On the Reduction of Adiabatic Dynamical Systems near Equilibrium Curves*

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Abstract

We consider adiabatic differential equations of the form $\varepsilon\, \mathrm{d}x/\mathrm{d}\tau = f(x,\tau)$, where ε is a small parameter. A few results on the behaviour of solutions close to an equilibrium curve of f are reviewed, including existence of tracking solutions, dynamic diagonalization and linearization, and invariant manifolds. We then point out some interesting connections between the effect of bifurcations, eigenvalues crossings and resonances.

Key words: adiabatic theory, slow–fast systems, invariant manifolds, bifurcation theory, dynamic bifurcations, eigenvalue crossing, resonance

1 Introduction

It frequently occurs that the dynamics of a physical system is governed by several time scales. Consider for instance an atom in a low frequency electric field, described by the Hamiltonian

$$H(q,p,t) = H_0(q,p) + q \cdot E(\varepsilon t), \tag{1}$$

where $H_0(q,p)$ is the Hamiltonian of the isolated atom, and $E(\varepsilon t)$ is an external electric field, which oscillates with a small frequency ε. The resulting equation of motion is a particular case of what we will call an **adiabatic system**,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x,\varepsilon t), \tag{2}$$

which is more conveniently written in the equivalent form

$$\varepsilon\frac{\mathrm{d}x}{\mathrm{d}\tau} = f(x,\tau), \tag{3}$$

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where $\tau = \varepsilon t$ is called **slow time**. This system is governed by two time scales, one for the unperturbed atom, which is of order 1, and one for the electric field, which is of order ε^{-1}. The ratio of these time scales can be made small by choosing a small enough value of the **adiabatic parameter** ε.

In other cases, the different time scales are intrinsic to the system, which can be described by a **slow-fast equation** of the form

$$\begin{aligned} \varepsilon \dot{x} &= f(x, y), \\ \dot{y} &= g(x, y). \end{aligned} \tag{4}$$

In this situation, x is called **fast variable** and y is called **slow variable**. A famous example is a perturbed integrable Hamiltonian system in action–angle variables,

$$H(I, \varphi) = H_0(I) + \varepsilon H_1(I, \varphi), \tag{5}$$

which is governed by the slow-fast equation

$$\begin{aligned} \varepsilon \, \mathrm{d}\varphi/\mathrm{d}\tau &= H_0'(I) + \varepsilon \partial_I H_1(I, \varphi), \\ \mathrm{d}I/\mathrm{d}\tau &= -\partial_\varphi H_1(I, \varphi). \end{aligned} \tag{6}$$

There are many more examples. Consider for instance two coupled "oscillators" with masses 1 and ε^2, described by the Hamiltonian

$$H(q_1, p_1, q_2, p_2) = \frac{1}{2} p_1^2 + V_1(q_1) + \frac{1}{2\varepsilon^2} p_2^2 + V_2(q_2) + \lambda q_1 q_2. \tag{7}$$

In this case, the light pendulum moves on a time scale ε^{-1}, and can be described by the fast variables $(q_2, \tilde{p}_2 = \varepsilon^{-1} p_2)$. The resulting slow–fast system is

$$\dot{q}_1 = p_1, \qquad \varepsilon \dot{q}_2 = \tilde{p}_2, \tag{8}$$

$$\dot{p}_1 = -V_1'(q_1) - \lambda q_2, \qquad \varepsilon \dot{\tilde{p}}_2 = -V_2'(q_2) - \lambda q_1. \tag{9}$$

The idea we would like to exploit is that for sufficiently small ε, the dynamics of the adiabatic system (3) or of the slow–fast system (4) should be close, in some sense, to the dynamics of the family of equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x, \lambda), \tag{10}$$

where λ is considered as a static parameter. This reduced system, being lower–dimensional (and autonomous), is easier to analyse than the original equation.

In this review, we only consider the adiabatic system (3), although certain results can be extended to the slow–fast system (4). The discussion is not limited to Hamiltonian systems, which means that the presented results are more general, but may not be optimal in that special case.

In Section 2, we discuss the situation when the reduced system (10) admits a hyperbolic equilibrium branch $x^\star(\lambda)$. In this case, the dynamics of (3) near the equilibrium can be analysed in detail, by showing the existence of adiabatic solutions, invariant manifolds, and dynamic normal forms. In Section 3, we comment some extensions to the elliptic case. In Section 4, we point out some interesting connections between bifurcations, eigenvalue crossings and resonances.

Detailed proofs of the results below can be found in [Berg] or other works which will be indicated. Some physical examples are discussed in [BK1, BK2].

2 Hyperbolic Case

2.1 Adiabatic Solutions

We consider the adiabatic differential equation

$$\varepsilon \dot{x} = f(x, \tau), \tag{11}$$

where $x \in \mathcal{D} \subset \mathbb{R}^n$, $\tau \in I \subset \mathbb{R}$, $f \in \mathcal{C}^k$ with $k \geqslant 2$, $\dot{x} = \mathrm{d}x/\mathrm{d}\tau$, and $\varepsilon > 0$ is a small parameter. In this section, we further assume that $x^\star(\tau)$ is a hyperbolic equilibrium branch of $f(x, \tau)$, that is,

$$\begin{aligned} f(x^\star(\tau), \tau) &= 0, \\ \partial_x f(x^\star(\tau), \tau) &= A(\tau), \end{aligned} \tag{12}$$

for all $\tau \in I$, where $A(\tau)$ has no purely imaginary eigenvalue. The first step is to show the existence of a particular solution of (11) which remains close to this equilibrium branch.

