LONG-TIME ASYMPTOTICS FOR THE COUPLED MAXWELL-LORENTZ EQUATIONS

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Abstract. We determine the long time behavior of solutions to the Maxwell-Lorentz equations, which describe a charge coupled to the electromagnetic field and subject to external time-independent potentials. The stationary solutions have vanishing magnetic field and a Coulomb type electrostatic field centered at the points of the set Z at which the external force vanishes. We prove that solutions of finite energy with bounded charge trajectory converge, in suitable local energy seminorms, to the set of stationary states in the long time limit $t \to \pm\infty$. If the set Z is discrete, this implies the convergence to a definite stationary state. For an unbounded particle trajectory, at least, the acceleration vanishes as $t \to \pm\infty$ and the Maxwell field as seen from the particle converges to the stationary Coulomb field.

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1 Introduction

We consider a single charge coupled to the Maxwell field and subject to prescribed time-independent external potentials. If $q(t) \in \mathbb{R}^3$ denotes the position of charge at time t, then the coupled Maxwell-Lorentz equations read

$$\begin{aligned}
\operatorname{div} E(x,t) = \rho(x-q(t))\,, \qquad & \operatorname{rot} E(x,t) = -\dot{B}(x,t), \\
\operatorname{div} B(x,t) = 0\,, \qquad & \operatorname{rot} B(x,t) = \dot{E}(x,t) + \rho(x-q(t))\dot{q}(t)\,, \\
\dot{q}(t) = (1+p^2(t))^{-1/2} p(t)\,, \qquad & \dot{p}(t) = E_{ex}(q(t)) + \dot{q}(t) \wedge B_{ex}(q(t)) \\
& + \int d^3x\, \rho(x-q(t))[E(x,t) + \dot{q}(t) \wedge B(x,t)]. \qquad (1.1)
\end{aligned}$$

Here and below all derivatives are understood in the sense of distributions. The last line is the Lorentz force equation and the first two lines are the inhomogeneous Maxwell equations. $E(x,t)$, $B(x,t)$ is the Maxwell field. $E_{ex} = -\nabla\phi_{ex}$, $B_{ex} = \operatorname{rot} A_{ex}$ are prescribed, time-independent external fields. ρ is the charge distribution of the particle. We use units such that the velocity of light $c = 1$, $\varepsilon_0 = 1$, and the mechanical rest mass of the particle $m_0 = 1$.

For a wealth of physical applications it suffices to split Eqs. (1.1) into two parts: Either one regards $q(t)$ as given and determines the Maxwell field E, B as the solution of an inhomogeneous wave equation. Or one considers the electromagnetic fields as given and solves for the motion of the charge. In both cases one may take the point charge limit and substitute $\rho(x-q)$ by $e\delta(x-q)$.

The perhaps most basic physical phenomenon where such a decoupling is no longer possible is radiative damping: an accelerated charge generates electromagnetic fields and thereby looses energy and momentum through radiation. Thus in the long-time limit the particle will either come to rest or, if permitted by the external potentials, move with uniform velocity. The main goal of our paper is to deduce such a qualitative behavior from Eqs. (1.1).

In the standard discussions of radiative damping one employs the Lorentz-Dirac equation, where the damping effects due to radiation are globally summarized through a sort of friction term. To allow a uniform motion it must be proportional to $\dddot{q}\,(t)$ rather than $\dot{q}(t)$. Thereby unphysical, so-called runaway, solutions appear which are eliminated through appropriate asymptotic conditions at $t = \pm\infty$. At present we regard the precise relationship between the Lorentz-Dirac equation and Eqs. (1.1) as an open problem. We refer to [7, 13, 18, 21] for an exhaustive treatment and only add one point to the discussion: It is argued that the Lorentz-Dirac equation is the point charge limit of Eqs. (1.1) [13, 21]. In this limit the electromagnetic mass of the charge tends to $+\infty$. To compensate, the (bare) mechanical rest mass m_0 tends to $-\infty$. However, even before taking any limits, if the rest mass $m_0 < 0$, then Eqs. (1.1) have already solutions increasing exponentially in time [4]. Thus the proposed limiting procedure is questionable.

Eqs. (1.1) are not Lorentz invariant, since we adopted a rigid charge distribution (the so-called Abraham model). It remains to be seen whether our techniques also apply to the Lorentz model, where the charge distribution is the same in each rest frame. A further obvious limitation is the restriction to a single charge. Since basically we exploit the conservation of energy, in the case of several particles only some qualitative information on the center of mass can be extracted, but not on the motion of individual particles.

We consider finite energy solutions to the Eqs. (1.1). The appropriate phase space will be introduced below, but first we note that the energy integral

$$\mathcal{H}(E,B,q,p) = (1+p^2)^{1/2} + \phi_{ex}(q) + \frac{1}{2}\int d^3x \left(|E(x)|^2 + |B(x)|^2\right) \tag{1.2}$$

is conserved along solution trajectories of (1.1). It is then natural to choose as phase space the set of all states with finite $\mathcal{H}$. In fact, (1.1) could be put into Hamiltonian form. In the canonical coordinates $\mathcal{H}$ takes then the role of the Hamiltonian of the system. We will not make use of the Hamiltonian structure, here.

For later convenience we state our assumptions on ϕ_{ex}, A_{ex}, and ρ. The external potentials satisfy

$$\phi_{ex}, A_{ex} \in C^\infty_{loc}(\mathbb{R}^3) \quad \text{and} \quad \inf_{q\in\mathbb{R}^3} \phi_{ex}(q) > -\infty. \tag{$P_{\min}$}$$

Alternatively, we will also use the following conditions

$$E_{ex}, B_{ex}, \nabla E_{ex}, \nabla B_{ex} \in C^\infty(\mathbb{R}^3) \quad \phi_{ex}, A_{ex} \in C^\infty(\mathbb{R}^3)\,, \tag{$P_{\max}$}$$

$$\lim_{|q|\to\infty} \phi_{ex}(q) = \infty\,. \tag{P_∞}$$

The charge distribution ρ is smooth, radially symmetric, and of compact support,

$$\rho \in C_0^\infty(\mathbb{R}^3)\,, \quad \rho(x) = \rho_r(|x|)\,, \quad \rho(x) = 0 \text{ for } |x| \geq R_\rho\,. \tag{C}$$

As noted in [10] a further important assumption is the Wiener condition

$$\hat\rho(k) = \int d^3x\, e^{ikx}\rho(x) \neq 0 \text{ for } k \in \mathbb{R}^3\,. \tag{W}$$

It ensures that all modes of the Maxwell field couple to the charge. In particular the total charge $\overline{\rho} = \hat\rho(0) \neq 0$. Charge distributions satisfying both (W) and (C) have been constructed in [10, §10]. Finally we state the condition

$$q_0 = \sup_{t\in\mathbb{R}} |q(t)| < \infty. \tag{Q}$$

To study the long-time behavior of the finite energy solutions to (1.1), clearly, the first step is to determine stationary solutions. For every $q \in \mathbb{R}^3$ we define the state $S_q = (E_q, B, q, p)$ by

$$S_q = (E_q, 0, q, 0), \quad E_q(x) = -\nabla\phi_q(x), \quad \phi_q(x) = \int \frac{d^3y}{4\pi|y-x|}\rho(y-q). \tag{1.3}$$

Let $Z = \{q \in \mathbb{R}^3 : \nabla\phi_{ex}(q) = 0\}$ be the set of points for which the external electrostatic force vanishes. The set of stationary solutions is simply given by

$$\mathcal{S} = \{S_q : q \in Z\}. \tag{1.4}$$

A second set of asymptotic solutions corresponds to the charge traveling with uniform velocity, v, when $E_{ex} = 0 = B_{ex}$. They are of the form

$$S_q^v(t) = (E^v(x-q-vt), B^v(x-q-vt), q+vt, p^v) \tag{1.5}$$

with $q \in \mathbb{R}^3$ and arbitrary velocity $v \in \mathcal{V} = \{v \in \mathbb{R}^3 : |v| < 1\}$, where

$$\begin{aligned} E^v(x) &= -\nabla\phi^v(x) + v\cdot\nabla\, A^v(x)\,, \quad B^v(x) = \operatorname{rot} A^v(x)\,, \\ \phi^v(x) &= \int d^3y\rho(y)(4\pi)^{-1}((1-v^2)(y-x)^2 + (v\cdot(y-x))^2)^{-1/2} \\ A^v(x) &= v\phi^v(x), \quad p^v = (1-v^2)^{-1/2}v\,. \end{aligned}$$

With these preliminaries we can summarize our main results, the precise theorems to be stated in the following section. Since only a finite amount of energy can be dissipated to infinity, we have $\ddot{q}(t) \to 0$ as $t \to \infty$. This is a crucial point of our asymptotic analysis. It follows that the fields are asymptotically Coulombic traveling waves in the sense

$$(E(q(t)+x,t), B(q(t)+x,t)) - (E^{v(t)}(x), B^{v(t)}(x)) \to 0 \text{ as } t \to \infty,$$

where $v(t) = \dot{q}(t)$. Since energy is conserved, the convergence here is in the sense of suitable local norms, cf. Section 2. To go further, several cases should be distinguished.

(i) The external fields decay at infinity and $|q(t)|$ is unbounded. In this case one expects that $\lim_{t\to\infty}\dot{q}(t) = v$ together with comoving Coulombic fields. For a scalar field such kind of asymptotics was studied in [9]. For the Maxwell-Lorentz system (1.1) some additional considerations are required which will not be touched upon here.
(ii) If $|q(t)|$ remains bounded, then the solution to (1.1) is attracted by the set $\mathcal{S}$. One would expect that the solution in fact converges to one specific S_q. To prove such a strengthned asymptotics we need
(iii) $|q(t)|$ is bounded and $\mathcal{S}$ is a discrete set. Then by continuity a definite stationary state S_q has to be approached as $t \to \infty$.

