/2} \phi \right)(e) \tag{5.8}$$

and $\Delta_n(u,v) = \{(\tau_1, \ldots, \tau_n)\, | u \le \tau_1 \le \tau_2 \le \cdots \le \tau_n \le v\}$.

Proof. To simplify notation, let $\xi_\tau = \theta_u^{-1}\theta_\tau$ for $u \le \tau \le 1$. Let us first assume that $\phi(x) = \langle h, \pi(x)w \rangle$, where $\pi : K \to End(W)$ is a finite-dimensional representation of K, $h \in W^*$ and $w \in W$. Set

$$f(\tau, x) = \left(e^{(v-\tau)s\Delta_K/2} \phi \right)(x) = \left\langle h, \pi(x) e^{(v-\tau)s\pi(\Delta_K)/2} w \right\rangle, \tag{5.9}$$

where $\pi(\Delta_K) = \sum_{k=1}^{\dim \mathfrak{k}} \pi(X_k)^2$ and by abuse of notation we are writing $\pi(X)$ for $\frac{d}{dt}|_0 \pi(e^{tX})$ when $X \in \mathfrak{k}$. The second equality in (5.9) follows from uniqueness of solutions to the heat equation. (See also [H1]). Notice that f solves the backward heat equation $\partial f(\tau, x)/\partial\tau + s\Delta f(\tau, x)/2 = 0$ with $f(v, x) = \phi(x)$. Therefore by the analog of (5.4) for ξ_τ,

$$f(\tau, \xi_\tau) = f(u, e) + \sum_{k=1}^{\dim \mathfrak{k}} \int_u^\tau X_k f(\sigma, \xi_\sigma)\, da_\sigma^k$$

Since Δ and X_k commute, it follows that $X_k f$ also satisfies the backward heat equation. Hence, the previous equation applies with f replaced by $X_k f$, namely

$$X_k f(\tau, \xi_\tau) = X_k f(u, e) + \sum_{l=1}^{\dim \mathfrak{k}} \int_u^\tau X_l X_k f(r, \xi_r)\, da_r^l.$$

Combining the two previous equations gives

$$f(\tau, \xi_\tau) = f(u, e) + \sum_{k=1}^{\dim \mathfrak{k}} \int_u^\tau X_k f(u, e)\, da_\sigma^k + \sum_{k,l=1}^{\dim \mathfrak{k}} \int_u^\tau \left(\int_u^\sigma X_l X_k f(r, \xi_r)\, da_r^l \right) da_\sigma^k.$$

Iterating this procedure gives (Remark 4.13)

$$\begin{aligned} \phi(\theta_u^{-1}\theta_v) &= \phi(\xi_v) = f(v, \xi_v) \\ &= \sum_{n=0}^{N} \sum_{k_1,\cdots,k_n=1}^{\dim \mathfrak{k}} \int_{\Delta_n(u,v)} \alpha_{k_1,\cdots,k_n}\, da_{\tau_1}^{k_1} \cdots da_{\tau_n}^{k_n} + R_N(u,v), \end{aligned}$$

where $\alpha_{k_1,\cdots,k_n}$ is defined by (5.8) and where

$$R_N(u,v) = \sum_{k_1,\cdots,k_{N+1}=1}^{\dim \mathfrak{k}} \int_u^v \int_u^{\tau_{N+1}} \cdots \int_u^{\tau_2} X_{k_1} \cdots X_{k_n} f(\tau_1, \xi_{\tau_1})\, da_{\tau_1} \cdots da_{\tau_{N+1}}.$$

Using the isometry property of iterated Itô integral together with the assumption that ϕ is a matrix entry of a finite-dimensional representation one shows that

$$\|R_N(u,v)\|^2_{L^2(\bar{\mathcal{A}},\tilde{P}_s)} \leq \frac{C^{N+1}}{(N+1)!} \to 0 \text{ as } N \to \infty.$$

Therefore, (5.7) holds when $\phi(x)$ is a linear combination of matrix elements of finite-dimensional representations of K.

We now consider general ϕ. Note that the distribution of $\theta_u^{-1}\theta_v$ is the same as that of θ_{v-u}, namely, the heat kernel measure $\rho_{(v-u)s}$. This measure has a smooth strictly positive density with respect to Haar measure on K, which by compactness is bounded and bounded away from zero. Thus by the Peter-Weyl theorem, there exist functions ϕ_n which are finite linear combinations of matrix entries such that $\phi_n \to \phi$ in $L^2\left(K, \rho_{(u-v)s}\right)$ and thus $\phi_n(\theta_u^{-1}\theta_v) \to \phi(\theta_u^{-1}\theta_v)$ in $L^2(\bar{\mathcal{A}}, \tilde{P}_s)$. The smoothness of the heat kernel shows that the map $\phi \to \left(X_{k_1}\cdots X_{k_n} e^{(v-\tau)s\Delta_K/2}\phi\right)(e)$ is a continuous linear functional on $L^2\left(K, \rho_{(v-u)s}\right)$. So passing to the limit gives the lemma in general. □

We have the following holomorphic analog of the previous lemma. Recall that $A_{s,t}^{K_\mathbb{C}} = (s-t/2)\,\Sigma X_k^2 + \frac{t}{2}\Sigma Y_k^2$, where $Y_k = JX_k$.

Lemma 5.8. *Suppose that Φ is a holomorphic function on $K_\mathbb{C}$ which is a finite linear combination of matrix entries. Then the holomorphic Wiener chaos expansion of $\Phi\left(\left(\theta_u^\mathbb{C}\right)^{-1}\theta_v^\mathbb{C}\right)$ is*

$$\Phi\left(\left(\theta_u^\mathbb{C}\right)^{-1}\theta_v^\mathbb{C}\right) = \sum_{n=0}^{\infty} \sum_{k_1,\cdots,k_n=1}^{\dim \mathfrak{k}} \int_{\Delta_n(u,v)} \beta_{k_1,\cdots,k_n}\, dc_{\tau_1}^{k_1}\cdots dc_{\tau_n}^{k_n}, \tag{5.10}$$

where

$$\beta_{k_1,\cdots,k_n} = \left(X_{k_1}\cdots X_{k_n} e^{(v-u)A_{s,t}^{K_\mathbb{C}}/2}\Phi\right)(e) \tag{5.11}$$

and $\Delta_n(u,v) = \{(\tau_1,\ldots,\tau_n)\,|u \leq \tau_1 \leq \tau_2 \leq \cdots \leq \tau_n \leq v\}$.

Proof. The argument is very similar to the preceding one. If $\Phi(g) = \langle h, \pi(g)w\rangle$, where π is a finite-dimensional holomorphic representation of $K_\mathbb{C}$, then we set

$$u(\tau,g) = e^{(v-\tau)A_{s,t}^{K_\mathbb{C}}/2}\Phi(g) = \left\langle h, \pi(g)\, e^{(v-\tau)\pi\left(A_{s,t}^{K_\mathbb{C}}\right)}w\right\rangle.$$

Again it may be verified that the second and third expressions are equal (see for example [H1]), with the second interpreted as convolution against the relevant heat kernel. Then $u(\tau,g)$ is holomorphic in g for each τ. Thus $Y_k u = iX_k u$, and the last two terms in (5.6) combine into one term involving integration against $da_\tau + idb_t = dc_\tau$. Iteration then proceeds as in the real case. The remainder estimate is similar as well after using the standard fact that the L^2 norm of $\pi\left(\left(\theta_u^\mathbb{C}\right)^{-1}\theta_v^\mathbb{C}\right)$ is bounded uniformly for $0 \leq u \leq v \leq 1$. The lemma in fact holds for all holomorphic Φ for which $\Phi\left(\left(\theta_u^\mathbb{C}\right)^{-1}\theta_v^\mathbb{C}\right)$ is square-integrable, but we will not require this. □

Proof of Theorem 5.2. Let $\phi \in L^2(K)$ and $f = \phi(\theta_u^{-1}\theta_v) \in L^2(\bar{\mathcal{A}}, \tilde{P}_s)$. We will show that there exists a unique holomorphic function Φ such that $\tilde{S}_{s,t}f(C) = \Phi\left(\left(\theta_u^\mathbb{C}\right)^{-1}\theta_v^\mathbb{C}\right)$ and such that $\Phi|_K = e^{(v-u)t\Delta_K/2}\phi$. Theorem 5.2 is the special case, $u=0$, $v=1$.