Theorem 1. *Let $x^\star(\tau)$ be defined on an interval $I \subset \mathbb{R}$, which need* not *be finite. We assume that there exist strictly positive constants c_0, a_0 and M such that*

$$\|f(x^\star(\tau) + y, \tau)\| \leqslant M \qquad \text{if } \|y\| \leqslant c_0, \tag{13}$$

$$|\mathrm{Re}\, a_j(\tau)| \geqslant a_0 \qquad \text{for each eigenvalue } a_j \text{ of } A(\tau), \tag{14}$$

uniformly for $\tau \in I$. Then there exist strictly positive constants c_k, c, C and ε_0 such that, when $0 < \varepsilon \leqslant \varepsilon_0$, the equation (11) admits a particular solution $\bar{x}(\tau)$ with the following properties:

1. *If $f(x, \tau) \in \mathcal{C}^2$, then*

$$\bar{x}(\tau) = x^\star(\tau) + \varepsilon R_1(\tau, \varepsilon), \tag{15}$$

with $\|R_1(\tau, \varepsilon)\| \leqslant c_1$ uniformly for $\tau \in I$.

2. *If $f(x, \tau) \in \mathcal{C}^k$, $k \geqslant 3$, then*

$$\bar{x}(\tau) = x^\star(\tau) + \sum_{j=1}^{k-2} \varepsilon^j x_j(\tau) + \varepsilon^{k-1} R_{k-1}(\tau, \varepsilon), \tag{16}$$

with $\|R_{k-1}(\tau, \varepsilon)\| \leqslant c_{k-1}$ uniformly for $\tau \in I$.

3. *If $f(x, \tau)$ is analytic in a complex neighborhood of $x^\star(\tau)$, then*

$$\bar{x}(\tau) = x^\star(\tau) + \sum_{j=1}^{N(\varepsilon)} \varepsilon^j x_j(\tau) + \mathrm{e}^{-1/C|\varepsilon|}\, R(\tau, \varepsilon), \tag{17}$$

with $\|R(\tau, \varepsilon)\| \leqslant c$ uniformly for $\tau \in I$ and $N(\varepsilon) = \mathcal{O}(1/\varepsilon)$.

In other words, this result means that if the equilibrium branch $x^\star(\tau)$ is uniformly hyperbolic, than there exists a particular solution of the adiabatic system tracking this branch at a distance of order ε. We call it an **adiabatic solution**. This solution can be expanded into powers of ε if f is sufficiently smooth, where the functions $x_j(\tau)$ can be 0 computed by an iterative scheme. In the ideal case, when f is analytic, it admits an asymptotic series in ε. This series is not convergent in general, but it admits an optimal 0 at exponentially small order.

This result has a rather long history, and appears to have been rediscovered several times (in fact it has been mainly studied in the context of slow–fast systems). The existence of a solution tracking an *attracting* equilibrium (i.e., such that all eigenvalues of A have a negative real part) is proved by Pontryagin and Rodygin [PR] using Lyapunov functions. A similar result is attributed to Tikhonov in [**VBK**]. The general hyperbolic case is treated by Fenichel [Fe], see also [Jo]. It is also related to the theory of shadowing. An alternative proof using Lyapunov functions is given in [Berg]. The exponentially small bound follows from an iterative scheme given by Neishtadt in [Ne]. An alternative proofs for maps, using Borel transformations, has been given by Baesens [Bæ].

In Theorem 1, we only prove the existence of an adiabatic solution, without any information on unicity. The forthcoming analysis of nearby solutions will show that if I is a finite interval, there exists an n–parameter family of adiabatic solutions close to any particular one. A particular solution may be selected by imposing special boundary conditions, either by letting I go to $\mathbb{R}$ and requiring the solution to be bounded, or by considering a periodic equation and imposing the same periodicity to the adiabatic solution.

2.2 Linear Systems

If $\bar{x}(\tau)$ is an adiabatic solution associated with the hyperbolic equilibrium branch $x^\star(\tau)$, the change of variables $x = \bar{x}(\tau) + y$ transforms (11) into $\varepsilon\dot{y} = A(\tau,\varepsilon)y + \mathcal{O}(\|y\|^2)$, where $A(\tau,\varepsilon) = \partial_x f(\bar{x}(\tau),\tau) = A(\tau) + \mathcal{O}(\varepsilon)$. Before dealing with nonlinear terms, we will analyse the linearized system

$$\varepsilon\dot{y} = A(\tau,\varepsilon)y. \tag{18}$$

The eigenvalues of $A(\tau,\varepsilon)$ can be indexed by continuous functions $a_j(\tau,\varepsilon)$, $j = 1,\dots,n$. Let us split them into two groups, and define their **real gap**

$$\gamma := \inf_{\substack{\tau\in I \\ 1\leqslant i\leqslant p \\ p+1\leqslant j\leqslant n}} \bigl|\mathrm{Re}\bigl(a_i(\tau) - a_j(\tau)\bigr)\bigr|, \tag{19}$$

for some p, $1 \leqslant p < n$. This gap is strictly positive, for instance, if the equilibrium is hyperbolic, and the first p eigenvalues have negative real part, while the others have positive real part. There are, however, other situations in which this gap is positive. Our main result is the following:

Theorem 2. *Assume that $A(\tau,\varepsilon)$ is of class $\mathcal{C}^3$ for $\tau\in I$, and that the real gap* (19) *is strictly positive. For sufficiently small ε and $\tau \in I$, there exists an invertible matrix $S(\tau,\varepsilon)$ such that* (18) *is equivalent to the equations*

$$y(\tau) = S(\tau,\varepsilon)z(\tau), \qquad \varepsilon\dot{z} = D(\tau,\varepsilon), \tag{20}$$

where $D(\tau,\varepsilon)$ is bloc–diagonal, with one bloc of size $p\times p$ and eigenvalues $a_j(\tau,\varepsilon)+\mathcal{O}(\varepsilon)$ for $j = 1,\dots,p$, and another bloc of size $(n-p)\times(n-p)$ and eigenvalues $a_j(\tau,\varepsilon)+\mathcal{O}(\varepsilon)$ for $j = p+1,\dots,n$.

The matrices $S(\tau,\varepsilon)$ and $D(\tau,\varepsilon)$ can be expanded into powers of ε. In particular, if $A(\tau,\varepsilon)$ is analytic in τ in a neighborhood of the (possibly unbounded) interval I, we have

$$\begin{aligned} S(\tau,\varepsilon) &= S_0(\tau) + \sum_{j=1}^{N(\varepsilon)} \varepsilon^j S_j(\tau) + \mathrm{e}^{-1/C|\varepsilon|}\, P(\tau,\varepsilon), \\ D(\tau,\varepsilon) &= D_0(\tau) + \sum_{j=1}^{N(\varepsilon)} \varepsilon^j D_j(\tau) + \mathrm{e}^{-1/C|\varepsilon|}\, Q(\tau,\varepsilon), \end{aligned} \tag{21}$$

where $C > 0$, $N = \mathcal{O}(1/\varepsilon)$, and all matrix elements of P and Q are bounded uniformly in τ and ε.

Corollary 1. *Assume that the eigenvalues of $A(\tau,\varepsilon)$ have uniformly disjoint real parts, that is,*

$$\inf_{\substack{\tau\in I \\ 1 \leqslant i<j \leqslant n}} \bigl|\mathrm{Re}\bigl(a_i(\tau) - a_j(\tau)\bigr)\bigr| > 0. \tag{22}$$

Then equation (18) can be diagonalized by a change of variables $y = S(\tau,\varepsilon)z$, and thus its principal solution can be written in the form

$$U(\tau,\tau_0) = S(\tau,\varepsilon) \begin{pmatrix} \mathrm{e}^{\alpha_1(\tau,\tau_0)/\varepsilon} & & 0 \\ & \ddots & \\ 0 & & \mathrm{e}^{\alpha_n(\tau,\tau_0)/\varepsilon} \end{pmatrix} S(\tau_0,\varepsilon)^{-1}, \tag{23}$$

where

$$\alpha_j(\tau,\tau_0) = \int_{\tau_0}^{\tau} a_j(s,\varepsilon)\,\mathrm{d}s + \mathcal{O}(\varepsilon). \tag{24}$$

Theorem 2 is similar to the adiabatic theorem of quantum mechanics, which states that if an eigenvalue of $A(\tau,\varepsilon)$ is spectrally isolated, than the associated eigenspace tends to be invariant in the limit $\varepsilon \to 0$, see for instance [**Wa**, Berry, JKP]. There is, however, a difference between Theorem 2 and those results. In our language, they show the existence of a transformation $y = S(\tau,\varepsilon)z$ such that the new system has small off–diagonal terms (of polynomial or exponential order, depending on the cases). This is sufficient in quantum mechanics, since solutions have a constant norm.

When the eigenvalues have different real parts, however, even exponentially small off–diagonal terms can lead to appreciable transition amplitudes. For this reason, it is needed to eliminate these terms totally. The transformation is constructed by imposing $S(\tau,\varepsilon)$ to be a solution of the differential equation

$$\varepsilon\dot{S} = AS - SD. \tag{25}$$

This system can be transformed in such a way that Theorem 1 can be applied, yielding the existence of the matrices S and D together with their asymptotic series and exponential bounds.

2.3 Invariant Manifolds

Let us return to the analysis of the motion near a hyperbolic equilibrium. After translating the coordinates to an adiabatic solution, we can separate the expanding and contracting parts of the linearization by applying Theorem 2. We thus end up with the system

$$\begin{aligned} \varepsilon \dot{u} &= D_{+}(\tau,\varepsilon)u + b_{+}(u,v,\tau,\varepsilon), \\ \varepsilon \dot{v} &= D_{-}(\tau,\varepsilon)v + b_{-}(u,v,\tau,\varepsilon), \end{aligned} \tag{26}$$

where D_+ has eigenvalues with a real part larger than some constant $a_0 > 0$, D_- has eigenvalues with a real part smaller than $-a_0$, and $b_\pm = \mathcal{O}(\|u\|^2 + \|v\|^2)$.