To guarantee, a priori, that $|q(t)|$ is bounded, an energy estimate can be used. Thus either P_∞ or a suitably small initial energy are sufficient conditions. If no such energy estimate is available, our method seems to fail. One example is a constant magnetic field, $B_{ex}(x) = B_0$, $E_{ex}(x) = 0$. The magnetic field confines and we expect the particle to come to rest as $t \to \infty$ at some position $q \in \mathbb{R}^3$. A similar example, only one with a discrete set $\mathcal{S}$, is a bounded space-periodic potential ϕ_{ex}, together with a sufficiently large initial energy. Again we expect convergence to some definite S_q, but cannot prove that $q(t)$ remains bounded.

Despite their physical importance the coupled Maxwell-Lorentz equation have received only little mathematical attention. In [4] the existence and smoothness of solutions are studied and in [2, 3] some aspects of the long-time asymptotics. Equations of a similar structure are the Maxwell-Vlasov equations, where the Lorentz force equation is replaced by the Vlasov equation. Its global existence theory is regarded as an outstanding open problem [5, 12].

The attraction to a set of stationary states as $t \to \infty$ is a familiar and widespread phenomenon. For dissipative systems energy is locally dissipated. This implies a stronger convergence than proved here, typically in some global energy norm [1]. However the convergence is only in the forward time direction, $t \to +\infty$, whereas for the Maxwell-Lorentz equations both time directions are on equal footing. For Hamiltonian systems, as the one studied here, energy can be transported to infinity. This mechanism has been first exploited for Hamiltonian linear wave equations [11, 19, 20] and later on for relativistic nonlinear wave equations either with a unique "zero" stationary solution [15, 17] or with small initial data [6, 8]. In all these cases the attractor consists only of the zero solution. In [10, 9] we study the long-time behavior of a particle coupled to a scalar wave field. While the general strategy is comparable to the one employed here, the details differ considerably, since the Maxwell-Lorentz equations are evolution equations with a constraint.

2 Main results

We first define a suitable phase space. A point in phase space is referred to as state. Let L^2 denote the real Hilbert space $L^2(\mathbb{R}^3, \mathbb{R}^3)$ with the norm $|\!|\cdot|\!|$. We introduce the Hilbert spaces $\mathcal{F} = L^2 \oplus L^2$ and $\mathcal{L} = \mathcal{F} \oplus \mathbb{R}^3 \oplus \mathbb{R}^3$ with finite norms

$$\|(E(x), B(x))\|_{\mathcal{L}} = |\!|E|\!| + |\!|B|\!| \,,$$
$$\|Y\|_{\mathcal{L}} = |\!|E|\!| + |\!|B|\!| + |q| + |p| \text{ for } Y = (E(x), B(x), q, p) \in \mathcal{L}. \tag{2.1}$$

$\mathcal{L}$ is the space of finite energy states , e.g. $S_q \in \mathcal{L}$. The energy functional $\mathcal{H}$ is continuous on the space $\mathcal{L}$. On $\mathcal{F}$ and $\mathcal{L}$ we define the local energy seminorms by

$$\|(E(x), B(x)\|_R = |E|_R + |B|_R$$
$$\|Y\|_R = |E|_R + |B|_R + |q| + |p| \text{ for } Y = (E(x), B(x), q, p) \tag{2.2}$$

for every $R > 0$, where $|\cdot|_R$ is the norm in $L^2(B_R)$, B_R the ball $\{x \in \mathbb{R}^3 : |x| < R\}$. Let us denote by $\mathcal{F}_F$, $\mathcal{L}_F$ the spaces $\mathcal{F}$, $\mathcal{L}$ equipped with the Fréchet topology induced by these seminorms. Note that the spaces $\mathcal{L}$ and $\mathcal{L}_F$, $\mathcal{F}_F$ are metrisable, but $\mathcal{L}_F$, $\mathcal{F}_F$ are not complete.

The system (1.1) is overdetermined. Therefore its actual phase space is a nonlinear submanifold of the linear space $\mathcal{L}$.

Definition 2.1 *(i) The phase space $\mathcal{M}$ for Maxwell-Lorentz equations (1.1) is the metric space of states $(E(x), B(x), q, p) \in \mathcal{L}$ satisfying the constraints,*

$$\operatorname{div} E(x) = \rho(x - q) \ \ \textit{and} \ \operatorname{div} B(x) = 0 \ \ for \ x \in \mathbb{R}^3. \tag{2.3}$$

The metric on $\mathcal{M}$ is induced through the embedding $\mathcal{M} \subset \mathcal{L}$.

(ii) $\mathcal{M}^\sigma$ for $0 \le \sigma \le 1$ is the set of the states $(E(x), B(x), q, p) \in \mathcal{M}$ such that $\nabla E(x)$,$\nabla B(x)$ are $L^\infty_{\rm loc}$ outside the ball B_{R^0} with some $R^0 = R^0(Y) > 0$ and

$$|E(x)| + |B(x)| + |x|\Big(|\nabla E(x)| + |\nabla B(x)|\Big) \le C^0 |x|^{-1-\sigma} \ \ for \ |x| > R^0. \tag{2.4}$$

(iii) $\mathcal{M}_F$ denotes the space $\mathcal{M}$ endowed with Fréchet topology induced through the embedding $\mathcal{M} \subset \mathcal{L}_F$.

Remarks (i) We denote by $\xrightarrow{\mathcal{F}_F}$ and $\xrightarrow{\mathcal{M}_F}$ the convergence in the Fréchet topology of the spaces $\mathcal{F}_F$ and $\mathcal{M}_F$ respectively.
(ii) $\mathcal{M}$ is a complete metric space, a nonlinear submanifold of $\mathcal{L}$. The spaces $\mathcal{M}, \mathcal{M}_F$ are metrisable.
(iii) All stationary states $S_q \in \mathcal{M}^1$ and $\mathcal{M}^1$ is dense in $\mathcal{M}$ by Lemma A.4. On the other hand, since the total charge $\overline{\rho} = \hat{\rho}(0) \ne 0$, $\mathcal{M}^\sigma = \emptyset$ for $\sigma > 1$ because of Gauss theorem. By the same reason $\operatorname{supp} E(x)$ cannot be a compact set in contrast to $\operatorname{supp} B(x)$.

Let us write the system (1.1) as a dynamical equation on $\mathcal{M}$

$$\dot{Y}(t) = F(Y(t)) \text{ for } t \in \mathbb{R}, \tag{2.5}$$

where $Y(t) = (E(x,t), B(x,t), q(t), p(t)) \in \mathcal{M}$.

Definition 2.2 $\mathcal{S} = \{S \in \mathcal{M} : F(S) = 0\}$ *is the set of all finite energy stationary states of the system (1.1).*

Proposition 2.3 *Let* $(P_{\min})$ *and* (C) *hold and* $Y^0 = (E^0(x), B^0(x), q^0, p^0) \in \mathcal{M}$. *Then*
(i) The system (1.1) has a unique solution $Y(t) = (E(x,t), B(x,t), q(t), p(t)) \in C(\mathbb{R}, \mathcal{M})$ *with* $Y(0) = Y^0$.
(ii) The energy is conserved, i.e.

$$\mathcal{H}(Y(t)) = \mathcal{H}(Y^0) \ \ for\ t \in \mathbb{R}. \tag{2.6}$$

(iii) If in addition, the external potential $\phi_{ex}(x)$ *is confining, i.e. if* (P_∞) *holds, then* (Q) *holds.*
(iv) The set $\mathcal{S}$ *is given by (1.4).*

We refer to the Appendix where also some uniform bounds for $|\dot{q}(t)|$, $|\ddot{q}(t)|$ and $|\dddot{q}(t)|$ are established.

Let $\mathcal{T}$ be a subset of a topological space $\mathcal{G}$, and $Y(t) \in \mathcal{T}$ for $t \in \mathbb{R}$. The convergence $Y(t) \xrightarrow{\mathcal{G}} \mathcal{T}$ as $t \to \infty$ means by definition that for every neighborhood $\mathcal{O}(\mathcal{T})$ of $\mathcal{T}$ in $\mathcal{G}$ there exist a $t_0 > 0$ such that $Y(t) \in \mathcal{O}(\mathcal{T})$ for $t > t_0$.

Definition 2.4 $\mathcal{T}$ *is a trapping subset in* $\mathcal{G}$, *if for every continuous curve* $Y(t) \in C(0, \infty; \mathcal{G})$ *with a precompact orbit* $O(Y) = \{Y(t) \in \mathcal{G} : t \in \mathbb{R}\}$ *the convergence* $Y(t) \xrightarrow{\mathcal{G}} \mathcal{T}$ *as* $t \to \infty$ *implies the convergence* $Y(t) \xrightarrow{\mathcal{G}} T$ *as* $t \to \infty$ *to some point* $T \in \mathcal{T}$.

For example every discrete set in $\mathbb{R}^3$ (i.e. a subset which does not have limit points in $\mathbb{R}^3$) is a trapping set in $\mathbb{R}^3$.

Our main result is the long-time asymptotics of the finite energy solutions to (1.1) in the Fréchet topology.