By standard density arguments it suffices to prove the theorem in the case where $\phi(x) = \langle h, \pi(x)w \rangle$ with $\pi : K_{\mathbb{C}} \to End(W)$ being a finite-dimensional holomorphic representation of $K_{\mathbb{C}}$, $h \in W^*$ and $w \in W$. In this case the holomorphic function in the statement of Theorem 5.2 is

$$\Phi(g) = (e^{(v-u)t\Delta_K/2}\phi)(g) = \left\langle h, \pi(g)e^{(v-u)t\pi(\Delta_K)/2}w \right\rangle.$$

By Lemma 5.7 and Theorem 4.7,

$$\tilde{S}_{s,t}(\phi(\theta_u^{-1}\theta_v)) = \sum_{n=0}^{\infty} \sum_{k_1,\cdots,k_n=1}^{\dim \mathfrak{k}} \int_{\Delta_n(u,v)} \alpha_{k_1,\cdots,k_n}\, dc_{\tau_1}^{k_1} \cdots dc_{\tau_n}^{k_n},$$

where $\alpha_{k_1,\cdots,k_n} = \left(X_{k_1} \cdots X_{k_n} e^{(v-u)s\Delta_K/2}\phi \right)(e)$. Lemma 5.8 will now give the theorem, provided that the coefficients $\beta_{k_1,\cdots,k_n}$ in Lemma 5.8 with Φ as above coincide with the coefficients $\alpha_{k_1,\cdots,k_n}$. So we require that

(5.12)
$$\left(X_{k_1} \cdots X_{k_n} e^{(v-u)s\Delta_K/2} \right) \phi(e) = \left(X_{k_1} \cdots X_{k_n} e^{(v-u)A_{s,t}/2} e^{(v-u)t\Delta_K/2}\phi \right)(e).$$

But $\Phi = e^{(v-u)t\Delta_K/2}\phi$ is holomorphic, and $A_{s,t}\Phi = (s-t)\,\Delta_K\Phi$ on holomorphic functions, so (5.12) holds. ∎

Proof of Theorem 5.4. Since we are assuming that K is compact, $\rho_s(x)$ will be bounded and bounded away from zero. Thus the L^2 norm with respect to $\rho_s(x)\,dx$ is finite if and only if the L^2 norm with respect to Haar measure is finite. It is an easy and standard result that $\rho_s(x)$ converges uniformly to the constant function 1, which establishes the first limit in the theorem.

Now, let δ_K denote Haar measure on K, viewed as a measure on $K_{\mathbb{C}}$. Then since $A_{s,t}$ is a left-invariant operator, we have formally

$$\nu_{s,t} = e^{A_{s,t}}(\delta_K).$$

But the two terms in the definition of $A_{s,t}$ commute, so

$$\nu_{s,t} = e^{t/2\sum JX_k^2} e^{(s-t/2)\sum X_k^2}(\delta_K).$$

Since δ_K is K-invariant, the exponential involving $\sum X_k^2$ has no effect, and the s-dependence vanishes.

The equivalence of square-integrability with respect to $\mu_{s,t}$ and ν_t is implied by the "averaging lemma" [H1, Lem. 11]. This is stated in [H1] for the case $s = t$, but the same proof applies in general. Using the commutativity of $\sum X_k^2$ and $\sum (JX_k)^2$,

$$\mu_{s,t} = e^{(s-t)\sum X_k^2} e^{t/2\sum JX_k^2} e^{t/2\sum X_k^2}(\delta_e) \quad \forall \quad s > t.$$

Thus

$$\mu_{s,t}(g) = \int_K \mu_{t,t}\left(gx^{-1}\right)\rho_{s-t}(x)\,dx,$$

from which it follows that $\lim_{s\to\infty} \mu_{s,t}(g) = \nu_t(g)$ for all g. Furthermore, applying the averaging lemma to $\mu_{t,t}$ we see that for all $s > t$, $\mu_{s,t}(g)$ is dominated by a constant (independent of s) times $\nu_t(g)$. So Dominated Convergence gives the second limit in the theorem. The methods of [H1] are sufficient to make all of this rigorous. ∎

6. General functions of the parallel transport

Recall that θ and $\theta^{\mathbb{C}}$ are the Itô maps satisfying the stochastic differential equations (5.1) and (5.2) of Definition 5.1.

Definition 6.1. *Let $W(K)$ denote the group of continuous path x with values in K, with time interval $[0,1]$ and satisfying $x_0 = e$. Define $W(K_{\mathbb{C}})$ similarly. Let $\tilde{\rho}_s$ be the* ***Wiener measure on*** *$W(K)$, that is, the law of the process $\theta_{\cdot}(A)$, where A is distributed as $\tilde{P}_s$. Similarly let $\tilde{\mu}_{s,t}$ be the* ***Wiener measure on*** *$W(K_{\mathbb{C}})$, the law of the process $\theta^{\mathbb{C}}_{\cdot}(C)$, where C is distributed as $\tilde{M}_{s,t}$.*

For each partition $\mathcal{P} = \{0 = \tau_0 < \tau_1 < \cdots < \tau_n = 1\}$ of $[0,1]$, let $\theta_{\mathcal{P}} = (\theta_{\tau_1}, \cdots, \theta_{\tau_n})$, $K^{\mathcal{P}} = K^n$ and $\rho_s^{\mathcal{P}}$ denote the law of $\theta_{\mathcal{P}}$, a probability measure on $K^{\mathcal{P}}$. Define $\theta^{\mathbb{C}}_{\mathcal{P}}$, $K^{\mathcal{P}}_{\mathbb{C}}$ and $\mu^{\mathcal{P}}_{s,t}$ similarly.

As in Theorem 5.3, let $\rho_s$denote the measure $\rho_s^{\mathcal{P}}$ on K, where $\mathcal{P} = \{0,1\}$. This measure has a smooth strictly positive density with respect to Haar measure, which we also call ρ_s. If $\mathbf{x} = (x_1, \cdots, x_n)$ is a typical element in $K^{\mathcal{P}}$, then (as is well known)

$$d\rho_s^{\mathcal{P}}(\mathbf{x}) = \prod_{i=1}^{n} \rho_{s\Delta_i\tau}(x_{i-1}^{-1}x_i)\, dx_i, \tag{6.1}$$

where $\Delta_i\tau = \tau_i - \tau_{i-1}$. As for ρ_s, we will also use $\rho_s^{\mathcal{P}}$ to denote the density on the right side of (6.1).

If $\mathcal{P}$ is a partition and $x \in W(K)$ and $g \in W(K_{\mathbb{C}})$, let $x_{\mathcal{P}} = (x_{\tau_1}, \cdots, x_{\tau_n}) \in K^{\mathcal{P}}$ and $g_{\mathcal{P}} = (g_{\tau_1}, \cdots, g_{\tau_n}) \in K^{\mathcal{P}}_{\mathbb{C}}$.

