On thermodynamic limits of entropy densities

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Abstract

We give some sufficient conditions which guarantee that the entropy density in the thermodynamic limit is equal to the thermodynamic limit of the entropy densities of finite-volume (local) Gibbs states. (to appear in Lett.Math.Phys.)

1 Introduction

In this note we give some results on the existence of the infinite-volume limit of the entropy density for (quantum) lattice systems, taken over a sequence of finite-volume (local) Gibbs states. The existence of this limit is not automatic; in fact one can without too much difficulties think of examples where the limit does not exist, due to possible coexistence of phases (equilibrium states) with different entropy densities, as happens for example in Potts models. Here we exclude this possibility by requiring the pressure (free energy density) function in the infinite-volume (thermodynamic) limit, which exists in great generality, to be differentiable as a function of (inverse) temperature. Note, however, that we do not need the local Gibbs states themselves to converge to a limit.

We will first give a short introduction and set up some notation for the type of quantum lattice systems we will be investigating. For an extensive treatment of the theory we refer to [8] or [3], see also [5].

Let $\mathbf{Z}^\nu$ be the $\nu-$dimensional lattice, and $\mathcal{A}_k (k \in \mathbf{Z}^\nu)$ be copies of $\mathrm{M}_d(\mathbf{C})$, the d×d full matrix algebra. For any bounded region $\Lambda \subset \mathbf{Z}^\nu$, the local algebra $\mathcal{A}_\Lambda$ is given by $\otimes_{x\in\Lambda}\mathcal{A}_x$. It follows from the definition that, for $\Lambda \subset \Lambda'$, we have $\mathcal{A}_{\Lambda'} = \mathcal{A}_\Lambda \otimes \mathcal{A}_{\Lambda'\setminus\Lambda}$, where $\Lambda' \setminus \Lambda$ is the complement of Λ in Λ'. We shall identify $\mathcal{A}_\Lambda$ with $\mathcal{A}_\Lambda \otimes 1_{\Lambda'\setminus\Lambda}$ in $\mathcal{A}_{\Lambda'\setminus\Lambda}$. The definition also implies that if Λ_1 and Λ_2 are disjoint, the elements of $\mathcal{A}_{\Lambda_1}$ commute with those of $\mathcal{A}_{\Lambda_2}$. The $\mathbf{C}^*$-algebra $\mathcal{A}$ for our investigation is the norm completion of the normed algebra $\mathcal{A}_\infty = \cup_\Lambda \mathcal{A}_\Lambda$, the union of all local algebras $\mathcal{A}_\Lambda$ associated with finite regions $\Lambda \subset \mathbf{Z}^\nu$. We note that the group $\mathbf{Z}^\nu$ of space translations is a subgroup of the automorphism group of $\mathcal{A}$ and we denote the action of this group by $A \in \mathcal{A}_\Lambda \mapsto \sigma_x A \in \mathcal{A}_{\Lambda+x}$, $x \in \mathbf{Z}^\nu$. For each finite subset Λ of $\mathbf{Z}^\nu$, we are given an element $\Phi(\Lambda) \in \mathcal{A}_\Lambda$, called an interaction potential, satisfying the following conditions:

$$
\begin{aligned}
(1)\ \Phi(\phi) &= 0, \\
(2)\ \sigma_x \Phi(\Lambda) &= \Phi(\Lambda + x) \quad : \textit{translation invariance}, \\
(3)\ \Phi(\Lambda)^* &= \Phi(\Lambda) \quad : \textit{self-adjointness}.
\end{aligned}
$$

Given a potential Φ and a finite subset $\Lambda \subset \mathbf{Z}^\nu$, the internal Hamiltonian confined to Λ takes the form

$$U_\Lambda \equiv \sum_{X \subseteq \Lambda} \Phi(X).$$

In the following, we will consider two different kinds of Banach spaces of interactions. For the first Banach space, we impose the following growth condition on Φ :

$$(4)\ \|\Phi\|_\circ \equiv \sum_{\Lambda \ni 0} \frac{\|\Phi(\Lambda)\|}{|\Lambda|} < \infty.$$

With this norm, the potentials under consideration form a real Banach space $\mathcal{B}_0$.