Theorem 2.5 *Let* $(P_{\min})$, (C), (W) *hold. Let* $Y(t) \in C(\mathbb{R}, \mathcal{M})$ *be the solution of the Maxwell-Lorentz equations (1.1) with initial state* $Y^0 \in \mathcal{M}^\sigma$ *for some* $\sigma > 1/2$. *We have*
(i) If either $(P_{\max})$ *or* (Q) *holds, then*

$$\ddot{q}(t) \to 0 \ as\ t \to \infty, \tag{2.7}$$

$$(E(q(t)+x,t), B(q(t)+x,t)) - (E^{v(t)}(x), B^{v(t)}(x)) \xrightarrow{\mathcal{F}_F} 0 \ as\ t \to \infty, \tag{2.8}$$

(ii) If (Q) *holds, in addition to (2.7) and (2.8), the orbit* $O(Y)$ *is precompact in* $\mathcal{M}_F$ *and*

$$Y(t) \xrightarrow{\mathcal{M}_F} \mathcal{S} \ as\ t \to \infty. \tag{2.9}$$

(iii) If in addition to (Q), Z *is a trapping set in* $\mathbb{R}^3$, *then there exist a stationary state* $S \in \mathcal{S}$ *depending on* Y^0 *such that*

$$Y(t) \xrightarrow{\mathcal{M}_F} S \ as\ t \to \infty. \tag{2.10}$$

Remarks (i) The convergence holds also for $t \to -\infty$.
(ii) The convergence (2.10) and (1.2) imply

$$\mathcal{H}(S) \le \mathcal{H}(Y(t)) \equiv \mathcal{H}(Y^0)\,, \quad t \in \mathbb{R}\,, \tag{2.11}$$

by the theorem of Fatou, similarly to a well known property of weak convergence in Hilbert and Banach spaces.
(iii) The convergence (2.9) and (2.10) cannot hold in the global norm of $\mathcal{M}$, in general, because of energy conservation (2.6) and because the energy functional $\mathcal{H}$ is continuous on $\mathcal{M}$.
(iv) Let $\phi_\infty = \liminf_{|q|\to\infty} \phi_{ex}(q)$. If $\mathcal{H}(Y^0) < 1 + \phi_\infty$, then (Q) holds by conservation of energy. For instance, (P_∞) implies then (Q).
(v) The assumption (C) can be weakened to finite differentiability and some decay of $\rho(x)$ at infinity.

As an input to our proof we establish in Section 3 a bound on the energy escaping to infinity and in Section 4 the asymptotics of the Maxwell field along the light cone. In Section 5 the limit (2.7) is then derived using a Wiener Tauberian theorem and in Section 6 the properties (2.8)–(2.10) are established. In the Appendices we collect existence theorems, integral representations, and a priori bounds for linear and nonlinear Maxwell dynamics.

3 Energy radiated to infinity

In this section we establish a lower bound on the total energy radiated to infinity in terms of the energy dissipation integral (3.2). Since the energy is bounded a priori, this integral has to be finite.

Let S^2 denote the unit sphere in $\mathbb{R}^3$ with surface area element $d^2\omega$. For $\omega \in S^2$ and $z \in \mathbb{R}^3$ let us set $z = \omega z_{||} + z_\perp$, where $z_{||} = \omega \cdot z$ and $z_\perp \perp \omega$, $\overline{\tau} = t + y_{||}$,

$$\overline{E}_{(r)}(\omega, t) = -\frac{1}{4\pi} \int d^3y\, \rho(y - q(\overline{\tau})) \frac{(1 - \dot{q}_{||}(\overline{\tau}))\ddot{q}_\perp(\overline{\tau}) + \ddot{q}_{||}(\overline{\tau})\dot{q}_\perp(\overline{\tau})}{(1 - \dot{q}_{||}(\overline{\tau}))^2}\,. \tag{3.1}$$

Remark $\overline{E}_{(r)}(\omega, t) \perp \omega$ for every $\omega \in S^2$ and $t \in \mathbb{R}$.

Proposition 3.1 *Let $(P_{\min})$, (C) hold and let $Y(t) \in C(\mathbb{R}, \mathcal{M})$ be the solution to (1.1) with initial state $Y^0 = Y(0) \in \mathcal{M}^\sigma$, $\sigma > 1/2$. Then*

$$\int_0^\infty dt \int_{S^2} d^2\omega |\overline{E}_{(r)}(\omega, t)|^2 < \infty. \tag{3.2}$$

Proof *Step 1.* The energy $\mathcal{H}_R(t)$ at time $t \in \mathbb{R}$ in the ball B_R with radius $R > |q(t)| + R_\rho$ is defined by

$$\mathcal{H}_R(t) = (1 + p^2(t))^{1/2} + \phi_{ex}(q(t)) + \frac{1}{2}\int_{B_R} d^3x \left(|E(x,t)|^2 + |B(x,t)|^2\right). \quad (3.3)$$

Let us fix $T > 0$ and consider the total energy $I_R(R, R+T)$ radiated from the ball B_R during the time interval $[R, R+T]$ with $R \gg T$:

$$I_R(R, R+T) = \mathcal{H}_R(R) - \mathcal{H}_R(R+T). \quad (3.4)$$

This energy is bounded apriori, because $\mathcal{H}_R(R)$ is bounded from above, while $\mathcal{H}_R(R+T)$ is bounded from below. Indeed, $\mathcal{H}_R(R) \le \mathcal{H}(Y(t)) = \mathcal{H}(Y^0)$ and $\mathcal{H}_R(R+T) \ge$const by $(P_{\min})$. Thus,

$$\mathcal{H}_R(R) - \mathcal{H}_R(R+T) \le I < \infty \quad (3.5)$$

with a constant I not depending on T and R.
Step 2. Let us assume $E^0(x), B^0(x) \in C^\infty$ for the simplicity. Then integrating by parts as in (A.40) we obtain

$$\frac{d}{dt}\mathcal{H}_R(t) = -\int_{\partial B_R} d^2x\, S(x,t) \cdot \omega(x) \quad (3.6)$$

for $t \in [R, R+T]$ and $R \gg T$, where S is the Poynting vector, $S(x,t) = E(x,t) \wedge B(x,t)$, $\omega(x) = x/|x|$, and d^2x is the surface area element of ∂B_R. Therefore (3.5) reads

$$\int_R^{R+T} dt \int_{\partial B_R} d^2x\, S(x,t) \cdot \omega(x) \le I. \quad (3.7)$$

Remark We assume $E^0(x), B^0(x) \in C^\infty$ everywhere in the rest of the proof. For general $E^0(x), B^0(x) \in L^2$ the bound (3.2) follows by standard smoothing reasons, since all constants in the proof are bounded for finite $|E^0|, |B^0|$, and C^0 from (2.4).
Step 3. To analyze (3.7) in the limit $R \to \infty$ we need the asymptotics of the fields $E(x,t)$ and $B(x,t)$ along the light cone $|x| \sim t$ in the limit $t \to \infty$. Let us identify for the moment $E(x,t)$ and $B(x,t)$ with the retarded fields $E_{(r)}(x,t)$ and $B_{(r)}(x,t)$. We state the Liénard-Wiechert type asymptotics in the following lemma, which will be proved in Section 4.

Lemma 3.2 *There exists a $T_r > 0$ such that the following asymptotics hold uniformly in $t \in [T_r, T]$ for every fixed $T > T_r$*

$$\begin{aligned} E_{(r)}(x, |x| + t) &= \overline{E}_{(r)}(\omega(x), t)|x|^{-1} + \mathcal{O}(|x|^{-2}), & (3.8)\\ B_{(r)}(x, |x| + t) &= \omega(x) \wedge \overline{E}_{(r)}(\omega(x), t)|x|^{-1} + \mathcal{O}(|x|^{-2}) & (3.9)\end{aligned}$$

as $|x| \to \infty$.

If we set $E = E_{(r)}, B = B_{(r)}$, then

$$\begin{aligned} S(x, |x| + t) &= \overline{E}_{(r)}(\omega(x), t) \wedge \Big(\omega(x) \wedge \overline{E}_{(r)}(\omega(x), t)\Big)|x|^{-2} + \mathcal{O}(|x|^{-3}) \\ &= \omega(x)|\overline{E}_{(r)}(\omega(x), t)|^2|x|^{-2} + \mathcal{O}(|x|^{-3}) \text{ for } t > T_r, \end{aligned} \tag{3.10}$$

since $\overline{E}_{(r)}(\omega, t))\perp\omega$. (3.7) implies then

$$\int_{T_r}^{T} dt \int_{S^2} d^2\omega \, |\overline{E}_{(r)}(\omega, t)|^2 \leq I + T\mathcal{O}(R^{-1}). \tag{3.11}$$

It remains to take the limit $R \to \infty$ and subsequently $T \to \infty$ to obtain (3.2).

Remark $E = E_{(r)}, B = B_{(r)}$ is a solution to the linear Maxwell system (A.1) iff the initial data $E^0 = B^0 = 0$. Such an initial state is however not in $\mathcal{M}$.

Step 4. To conclude the proof, we substitute in (3.7) $E = E_{(r)} + E_{(0)}, B = B_{(r)} + B_{(0)}$ as defined in (A.23) . Then

$$\int_{R}^{R+T} dt \int_{\partial B_R} d^2x \, \omega(x) \cdot (E_{(r)} \wedge B_{(r)} + E_{(0)} \wedge B_{(r)} + E_{(r)} \wedge B_{(0)} + E_{(0)} \wedge B_{(0)}) \leq I. \tag{3.12}$$

The retarded fields $E_{(r)}, B_{(r)}$ originate in the charge and current densities $\rho(x-q(t))$ and $\rho(x-q(t))\dot{q}(t)$, cf. (A.25), (A.26). On the other hand, $E_{(0)}, B_{(0)}$ are due to the initial fields E^0, B^0, cf. (A.24), which are controlled by the bounds (2.4) as stated in following lemma, which will be proved also in Section 4.

Lemma 3.3 *Let $(E^0, B^0, q^0, p^0) \in \mathcal{M}^\sigma$ with $\sigma > 1/2$. Then there exist $I_0 < \infty$ and $T_0 > 0$ such that for every $R > 0$ and $T > T_0$*

$$\int_{R+T_0}^{R+T} dt \int_{\partial B_R} d^2x \left(|B_{(0)}(x, t)|^2 + |E_{(0)}(x, t)|^2\right) \leq I_0. \tag{3.13}$$

Using Cauchy-Schwarz in (3.12), this lemma together with (3.8), (3.9) imply that for every $T > \overline{T} = \max(T_r, T_0)$,

$$\int_{\overline{T}}^{T} dt \int_{S^2} d^2\omega \, |\overline{E}_{(r)}(\omega, t)|^2 \leq I_1 + T\mathcal{O}(R^{-1}). \tag{3.14}$$

with a constant $I_1 < \infty$ not depending on T and R. □

Remark The asymptotics (3.8), (3.9) for a charge distribution generalizes the decomposition into far and near fields of the Liénard-Wiechert potentials for a point charge and the leading term of the asymptotics (3.10) generalizes the Larmor-Liénard expression for the power of radiation [16].