Definition 6.2. *Let $\mathcal{P} = \{0 = \tau_0 < \tau_1 < \cdots < \tau_n = 1\}$ be a partition of $[0,1]$. A function $f \in L^2(W(K), \tilde{\rho}_s)$ is said to be a* ***cylinder function based on*** *$\mathcal{P}$ if f is the form $f(x) = \phi(x_{\mathcal{P}})$ for some measurable function $\phi : K^{\mathcal{P}} \to \mathbb{C}$.*

Similarly, we say a function $F \in L^2(W(K_{\mathbb{C}}), \tilde{\mu}_{s,t})$ is a ***holomorphic cylinder function based on*** *$\mathcal{P}$ provided that F is of the form $F(g) = \Phi(g_{\mathcal{P}})$, where Φ is a holomorphic function on $K^{\mathcal{P}}_{\mathbb{C}}$.*

The ***holomorphic subspace of*** *$L^2(W(K_{\mathbb{C}}), \tilde{\mu}_{s,t})$, denoted $\mathcal{H}L^2(W(K_{\mathbb{C}}), \tilde{\mu}_{s,t})$, is the L^2 closure of the L^2 holomorphic cylinder functions.*

Theorem 6.3. *There exists a unique isometric isomorphism $\tilde{B}_{s,t} : L^2(W(K), \tilde{\rho}_s) \to \mathcal{H}L^2(W(K_{\mathbb{C}}), \tilde{\mu}_{s,t})$ such that for all partitions $\mathcal{P}$ and all cylinder functions*

$$f(x) = \phi(x_{\mathcal{P}})$$

based on $\mathcal{P}$, $\tilde{B}_{s,t}$ is of the form

$$\tilde{B}_{s,t}f(g) = \Phi(g_{\mathcal{P}})$$

where Φ is holomorphic on $K^{\mathcal{P}}_{\mathbb{C}}$ which is determined uniquely by the condition that

$$\Phi(\mathbf{g}) = \int_{K^{\mathcal{P}}} \rho_t^{\mathcal{P}}\left(\mathbf{g}\mathbf{x}^{-1}\right) \phi(\mathbf{x})\, d\mathbf{x} \tag{6.2}$$

for all $\mathbf{g} \in K^{\mathcal{P}}$. If $F \in \mathcal{H}L^2(W(K_{\mathbb{C}}), \tilde{\mu}_{s,t})$ is a holomorphic cylinder function, then $\tilde{B}_{s,t}^{-1}F$ is a cylinder function based on the same partition.

The following diagram is well defined and commutative, and all maps are one-to-one, onto, and isometric.

$$\begin{array}{ccc} L^2(\bar{\mathcal{A}},\tilde{P}_s) & \overset{\tilde{S}_{s,t}}{\to} & \mathcal{H}L^2(\bar{\mathcal{A}}_{\mathbb{C}},\tilde{M}_{s,t}) \\ \uparrow\theta & & \uparrow\theta^{\mathbb{C}} \\ L^2\left(W\left(K\right),\tilde{\rho}_s\right) & \overset{\tilde{B}_{s,t}}{\to} & \mathcal{H}L^2\left(W\left(K_{\mathbb{C}}\right),\tilde{\mu}_{s,t}\right), \end{array}$$

where θ and $\theta^{\mathbb{C}}$ are being used here to denote the unitary maps, $f\in L^2\left(W\left(K\right),\tilde{\rho}_s\right)\to f\circ\theta\in L^2(\bar{\mathcal{A}},\tilde{P}_s)$ and $F\in L^2\left(W\left(K_{\mathbb{C}}\right),\tilde{\mu}_{s,t}\right)\to F\circ\theta^{\mathbb{C}}\in L^2(\bar{\mathcal{A}}_{\mathbb{C}},\tilde{M}_{s,t})$, respectively.

Note that the vertical arrow on the right side is not obviously well defined, since composition with $\theta^{\mathbb{C}}$ does not take cylinder functions to cylinder functions. Theorems 5.2 and 5.3 are special cases in which the partition is $\mathcal{P}=\{0=\tau_0<\tau_1=1\}$. The $s=t$ case of this theorem is part of Theorem 17 of [HS]. The theorem implies that if $f\in L^2(\bar{\mathcal{A}},\tilde{P}_s)$ is a function of the parallel transport θ at a finite number of times $\tau_1,\cdots,\tau_n$, then $\tilde{S}_{s,t}f$ is a function of the complex parallel transport at the same times $\tau_1,\cdots,\tau_n$.

The heuristic argument for Theorem 5.2 applies just as well when applying $\tilde{S}_{s,t}$ to a function of the form $\phi\left(\theta_{\tau_1}\left(A\right),\cdots\theta_{\tau_n}\left(A\right)\right)$, and so provides a heuristic argument for the commutative diagram in Theorem 6.3. Isometricity of $\tilde{B}_{s,t}$ would then follow from the isometricity of $\tilde{S}_{s,t}$. Appendix A provides another heuristic argument for Theorem 6.3. See especially Example 8.8 and Theorem 8.9. The actual proof of Theorem 6.3 will be by reduction to Theorem 5.2, using the following result. (See also [AHS, Prop. 3.3.1].)

Proposition 6.4 (Factorization Proposition). *For $0\le l<m\le 1$, let $\mathcal{F}_{[l,m]}$ be the σ-algebra in $\bar{\mathcal{A}}$ generated by a_s-a_l, with $s\in[l,m]$. Suppose $l\in(0,1)$, $f\in L^2(\bar{\mathcal{A}},\mathcal{F}_{[0,l]},\tilde{P}_s)$ and $g\in L^2(\bar{\mathcal{A}},\mathcal{F}_{[l,1]},\tilde{P}_s)$. Then $fg\in L^2(\bar{\mathcal{A}},\tilde{P}_s)$ and*

$$\tilde{S}_{s,t}(fg)=\tilde{S}_{s,t}(f)\tilde{S}_{s,t}(g).$$

Proof. First suppose that

$$f\left(A\right)=\phi\left(\left\langle e_1,A\right\rangle,\cdots\left\langle e_d,A\right\rangle\right)\text{ and }g\left(A\right)=\psi\left(\left\langle u_1,A\right\rangle,\cdots\left\langle u_k,A\right\rangle\right)$$

where $\{e_1,\cdots,e_d\}$ and $\{u_1,\cdots,u_k\}$ are orthonormal subsets of $\mathcal{A}$ which are contained in $\mathcal{A}\cap C^\infty([0,c];\mathfrak{k})$ and $\mathcal{A}\cap C^\infty([c,1];\mathfrak{k})$ respectively. Then

$$\begin{aligned} \langle e_i,A\rangle &= -\int_0^c\langle e_i'(\tau),a_\tau(A)\rangle\,d\tau,\\ \langle u_i,A\rangle &= -\int_c^1\langle u_i'(\tau),a_\tau(A)-a_c(A)\rangle\,d\tau. \end{aligned}$$

Approximating the integrals by Riemann sums, one shows that these expressions are $\mathcal{F}_{[0,c]}$- and $\mathcal{F}_{[c,1]}$-measurable, respectively. Therefore f and g are cylinder functions that are $\mathcal{F}_{[0,c]}$- and $\mathcal{F}_{[c,1]}$-measurable, respectively. Since each e_i is orthogonal to each u_j, $\{e_1,\cdots,e_d,u_1,\cdots,u_k\}$ is an orthonormal set and $\mathcal{F}_{[0,c]}$ and $\mathcal{F}_{[c,1]}$ are $\tilde{P}_t$ independent σ-fields. The heat kernel on $\mathbb{R}^{d+k}$ factors, so applying the finite-dimensional transform $S_{s,t}$ (3.1) to the function $\phi\left(x\right)\psi\left(y\right)$ gives $S_{s,t}\left(\phi\right)S_{s,t}\left(\psi\right)$. Hence $\tilde{S}_{s,t}(fg)=\tilde{S}_{s,t}(f)\tilde{S}_{s,t}(g)$.