For the second Banach space, we recall the notion of a "standard potential" [1]. Let ω be a product state of $\mathcal{A}$, namely $\omega(aa') = \omega(a)\omega(a')$ for $a \in \mathcal{A}_\Lambda, a' \in \mathcal{A}_{\Lambda^c}$ for any finite subset Λ. Let C^ω_Λ be the conditional expectation from $\mathcal{A}$ onto $\mathcal{A}_\Lambda$. Namely, for each $a \in \mathcal{A}$, $C^\omega_\Lambda(a)$ is defined as the element of $\mathcal{A}_\Lambda$ satisfying $\omega(C^\omega_\Lambda(a)b) = \omega(ab)$ for all $b \in \mathcal{A}_\Lambda$. We assume

$$(4)' \qquad C^\omega_I\,(\Phi(\Lambda)) = 0 \ \text{ unless } I \supseteq \Lambda \ : \textit{standardness}.$$

Finally, for each finite subset $\Lambda \subset \mathbf{Z}^\nu$ we assume the existence of the surface energy connecting inside and outside of Λ, i.e.

$$(5)'\ W_\Lambda \equiv \left\{ \sum \Phi(X) : X \cap \Lambda \neq \emptyset, X \cap \Lambda^c \neq \emptyset \right\} \in \mathcal{A},$$

where on the right-hand side the finite sum over $X \subset \Lambda'$ is first taken and the limit $\Lambda' \nearrow \mathbf{Z}^\nu$ is then taken. This limit is assumed to converge in norm. The total interaction Hamiltonian of Λ is defined by

$$H_\Lambda = U_\Lambda + W_\Lambda.$$

We then define

$$\|\Phi\| \equiv \sup\{\|H_n\| \quad : n \in \mathbf{Z}^\nu\} \, (= \|H_n\|) \, ,$$

where $H_n = H_\Lambda$ for the one-point set $\Lambda = \{n\}$ and $\|H_n\|$ is independent of $n \in \mathbf{Z}^\nu$ due to the assumed translation invariance. With this norm, the potentials under consideration form a real Banach space $\mathcal{B}$.

For each $\Phi \in \mathcal{B}_o$, let us define $A_\Phi \in \mathcal{A}$ by

$$A_\Phi \equiv \sum_{\Lambda \ni 0} \frac{\Phi(\Lambda)}{|\Lambda|}.$$

By definition, A_Φ is linear in Φ and

$$\|A_\Phi\| \leq \|\Phi\|_o. \tag{1}$$

For $\Phi \in \mathcal{B}$, the following estimates have been obtained [1]

$$\|U_\Lambda\| \quad \leq \quad \|H_\Lambda\| \leq \sum_{n \in \Lambda} \|H_n\| \leq |\Lambda| \cdot \|\Phi\| \tag{2}$$

$$\lim_{\Lambda \nearrow \mathbf{Z}^\nu} \frac{\|W_\Lambda\|}{|\Lambda|} = 0 \tag{3}$$

where the limit of Λ tending to $\mathbf{Z}^\nu$ is the van Hove limit.

Let Tr_Λ denote the canonical trace of $\mathcal{A}_\Lambda$, where the term "canonical" means that $\mathrm{Tr}_\Lambda(e) = 1$ for each minimal projection $e \in \mathcal{A}_\Lambda$.

For each finite subset $\Lambda \subset \mathbf{Z}^\nu$, the partition function is given by:

$$Z_\Lambda(\beta\Phi) \equiv \mathrm{Tr}_\Lambda \left(e^{-\beta U_\Lambda}\right) . \tag{4}$$

Let $P_\Lambda(\beta\Phi) \equiv \frac{\log Z_\Lambda(\beta\Phi)}{|\Lambda|}$, then for $\Phi \in \mathcal{B}_o$(or $\mathcal{B}$) the following basic results are known.

$$(i) \qquad P(\beta\Phi) \equiv \lim_{\Lambda \mapsto \mathbf{Z}^\nu} P_\Lambda(\beta\Phi) \ \text{ exists in the van Hove limit,} \tag{5}$$

$$(ii) \qquad |P(\Phi - \Psi)| \leq \|\Phi - \Psi\|_o \ \ (\|\Phi - \Psi\|), \tag{6}$$

$$(iii) \qquad P(\Phi) \text{ is a convex function of } \Phi. \tag{7}$$

We call $P(\beta\Phi)$ the (thermodynamic) pressure of Φ at β. If ω is a state of $\mathcal{A}$, then ω_Λ denotes its restriction to $\mathcal{A}_\Lambda$ for each finite subset of $\Lambda \subset \mathbf{Z}^\nu$. The energy and entropy of this finite system Λ are given by

$$\begin{aligned} E_\Lambda(\omega) &= \omega(U_\Lambda), \\ S_\Lambda(\omega) &\equiv -\mathrm{Tr}_\Lambda D_\Lambda \log D_\Lambda . \end{aligned}$$