This formulation is not yet satisfactory, since the manifolds $u = 0$ and $v = 0$ are not invariant, so that we are not able to decide in which direction a solution initially close to $u = 0$ will leave this region. This question can be solved by introducing invariant manifolds, generalizing the stable manifold theorem for autonomous differential equations.

Theorem 3. *Assume* (26) *is of class* $\mathcal{C}^2$, *and* ε *is small enough.*

1. *In a neighborhood of* $u = 0$ *and* $v_1 = 0$, *there exist continuous functions* $\eta(u,\tau,\varepsilon) = \mathcal{O}(\|u\|^2)$ *and* $\xi(v_1,\tau,\varepsilon) = \mathcal{O}(\|v_1\|^2)$, *such that the successive changes of variables* $v = \eta(u,\tau,\varepsilon) + v_1$ *and* $u = \xi(v_1,\tau,\varepsilon) + u_1$ *transform* (26) *into*

$$\begin{aligned} \varepsilon \dot{u}_1 &= \big[D_{+}(\tau,\varepsilon) + B_{+}(u_1,v_1,\tau,\varepsilon)\big]u_1 \\ \varepsilon \dot{v}_1 &= \big[D_{-}(\tau,\varepsilon) + B_{-}(u_1,v_1,\tau,\varepsilon)\big]v_1, \end{aligned} \tag{27}$$

   where $B_\pm = \mathcal{O}(\|u_1\| + \|v_1\|)$.

2. *If* (26) *is* $\mathcal{C}^k$, $k \geqslant 2$, η *admits an expansion*

$$\eta(u,\tau,\varepsilon) = \sum_{j=0}^{k-2} \varepsilon^j \eta_j(u,\tau) + \varepsilon^{k-1}\rho_{k-1}(u,\tau,\varepsilon), \tag{28}$$

   where $\|\rho_{k-1}(u,\tau,\varepsilon)\| \leqslant c_{k-1}\|u\|^2$ *and* $\xi(v_1,\tau,\varepsilon)$ *admits a similar expansion.*

3. *If* (26) *is analytic in an open complex set,* η *admits an expansion*

$$\eta(u,\tau,\varepsilon) = \sum_{j=0}^{N(\varepsilon)} \varepsilon^j \eta_j(u,\tau) + \mathrm{e}^{-1/C|\varepsilon|}\,\rho(u,\tau,\varepsilon), \tag{29}$$

   where $N(\varepsilon) = \mathcal{O}(1/\varepsilon)$, $\|\rho(u,\tau,\varepsilon)\| \leqslant c\|u\|^2$ *and similarly for* ξ.

This result has the following interpretation: the manifolds with parametric equation $v = \eta(u,\tau,\varepsilon)$ and $u = \xi(v_1,\tau,\varepsilon)$ define, respectively, a local unstable and stable adiabatic manifold, on which the motion is, respectively, expanding and contracting. The stable manifold separates the neighborhood of the adiabatic solution into two regions, from which trajectories escape in different directions.

The existence of invariant manifolds for non-autonomous equations has been proved in some particular cases in [**Hal**]. Results similar to point 1. have been obtained in [Fe], see also [Jo].

2.4 Dynamic Linearization

On the invariant manifold $v = \eta(u, \tau, \varepsilon)$, equation (26) becomes

$$\varepsilon \dot{u} = D_+(\tau, \varepsilon) u + \beta_+(u, \tau, \varepsilon), \tag{30}$$

where $\beta_+(u, \tau, \varepsilon) = b_+(u, \eta, \tau, \varepsilon) = \mathcal{O}(\|u\|^2)$. To simplify this equation further, it would be ideal to be able to remove the 0 term β_+ completely. This turns out to be possible under more restrictive conditions on the eigenvalues of the matrix D_+.

Theorem 4. *Assume that there exists a positive integer N such that D_+ and β_+ are $\mathcal{C}^N$ functions of τ, u and ε, and that the eigenvalues $d_1(\tau, \varepsilon), \dots, d_m(\tau, \varepsilon)$ of D_+ satisfy the relations*

$$\begin{aligned} 0 < \operatorname{Re} d_1(\tau, \varepsilon) < \ldots < \operatorname{Re} d_n(\tau, \varepsilon), \\ N \operatorname{Re} d_1(\tau, \varepsilon) > \operatorname{Re} d_n(\tau, \varepsilon) \end{aligned} \tag{31}$$

as well as the non–resonance conditions

$$\sum_{j=1}^{m} p_j d_j(\tau, \varepsilon) \neq d_k(\tau, \varepsilon), \qquad p_j \geqslant 0,\ 2 \leqslant \sum_j p_j \leqslant N,\ k = 1, \dots m \tag{32}$$

uniformly in τ, ε. Then there exists, for small ε and $\|u_1\|$, a function $h(u_1, \tau, \varepsilon) = \mathcal{O}(\|u_1\|^2)$ such that the change of variables $u = u_1 + h(u_1, \tau, \varepsilon)$ transforms equation (30) *into its linearization*

$$\varepsilon \dot{u}_1 = D_+(\tau, \varepsilon) u_1. \tag{33}$$

The function $h(u_1, \tau, \varepsilon)$ can be expanded into powers of u and ε up to order N.