For general $f\in L^2(\bar{\mathcal{A}}_{\mathbb{C}},\mathcal{F}_{[0,c]},\tilde{P}_s)$ and $g\in L^2(\bar{\mathcal{A}}_{\mathbb{C}},\mathcal{F}_{[c,1]},\tilde{P}_s)$, choose cylinder functions f_n and g_n as above such that $f_n\to f$ in $L^2(\bar{\mathcal{A}}_{\mathbb{C}},\mathcal{F}_{[0,c]},\tilde{P}_s)$ and $g_n\to g$

in $L^2(\bar{\mathcal{A}}_{\mathbb{C}}, \mathcal{F}_{[c,1]}, \tilde{P}_s)$. Because of the independence of $\mathcal{F}_{[0,c]}$ and $\mathcal{F}_{[c,1]}$, $f_n g_n \to fg$ in $L^2(\bar{\mathcal{A}}_{\mathbb{C}}, \mathcal{F}_{[0,1]}, \tilde{P}_s)$. Furthermore, it is easily seen that $\tilde{S}_{s,t}(f_n)$ and $\tilde{S}_{s,t}(g_n)$ are measurable with respect to independent σ-algebras in $\bar{\mathcal{A}}_{\mathbb{C}}$. Thus by the isometry property of $\tilde{S}_{s,t}$,

$$\tilde{S}_{s,t}(fg) = \lim_{n\to\infty} \tilde{S}_{s,t}(f_n g_n) = \lim_{n\to\infty} \tilde{S}_{s,t}(f_n)\tilde{S}_{s,t}(g_n) = \tilde{S}_{s,t}(f)\tilde{S}_{s,t}(g).$$

□

Proof of Theorem 6.3. Our strategy is to use Theorem 5.2 and the Factorization Proposition to compute $\tilde{S}_{s,t}$ on functions of the form $\phi(\theta_{\tau_1}, \cdots, \theta_{\tau_n})$. The result shows that $\tilde{S}_{s,t}(f\circ\theta) = \left(\tilde{B}_{s,t}f\right)\circ\theta_{\mathbb{C}}$ and hence that $\tilde{B}_{s,t}$ is isometric because $\tilde{S}_{s,t}$ is isometric. A surjectivity argument for $\tilde{B}_{s,t}$ then establishes the well-definedness and commutativity of the diagram.

It is easily seen that there is at most one isometric isomorphism having the given action on cylinder functions. We will now prove the existence of $\tilde{B}_{s,t}$ and establish the commutative diagram in the Theorem.

Suppose $f \in L^2(\bar{\mathcal{A}}, \tilde{P}_s)$ is of the form

$$f = \psi_1(\theta_{\tau_1})\,\psi_2\left(\theta_{\tau_1}^{-1}\theta_{\tau_2}\right)\cdots\psi_n\left(\theta_{\tau_{n-1}}^{-1}\theta_{\tau_n}\right)$$

where $\psi_1, \cdots, \psi_n$ are functions on K. Then $\psi_i\left(\theta_{\tau_{i-1}}^{-1}\theta_{\tau_i}\right)$ is $\mathcal{F}_{[\tau_{i-1},\tau_i]}$-measurable. So by the strong form of Theorem 5.2 proved in Section 5.3 and by the Factorization Proposition 6.4 (extended by induction to hold for products of n factors),

$$\tilde{S}_{s,t}f = \Psi_1\left(\theta_{\tau_1}^{\mathbb{C}}\right)\Psi_2\left(\left(\theta_{\tau_1}^{\mathbb{C}}\right)^{-1}\theta_{\tau_2}^{\mathbb{C}}\right)\cdots\Psi_n\left(\left(\theta_{\tau_{n-1}}^{\mathbb{C}}\right)^{-1}\theta_{\tau_n}^{\mathbb{C}}\right), \tag{6.3}$$

where Ψ_i is a holomorphic function on $K_{\mathbb{C}}$ whose restriction to K is $e^{(\tau_i-\tau_{i-1})t\Delta_K/2}\psi_i$.

Now suppose $f \in L^2(\bar{\mathcal{A}}, \tilde{P}_s)$ is any function of the form

$$f = \psi\left(\theta_{\tau_1}, \theta_{\tau_1}^{-1}\theta_{\tau_2}, \cdots, \theta_{\tau_{n-1}}^{-1}\theta_{\tau_n}\right) \tag{6.4}$$

with $\psi \in L^2\left(K^n, \rho_{s\Delta_1\tau}\times\cdots\times\rho_{s\Delta_n\tau}\right)$. Then we claim that

$$\tilde{S}_{s,t}f = \Psi\left(\theta_{\tau_1}^{\mathbb{C}}, \left(\theta_{\tau_1}^{\mathbb{C}}\right)^{-1}\theta_{\tau_2}^{\mathbb{C}}, \cdots, \left(\theta_{\tau_{n-1}}^{\mathbb{C}}\right)^{-1}\theta_{\tau_n}^{\mathbb{C}}\right) \tag{6.5}$$

where Ψ is the unique holomorphic function on $K_{\mathbb{C}}^n$ whose restriction to K^n is given by

(6.6)

$$\Psi(a_1, \cdots, a_n) = \int_K \cdots \int_K \rho_{t\Delta_1\tau}\left(a_1 b_1^{-1}\right)\cdots\rho_{t\Delta_n\tau}\left(a_n b_n^{-1}\right)\psi(b_1, \cdots, b_n)\, db_1\, db_2 \cdots db_n.$$

If ψ is a product function then this assertion is simply (6.3); since linear combinations of product functions are dense in $L^2\left(K^n, \rho_{s\Delta_1\tau}\times\cdots\times\rho_{s\Delta\tau_n}\right)$ the assertion holds in general. (Recall (6.1).)

The equations (6.4)-(6.6) express the action of $\tilde{S}_{s,t}$ on cylinder functions in terms of the "incremental coordinates" $\theta_{\tau_{i-1}}^{-1}\theta_{\tau_i}$. We wish to have in addition a formula for the action of $\tilde{S}_{s,t}$ on cylinder functions in terms of the "direct coordinates" $\theta_{\tau_1}, \cdots, \theta_{\tau_n}$. (See the proof of Theorem 3 in [HS, Sect. 3.2].) So suppose f is any cylinder function based on the partion $\mathcal{P}$:

$$f = \phi(\theta_{\tau_1}, \cdots, \theta_{\tau_n}) \tag{6.7}$$

with $\phi \in L^2\left(K^{\mathcal{P}}, \rho_s^{\mathcal{P}}\right)$. Then we claim that

$$\tilde{S}_{s,t} f = \Phi\left(\theta_{\tau_1}^{\mathbb{C}}, \theta_{\tau_2}^{\mathbb{C}}, \cdots, \theta_{\tau_n}^{\mathbb{C}}\right), \tag{6.8}$$

where Φ is the unique holomorphic function on $K_{\mathbb{C}}^{\mathcal{P}}$ such that

$$\Phi(\mathbf{g}) = \int_{K^{\mathcal{P}}} \rho_t^{\mathcal{P}}\left(\mathbf{g}\mathbf{x}^{-1}\right) \phi(\mathbf{x})\, d\mathbf{x}, \tag{6.9}$$

for all $\mathbf{g} \in K^{\mathcal{P}}$. Explicitly, by (6.1) this means that

(6.10)

$$\Phi\left(g_1, \cdots, g_n\right) = \int_{K^n} \left[\prod_{i=1}^n \rho_{t\Delta_i\tau}\left(\left(g_{i-1} x_{i-1}^{-1}\right)^{-1} g_i x_i^{-1}\right)\right] \phi\left(x_1, \cdots, x_n\right) dx_1 \cdots dx_n.$$

(Here $g_0 = x_0 = e$.)