4 Liénard-Wiechert asymptotics along the light cone

We prove Lemmas 3.2 and 3.3. Let us note that both lemmas provide the field asymptotics in the wave zone along the light cone $|x| \sim t \to \infty$.

Proof of Lemma 3.2 *Step 1* . Let $T_r = |q^0| + R_\rho$. The integrands in (A.28) and (A.31) have bounded supports in y uniformly for bounded $t - |x|$ because of the estimate (A.36). (A.28) implies then for bounded $t - |x| > T_0$

$$\begin{aligned} E_{(r)}(x,t) &= \int \frac{d^3y}{4\pi|x-y|}\{\dot q(\tau)\cdot\nabla\rho(y-q(\tau))\,[-n+\dot q(\tau)] - \rho(y-q(\tau))\ddot q(\tau)\} \\ &+ \mathcal{O}(|x|^{-2}), \end{aligned} \tag{4.15}$$

where $\tau = t - |x-y|$ and $n = \dfrac{x-y}{|x-y|}$. Similarly (A.31) implies for bounded $t - |x| > T_r$

$$\begin{aligned} B_{(r)}(x,t) &= \int \frac{d^3y}{4\pi|x-y|} n\wedge\{\dot q(\tau)\cdot\nabla\rho(y-q(\tau))\,\dot q(\tau) - \rho(y-q(\tau))\,\ddot q(\tau)\} \\ &+ \mathcal{O}(|x|^{-2}) \;=\; \omega(x)\wedge E_{(r)}(x,t) + \mathcal{O}(|x|^{-2}), \end{aligned} \tag{4.16}$$

since $n = \omega(x) + \mathcal{O}(|x|^{-1})$ for bounded y. In (4.15), (4.16) we substitute $|x| + t$ instead of t. Then these expressions are valid for $t > T_r$. τ becomes $|x| + t - |x-y|$ and, uniformly in bounded t and $|y|$,

$$\tau = |x|+t-|x-y| = t+\omega(x)\cdot y+\mathcal{O}(|x|^{-1}) = \overline{\tau}+\mathcal{O}(|x|^{-1}),\;\; |x-y| = |x|+\mathcal{O}(1). \tag{4.17}$$

Hence (4.15), (4.16) imply (3.8), (3.9) with

$$\overline{E}_{(r)}(\omega,t) = \frac{1}{4\pi}\int d^3y\,\{\dot q(\overline{\tau})\cdot\nabla\rho(y-q(\overline{\tau}))\,[-\omega+\dot q(\overline{\tau})] - \rho(y-q(\overline{\tau}))\ddot q(\overline{\tau})\}. \tag{4.18}$$

To complete the proof of (3.8), (3.9) we only have to identify this expression with (3.1) through a partial integration.

Step 2. For the partial integration we note

$$\dot q(\overline{\tau})\cdot\nabla_y\rho(y-q(\overline{\tau})) = \dot q(\overline{\tau})\cdot\nabla\rho(y-q(\overline{\tau}))\,(1-\omega\cdot\dot q(\overline{\tau})),$$

since $\overline{\tau} = t + y_{||} = t + \omega\cdot y$. Then the first summand in the integrand of (4.18) yields

$$\begin{aligned} &\int d^3y\,\dot q(\overline{\tau})\cdot\nabla\rho(y-q(\overline{\tau}))\,[-\omega+\dot q(\overline{\tau})] \\ &= \int d^3y\,\dot q(\overline{\tau})\cdot\nabla_y\rho(y-q(\overline{\tau}))\;(1-\omega\cdot\dot q(\overline{\tau}))^{-1}[-\omega+\dot q(\overline{\tau})] \\ &= -\int d^3y\,\rho(y-q(\overline{\tau}))\frac{\partial}{\partial y_\alpha}\{(1-\omega\cdot\dot q(\overline{\tau}))^{-1}\dot q_\alpha(\overline{\tau})[-\omega+\dot q(\overline{\tau})]\}, \end{aligned} \tag{4.19}$$

summing over repeated indices. Differentiating we obtain

$$\begin{aligned}
&\frac{\partial}{\partial y_\alpha}\{(1-\omega\cdot\dot q)^{-1}\dot q_\alpha[-\omega+\dot q]\}\\
&=(1-\omega\cdot\dot q)^{-2}[(\ddot q_\alpha\omega_\alpha[-\omega+\dot q]+\dot q_\alpha\ddot q\omega_\alpha)(1-\omega\cdot\dot q)+\dot q_\alpha[-\omega+\dot q]\omega\cdot\ddot q\omega_\alpha]\\
&=(1-\dot q_{\|})^{-2}[(\ddot q_{\|}[-\omega+\dot q]+\dot q_{\|}\ddot q)(1-\dot q_{\|})+\dot q_{\|}\ddot q_{\|}[-\omega+\dot q]]\\
&=(1-\dot q_{\|})^{-2}[\ddot q_{\|}[-\omega+\dot q]+(\dot q_{\|}-\dot q_{\|}^2)\ddot q] \qquad (4.20)
\end{aligned}$$

and (4.18) becomes

$$\overline{E}_{(r)}(\omega,t)=-\frac{1}{4\pi}\int d^3y\,\rho(y-q(\overline{\tau}))\frac{\ddot q_{\|}[-\omega+\dot q]+(\dot q_{\|}-\dot q_{\|}^2)\ddot q+(1-\dot q_{\|})^2\ddot q}{(1-\dot q_{\|})^2}\,, \qquad (4.21)$$

which agrees with (3.1). □

Proof of Lemma 3.3 We prove (3.13) with $T_0=R^0=R^0(Y^0)$, cf. Definition 2.1 (ii). The representations (A.24) and the bounds (A.6) together with assumption (2.4) for Y^0 imply that

$$|E_{(0)}(x,t)|+|B_{(0)}(x,t)|\le C\sum_{k=0,1}t^{k-2}\int_{S_t(x)}d^2y\,|y|^{-1-\sigma-k} \qquad (4.22)$$

for $t>R^0+|x|$, where $S_t(x)=\{y\in\mathbb{R}^3:\,|y-x|=t\}$. We can always adjust σ such that $\sigma+k\neq 1$. Then, by explicit computation, a typical term reads

$$I^k(x,t):=\int_{S_t(x)}d^2y\,|y|^{-1-\sigma-k}=\frac{2\pi t}{|x|(1-\sigma-k)}\Big((t+|x|)^{1-\sigma-k}-(t-|x|)^{1-\sigma-k}\Big). \qquad (4.23)$$

Hence the contribution of the corresponding term to the left hand side of (3.13) can be majorized by

$$\begin{aligned}
J^k_{R,T}:=&\quad C_1\int_{R+R^0}^{R+T}dt\int_{\partial B_R}d^2x\,\left|t^{k-2}I^k(x,t)\right|^2\\
=&\quad C_2\int_{R^0}^{T}ds\,(R+s)^{2(k-1)}\left|(2R+s)^{1-\sigma-k}\;-\;s^{1-\sigma-k}\right|^2. \qquad (4.24)
\end{aligned}$$

We may adjust σ slightly larger than 1/2. Then for $k=0$, we have $\sigma+k\le 1$ and (4.24) implies

$$J^k_{R,T}\le C\int_{R^0}^{T}ds\,(R+s)^{-2\sigma}\le J^k<\infty\text{ for }R\ge 0,T\ge R^0.$$

For $k=1$ the bound (4.24) implies

$$J^k_{R,T}\le C\int_{R^0}^{T}ds\,s^{-2\sigma}\le J^k<\infty\text{ for }R\ge 0,T\ge R^0.$$

$\square$

Remark The representations (A.24) and the bounds (A.6) together with (2.4) imply that for every $R > 0$

$$\max_{|x|\le R}\Big(|E_{(0)}(x,t)| + |B_{(0)}(x,t)|\Big) = \mathcal{O}(t^{-1-\sigma}) \text{ as } t \to \infty. \tag{4.25}$$

5 Relaxation of the particle acceleration and velocity

In this section we will deduce from Proposition 3.1 that $\ddot{q}(t)$, $\dot{q}(t) \to 0$ as $t \to \infty$ provided we require in addition the Wiener condition (W) and either $(P_{\max})$ or (Q).

Either $(P_{\max})$ or (Q) imply the bounds (A.36) and (A.37) of Proposition A.5 (v) and (vi). Hence, the function $\overline{E}_{(r)}(\omega,t)$ is globally Lipschitz continuous in ω and t. Thus (3.2) implies

$$\lim_{t\to\infty} \overline{E}_{(r)}(\omega,t) = 0 \tag{5.1}$$

uniformly in $\omega \in S^2$. Let us analyze the structure of the integrand in (3.1). The main observation is that the integration over the planes $y_{||} = const$ can be performed, since $\overline{\tau} = t + y_{||} = t + \omega\cdot y$ depends on $y_{||}$ only. Therefore (3.1) can be rewritten as a one-dimensional convolution. Let $r(t) = q_{||}(t) \in \mathbb{R}$, $s = y_{||}$, and $\rho_a(q_3) = \int dq_1 dq_2\, \rho(q_1,q_2,q_3)$. Then $\overline{\tau} = t + s$ and (3.1) becomes

$$\begin{aligned}
&\overline{E}_{(r)}(\omega,t)\\
&= -\frac{1}{4\pi}\int ds\, \rho_a(s - r(t+s))\,\frac{(1-\dot{q}_{||}(t+s))\ddot{q}_\perp(t+s) + \ddot{q}_{||}(t+s)\dot{q}_\perp(t+s)}{(1-\dot{q}_{||}(t+s))^2}\\
&= -\frac{1}{4\pi}\int d\tau\, \rho_a(t - (\tau - r(\tau)))\,\frac{(1-\dot{q}_{||}(\tau))\ddot{q}_\perp(\tau) + \ddot{q}_{||}(\tau)\dot{q}_\perp(\tau)}{(1-\dot{q}_{||}(\tau))^2}\\
&= -\frac{1}{4\pi}\int d\theta\, \rho_a(t-\theta) g_\omega(\theta) = \rho_a * g_\omega(t)
\end{aligned} \tag{5.2}$$

with the substitution $\theta = \tau - r(\tau)$, which is a nondegenerate diffeomorphism since $|\dot{r}| \le q_1 < 1$ by (A.36), and with the definition

$$g_\omega(\theta) = -\frac{1}{4\pi}\frac{(1-\dot{q}_{||}(\tau(\theta)))\ddot{q}_\perp(\tau(\theta)) + \ddot{q}_{||}(\tau(\theta))\dot{q}_\perp(\tau(\theta))}{(1-\dot{q}_{||}(\tau(\theta)))^2}\,.$$