A similar result is valid for the contracting part of the flow, when all inequalities in (31) are reversed (this is obtained simply by reversing the direction of time). This result is an extension to the non–autonomous case of a result by Sternberg and Chen [Ch, **Har**] (in fact, of the simpler case discussed in [St]). Once the equation has been linearized, it can be studied by Corollary 1.

Theorems 1 to 4 provide a rather complete picture of the flow in a vicinity of the equilibrium curve, under some conditions on the linearization around this curve. Each result shows the existence of a transformation which simplifies the equation near the equilibrium, in such a way that it becomes solvable in certain cases. The transformations can only be constructed approximately (at best up to an exponentially small remainder), but this is often sufficient for answering many questions of physical interest (see [BK1] for a concrete example).

These results leave open the question of what happens when the conditions on the linear part are not verified. In the next two sections, we will show how to analyse some of these exceptions.

3 Elliptic Case

Some results of the previous section can be extended to the case of elliptic equilibria, with, however, some restrictions due to the possibility of resonance. We consider again the adiabatic differential equation

$$\varepsilon \dot{x} = f(x, \tau), \tag{34}$$

where $x \in \mathcal{D} \subset \mathbb{R}^n$, $\tau \in I \subset \mathbb{R}$, $f \in \mathcal{C}^k$ with $k \geqslant 3$, $\dot{x} = \mathrm{d}x/\mathrm{d}\tau$, and $\varepsilon > 0$ is a small parameter. We now assume that $x^\star(\tau)$ is a smooth curve such that

$$\begin{aligned} f(x^\star(\tau),\tau) &= 0, \\ \partial_x f(x^\star(\tau),\tau) &= A(\tau), \end{aligned} \tag{35}$$

for all $\tau \in I$, where the eigenvalues of $A(\tau)$ have a vanishing real part. Then the following results correspond respectively to Theorems 1 and 2.

Theorem 5. *Let I be a* bounded *interval. Assume that the eigenvalues $a_j(\tau) = \mathrm{i}\,\omega_j(\tau)$ are all imaginary and distinct. For sufficiently small ε, equation* (34) *admits a particular solution*

$$\bar{x}(\tau) = x^\star(\tau) + \varepsilon r_1(\tau,\varepsilon), \tag{36}$$

where $\|r_1(\tau,\varepsilon)\| \leqslant c_1$ uniformly for $\tau \in I$. As in Theorem 1, if f and $x^\star$ are sufficiently smooth, this solution can be expanded into powers of ε, up to exponentially small order in the analytic case.

Theorem 6. *Let I be a* bounded *interval, and consider for $\tau \in I$ the equation*

$$\varepsilon \dot{y} = A(\tau,\varepsilon) y, \tag{37}$$

where $A(\tau,\varepsilon) \in \mathcal{C}^3$ has eigenvalues $a_j(\tau,\varepsilon) = \mathrm{i}\,\omega_j(\tau) + \mathcal{O}(\varepsilon)$, with the ω_j all real and distinct for $\tau \in I$. For sufficiently small ε, there exist an invertible matrix-valued function $S(\tau,\varepsilon)$ and scalar functions

$$\phi_j(\tau,\tau_0) = \int_{\tau_0}^{\tau} \omega_j(s)\,\mathrm{d}s + \mathcal{O}(\varepsilon), \qquad j = 1,\dots,n \tag{38}$$

such that the principal solution of (37) *can be written in the form*

$$U(\tau,\tau_0) = S(\tau,\varepsilon) \begin{pmatrix} \mathrm{e}^{\mathrm{i}\,\phi_1(\tau,\tau_0)/\varepsilon} & & 0 \\ & \ddots & \\ 0 & & \mathrm{e}^{\mathrm{i}\,\phi_n(\tau,\tau_0)/\varepsilon} \end{pmatrix} S(\tau_0,\varepsilon)^{-1} \tag{39}$$

for $\tau_0, \tau \in I$. If $A(\tau,\varepsilon)$ is sufficiently smooth, the functions S and ϕ_j can be expanded into powers of ε, up to exponentially small order in the analytic case.

The proofs of these two theorems are, in fact, closely related, and can be carried out by induction on the dimension n. The following example shows that without further assumptions, the boundedness of the interval I is a necessary condition. Consider, indeed, the simple equation

$$\varepsilon \dot{x} = \mathrm{i}\,x + h(\tau), \tag{40}$$

which admits the explicit solution

$$x(\tau) = \mathrm{e}^{\mathrm{i}\,\tau/\varepsilon}\,x(0) + \mathrm{e}^{\mathrm{i}\,\tau/\varepsilon}\,\frac{1}{\varepsilon}\int_0^{\tau} \mathrm{e}^{-\mathrm{i}\,s/\varepsilon}\,h(s)\,\mathrm{d}s. \tag{41}$$

If $h(\tau)$ is twice differentiable, integrations by part show that

$$x(\tau) = x_1(\tau,\varepsilon) + \mathrm{e}^{\mathrm{i}\,\tau/\varepsilon}\big[x(0) - x_1(0,\varepsilon)\big] - \varepsilon\,\mathrm{e}^{\mathrm{i}\,\tau/\varepsilon}\int_0^\tau \mathrm{e}^{-\mathrm{i}\,s/\varepsilon}\,h''(s)\,\mathrm{d}s, \tag{42}$$

where $x_1(\tau,\varepsilon) = \mathrm{i}\,h(\tau) + \varepsilon h'(\tau)$. If $h''(\tau)$ is a sufficiently wild function, such as $\mathrm{e}^{\mathrm{i}\,/s}$, the last term may grow linearly with τ. Another way to look at this example is to consider a periodic $h(\tau)$,

$$h(\tau) = \sum_{p=-\infty}^{\infty} \hat{h}(p)\,\mathrm{e}^{\mathrm{i}\,p\tau}\,. \tag{43}$$