To verify this, we note that f can be expressed in the form (6.4) with

$$\psi\left(a_1, \cdots, a_n\right) = \phi\left(a_1, a_1 a_2, \cdots, a_1 a_2 \cdots a_n\right),$$

in which case ϕ can be expressed in terms of ψ by

$$\phi\left(x_1, \cdots, x_n\right) = \psi\left(x_1, x_1^{-1} x_2, \cdots, x_{n-1}^{-1} x_n\right).$$

Thus $\tilde{S}_{s,t} f$ is given by (6.5). But then $\tilde{S}_{s,t} f$ can be expressed in the form (6.8), where

$$\Phi\left(g_1, \cdots, g_n\right) = \Psi\left(g_1, g_1^{-1} g_2, \cdots, g_{n-1}^{-1} g_n\right),$$

with Ψ given in (6.6). We need only verify the relationship between Φ and ϕ. Putting together the definitions we get

$$\Phi\left(g_1, \cdots, g_n\right) = \int_{K^n} \left[\prod \rho_{t\Delta_i\tau}\left(g_{i-1}^{-1} g_i b_i^{-1}\right)\right] \psi\left(b_1, \cdots, b_n\right) db_1 \cdots db_n. \tag{6.11}$$

We then make successive changes of variable $\left(x_1, \cdots, x_n\right) = \left(b_1, b_1 b_2, \cdots, b_1 b_2 \cdots b_n\right)$, so that $b_i = x_{i-1}^{-1} x_i$. Since the heat kernel on K is a class function, we have

$$\rho_{t\Delta_i\tau}\left(g_{i-1}^{-1} g_i b_i^{-1}\right) = \rho_{t\Delta_i\tau}\left(g_{i-1}^{-1} g_i x_i^{-1} x_{i-1}\right) = \rho_{t\Delta_i\tau}\left(\left(g_{i-1} x_{i-1}^{-1}\right)^{-1} g_i x_i^{-1}\right).$$

Thus (6.11) agrees with (6.10).

Now recall that θ is a measure-theoretic isomorphism of $(\bar{\mathcal{A}}, \tilde{P}_s)$ with $\left(W(K), \tilde{\rho}_s\right)$ and $\theta^{\mathbb{C}}$ is an isomorphism of $(\bar{\mathcal{A}}_{\mathbb{C}}, \tilde{M}_{s,t})$ with $\left(W\left(K_{\mathbb{C}}\right), \tilde{\mu}_{s,t}\right)$. So let us now *define* a transform $\tilde{B}_{s,t}$ by

$$\tilde{B}_{s,t} f = \left[\tilde{S}_{s,t}\left(f \circ \theta\right)\right] \circ \left(\theta^{\mathbb{C}}\right)^{-1}.$$

This is a well-defined isometric map of $L^2\left(W(K), \tilde{\rho}_s\right)$ into $L^2\left(W\left(K_{\mathbb{C}}\right), \tilde{\mu}_{s,t}\right)$. But (6.7)-(6.10) tell us that $\tilde{B}_{s,t}$ takes cylinder functions to holomorphic cylinder functions. Since cylinder functions are dense in $L^2\left(W(K), \tilde{\rho}_s\right)$ then by the definition of $\mathcal{H}L^2\left(W\left(K_{\mathbb{C}}\right), \tilde{\mu}_{s,t}\right)$, $\tilde{B}_{s,t}$ maps into the holomorphic subspace. Equations (6.7)-(6.10) assert that $\tilde{B}_{s,t}$ satisfies (6.2) of the theorem. By the definition of $\tilde{B}_{s,t}$, the diagram in the theorem would commute if the $\mathcal{H}L^2$'s were replaced by L^2's.

It remains then to show that composition with $\theta^{\mathbb{C}}$ takes $\mathcal{H}L^2\left(W\left(K_{\mathbb{C}}\right), \tilde{\mu}_{s,t}\right)$ onto $\mathcal{H}L^2(\bar{\mathcal{A}}_{\mathbb{C}}, \tilde{M}_{s,t})$, and that $\tilde{B}_{s,t}$ maps *onto* $\mathcal{H}L^2\left(W\left(K_{\mathbb{C}}\right), \tilde{\mu}_{s,t}\right)$. We address the second point first. We compute $\tilde{B}_{s,t}$ in "incremental coordinates" using (6.4)-(6.6).

So we are applying the heat equation in the "increments" $x_{i-1}^{-1}x_i$. Since in incremental coordinates both $\rho_s^{\mathcal{P}}$ and $\mu_{s,t}^{\mathcal{P}}$ factor as product measures, the surjectivity argument in Theorem 5.3 (using the method of [H1]) applies to show that every L^2 holomorphic cylinder function $\Phi(g_{\mathcal{P}})$ comes from an L^2 cylinder function $\phi(x_{\mathcal{P}})$. For the details of this argument see again the proof of Theorem 3 in [HS]. Since holomorphic cylinder functions are dense by definition in $\mathcal{H}L^2(W(K_{\mathbb{C}}), \tilde{\mu}_{s,t})$, $\tilde{B}_{s,t}$ is surjective.

Now that we know $\tilde{B}_{s,t}$ is surjective, we may show that composition with $\theta^{\mathbb{C}}$ takes the holomorphic subspace onto the holomorphic subspace. If $F \in \mathcal{H}L^2(W(K_{\mathbb{C}}), \tilde{\mu}_{s,t})$, then letting $f = \tilde{B}_{s,t}^{-1}F$, we have

$$F \circ \theta^{\mathbb{C}} = \left(\tilde{B}_{s,t} f\right) \circ \theta^{\mathbb{C}} = \tilde{S}_{s,t}(f \circ \theta).$$

But $\tilde{S}_{s,t}$ maps into the holomorphic subspace, so $F \circ \theta^{\mathbb{C}} \in \mathcal{H}L^2(\bar{\mathcal{A}}_{\mathbb{C}}, \tilde{M}_{s,t})$. A similar argument using the surjectivity of $\tilde{S}_{s,t}$ shows that if $F \in \mathcal{H}L^2(\bar{\mathcal{A}}_{\mathbb{C}}, \tilde{M}_{s,t})$ then $F \circ \left(\theta^{\mathbb{C}}\right)^{-1} \in \mathcal{H}L^2(\bar{\mathcal{A}}_{\mathbb{C}}, \tilde{M}_{s,t})$. □

7. Another proof of Theorem 5.2

In this section, we sketch another method for proving Theorem 5.2. The idea is to approximate $h(A)$ by cylinder functions and to compute $\tilde{S}_{s,t}$ by first principles. So for a partition $\mathcal{P}$ we make a piecewise-linear approximation $a^{\mathcal{P}}$ to the Brownian motion a, and then apply the deterministic Itô map to $a^{\mathcal{P}}$. This gives an approximation $h^{\mathcal{P}}(A)$ to the holonomy $h(A)$, given explicitly by

$$h^{\mathcal{P}}(A) = e^{\Delta_1(a)} e^{\Delta_2(a)} \cdots e^{\Delta_N(a)}, \tag{7.1}$$

where $\Delta_i(a) = a_{\tau_i} - a_{\tau_{i-1}}$ is the ith increment of a. Standard approximation results (e.g., [IW, Thm. VI.7.2]) show that for any finite-dimensional irreducible representation π of K,

$$\lim_{|\mathcal{P}| \to 0} \pi \circ h^{\mathcal{P}} = \pi \circ h$$

in $L^2(\bar{\mathcal{A}}, \tilde{P}_s)$ where $|\mathcal{P}|$ is the partition size. The proof relies on the fact that $\pi \circ \theta$ satisfies its own (matrix-valued) stochastic differential equation.