Let us extend $q(t)$ smoothly to zero for $t < 0$. Then $\rho_a * g_\omega\,(t)$ is defined for all t and agrees with $\overline{E}_{(r)}(\omega,t)$ for sufficiently large t. Hence (5.1) reads as the convolution limit

$$\lim_{t\to\infty} \rho_a * g_\omega(t) = 0. \tag{5.3}$$

Note that (A.36) and (A.37) with $k = 2, 3$ imply that $g'_\omega(\theta)$ is bounded. Hence (5.3) and (W) imply by Pitt's extension to Wiener's Tauberian Theorem, cf. [14, Thm. 9.7(b)],

$$g_\omega(\theta) \to 0 \ \text{ as } \theta \to \infty. \tag{5.4}$$

Since $\theta(t) \to \infty$ as $t \to \infty$, we conclude

$$(1 - \dot{q}_{||}(t))\ddot{q}_\perp(t) + \ddot{q}_{||}(t)\dot{q}_\perp(t) \to 0 \ \text{ as } t \to \infty. \tag{5.5}$$

Replacing ω by $-\omega$ we get $(1 + \dot{q}_{||}(t))\ddot{q}_\perp(t) - \ddot{q}_{||}(t)\dot{q}_\perp(t) \to 0$, and for the sum $2\ddot{q}_\perp(t) \to 0$. Since $\omega \in S^2$ is arbitrary, we have proved

Lemma 5.1 *Let the assumptions of Proposition 3.1 hold. If the Wiener condition* (W) *and either* $(P_{\max})$ *or* (Q) *are satisfied, then*

$$\lim_{t\to\infty} \ddot{q}(t) = 0. \tag{5.6}$$

Remarks. (i) For a point charge $\rho(x) = \delta(x)$, (5.3) implies (5.4) directly.
(ii) Parseval's identity, (5.2), and (3.2) yield

$$\int_{S^2} d^2\omega \int d\xi \, |\hat{\rho}_a(\xi)\hat{g}_\omega(\xi)|^2 < \infty.$$

If $|\hat{\rho}_a(\xi)| \geq C > 0$, then $\int d^2\omega \int dt \, |g_\omega(t)|^2 < \infty$, and (5.4) would follow from the Lipschitz continuity of g_ω. Thus, the main difficulty results from the rapid decay of the Fourier transform ("symbol") $\hat{\rho}_a$, due to the smoothness of the kernel ρ_a.
(iii) The condition (W) seems to be necessary. If (W) is violated, then $\hat{\rho}_a(\nu) = 0$ for some $\nu \in \mathbb{R}$, $\omega \in S^2$, and with the choice $g(t) = \exp(i\nu t)$ we have $\rho_a * g(t) = 0$ whereas g does not decay to zero.

Corollary 5.2 *Let the assumptions of Theorem 2.5 and let* (Q) *hold. Then*

$$\lim_{t\to\infty} \dot{q}(t) = 0. \tag{5.7}$$

Proof Since $|q(t)| \leq q_0$ by assumption (Q), (5.6) implies (5.7). □

6 Long-time asymptotics

We prove Theorem 2.5.

6.1 Attraction to the set of solitons

We prove Theorem 2.5 (i). The relaxation of the acceleration (5.6) has been proved in Lemma 5.1. It remains to derive (2.8) for $t \to \infty$. We use the integral representations (A.23)–(A.25) and (A.28), (A.31) for the solution $(E(x,t), B(x,t))$ and for the solitons $(E^v(x-vt), B^v(x-vt))$ from (1.5). Firstly, we substitute (E^v, B^v) in (A.28), (A.31) and obtain for $v \in \mathcal{V}$,

$$E^v(x) = \int \frac{d^3y}{4\pi|x-y|}\{\frac{n}{|x-y|}\rho(y-v\overline{\tau}) + v\cdot\nabla\rho(y-v\overline{\tau})\,[-n+v]\}, \quad (6.1)$$

$$B^v(x) = \int \frac{d^3y}{4\pi|x-y|} n\wedge\{-\frac{1}{|x-y|}\rho(y-v\overline{\tau})v + v\cdot\nabla\rho(y-v\overline{\tau})\,v\}, \quad (6.2)$$

where $n = \dfrac{x-y}{|x-y|}$ and $\overline{\tau} = -|x-y|$. These representations, together with (A.28) and (A.31) for (E, B), imply by (5.6)

$$(E_{(r)}(q(t)+x,t), B_{(r)}(q(t)+x,t)) - (E^{v(t)}(x), B^{v(t)}(x)) \xrightarrow{\mathcal{F}_F} 0 \text{ as } t\to\infty. \quad (6.3)$$

Therefore (2.8) would follow from

$$(E_{(0)}(q(t)+x,t), B_{(0)}(q(t)+x,t)) \xrightarrow{\mathcal{F}_F} 0 \text{ as } t\to\infty. \quad (6.4)$$

By definition, this means that for every $R > 0$

$$\int_{|x|<R} d^3x \left(|E_{(0)}(q(t)+x,t)|^2 + |B_{(0)}(q(t)+x,t))|^2\right) \to 0 \text{ as } t\to\infty. \quad (6.5)$$

The estimates (4.22), (4.23) imply

$$|E_{(0)}(q(t)+x,t)| + |B_{(0)}(q(t)+x,t))| \quad (6.6)$$

$$\le C\sum_{k=0,1}\frac{t^{k-1}}{|x|}\left((t+|x|)^{1-k-\sigma} - (t-|x|)^{1-k-\sigma}\right). \quad (6.7)$$

Setting here $q(t)+x$ instead of x and taking into account that $|q(t)| \le q_1 t + const$ with $0 < q_1 < 1$, we get

$$\max_{|x|\le R}\left(|E_{(0)}(q(t)+x,t)| + |B_{(0)}(q(t)+x,t)|\right) = \mathcal{O}(t^{-1-\sigma}) \text{ as } t\to\infty\,. \quad (6.8)$$

Therefore (6.5) follows. □

6.2 Attraction to the set of stationary states

We prove Theorem 2.5 (ii). First we construct a compact attracting set $\mathcal{A}$ for the trajectory $Y(t)$ under consideration.

Definition 6.1 *Let $\mathcal{A} = \{S_q : q \in \mathbb{R}^3,\ |q| \le q_0\}$, where S_q and q_0 are defined in (1.3) and in (Q), respectively.*

Since $\mathcal{A}$ is homeomorphic to a closed ball in $\mathbb{R}^3$, $\mathcal{A}$ is compact in $\mathcal{M}_F$.

Lemma 6.2 *Let the assumptions of Theorem 2.5 hold. Then $Y(t) \xrightarrow{\mathcal{M}_F} \mathcal{A}$ as $t \to \infty$.*

Proof. It suffices to verify that for every $R > 0$

$$|||Y(t) - S_{q(t)}|||_R = |E(t) - E_{q(t)}|_R + |B(t)|_R + |p(t)| \to 0 \text{ as } t \to \infty. \quad (6.9)$$

Let us estimate each term separately.
(i) (5.7) implies $|p(t)| \to 0$ as $t \to \infty$.
(ii) Let us denote $R_\infty = q_0 + R_\rho$. Then the representations (A.31) imply

$$|B_{(r)}(x,t)| \le C_R \max_{t-R-R_\infty \le \tau \le t} \Big(|\ddot{q}(\tau)| + |\dot{q}(\tau)| + |\ddot{q}(\tau)|^2 + |\dot{q}(\tau)|^2\Big)$$

for $t > R + R_\infty$ and $|x| < R$. Therefore (2.7) and (5.7) imply that $|B_{(r)}(t)|_R \to 0$ as $t \to \infty$. Then also $|B(t)|_R \to 0$ by (A.23) and (4.25).
(iii) (A.28) implies that for $t > R + R_\infty$ and $|x| < R$

$$\begin{aligned} |E_{(r)}(x,t) \ &- \ E_{q(t)}(x)| \le C_R \max_{t-R-R_\infty \le \tau \le t} \Big(|\ddot{q}(\tau)| + |\dot{q}(\tau)| + |\ddot{q}(\tau)|^2 + |\dot{q}(\tau)|^2\Big) \\ &+ \ C \int_{|y|<R_\infty} \frac{d^3y}{|x-y|^2} \Big|\rho(y - q(t - |x-y|)) - \rho(y - q(t))\Big|. \end{aligned}$$

The difference $\rho(y - q(t - |x-y|)) - \rho(y - q(t))$ can be written as an integral depending only on $\dot{q}(\tau)$ for $\tau \in [t - R - R_\infty, t]$, which tends to zero as $t \to \infty$ uniformly in $x \in B_R$ by (5.7). Hence $|E_{(r)}(t) - E_{q(t)}|_R \to 0$ as $t \to \infty$. Similarly, $|E(t) - E_{q(t)}|_R \to 0$ by (A.23) and (4.25). □

Lemma 6.2 implies that the orbit $O(Y)$ is precompact in $\mathcal{M}_F$. Therefore, the following lemma implies (2.9). We denote by $\Omega(Y)$ the omega-set of the trajectory $Y(t)$ in the Fréchet topology of the space $\mathcal{M}_F$: $\overline{Y} \in \Omega(Y)$ if and only if $Y(t_k) \xrightarrow{\mathcal{M}_F} \overline{Y}$ for some sub-sequence $t_k \to \infty$.