Let q be the closest integer to $1/\varepsilon$. The solution can be written

$$x(\tau) = \sum_{p\neq q} \frac{\mathrm{i}\,\hat{h}(p)}{1-p\varepsilon}\,\mathrm{e}^{\mathrm{i}\,p\tau} + \mathrm{e}^{\mathrm{i}\,\tau/\varepsilon}\Big[c + \hat{h}(q)\frac{1}{\varepsilon}\int_0^\tau \mathrm{e}^{\mathrm{i}\,s(q-1/\varepsilon)}\,\mathrm{d}s\Big], \tag{44}$$

where c is an integration constant. In particular, when $q = 1/\varepsilon$, there is a resonance and the last term grows as $\hat{h}(1/\varepsilon)\tau/\varepsilon$. Its amplitude depends again on the smoothness of h, in general it will grow as $\varepsilon^k\tau$ if $h(\tau)$ is of class $\mathcal{C}^k$.

These results may of course be substantially improved under additional hypotheses. For instance, it is well known that the principal solution is unitary if the matrix $A(\tau,\varepsilon)$ is anti–hermitian. And if the system is Hamiltonian, it becomes possible to apply KAM theory, see for instance [Ar]. A dynamic linearization is probably possible under appropriate Diophantine conditions on the eigenvalues.

4 Bifurcations, Crossings and Resonances

The results of Section 2 fail when certain "hyperbolicity" conditions are not satisfied: Theorem 1 on existence of adiabatic solutions fails when eigenvalues of the linearization cross the imaginary axis, i.e., in case of **bifurcations**; Theorem 2 on dynamic diagonalization fails in situations of **eigenvalue crossing**; and Theorem 4 on dynamic linearization fails in case of **resonance** between eigenvalues. It turns out that there exist certain connections between these phenomena, which allow to treat them in a unified way. These connections have not, to the best of our knowledge, been exploited to the present date. We present here some of the main ideas, more detailed results can be found in [Berg].

4.1 Bifurcations

Assume that the origin is a bifurcation point of the adiabatic system, at which exactly one eigenvalue vanishes. It is possible to extend the center manifold theorem in order to reduce the system to a one–dimensional one. Let us thus consider the equation

$$\varepsilon\dot{x} = f(x,\tau) = \sum_{n,m\,\geqslant\,0} c_{nm}x^n\tau^m \tag{45}$$

in a neighborhood of the origin, where x is a scalar variable. For simplicity, we assume $f(x,\tau)$ to be analytic. The origin is a bifurcation point if $c_{00} = c_{10} = 0$.

Let $x^\star(\tau)$ be an equilibrium branch of (45) reaching the origin. It appears that the behaviour of solutions near the bifurcation point is controlled by two rational numbers q and p, defined by the relations

$$|x^\star(\tau)| \asymp |\tau|^q, \qquad |\partial_x f(x^\star(\tau),\tau)| \asymp |\tau|^p, \tag{46}$$

where the notation $x \asymp y$ means that $c_- x \leqslant y \leqslant c_+ x$ for two positive constants $c_\pm$ independent of τ and ε. The numbers q and p can be determined graphically by **Newton's polygon**, which is constructed with the convex envelope of points (n,m) such that $c_{nm} \neq 0$: $-q$ is the slope of a tangent to the polygon, and p its ordinate at 1.

Assume that $x^\star(\tau) \asymp |\tau|^q$ is a stable decreasing branch reaching the origin, such that $f(x^\star + y, \tau)$ is negative for small positive y. Then we can show that generically, the adiabatic solution $\bar{x}(\tau)$ tracking this branch obeys the scaling relation

$$\bar{x}(\tau) - x^\star(\tau) \asymp \begin{cases} \varepsilon|\tau|^{q-p-1} & \text{for } \tau \leqslant -\varepsilon^{\frac{1}{p+1}}, \\ \varepsilon^{\frac{q}{p+1}} & \text{for } -\varepsilon^{\frac{1}{p+1}} \leqslant \tau \leqslant 0. \end{cases} \tag{47}$$

The passage through the bifurcation point has two important effects. One of them is that after crossing this point, the solution may follow one of several outgoing branches, or quickly leave the vicinity of the bifurcation point, which may lead to hysteresis when the parameter is varied periodically. The other effect is the nontrivial scaling behaviour of the solutions with ε, which is also reflected by properties such as the surface of hysteresis cycles.

These phenomena belong to the field of **dynamic bifurcations**, which has been studied by several authors in recent years. See [**Ben**] for a review, and [JGRM, HL&, GBS] for the study of scaling laws in some particular cases. In [Berg], we developed a new method to study these scaling laws from a qualitative point of view.