Here D_Λ denotes the density matrix of ω_Λ with respect to the canonical trace Tr_Λ on A_Λ. Let $\mathcal{A}^*_{+,1,inv}$ be the set of translation invariant states. We may define the energy density and the entropy density of the translation invariant state $\omega \in \mathcal{A}^*_{+,1,inv}$ to be

$$e_\Phi(\omega) \equiv \lim_{\Lambda \nearrow \mathbf{Z}^\nu} \frac{E_\Lambda(\omega)}{|\Lambda|}, \quad s(\omega) \equiv \lim_{\Lambda \nearrow \mathbf{Z}^\nu} \frac{S_\Lambda(\omega)}{|\Lambda|}.$$

It has been shown that for Φ in $\mathcal{B}_0$ or $\mathcal{B}$, $e_\Phi(\omega)$ exists and

$$\begin{aligned} |e_\Phi(\omega)| &\leq \|\Phi\|_o(\|\Phi\|), \\ e_{c\Phi+d\Psi}(\omega) &= c \cdot e_\Phi(\omega) + d \cdot e_\Psi(\omega) \quad : linearity, \\ e_\Phi(\lambda\omega + (1-\lambda)\mu) &= \lambda e_\Phi(\omega) + (1-\lambda) e_\Phi(\mu) \quad : affinity. \end{aligned}$$

Furthermore, especially in the case of $\Phi \in \mathcal{B}_o$ we have

$$e_\Phi(\omega) = \omega(\mathcal{A}_\Phi).$$

¿From the strong subadditivity property, it has been shown that the limit entropy density for translation invariant states exists and is given by the infimum over all finite subsets $\Lambda \subset \mathbf{Z}^\nu$. It has also been shown that the function $\omega \in \mathcal{A}^*_{+,1,inv} \to s(\omega)$ is affine and upper semi-continuous with respect to the weak* topology.

The "Gibbs Variational Principle" is used to define the equilibrium states, that is $P(\beta\Phi)$ is obtained by the following variational equality:

$$P(\beta\Phi) = sup_{\omega \in \mathcal{A}^*_{+,1,inv}}[s(\omega) - \beta \cdot e_\Phi(\omega)]. \tag{8}$$

Furthermore, if $\rho \in \mathcal{A}^*_{+,1,inv}$ satisfies

$$P(\beta\Phi) = s(\rho) - \beta \cdot e_\Phi(\rho), \tag{9}$$

then ρ is called an invariant equilibrium state for Φ at inverse temperature β. (In the case of $\mathcal{B}_0$ see for example [3], and in the case of $\mathcal{B}$ the above results have been shown in [2].) Denote the set of all invariant equilibrium states as $I_{\beta\Phi}$. $I_{\beta\Phi}$ is non empty due to the weak* upper semi-continuity of $s(\omega) - \beta \cdot e_\Phi(\omega)$ in ω.

Let $\varphi_\Lambda^{\beta,c}$ denote the local Gibbs state in Λ, that is,

$$\varphi_\Lambda^{\beta,c}(A) \equiv \frac{\mathrm{Tr}_\Lambda(Ae^{-\beta \cdot U_\Lambda})}{\mathrm{Tr}_\Lambda(e^{-\beta \cdot U_\Lambda})} \quad (A \in \mathcal{A}_\Lambda).$$

2 Results and Proofs

We have the following Theorem for a lattice of arbitrary dimension.

Theorem 2.1 (Main Theorem) *Assume that the pressure is differentiable w.r.t. the inverse temperature at a specific value β, then for the same β,*

$$s(\varphi) = \lim_{\Lambda \nearrow \mathbf{Z}^\nu} \frac{S_\Lambda(\varphi_\Lambda^{\beta,c})}{|\Lambda|},$$

where φ is any translation invariant equilibrium state, $\varphi_\Lambda^{\beta,c}$ is the local Gibbs state on $\mathcal{A}_\Lambda$, and the limit is taken in the van Hove sense.

This theorem generalizes a result of [6]. Note that in contrast to that situation, here we incorporate also cases where there exist more than one equilibrium states (phase transitions). Moreover, our assumption here does not exclude the possibility that thermodynamic limit points of local Gibbs states $\{\varphi_\Lambda^{\beta,c}\}_\Lambda$ are not unique. In [6], the essential part of the proof was the estimation of the analytic continuation of Connes-Cocycle and we evaluated the difference between local Gibbs states and the restrictions to the local subalgebras of a global Gibbs(Araki-Gibbs) state.