Lemma 6.3 *$\Omega(Y)$ is a subset of $\mathcal{S}$.*

Proof $\Omega(Y) \subset \mathcal{A}$ by Lemma 6.2. Moreover, the set $\Omega(Y)$ is invariant with respect to W_t, $t \in \mathbb{R}$, because of the continuity of W_t in $\mathcal{M}_F$. This means for every $\overline{Y} \in \Omega(Y)$ there exists a C^2-curve $t \mapsto Q(t) \in \mathbb{R}^3$ such that $W_t\overline{Y} = S_{Q(t)}$. Thus $S_{Q(t)}$ is a solution to (1.1). Therefore $\dot{Q}(t) = 0$, i.e. $Q(t) \equiv q \in Z$ and $\overline{Y} = S_q \in \mathcal{S}$. □

6.3 Attraction to stationary states

We prove Theorem 2.5 (iii). The set Z is the image of the set $\mathcal{S}$ under the map $I: (E, B, q, p) \mapsto q$. As a map from $\mathcal{M}_F$ to $\mathbb{R}^3$ it is continuous and an injection on $\mathcal{S}$. Since Z is a trapping subset in $\mathbb{R}^3$, $\mathcal{S}$ is a trapping subset in $\mathcal{M}_F$. Hence (2.9) implies (2.10). □

A Appendix. Existence of dynamics

A.1 Linear Maxwell dynamics

We state in an appropriate form a convolution representation for solutions to the Cauchy problem for the linear Maxwell system with prescribed charges and currents,

$$\begin{aligned}
&\operatorname{div} E(x,t) = \rho(x,t), && \operatorname{rot} E(x,t) = -\dot{B}(x,t),\\
&\operatorname{div} B(x,t) = 0, && \operatorname{rot} B(x,t) = \dot{E}(x,t) + j(x,t),\\
&E|_{t=0} = E^0(x), && B|_{t=0} = B^0(x).
\end{aligned} \tag{A.1}$$

We assume $(E^0(x), B^0(x)) \in L^2 \oplus L^2$, where $L^2 = L^2(\mathbb{R}^3, \mathbb{R}^3)$, $\rho(x,t) \in C(\mathbb{R}, L^2(\mathbb{R}^3))$, $j(x,t) \in C(\mathbb{R}, H)$ and also $(E(x,t), B(x,t)) \in C(\mathbb{R}, L^2 \oplus L^2)$. Then the system (A.1) leads to the identities

$$\operatorname{div} E^0(x) = \rho(x,0) \text{ and } \operatorname{div} B^0(x) = 0 \text{ for } x \in \mathbb{R}^3, \tag{A.2}$$

$$\dot{\rho}(x,t) + \operatorname{div} j(x,t) = 0 \text{ for } x \in \mathbb{R}^3,\ t \in \mathbb{R}, \tag{A.3}$$

which are necessary constraints for the existence of solutions to the overdetermined system (A.1). Note that the charge and current densities in (1.1), $\rho(x,t) = \rho(x - q(t))$, $j(x,t) = \rho(x - q(t))\dot{q}(t)$, automatically satisfy the continuity equation (A.3). Let T be an arbitrary positive number.

Definition A.1 *Let $C_T = C(0,T; L^2(\mathbb{R}^3))$, $\overline{C}_T = C(0,T; L^2)$ and $D_T = \{(E^0, B^0, \rho, j) \in L^2 \oplus L^2 \oplus C_T \oplus \overline{C}_T :$ (A.2) and (A.3) hold for $0 \le t \le T\}$.*

Note that D_T is a linear Banach space.

Lemma A.2 *Let $E^0(x), B^0(x)$ and $\rho(x,t), j(x,t)$ satisfy the constraints (A.2), (A.3). Then*
(i) The Cauchy problem (A.1) has a unique solution $(E(x,t), B(x,t)) \in C(\mathbb{R}, L^2 \oplus L^2)$.
(ii) For every $T > 0$ the map $(E^0, B^0, \rho, j) \mapsto (E(x,t), B(x,t))|_{0 \le t \le T}$ is a

linear continuous operator $D_T \to \overline{C}_T \oplus \overline{C}_T$ with norm $\mathcal{O}(T)$.
(iii) The convolution representation holds

$$\begin{pmatrix} E(x,t) \\ B(x,t) \end{pmatrix} = m_t * \begin{pmatrix} E^0 \\ B^0 \end{pmatrix} + \int_0^t ds\, g_{t-s} * \begin{pmatrix} \rho(s) \\ j(s) \end{pmatrix} \quad for\ t \in \mathbb{R}, \tag{A.4}$$

where $\rho(s) = \rho(x,s)$ and $j(s) = j(x,s)$, and m_t and g_t are 6×6- and 6×4-matrix valued distributions concentrated on the sphere $|x| = |t|$,

$$m_t(x) = 0 \ \ and \ \ g_t(x) = 0 \ \ for\ |x| \neq |t|. \tag{A.5}$$

(iv) Let $E^0(y), B^0(y)$ be C^1 functions in a region $\{y \in \mathbb{R}^3 : |y| > R^0\}$ with some $R^0 \geq 0$. Then the convolution $m_t * \begin{pmatrix} E^0 \\ B^0 \end{pmatrix}(x)$ is a continuous function in the region $\{x \in \mathbb{R}^3 : |x| > R^0 + |t|\}$ and the following bounds hold,

$$\begin{aligned} \left| m_t * \begin{pmatrix} E^0 \\ B^0 \end{pmatrix}(x) \right| &\leq C_0 |t|^{-2} \int_{|y-x|=|t|} d^3y\, (|E^0(y)| + |B^0(y)|) \\ &+ C_1 |t|^{-1} \int_{|y-x|=|t|} d^3y\, (|\nabla E^0(y)| + |\nabla B^0(y)|). \end{aligned} \tag{A.6}$$

Proof *ad (i) and (ii)* We introduce the complex field $C(x,t) = E(x,t) + iB(x,t)$ and rewrite (A.1) as

$$\dot{C}(x,t) = -i \operatorname{rot} C(x,t) - j(x,t), \ \ C|_{t=0} = C^0(x), \tag{A.7}$$

$$\operatorname{div} C(x,t) = \rho(x,t), \tag{A.8}$$

where $C^0(x) = E^0(x) + iB^0(x)$. In general, we denote the Fourier transform of some function f by

$$\hat{f}(k) = (\mathcal{F}f)(k) = \int d^3k\ \exp(ik \cdot x) f(x)\,. \tag{A.9}$$

Then (A.7, (A.8 in Fourier space read

$$\dot{\hat{C}}(k,t) = m(k)\hat{C}(k,t) - \hat{j}(k,t), \ \ \hat{C}|_{t=0} = \hat{C}^0(k), \tag{A.10}$$

$$-ik \cdot \hat{C}(k,t) = \hat{\rho}(k,t), \tag{A.11}$$

where $m(k)$ denotes the 3×3 skew-adjoint matrix of the operator $-k\wedge$ in $\mathbb{C}^3$. The solution $\hat{C}(k,t)$ is defined uniquely from the first equation (A.10) of the overdetermined system (A.10), (A.11),

$$\hat{C}(k,t) = \exp(m(k)t)\hat{C}^0(k) - \int_0^t ds\ \exp(m(k)(t-s))\hat{j}(k,s) \text{ for } k \in \mathbb{R}^3. \tag{A.12}$$

We still have to show that (A.12) satisfies the constraints (A.11). Indeed, the Fourier transformed equations (A.2), (A.3) are

$$-ik\cdot\hat{C}^0(k)=\hat{\rho}(k,0)\text{ for }k\in\mathbb{R}^3, \tag{A.13}$$
$$\dot{\hat{\rho}}(k,t)-ik\cdot\hat{j}(k,t)=0\text{ for }k\in\mathbb{R}^3,\ t\in\mathbb{R}. \tag{A.14}$$

With $d(k,t)=-ik\cdot\hat{C}(k,t)$ they imply

$$d(k,0)=-ik\cdot\hat{C}^0(k)=\hat{\rho}(k,0)\ \text{ and }\dot{d}(k,t)=-ik\cdot\hat{j}(k,t)=\dot{\hat{\rho}}(k,t). \tag{A.15}$$

Since $m(k)$ is a skew-adjoint matrix,

$$|\exp(m(k)t)|=1\text{ for }k\in\mathbb{R}^3\text{ and }t\in\mathbb{R}.$$

Therefore (i) and (ii) are a consequence of (A.12).

ad (iii) and (iv) In order to check (A.5) and (A.6), we have to transform (A.12) back to position space . We have $m=m(k)=-k\wedge$, $m^2=-k^2+|k><k|$, $m^3=-|k|^2m,\ldots$. Hence

$$m^{2j+1}=(-1)^j|k|^{2j}m=(-1)^j\frac{m}{|k|}|k|^{2j+1}\text{ for }j\geq 0,$$
$$m^{2j}=m^{2j-1}m=(-1)^{j-1}|k|^{2j-2}m^2=-(-1)^j\left(\frac{m}{|k|}\right)^2|k|^{2j}\text{ for }j\geq 1,$$

which yields by Euler's trick for the exponential

$$\begin{aligned}\exp(m(k)t)&=\sum_0^\infty(mt)^n/n!=\sum_0^\infty(mt)^{2j}/(2j)!+\sum_0^\infty(mt)^{2j+1}/(2j+1)!\\&=1+\left(\frac{m}{|k|}\right)^2(1-\cos|k|t)+\frac{m}{|k|}\sin|k|t\\&=\cos|k|t+m\frac{\sin|k|t}{|k|}+(1-\cos|k|t)\frac{|k><k|}{|k|^2}.\end{aligned}$$