4.2 Eigenvalue Crossings

We saw in Section 2.2 that a linear equation of the form

$$\varepsilon\dot{y} = A(\tau)y \tag{48}$$

could be (bloc–)diagonalized by a linear change of variables $y = S(\tau,\varepsilon)z$, where S satisfies an equation of the form

$$\varepsilon\dot{S} = AS - SD. \tag{49}$$

This system admits bounded solutions provided the eigenvalues' real parts don't cross.

There are several types of crossing. If $A(\tau)$ has no particular symmetry, it is generically not diagonalizable at the crossing point. This is a well-known turning point problem reducible to Airy's equation [**Wa**, **Ol**]. Another situation arises when, for instance, the matrix $A(\tau)$ is symmetric and thus remains diagonalizable at the crossing point. Let us discuss the example of the matrix

$$A(\tau) = a(\tau)\begin{pmatrix} \cos 2\theta(\tau) & \sin 2\theta(\tau) \\ \sin 2\theta(\tau) & -\cos 2\theta(\tau) \end{pmatrix}, \tag{50}$$

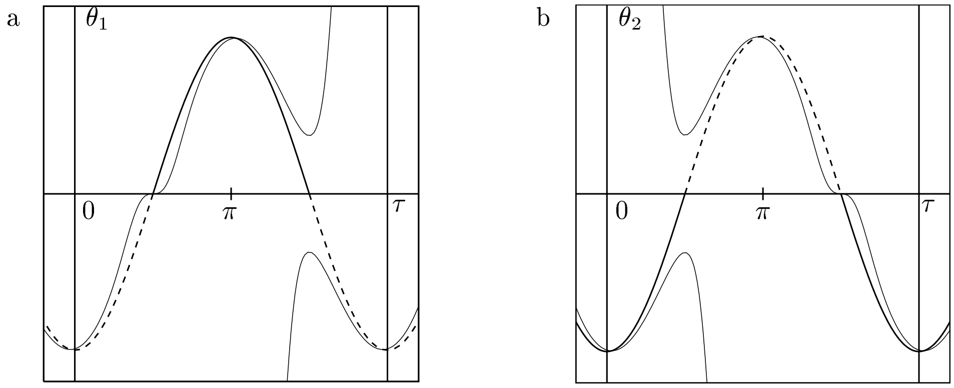

FIGURE 1. Solutions (thin lines) of (52) and (53) when $a(\tau) = \theta(\tau) = -\cos\tau$. In both cases, one can construct a particular solution remaining close to the static equilibrium $\theta(\tau)$ (thick lines, where the solid lines indicate stable branches and the dashed lines unstable ones), admitting a discontinuity of order $\sqrt{\varepsilon}$ at $\tau = \frac{\pi}{2}$ or $\tau = \frac{3\pi}{2}$.

which has eigenvalues $\pm a(\tau)$ and eigenvectors $(\cos\theta, \sin\theta)$ and $(-\sin\theta, \cos\theta)$. We try to solve (49) with the Ansatz

$$S = \begin{pmatrix} \cos\theta_1(\tau) & -\sin\theta_2(\tau) \\ \sin\theta_1(\tau) & \cos\theta_2(\tau) \end{pmatrix}, \qquad D = \begin{pmatrix} d_1(\tau) & 0 \\ 0 & d_2(\tau) \end{pmatrix}, \tag{51}$$

which yields the equations

$$\varepsilon\dot{\theta}_1 = -a(\tau)\sin 2(\theta_1 - \theta(\tau)), \qquad d_1(\tau) = a(\tau)\cos 2(\theta_1 - \theta(\tau)), \tag{52}$$

$$\varepsilon\dot{\theta}_2 = a(\tau)\sin 2(\theta_2 - \theta(\tau)), \qquad d_2(\tau) = -a(\tau)\cos 2(\theta_2 - \theta(\tau)). \tag{53}$$

Two of these equations are adiabatic equations for θ_1 and θ_2, while the other two determine the diagonal matrix $D(\tau)$. As long as $a(\tau) \neq 0$, Theorem 1 shows that the adiabatic equations admit solutions $\theta_j(\tau) = \theta(\tau) + \mathcal{O}(\varepsilon)$, and thus (48) can be diagonalized.

Assume now that $a(\tau)$ vanishes, so that the eigenvalues of $A(\tau)$ cross. We can translate τ and θ in such a way that $a(0) = \theta(0) = 0$, so that (52) and (53) admit bifurcation points at the origin. The discussion of the previous subsection applies, with exponents q and p defined by the relations $|\theta(\tau)| \asymp |\tau|^q$ and $|a(\tau)| \asymp |\tau|^p$. In the most generic case, $q = p = 1$, which means in particular that the equations admit solutions $\theta_j(\tau)$ tracking $\theta(\tau)$ at a distance scaling as $\sqrt{\varepsilon}$ when τ approaches 0. For positive time, the behaviour of solutions depends on the signs of a and θ. It turns out that in this generic case, the solution of one equation keeps tracking the equilibrium branch, while the solution of the other one escapes the vicinity of the bifurcation point (Fig. 1).