Meanwhile, applying the Segal-Bargmann transform to the (matrix-valued) cylinder function $\pi \circ h^{\mathcal{P}}$ gives

$$\tilde{S}_{s,t}\left(\pi \circ h^{\mathcal{P}}\right)(C) = \int_{\bar{\mathcal{A}}} \pi\left(e^{\Delta_1(c+a)} \cdots e^{\Delta_N(c+a)}\right) d\tilde{P}_s(A) \tag{7.2}$$

where π now refers to the holomorphic extension of π to $K_{\mathbb{C}}$. Unfortunately the authors have not been able to find in the current literature similar L^2 convergence results which are applicable when K is replaced by $K_{\mathbb{C}}$. This is because for a noncompact group the vector fields entering in the stochastic differential equations are not bounded as is required in all references known to the authors. Nevertheless, it is possible to show [DHu] by essentially standard arguments that

$$\lim_{|\mathcal{P}| \to 0} \pi\left(e^{\Delta_1(c+a)} \cdots e^{\Delta_N(c+a)}\right) = \pi\left(h_{\mathbb{C}}(A+C)\right)$$

in $L^2\left(\bar{\mathcal{A}} \times \bar{\mathcal{A}}_{\mathbb{C}}, \tilde{P}_s \times \tilde{M}_{s,t}\right)$, where $h_{\mathbb{C}}(A+C) = g_1$, g_τ being the $K_{\mathbb{C}}$-valued solution to the stochastic differential equation, $dg_\tau = g_\tau \circ d(a+c)$.

Because conditional expectations are contractive, we may interchange the $|\mathcal{P}| \to 0$ limit with the integral in (7.2) to find that

$$\tilde{S}_{s,t}\left(\pi \circ h\right)(C) = \int_{\bar{\mathcal{A}}} \pi\left(h_{\mathbb{C}}\left(A + C\right)\right) \, d\tilde{P}_t\left(A\right). \tag{7.3}$$

To compute this integral we write out in Itô form the matrix-valued s.d.e. satisfied by $\theta^\pi := \pi\left(\theta^{\mathbb{C}}\left(A + C\right)\right)$:

(7.4)

$$\theta^\pi_\tau = I + \int_0^\tau \theta^\pi_\sigma \, da^\pi_\sigma + \frac{t}{2}\int_0^\tau \theta^\pi_\sigma \pi\left(\Delta_K\right) \, d\sigma + \int_0^\tau \theta^\pi_\sigma \, dc^\pi_\sigma + \frac{s-t}{2}\int_0^\tau \theta^\pi_\sigma \pi\left(\Delta_K\right) \, d\sigma.$$

Here $a^\pi_\tau = \pi\left(a_\tau\right)$ and $c^\pi_\tau = \pi\left(c_\tau\right)$, where as above $\pi(\xi) = d\pi(e^{t\xi})/dt|_{t=0}$ for $\xi \in \mathfrak{k}_{\mathbb{C}}$. (We are using π for both the representation on $K_{\mathbb{C}}$ and the induced representation on $\mathfrak{k}_{\mathbb{C}}$.) We now wish to take the expectation in A with C fixed. So let $\bar{\theta}^\pi_\tau\left(C\right) = \int \theta^\pi_\tau\left(A + C\right) \, d\tilde{P}_t\left(A\right)$. By the Martingale property of stochastic integrals, the term $\int_0^\tau \theta^\pi_\sigma \, da^\pi_\sigma$ integrates to zero so that

$$\begin{aligned} \bar{\theta}^\pi_\tau\left(C\right) &= I + \frac{t}{2}\int_0^\tau \bar{\theta}^\pi_\sigma \pi\left(\Delta_K\right) \, d\sigma + \int_0^\tau \bar{\theta}^\pi_\sigma \, dc^\pi_\sigma + \frac{s-t}{2}\int_0^\tau \bar{\theta}^\pi_\sigma \pi\left(\Delta_K\right) \, d\sigma \\ &= I + \frac{t}{2}\int_0^\tau \bar{\theta}^\pi_\sigma \pi\left(\Delta_K\right) \, d\sigma + \int_0^\tau \bar{\theta}^\pi_\sigma \circ dc^\pi_\sigma. \end{aligned} \tag{7.5}$$

Since π is assumed irreducible, $\pi\left(\Delta_K\right)$ is simply a multiple of the identity, and so it is easily verified that the (unique) solution to (7.5) is

$$\bar{\theta}^\pi_\tau = e^{\tau t \pi(\Delta_K)/2} \pi\left(\theta^{\mathbb{C}}_\tau\left(C\right)\right).$$

Putting $\tau = 1$ and recalling (7.3) we see that Theorem 5.2 holds for the matrix entries of π. Using the Peter-Weyl theorem it follows that Theorem 5.2 holds in general.

Note that the second of two proofs of [HS, Lem. 24] mistakenly applies [IW] even on $K_{\mathbb{C}}$; that proof is therefore incomplete. The convergence results in [DHu] correct this error.

8. Appendix A: Laplacians on $\mathcal{A}$ and $H(K)$

Let $\langle \cdot, \cdot \rangle$ denote the unique bi-invariant Riemannian metric on K which agrees with the given Ad-K-invariant inner product on $\mathfrak{k}$. For $v \in T\left(K\right)$ we will simply write $|v|^2$ for $\langle v, v \rangle$. Let $H(K)$ denote the finite-energy paths in K. Explicitly, $H(K)$ is the collection of absolutely continuous paths $x : [0, 1] \to K$ such that $x(0) = e$ and $\int_0^1 |\dot{x}(\tau)|^2 d\tau < \infty$. It is well known that $H(K)$ is a Hilbert Lie group under pointwise multiplication and that the map

$$(x, h) \in H(K) \times H(\mathfrak{k}) \to R_{x*}h \in T\left(H\left(K\right)\right)$$

is a trivialization of the tangent bundle of $H(K)$. (We are using $R_x : H(K) \to H(K)$ to denote right multiplication by x.) This trivialization induces a right-invariant Riemannian metric $(\cdot, \cdot)$ on $H(K)$ given explicitly by

$$(R_{x*}h, R_{x*}h) = \int_0^1 \left\langle \dot{h}(\tau), \dot{h}(\tau) \right\rangle \, d\tau \quad \forall x \in H(K) \text{ and } h \in H(\mathfrak{k}).$$

The following theorem appears (in a disguised form) in Theorem 3.14 and Lemma 3.15 of Gross [G].

Theorem 8.1. *Let $\theta : \mathcal{A} \to H(K)$ denote the deterministic solution to (5.1), i.e., $d\theta_\tau(A)/d\tau = \theta_\tau(A)A_\tau$ with $\theta_0(A) = e$. Then θ is an isometric isomorphism of infinite-dimensional Riemannian manifolds. In particular, $H(K)$ is flat.*

Proof. For simplicity of notation, we will assume, without loss of generality since K is compact, that K is a matrix group. In order to compute the differential of θ, let $x(\tau, s) = \theta_\tau(A + sB)$, $x(\tau) = x(\tau, 0) = \theta_\tau(A)$ and $h(\tau) = x'(\tau, 0)x(\tau)^{-1}$. To simplify the exposition, first assume that A and B are C^1. Then by smooth dependence of ordinary differential equations on parameters, h is differentiable and satisfies

$$\begin{aligned}\dot{h}(\tau) &= \dot{x}'(\tau,0)x(\tau)^{-1} - x'(\tau,0)A(\tau)x(\tau)^{-1} \\ &= \left.\frac{d}{ds}\right|_{s=0} (x(\tau,s)\,[A(\tau) + sB(\tau)])\, x(\tau)^{-1} - x'(\tau,0)A(\tau)x(\tau)^{-1} \\ &= Ad_{x(\tau)}B(\tau) = Ad_{\theta_\tau(A)}B(\tau).\end{aligned}$$

Here $\cdot$ indicates a derivative with respect to τ and $'$ a derivative with respect to s. The above equation says that

$$\theta_* B_A = R_{\theta(A)*} \int_0^{\cdot} Ad_{\theta_\tau(A)} B(\tau) d\tau. \tag{8.1}$$

and therefore

$$\begin{aligned}(\theta_* B_A, \theta_* B_A) &= (x'(\cdot,0), x'(\cdot,0)) = \int_0^1 \langle \dot{h}(\tau), \dot{h}(\tau)\rangle \, d\tau \\ &= \int_0^1 |Ad_{x(\tau)}B(\tau)|^2 d\tau = \int_0^1 |B(\tau)|^2 d\tau = (B,B)_{\mathcal{A}}. \end{aligned}\tag{8.2}$$

Since the map $\theta : \mathcal{A} \to H(K)$ is smooth as a mapping of infinite-dimensional Hilbert manifolds (see for example [P] or [D1]), both (8.1) and (8.2) extend by continuity to $A, B \in \mathcal{A}$. ☐

Definition 8.2 (Directional Derivatives)**.** *For a function $F : \mathcal{A} \to \mathbb{C}$ and $A, B \in \mathcal{A}$ let*

$$\partial_B F(A) = \left.\frac{d}{dt}\right|_0 F(A + tB)$$

provided the limit exists. For a function $f : H(\mathfrak{k}) \to \mathbb{C}$ and $x \in H(K)$ and $h \in H(\mathfrak{k})$, let

$$\partial_h f(x) = \left.\frac{d}{dt}\right|_0 f\left(e^{th}x\right),$$

provided the limit exists, where $\left(e^{th}x\right)(s) = e^{th(s)}x(s)$ for all $s \in [0,1]$.