Denoting $\hat{K}_t(k)=\sin|k|t/|k|$, $\hat{m}_t(k)=\dot{\hat{K}}_t(k)+m\hat{K}_t(k)$ and $\hat{D}_t(k)=1-\cos|k|t$ we obtain

$$\exp(m(k)t)=\hat{m}_t(k)+|k>\frac{\hat{D}_t(k)}{|k|^2}<k|. \tag{A.16}$$

Inserting into (A.12) and using the constraints (A.13) and (A.14)

$$\hat{C}(k,t)\ =\ \hat{m}_t(k)\hat{C}^0(k)+i|k>\frac{\hat{D}_t(k)}{|k|^2}\hat{\rho}(k,0) \tag{A.17}$$
$$-\int_0^t ds\,(\hat{m}_{t-s}(k)\hat{j}(k,s)-i|k>\frac{\hat{D}_{t-s}(k)}{|k|^2}\dot{\hat{\rho}}(k,s)), \tag{A.18}$$

which through integration by parts becomes

$$\hat{C}(k,t) = \hat{m}_t(k)\hat{C}^0(k) - \int_0^t ds\, (\hat{m}_{t-s}(k)\hat{j}(k,s) - i|k> \frac{\dot{\hat{D}}_{t-s}(k)}{|k|^2}\hat{\rho}(k,s)). \quad (A.19)$$

Using $\dot{\hat{D}}_t(k) = |k|\sin|k|t = |k|^2\hat{K}_t(k)$, we get

$$\tilde{m}_t(x) = \mathcal{F}^{-1}\hat{m}_t(k) = \dot{K}_t(x) - i\mathrm{rot}\circ K_t(x), \quad (A.20)$$
$$\tilde{g}_t(x) = \mathcal{F}^{-1}\Big(i|k> \hat{K}_t(k),\ -\hat{m}_t(k)\Big) = (-\nabla K_t(x),\ -\tilde{m}_t(x)). \quad (A.21)$$

(A.19) implies then (A.4) in the "complex" form

$$C(x,t) = \tilde{m}_t * C^0 + \int_0^t ds\, \tilde{g}_{t-s} * \left(\begin{array}{c} \rho(s) \\ j(s) \end{array}\right) \quad \text{for } t \in \mathbb{R}. \quad (A.22)$$

Separating into real and imaginary parts we obtain

$$E(x,t) = E_{(r)}(x,t) + E_{(0)}(x,t), \quad B(x,t) = B_{(r)}(x,t) + B_{(0)}(x,t), \quad (A.23)$$

$$\left(\begin{array}{c} E_{(0)}(x,t) \\ B_{(0)}(x,t) \end{array}\right) = m_t * \left(\begin{array}{c} E^0 \\ B^0 \end{array}\right) = \left(\begin{array}{cc} \dot{K}_t & 0 \\ -K_t & \dot{K}_t \end{array}\right) * \left(\begin{array}{c} E^0 \\ B^0 \end{array}\right), \quad (A.24)$$

$$\left(\begin{array}{c} E_{(r)}(x,t) \\ B_{(r)}(x,t) \end{array}\right) = \int_0^t ds \left(\begin{array}{cc} -\nabla K_{t-s} & -\dot{K}_{t-s} \\ 0 & \mathrm{rot}\circ K_t \end{array}\right) * \left(\begin{array}{c} \rho(s) \\ j(s) \end{array}\right). \quad (A.25)$$

Here $K_t(x)$ denotes the Kirchhoff kernel

$$K_t(x) = \mathcal{F}^{-1}\hat{K}_t(k) = \frac{1}{4\pi t}\delta(|x| - |t|). \quad (A.26)$$

(A.4)–(A.6) follow now immediately. □

Remark (A.25) coincides with standard representation of the "retarded" fields $E_{(r)}(x,t)$ and $B_{(r)}(x,t)$ through the Kirchhoff retarded potentials [16]

$$\begin{array}{l} E_{(r)}(x,t) = -\nabla\phi(x,t) - \dot{A}(x,t), \quad B_{(r)}(x,t) = \mathrm{rot}\, A(x,t), \\ \phi(x,t) = \int d^3y \frac{\Theta(\tau)}{4\pi|x-y|}\rho(y,\tau), \quad A(x,t) = \int d^3y \frac{\Theta(\tau)}{4\pi|x-y|}j(y,\tau), \end{array} \quad (A.27)$$

where $\tau = t - |x-y|$. We emphasize, that $(E_{(r)}, B_{(r)})$ is not a solution to Maxwell equations (A.1) with prescribed ρ and j, since $E_{(r)}|_{t=0} = 0$ and hence $\mathrm{div}\, E = \rho$ is not satisfied at $t = 0$. For the same reason, $(E_{(0)}, B_{(0)})$ is not a solution to the Maxwell equations (A.1) with $\rho = 0, j = 0$.

A.2 Liénard-Wiechert representations

Let us consider a solution $Y(t) = (E(x,t), B(x,t), q(t), p(t)) \in C(\mathbb{R}, \mathcal{M})$ to the Maxwell-Lorentz system (1.1) with an initial state $Y^0 = (E^0(x), B^0(x), q^0, p^0) \in \mathcal{M}$. Then $(E(x,t),\ B(x,t)) \in C(\mathbb{R}, H \oplus H)$ is a solution to the Maxwell system (A.1) with $\rho(y,\tau) = \rho(y - q(\tau))$ and $j(y,\tau) = \rho(y - q(\tau))\dot q(\tau)$ and $q(t) \in C(\mathbb{R})$. Since the constraints (A.2) and (A.3) are satisfied, all assumptions of Lemma A.2 hold. The uniqueness from Lemma A.2 (i) implies then the representations (A.23)-(A.25). Therefore (A.27) implies the standard Liénard-Wiechert representations [16] for the corresponding "retarded" fields $E_{(r)}$ and $B_{(r)}$ in the region $t - |x| > T_r = |q^0| + R_\rho$,

$$\begin{aligned} E_{(r)}(x,t) &= \int \frac{d^3y}{4\pi|x-y|}\{\frac{n}{|x-y|}\rho(y-q(\tau)) \\ &+ \dot q(\tau)\cdot\nabla\rho(y-q(\tau))\,[-n+\dot q(\tau)] - \rho(y-q(\tau))\ddot q(\tau)\}, \quad \text{(A.28)} \\ B_{(r)}(x,t) &= \int \frac{d^3y}{4\pi|x-y|}\{-\frac{n}{|x-y|}\wedge\rho(y-q(\tau))\dot q(\tau) \\ &+ \mathrm{rot}_x(\rho(y-q(\tau))\dot q(\tau))\}, \quad \text{(A.29)} \end{aligned}$$

where $\tau = t - |x-y|$ and $n = \dfrac{x-y}{|x-y|}$. Evidently

$$\mathrm{rot}_x(\rho(y-q(\tau))\dot q(\tau)) = \nabla_x\rho\wedge\dot q + \rho\,\mathrm{rot}_x\dot q = -\dot q\cdot\nabla\rho\,\nabla_x\tau\wedge\dot q + \rho\,\nabla_x\tau\wedge\ddot q \quad \text{(A.30)}$$

and $\nabla_x\tau = -n$. Then (A.29) becomes

$$\begin{aligned} B_{(r)}(x,t) = \quad & \int \frac{d^3y}{4\pi|x-y|} n\wedge\{-\frac{1}{|x-y|}\rho(y-q(\tau))\dot q(\tau) \\ & +\dot q(\tau)\cdot\nabla\rho(y-q(\tau))\,\dot q(\tau) - \rho(y-q(\tau))\,\ddot q(\tau)\}. \quad \text{(A.31)} \end{aligned}$$

Remark By a partial integration in (A.28) the integrands in (A.28) and (A.31) can be transformed to satisfy the standard identity $\{\dots\}_B = n\wedge\{\dots\}_E$.

A.3 Smooth approximations to transverse fields

Definition A.3 *$\mathcal{M}_\infty$ is the subspace of the states $(E(x), B(x), q, p) \in \mathcal{M}$ with $E(x),\ B(x) \in (C^\infty(\mathbb{R}^3))^3$ and satisfying the estimates*

$$|\partial_x^\alpha E(x)| + |\partial_x^\alpha B(x)| \le \frac{C_\alpha}{(|x|+1)^{2+|\alpha|}} \ \ for\ x\in\mathbb{R}^3 \quad \text{(A.32)}$$

with every $\alpha = (\alpha_1,\alpha_2,\alpha_3)$, $\alpha_j = 0,1,2,\dots$.

Lemma A.4 *$\mathcal{M}_\infty$ is a dense subspace of $\mathcal{M}$.*

Proof By Definition 2.1, for $(E(x), B(x), q, p) \in \mathcal{M}$ the constraints (2.3) are satisfied. The bounds (A.32) hold for the field $E_q(x) = -\nabla\phi_q(x)$ from (1.3) and this example shows that $2 + |\alpha|$ is the optimal degree in (A.32). Thus the problem is to smoothly approximate purely transverse fields $E(x) - E_q(x)$ and $B(x)$. We will carry this out for the magnetic field $B(x)$, say. Fourier transformed the transversality condition is $k \cdot \hat{B}(k) = 0$ for $k \in \mathbb{R}^3$. To approximate $B(x)$ we first define for $\varepsilon > 0$

$$\hat{B}_\varepsilon(k) = \begin{cases} \hat{B}(k) & \text{for } \varepsilon < |k| < 1/\varepsilon, \\ 0 & \text{otherwise.} \end{cases} \tag{A.33}$$

Then $B_\varepsilon \in L^2 \cap (C^\infty(\mathbb{R}^3))^3$ and $\|B - B_\varepsilon\| \to 0$ as $\varepsilon \to 0$. To provide a rapid decay at infinity we smoothen out $\hat{B}_\varepsilon(k)$ respecting its transversality. (A.33) implies that the support of $\hat{B}_\varepsilon(k)$ can be covered by a finite number of the sets diffeomorphic to a cube through the polar coordinate mapping $k \mapsto \tilde{k} = (\omega, r) = (k/|k|, |k|)$. Then the vector field $\hat{B}_\varepsilon(k)$, transformed under this mapping, lies in the planes $r = const$. Hence averaging with a smooth δ-sequence in the $\tilde{k}$-coordinates preserves such a property. It remains transform this averaged vector field back to $k = \omega r$. $\square$

A.4 Maxwell-Lorentz dynamics

We prove Proposition 2.3. For $Y(t) = (E(x,t), B(x,t), q(t), p(t)) \in C(\mathbb{R}, \mathcal{M})$ the Maxwell-Lorentz equations (1.1) can be written as a dynamical system

$$\begin{aligned} \dot{E} &= \operatorname{rot} B - \rho(x-q)\dot{q}, \quad & \dot{B} &= -\operatorname{rot} E, \\ \dot{q} &= \frac{p}{\sqrt{1+p^2}}, & \dot{p} &= < E + E_{ex} + \dot{q} \wedge (B + B_{ex}), \rho(x-q) >, \end{aligned} \tag{A.34}$$

where all derivatives are understood in the sense of distributions. Let us write the Cauchy problem for the system in the form (2.5)

$$\dot{Y}(t) = F(Y(t)), \quad t \in \mathbb{R}, \quad Y(0) = Y^0, \tag{A.35}$$

where $Y^0 \in \mathcal{M}$.