In this paper, we do not need the Tomita-Takesaki theory which is a crucial point to the Araki-Gibbs condition. To prove our main theorem, we need the following general result about the analysis of the convex functions. (See for example [8] Proposition I.3.2, or [9] Lemma 9 .)

Lemma 2.2 *Let $\{F_\alpha\}$ be a net of convex functions on $\mathbf{R}$ converging pointwise to a function F. Then F is convex and for any fixed $x_\circ$*

$$\big(D^- F\big)(x_\circ) \leq \underline{\lim} D^- F_\alpha(x_\circ) \leq \overline{\lim} D^+ F_\alpha(x_\circ) \leq \big(D^+\big)F(x_\circ).$$

In particular, if F is differentiable at $x_\circ$, then

$$\big(DF\big)(x_\circ) = \lim D^- F_\alpha(x_\circ) = \lim D^+ F_\alpha(x_\circ).$$

Here $D^\pm$ denote the right and left derivatives.

Proof of Main Theorem:
By (9) and linearity of the energy density w.r.t. potentials, for every $\Psi \in \mathcal{B}_0(\mathcal{B})$ we have

$$\begin{aligned} P(\beta\Phi + \Psi) &\geq s(\varphi) - e_{\beta\Phi+\Psi}(\varphi) \\ &= s(\varphi) - \beta \cdot e_\Phi(\varphi) - e_\Psi(\varphi) \\ &= P(\beta\Phi) - e_\Psi(\varphi). \end{aligned}$$

For $\Psi = \pm\lambda\Phi(\lambda > 0)$, we obtain the following estimate from this inequality

$$\frac{\{P\big((\beta+\lambda)\Phi\big) - P\big(\beta\Phi\big)\}}{\lambda} \geq -e_\Phi(\varphi) \geq \frac{\{P\big((\beta-\lambda)\Phi\big) - P\big(\beta\Phi\big)\}}{-\lambda}.$$

If $P(\beta\Phi)$ is β differentiable at a specific β, we obtain

$$-\frac{d}{d\beta}P(\beta\Phi) = e_{\Phi}(\varphi). \tag{10}$$

For each finite subsystem $\mathcal{A}_\Lambda$, we have

$$\frac{S_\Lambda(\varphi_\Lambda^{\beta,c})}{|\Lambda|} = \beta \cdot \frac{E_\Lambda(\varphi_\Lambda^{\beta,c})}{|\Lambda|} + P_\Lambda(\beta\Phi).$$

It is easy to check that

$$-\frac{d}{d\beta}P_\Lambda(\beta\Phi) = \frac{E_\Lambda(\varphi_\Lambda^{\beta,c})}{|\Lambda|}. \tag{11}$$

We note that the sequence of convex functions $\{P_\Lambda(\beta\Phi)\}$, differentiable w.r.t. β, converges to the convex function $P(\beta\Phi)$.
Because by assumption $P(\beta\Phi)$ is differentiable w.r.t. β, lemma 2.2. implies

$$\frac{d}{d\beta}P_\Lambda(\beta\Phi) \mapsto \frac{d}{d\beta}P(\beta\Phi): \quad \text{as } \Lambda \nearrow \mathbf{Z}^\nu. \tag{12}$$

From (10),(11), and (12) we obtain

$$\frac{E_\Lambda(\varphi_\Lambda^{\beta,c})}{|\Lambda|} \mapsto e_\Phi(\varphi): \quad \text{as } \Lambda \nearrow \mathbf{Z}^\nu.$$

We have

$$\begin{aligned} s(\varphi) &= \beta \cdot e_\Phi(\varphi) + P(\beta\Phi) \\ &= \lim_{\Lambda \nearrow \mathbf{Z}^\nu} \left(\beta \cdot \frac{E_\Lambda(\varphi_\Lambda^{\beta,c})}{|\Lambda|} + P_\Lambda(\beta\Phi) \right) \\ &= \lim_{\Lambda \nearrow \mathbf{Z}^\nu} \frac{S_\Lambda(\varphi_\Lambda^{\beta,c})}{|\Lambda|}. \end{aligned}$$

□

Corollary 2.3 *If Φ has a unique invariant equilibrium state φ at β, then*

$$s(\varphi) = \lim_{\Lambda \nearrow \mathbf{Z}^\nu} \frac{S_\Lambda(\varphi_\Lambda^{\beta,c})}{|\Lambda|}.$$