Definition 8.3 (Hessians)**.** *Suppose that both $F : \mathcal{A} \to \mathbb{C}$ and $f : H(K) \to \mathbb{C}$ are twice continuously differentiable at $A \in \mathcal{A}$ and $x \in K$, respectively. The Hessians of F and f at A and x, respectively, are the quadratic forms $D^2F(A)$ on $\mathcal{A}$ and $D^2 f(x)$ on $H(K)$ defined by*

$$D^2F(A)(B_1, B_2) = (\partial_{B_1}\partial_{B_2}F)(A)$$

and

$$D^2 f(x)(h_1, h_2) = (\partial_{h_1}\partial_{h_2} f)(x).$$

Notice that $D^2 f(x)(h_1, h_2)$ is ***not*** *symmetric in h_1 and h_2, a reflection of the fact that we did not use the Levi-Civita connection on $H(K)$ to define $D^2 f$. See Remark 8.6 below.*

Using the above notation, it would be natural to define $\Delta_{\mathcal{A}} F(A)$ and $\Delta_{H(K)} f(x)$ by

$$\Delta_{\mathcal{A}} F(A) = \mathrm{tr}_{\mathcal{A}} D^2 F(A) \quad \text{and} \quad \Delta_{H(K)} f(x) = \mathrm{tr}_{H(\mathfrak{k})} D^2 f(x), \tag{8.3}$$

provided that the quadratic forms $D^2 F(A)$ and $D^2 f(x)$ were trace-class. The above definitions certainly would be suitable if F and f were smooth cylinder functions on $\mathcal{A}$ and $H(K)$, respectively. However, the definition in (8.3) is too restrictive for our purposes. In particular, we are interested in *non-cylinder* functions on $\mathcal{A}$ of the form $F = f \circ \theta$, where f is a cylinder function on $H(K)$. For such a function F, $D^2 F(A)$ is typically not trace-class and hence the $\Delta_{\mathcal{A}} F$ would not be defined. Definition 8.5 below overcomes this problem by using a more inclusive notion of the trace of $D^2 F(A)$.

Notation 8.4. *Let $\beta = \{e_i\}_{i=1}^d$ be an orthonormal basis for $\mathfrak{k}$, Γ an orthonormal basis of $L^2([0,1];\mathbb{R})$, and γ the orthonormal basis of $H(\mathbb{R})$ given by*

$$\gamma = \{v(\cdot) = \int_0^{\cdot} V(\tau) d\tau | V \in \Gamma\}.$$

Notice that $\Gamma\beta := \{V e_i | V \in \Gamma \text{ and } i = 1, \dots, d\}$ and $\gamma\beta := \{v e_i | v \in \gamma \text{ and } i = 1, \dots, d\}$ are orthonormal bases for $\mathcal{A}$ and $H(\mathfrak{k})$ respectively.

Definition 8.5 (Laplacians)**.** *Let $F : \mathcal{A} \to \mathbb{C}$, $f : H(K) \to \mathbb{R}$, $A \in \mathcal{A}$ and $x \in K$. Then*

$$(\Delta_{\mathcal{A}} F)(A) = \sum_{V \in \Gamma} \left(\sum_{i=1}^{\dim \mathfrak{k}} D^2 F(A)(V e_i, V e_i) \right) = \sum_{V \in \Gamma} \left(\sum_{i=1}^{\dim \mathfrak{k}} (\partial^2_{V e_i} F)(A) \right) \tag{8.4}$$

and

$$(\Delta_{H(K)} f)(x) = \sum_{v \in \gamma} \left(\sum_{i=1}^{\dim \mathfrak{k}} D^2 f(x)(v e_i, v e_i) \right) = \sum_{v \in \gamma} \left(\sum_{i=1}^{\dim \mathfrak{k}} (\partial^2_{v e_i} f)(x) \right), \tag{8.5}$$

provided the derivatives and the sums exist and are independent of the choice of bases.

Remark 8.6. This two-step procedure for defining an infinite-dimensional trace appears already in Freed [F] and [DL]. Moreover, it is shown in [DL] that $D^2 f(x)(v e_i, v e_i)$ is the same as $\mathrm{Hess} f(x)(v e_i, v e_i)$ where $\mathrm{Hess} f$ denotes the Hessian of f relative to the Levi-Civita connection on $H(K)$. So despite the fact that $\mathrm{Hess} f(x)$ is **not** trace-class ([DL, Remark 3.13]) when f is a cylinder function on $H(K)$, it is reasonable to interpret $\Delta_{H(K)}$ defined in (8.5) as the Levi-Civita Laplacian.

Definition 8.7. *Given a partition $\mathcal{P} = \{0 = \tau_0 < \tau_1 < \cdots < \tau_n = 1\}$ of $[0,1]$, let $K^{\mathcal{P}} = K^n$ and $x_{\mathcal{P}} = (x_{\tau_1}, x_{\tau_2}, \dots, x_{\tau_n}) \in K^{\mathcal{P}}$ for all $x \in H(K)$. We also define a second-order elliptic operator $\Delta_{\mathcal{P}}$ acting on $C^\infty(K^{\mathcal{P}})$ by*

$$(\Delta_{\mathcal{P}} \phi)(x_1, x_2, \dots, x_n) = \sum_{i,j=1}^{n} \sum_{m=1}^{\dim \mathfrak{k}} \min(\tau_i, \tau_j) \left(D^{(i)}_{e_m} D^{(j)}_{e_m} \phi \right)(x_1, x_2, \dots, x_n)$$

where

$$\left(D_A^{(i)}\phi\right)(x_1,x_2,\dots,x_n)=\frac{d}{ds}\Big|_{s=0}\phi(x_1,x_2,\dots,x_{i-1},e^{sA}x_i,x_{i+1},\dots,x_n)$$

for all $A\in\mathfrak{k}$ and $i=1,\dots,n$.

Example 8.8. Suppose that $\mathcal{P}=\{0=\tau_0<\tau_1<\cdots<\tau_n=1\}$ is a partition of $[0,1]$ and $f:H(K)\to\mathbb{C}$ is a smooth cylinder function of the form $f(x)=\phi(x_{\mathcal{P}})$ where $\phi:K^n\to\mathbb{C}$ is a smooth function.

1. Then $\Delta_{H(K)}f$ exists and

$$\left(\Delta_{H(K)}f\right)(x)=\left(\Delta_{\mathcal{P}}\phi\right)(x_{\mathcal{P}}) \tag{8.6}$$

for all $x\in H(K)$. See the proof of Proposition 4.19 in [DL] for details.