Proposition A.5 *Let (C) and $(P_{\min})$ hold, and $Y^0 = (E^0, B^0, q^0, p^0) \in \mathcal{M}$. Then*
(i) The Cauchy problem (A.35) has a unique solution $Y(t) \in C(\mathbb{R}, \mathcal{M})$.
(ii) For every $t \in \mathbb{R}$ the map $W_t : Y^0 \mapsto Y(t)$ is continuous both on $\mathcal{M}$ and $\mathcal{M}_F$.

(iii) The energy conservation (2.6) holds.
(iv) The speed is bounded,

$$|\dot{q}(t)| \le q_1 < 1 \; for \; t \in \mathbb{R}. \tag{A.36}$$

(v) If (Q) holds, then there are constants $q_k > 0$, $k = 2, 3$ depending only on the initial data, such that

$$|\ddot{q}(t)| \le q_2 \quad and \quad |\dddot{q}(t)| \le q_3 \;\; for \; t \in \mathbb{R}. \tag{A.37}$$

(vi) ($P_{\max}$) implies (A.37).
(vii) (P_∞) implies (Q).

This Proposition is established by the traditional contraction mapping reasoning, eliminating the "unbounded" Maxwellian part of the dynamics by Lemma A.2. A similar proposition is proved in [10] for (1.1) with a scalar field instead of the Maxwell field.

Proof of Proposition A.5 *ad (i)-(iii)* Let us fix an arbitrary $b > 0$ and prove first the existence and uniqueness of $Y(t) \in C(-\varepsilon, \varepsilon; \mathcal{M})$ satisfying (i)-(iii) for $\|Y^0\|_{\mathcal{H}} \le b$ and $|t| \le \varepsilon = \varepsilon(b)$ with some sufficiently small $\varepsilon = \varepsilon(b) > 0$.
Step 1. *There exists a unique $Y(t) \in C(-\varepsilon, \varepsilon; \mathcal{M})$ satisfying (i) and (ii).*

Lemma A.2 implies that the Maxwell-Lorentz system (1.1) for $Y(t) \in C(\mathbb{R}, \mathcal{M})$ is equivalent to the equations for the particle

$$\begin{aligned} \dot{q}(t) &= p(t)/(1+p^2(t))^{1/2}, \\ \dot{p}(t) &= < E(x,t) + E_{ex}(x) + \dot{q} \wedge (B(x,t) + B_{ex}(x)), \rho(x-q) > , \end{aligned} \tag{A.38}$$

together with the expressions (A.4) for $E(x,t)$, $B(x,t)$, where $\rho(x,t) = \rho(x - q(t))$ and $j(x,t) = \rho(x - q(t))\dot{q}(t)$. Inserting these expressions for E, B into (A.38) we reduce the Cauchy problem (A.35) to

$$\dot{u}(t) = f_t(I_t u), u(0) = u^0, \tag{A.39}$$

where $u(t) = (q(t), p(t))$, $u^0 = (q^0, p^0)$ and $I_t u$ means the restriction of the function $u(t)$ to the interval $[0, t]$. Lemma A.2 (ii) and ($P_{\min}$) imply that the map $I_t u \mapsto f_t(I_t u)$ is locally Lipschitz continuous in the space $C(0, T; \mathbb{R}^6)$ for every $T > 0$. Hence, by the contraction mapping principle, the Cauchy problem (A.39) has a unique local solution $u(\cdot) = (q(\cdot), p(\cdot)) \in C(-\varepsilon, \varepsilon; \mathbb{R}^6)$ with $\varepsilon > 0$ depending only on b. It remains to define $Y(t) = (E(x,t), B(x,t), q(t), p(t))$ where $E(x,t)$ and $B(x,t)$ are given by (A.4) with $\rho(x,t) = \rho(x - q(t))$ and $j(x,t) = \rho(x - q(t))\dot{q}(t)$. Thus, (A.35) has a unique local solution $Y(\cdot) \in C([-\varepsilon, \varepsilon], \mathcal{M})$ with $\varepsilon > 0$ depending only on b.

Step 2. *W_t is a continuous map in $\mathcal{M}$ and in $\mathcal{M}_F$.*

The continuity of the map $W_t : Y^0 \mapsto Y(t)$ in $\mathcal{M}$ for $|t| \le \varepsilon$ and $\|Y^0\|_{\mathcal{H}} \le b$ follows from the continuity of $w_t : u^0 \mapsto u(t)$ and from Lemma A.2 (ii). To prove continuity of W_t in $\mathcal{M}_F$, let us consider Picard's successive approximation scheme

$$u^N(t) = u^0 + \int_0^t ds\, f_s(I_s u^{N-1}), \quad N = 1, 2, \ldots$$

The equation for $\dot q^N$ in this system implies $|\dot q^N(t)| < 1$ and therefore $|q(t)| < |q^0| + |t|$. We fix now $t \in \mathbb{R}$. Then from the integral representation (A.4) we conclude that every Picard's approximation $u^N(t)$ and hence the solution $u(t)$ depends only on the initial data $(E^0(x'),\, B^0(x'),\, q^0\,,\, p^0)$ with $|x'-q^0| < |t|+R_\rho$. Therefore (A.4) implies that the fields $E(x,t)$, $B(x,t)$ in a neighborhood of a point (x,t) depend only on the initial data $(E^0(x'),\, B^0(x'),\, q^0\,,\, p^0)$ with $|x'| < 2|t| + R_\rho + |q^0|$, $|x' - x| \le |t|$. Thus the continuity of W_t in $\mathcal{M}_F$ follows from the continuity in $\mathcal{M}$.

Step 3. *The energy conservation (2.6) holds.*

Energy conservation is provided first for a dense subset of smooth $Y^0 \in \mathcal{M}_\infty$ (Definition A.3 and Lemma A.4) and afterwards extended to all of $Y^0 \in \mathcal{M}$ by density and continuity. The system (A.34) for $Y(t) \in C(-\varepsilon, \varepsilon; \mathcal{M})$ implies a convolution representation (A.4). Then (A.5) and the bounds (A.6), (A.32) imply that $E(x,t), B(x,t) \in C^1(]\varepsilon, \varepsilon[\times \mathbb{R}^3)$ and $|E(x,t)| + |B(x,t)| \sim |x|^{-2}$. Therefore (2.6) follows by partial integration:

$$\begin{aligned}
\frac{d}{dt}\mathcal{H}(Y(t)) \;&=\; \frac{p\cdot\dot p}{\sqrt{1+p^2}} + < E, \dot E > + < B, \dot B > \\
&- < \phi_{ex}(x), \dot q \cdot \nabla\rho(x-q) > = \dot q \cdot < E + E_{ex} + \dot q \wedge (B + B_{ex}), \rho(x-q) > \\
&+ < E, \mathrm{rot} B - \rho(x-q)\dot q > - < B, \mathrm{rot} E > - \dot q \cdot < E_{ex}(x), \rho(x-q) > \\
&= < E, \mathrm{rot} B > - < B, \mathrm{rot} E > = -\lim_{R\to\infty} \int\limits_{|x|=R} d^2x\, (E \wedge B)\cdot \frac{x}{|x|} \;=\; 0.
\end{aligned}$$

Step 4. *The global solution exists.*

For $|t| < \varepsilon$, we have

$$\sqrt{1 + p^2(t)} + \frac{1}{2}\|E(x,t)\|^2 + \frac{1}{2}\|B(x,t)\|^2 + \phi_{ex}(q) = \mathcal{H}(Y^0). \tag{A.40}$$

Therefore $(P_{\min})$ implies the apriori estimate

$$\|E(x,t)\| + \|B(x,t)\| + |p(t)| \le \overline{h} \tag{A.41}$$

with $\overline{h}$ depending only on the initial data Y^0. This apriori estimate implies that the properties (i)–(iii) for arbitrary $t \in \mathbb{R}$ follow from the same properties for small $|t|$.

ad (iv) (A.41) implies $|p(t)| \le p_0 < \infty$. Hence

$$|\dot q(t)|/(1 - \dot q^2(t))^{1/2} = |p(t)| \le p_0 < \infty,$$

which yields $|\dot{q}(t)| \le q_1 < 1$.

ad (v) The last equation in (A.34) and (Q), $(P_{\min})$, (A.36) imply (A.37) for $\ddot{q}$. Differentiating the last equation in (A.34) and using $|q^{(k)}(t)| \le q_k$ with $k = 0, 1, 2$, and $(P_{\min})$ again, we finally obtain $|\dddot{q}(t)| \le q_3 < \infty$ for $t \in \mathbb{R}$.

ad (vi) $(P_{\max})$ and the last equation in (A.34) provides (A.37) as above.

ad (vii) (P_∞) implies (Q) by (A.40). $\square$

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