Proof. If l is a tangent of $P(\beta\Phi)$ as a function of β, i.e. for every real number β',

$$P(\beta'\Phi) - P(\beta\Phi) \geq l\big((\beta' - \beta) \cdot \Phi\big),$$

then by the Hahn-Banach extension theorem, l can be extended to a (not necessarily unique) tangent plane of the graph of $\Psi \in \mathcal{B}_0(\text{or } \mathcal{B}) \mapsto P(\Psi)$ at $\Psi = \beta\Phi$. But by the assumed uniqueness of the translation invariant equilibrium state, the tangent plane of $\Psi \in \mathcal{B}_0(\mathcal{B}) \mapsto P(\Psi)$ *is* unique and hence l has to be unique. This implies the differentiability of $P(\beta\Phi)$ w.r.t. β and hence from Theorem 1.1 our claim follows.

□

To get a more direct connection with the set-up of [6], we follow the Araki's construction of standard potentials for the case where time translation automorphisms are given [1]. We start from the assumption of the existence of a strongly continuous one-parameter group of *-automorphisms α_t of $\mathcal{A}$. Let δ be the generator of α_t and suppose that each finite type-I $*$-subalgebra $\mathcal{A}_\Lambda$ ($\Lambda \subset \mathbf{Z}^\nu$) is contained in the domain $D(\delta)$ of of the generator δ. By a result of Sakai in [7], for each finite $\Lambda \subset \mathbf{Z}^\nu$, there exists $H_\Lambda = H_\Lambda^* \in \mathcal{A}$ such that

$$\delta(a) = i[H_\Lambda, a] = i(H_\Lambda \cdot a - a \cdot H_\Lambda), \quad a \in \mathcal{A}_\Lambda. \tag{13}$$

Such an H_Λ is determined by δ, up to adding elements in A_{Λ^c}. Replacing H_Λ by $H_\Lambda - C^\omega_{\Lambda^c}(H_\Lambda)$, it is always possible to choose H_Λ in (13) satisfying

$$C^\omega_{\Lambda^c}(H_\Lambda) = 0. \tag{14}$$

The conditions (13) and (14) uniquely determine H_Λ. Let

$$U_\Lambda \equiv C^\omega_\Lambda(H_\Lambda), \quad W_\Lambda \equiv H_\Lambda - U_\Lambda. \tag{15}$$

The potential Φ is then defined recursively (from small Λ to large Λ').

$$\Phi(\Lambda) \equiv U_\Lambda - \sum_{X \subset \Lambda, X \neq \Lambda} \Phi(X). \tag{16}$$

¿From this definition, the function Φ satisfies the conditions (1),(3),(4)$'$,and (5)$'$. We denote by Φ^δ the Φ constructed from δ. Now we suppose that α_t is translation invariant, i.e. $\alpha_t\sigma_x = \sigma_x\alpha_t$ for each $x \in \mathbf{Z}^\nu$, then it is easy to see that Φ satisfies (2). This Φ^δ automatically belongs to $\mathcal{B}$ as pointed out to us by Araki. We recall the following well-known Theorem .

Theorem 2.4 *Let α_t be a translation invariant one-parameter group of *-automorphisms of $\mathcal{A}$, and let δ be the generator of α. Assume that the domain of δ contains $\mathcal{A}_\infty = \cup_\Lambda \mathcal{A}_\Lambda$, the union of all local algebras. If ρ is an invariant equilibrium state, i.e. $\rho \in I_{\beta\Phi}$, then ρ is an (α_t, β)-KMS state.*

Proof : See Theorem 6.2.42 in [3]. In this theorem, the potential is considered to be an exponentially decreasing function in the many-body terms, i.e. for some fixed $r > 0$

$$\|\Phi\|_r \equiv \sum_{\Lambda \ni 0} e^{r|\Lambda|} \|\Phi(\Lambda)\| < \infty.$$

But we can obviously use the same arguments if we consider $\mathcal{B}$ instead of $\mathcal{B}_r$ (the Banach space of potentials with the above norm $\|\cdot\|_r$).

Corollary 2.5 *Assume that the same conditions hold as in theorem 2.4. If φ is a unique (α_t, β)-KMS state, then*

$$s(\varphi) = \lim_{\Lambda \nearrow \mathbf{Z}^\nu} \frac{S_\Lambda(\varphi_\Lambda^{\beta,c})}{|\Lambda|}.$$

Proof : By theorem 2.4, there exists a unique equilibrium state. Thus from Corollary 2.3, our assertion follows.

□

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