2. For $x\in H(K)$ let

$$x'_{\mathcal{P}}=(x_{\tau_1},x_{\tau_1}^{-1}x_{\tau_2},\dots,x_{\tau_{n-1}}^{-1}x_{\tau_n})$$

be the "incremental coordinates" of x relative to the partition $\mathcal{P}$. If $f:H(K)\to\mathbb{C}$ is a smooth cylinder function of the form $f(x)=\psi(x'_{\mathcal{P}})$ with $\psi:K^n\to\mathbb{C}$ being a smooth function, then

$$\Delta_{H(K)}f(x)=\sum_{i=1}^{n}(\tau_i-\tau_{i-1})\left(\Delta_K^{(i)}\psi\right)(x'_{\mathcal{P}}). \tag{8.7}$$

Here $\Delta_K^{(i)}\psi$ denotes Δ_K acting on the ith variable of ψ while holding the remaining variables fixed. This could be proved by a finite-dimensional calculation showing that

$$(\Delta_{\mathcal{P}}\phi)(x_1,x_2,\dots,x_n)=\sum_{i=1}^{n}(\tau_i-\tau_{i-1})\left(\Delta_K^{(i)}\psi\right)(x_1,x_1^{-1}x_2,\dots,x_{n-1}^{-1}x_n)$$

where $\phi(x_1,x_2,\dots,x_n)=\psi\left(x_1,x_1^{-1}x_2,\dots,x_{n-1}^{-1}x_n\right)$ or by a calculation similar to the proof of Proposition 4.19 in [DL].

Since $\theta:\mathcal{A}\to H(K)$ is an isometry by Theorem 8.1 and $\Delta_{\mathcal{A}}$ and $\Delta_{H(K)}$ deserve to be thought of as Laplace-Beltrami operator on $\mathcal{A}$ and $H(K)$ respectively (see Remark 8.6 above), we should expect that $\Delta_{\mathcal{A}}(f\circ\theta)=(\Delta_{H(K)}f)\circ\theta$ for all "nice" functions f on $H(K)$. This would certainly be true in finite dimensions. It remains true in this infinite-dimensional context when we use the "two step" trace in Definition 8.5 of the Laplacians.

Theorem 8.9. *Suppose that $f:H(K)\to\mathbb{C}$ is a smooth cylinder function. Then $\Delta_{\mathcal{A}}(f\circ\theta)$ and $\Delta_{H(K)}f$ exist and*

$$\Delta_{\mathcal{A}}(f\circ\theta)=(\Delta_{H(K)}f)\circ\theta. \tag{8.8}$$

In particular if ϕ is a smooth function on K, then

$$\Delta_{\mathcal{A}}(\phi\circ h)=(\Delta_K\phi)\circ h. \tag{8.9}$$

Recall that $h(A)=\theta_1(A)$. The following lemma will be needed in the proof of this theorem.

Lemma 8.10. *Let $a, b \in \mathfrak{k}$, $x \in H(K)$ and $k : [0,1]^2 \to \mathfrak{k}$ be given by $k(r,\tau) = [Ad_{x(r)}a, Ad_{x(\tau)}b]$. Then, for $t \in [0,1]$,*

$$\sum_{V\in\Gamma}\int_{[0,t]^2} 1_{r\le\tau}k(r,\tau)V(r)V(\tau)\,drd\tau = \frac{1}{2}\int_0^t k(\tau,\tau)\,d\tau. \tag{8.10}$$

Proof. Let S_t be the left member of (8.10) and for $V \in \Gamma$ let $v(\cdot) = \int_0^\cdot V(\tau)d\tau$. Also define

$$A_V(t) = \int_{[0,t]^2} 1_{r\le\tau}k(r,\tau)V(r)V(\tau)\,drd\tau,$$

so that $S_t = \sum_{V\in\Gamma} A_V(t)$. Integration by parts shows:

$$\begin{aligned}
A_V(t) &= \int_0^t k(\tau,\tau)v(\tau)V(\tau)d\tau - \int_{[0,t]^2} 1_{r\le\tau}k_r(r,\tau)v(r)V(\tau)\,drd\tau \\
&= \frac{1}{2}\int_0^t k(\tau,\tau)\frac{dv^2(\tau)}{d\tau}\,d\tau + \int_{[0,t]^2} 1_{r\le\tau}k_{r,\tau}(r,\tau)v(r)v(\tau)\,drd\tau \\
&\quad - \int_0^t \left(k_r(r,t)v(r)v(t) - k_r(r,r)v(r)v(r)\right)dr \\
&= \frac{1}{2}k(t,t)v^2(t) - \frac{1}{2}\int_0^t \frac{dk(\tau,\tau)}{d\tau}v^2(\tau)\,d\tau + \int_{[0,t]^2} 1_{r\le\tau}k_{r,\tau}(r,\tau)v(r)v(\tau)\,drd\tau \\
&\quad - \int_0^t \left(k_r(r,t)v(r)v(t) - k_r(r,r)v(r)v(r)\right)dr,
\end{aligned}$$

where $k_r = \partial k/\partial r$ and $k_{r,\tau} = \partial^2 k/\partial r\partial\tau$. Summing this equation on $V \in \Gamma$ implies that

$$\begin{aligned}
S_t &= \frac{1}{2}k(t,t)t - \frac{1}{2}\int_0^t \frac{dk(\tau,\tau)}{d\tau}\tau\,d\tau + \int_{[0,t]^2} 1_{r\le\tau}k_{r,\tau}(r,\tau)r\,drd\tau \\
&\quad - \int_0^t \left(k_r(r,t)r - k_r(r,r)r\right)dr,
\end{aligned}$$

wherein we have used the identity $\sum_{V\in\Gamma} v(r)v(\tau) = \min(r,\tau)$. (This identity is a consequence of the reproducing kernel property of $\min(r,\tau)$ and Bessel's equality, see for example the proof of Lemma 3.8 in [DL].) An integration by parts on the second and third terms above shows that $S_t = \frac{1}{2}\int_0^t k(\tau,\tau)\,d\tau$. □

Proof of Theorem 8.9. Let $A, B \in \mathcal{A}$, $x = \theta(A)$, and $w(\cdot) = \int_0^\cdot Ad_{\theta_\tau(A)}B(\tau)d\tau$. By (8.1) and the chain rule, $\partial_B(f\circ\theta)(A) = (\partial_w f)(\theta(A))$ and

$$\partial_B^2(f\circ\theta) = (\partial_w^2 f)(x) + \partial_{\int_{[0,\cdot]^2} 1_{r\le\tau}[Ad_{x(r)}B(r), Ad_{x(\tau)}B(\tau)]\,drd\tau} f(x). \tag{8.11}$$

Because of the second term on the right side of (8.11) it may be seen that $D^2(f\circ\theta)$ is not trace-class–see [DL, Remark 3.13]. On the other hand the two-step trace does exist. To compute this trace, let $B = Ve_i \in \Gamma\beta$ and sum (8.11) on $i = 1, \ldots, d$ and then on $V \in \Gamma$. Letting

$$k(r,\tau) = \sum_{i=1}^{\dim\mathfrak{k}} [Ad_{x(r)}e_i, Ad_{x(\tau)}e_i] \tag{8.12}$$

and noting that $k(\tau,\tau)=0$, we may apply Lemma 8.10 to find

$$\Delta_{\mathcal{A}}(f\circ\theta)(A)=\sum_{V\in\Gamma}\sum_{i=1}^{\dim\mathfrak{k}}\left(\partial^2_{\int_0^{\cdot} Ad_{x(\tau)}V(\tau)e_i d\tau}f\right)(x). \tag{8.13}$$

Since Ad_{x_τ} is an isometry on $\mathfrak{k}$, $\{\int_0^{\cdot} Ad_{x_\tau}V(\tau)e_i d\tau : V\in\Gamma, i=1,\ldots,d\}$ is an orthonormal basis for $H(\mathfrak{k})$. Therefore by Example 8.8, the sum appearing in (8.13) is precisely $\Delta_{H(K)}f(x)$. □

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