The result is that one of these solutions, say $\theta_2(\tau)$, must admit a discontinuity of order $\sqrt{\varepsilon}$ at the bifurcation point in order to remain close to $\theta(\tau)$. If $\tau_0 < 0 < \tau$, the principal solution of (48) thus takes the form

$$U(\tau,\tau_0) = S(\tau) \begin{pmatrix} e^{\delta_1(\tau,0)/\varepsilon} & 0 \\ 0 & e^{\delta_2(\tau,0)/\varepsilon} \end{pmatrix} T \begin{pmatrix} e^{\delta_1(0,\tau_0)/\varepsilon} & 0 \\ 0 & e^{\delta_2(0,\tau_0)/\varepsilon} \end{pmatrix} S(\tau_0)^{-1}, \tag{54}$$

where $\delta_j(\tau,\tau_0) = \int_{\tau_0}^{\tau} d_j(s)\,ds + \mathcal{O}(\sqrt{\varepsilon})$, and T is an unavoidable transition matrix given by

$$T = S(0+)^{-1}S(0-) = \begin{pmatrix} 1 + \mathcal{O}(\sqrt{\varepsilon}) & \sin(\theta_2^+ - \theta_2^-) + \mathcal{O}(\varepsilon) \\ 0 & 1 \end{pmatrix}, \tag{55}$$

where $\theta_2^{\pm} = \theta_2(0\pm)$. Because of this transition matrix, there is only one invariant subspace surviving the eigenvalue crossing.

4.3 Dynamic Normal Forms and Resonances

Let us consider again the equation

$$\varepsilon\dot{u} = D(\tau)u + b(u,\tau), \qquad b(u,\tau) = \mathcal{O}(\|u\|^2). \tag{56}$$

Theorem 4 asserts that the nonlinear term can be removed by a change of variables, under certain conditions on the eigenvalues $d_1(\tau),\dots,d_m(\tau)$ of $D(\tau)$. These conditions can be relaxed somewhat. Let us still assume the existence of an integer N such that (56) is of class $\mathcal{C}^N$, and

$$N \min_j d_j(\tau) > \max_j d_j(\tau) > 0 \tag{57}$$

uniformly for $\tau \in I$. Let us however allow a relation of the following kind to hold for certain values of τ:

$$\sum_{j=1}^m d_j(\tau)p_j = d_k(\tau), \qquad 2 \leqslant \sum_{j=1}^m p_j \leqslant N, \tag{58}$$

where the p_j are positive integers. In such a case, the m–tuple $p = (p_1,\dots,p_m)$ is called **resonant** with k at time τ. Let e_j denote the j^{th} canonical basis vector in $\mathbb{R}^m$, and $u^p := u_1^{p_1}\dots u_m^{p_m}$. Theorem 4 can be extended to show the existence of a local transformation $u = v + h(v,\tau,\varepsilon)$ changing (56) into

$$\varepsilon\dot{v} = D(\tau)v + \sum_{(p,k)\text{ resonant}} c_{p,k}(\tau)v^p e_k, \tag{59}$$

where the sum extends over all (p,k) such that relation (58) is satisfied for *some* value of $\tau \in I$. We call (59) the **dynamic normal form** of (56). It is possible to remove even those resonant terms by a change of variables admitting some irregularity at the resonance time, just as linear systems undergoing eigenvalue crossing can be diagonalized by a transformation admitting an irregularity at the crossing time, which can be analysed as a bifurcation problem.

As an illustration, consider the two–dimensional case with eigenvalues $d_1(\tau) = 1$ and $d_2(\tau) = 2 + \tau$. There is a resonance between $p = (2,0)$ and $k = 2$ and $\tau = 0$. Thus the dynamic normal form close to $\tau = 0$ reads

$$\begin{aligned} \varepsilon\dot{v}_1 &= v_1, \\ \varepsilon\dot{v}_2 &= (2+\tau)v_2 + c(\tau)v_1^2. \end{aligned} \tag{60}$$

Let us forget that this equation can be solved exactly, and observe that the nonlinear term can be removed by a change of variables $v_2 = w_2 + y(\tau)v_1^2$, where $y(\tau)$ satisfies the differential equation

$$\varepsilon\dot{y} = \tau y + c(\tau). \tag{61}$$

This linear equation can be solved exactly. The question is whether one can construct a *bounded* solution. In fact, one can choose two particular solutions, one of which is bounded for positive times, and the other one bounded for negative times. The discontinuity at $\tau = 0$ is of order $\varepsilon^{-1/2}$ (this factor has a small effect if x is of order $\varepsilon^{1/4}$). If we denote the linearizing change of variables by $v = H(\tau)(w)$, we obtain that for $\tau_0 < 0 < \tau$,

$$v(\tau) = H(\tau) \circ U(\tau, 0) \circ H(0+)^{-1} \circ H(0-) \circ U(0, \tau_0) \circ H(\tau_0)^{-1}(v(\tau_0)), \tag{62}$$

where U is the principal solution of the linearized equation. The effect of the resonance appears in the transition term $H(0+)^{-1} \circ H(0-)$. Note that if τ is replaced by $-\tau$ in (60), it is possible to construct a bounded solution of (61), and this transition term reduces to identity.

The relation with bifurcation problems is that the change of variables

$$x = \varepsilon y - C(\tau), \qquad C(\tau) = \int_0^\tau c(s)\, \mathrm{d}s \tag{63}$$

transforms (61) into

$$\varepsilon \dot{x} = \tau(x + C(\tau)). \tag{64}$$

This equation admits an equilibrium $x = -C(\tau)$ bifurcating at $\tau = 0$.

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