The Dirac equation without spinors

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In the first part of the paper we give a tensor version of the Dirac equation. In the second part we formulate and analyse a simple model equation which for weak external fields appears to have properties similar to those of the 2–dimensional Dirac equation.

1. Introduction

The paper is an attempt at analysing the geometrical meaning of the Dirac equation

$$\gamma^{\varkappa}(i\nabla - eA)_{\varkappa}\psi = m\psi, \tag{1.1}$$

and at presenting this geometry in a mathematical language understandable to specialists in partial differential equations. The paper consists of two parts.

In the first part we give a tensor interpretation of bispinors (Section 3) and reformulate the Dirac equation in tensor form (Section 4). In Section 5 we discuss the simplifications which occur when we move from $\mathbb{M}^4$ (3–dimensional Dirac equation) to $\mathbb{M}^3$ (2–dimensional Dirac equation). Note that the fact that the Dirac equation can, in principle, be written in tensor form is not new; see, for example, [BT].

In the second part we adopt the point of view that the Dirac equation is not a precise equation, but a first approximation in an asymptotic process described by Feynman diagrams, and it should be used only when the external field A is weak, smooth, and slowly varying. We suggest the following model equation in $\mathbb{M}^3$:

$$\operatorname{curl}_A u = \pm m u\,. \tag{1.2}$$

Here u is a complex valued vector function, and curl_A is the usual curl corrected by the electromagnetic field; see next section for precise definition. We show that (1.2) has properties very similar to those of the two–dimensional Dirac equation. In Section 6 we give a mathematical meaning to "weak, smooth, and slowly varying" by introducing asymptotic scaling in terms of a small dimensionless parameter α (fine structure constant) and perform a formal asymptotic analysis of (1.2). This formal asymptotic analysis reduces (1.2) to the Pauli equation. Mathematically rigorous (but, inevitably, more restrictive) results are presented in Section 7.

Finally, in Section 8 we point out the remarkable similarity between (1.2) and the Maxwell equations.

2. Principal notation

Our notation mostly follows [LL4], with minor modifications.

By $\mathbb{M}^n$ we denote Minkowski n–space with temporal coordinate x^0, spatial coordinates $(x^1, \ldots, x^{n-1})$, and metric $g_{\mu\nu} = \mathrm{diag}(+1, -1, \ldots, -1)$. By $\mathbb{R}^m$ we denote Euclidean m–space with spatial coordinates $(x^1, \ldots, x^m)$, and metric $\mathrm{diag}(-1, \ldots, -1)$. We use Greek letters for tensor indices (irrespectively of whether we are in Minkowski or Euclidean space), and repeated indices imply summation. For vectors in $\mathbb{M}^n$ or $\mathbb{R}^m$ we denote $\langle u, v\rangle := u_\mu v^\mu$. Bold type indicates the contravariant representation of a Euclidean vector: $\mathbf{u} = (u^1, \ldots, u^m) = (-u_1, \ldots, -u_m)$. We write $\mathbf{u}\cdot\mathbf{v} := u^\mu v^\mu = -\langle u, v\rangle$.

We use the "overline" to denote complex conjugation, $\,^T$ to denote transposition, and $\,^*$ to denote Hermitian conjugation (combination of complex conjugation and transposition). Thus, the Dirac conjugate of a bispinor ψ in our notation is $\psi^*\gamma^0$. The norm sign without any additional indices indicates the L^2–norm of a function; say, for a vector function $\mathbf{u} : \mathbb{R}^m \to \mathbb{C}^m$ we have $\|\mathbf{u}\|^2 = \int \overline{\mathbf{u}}\cdot\mathbf{u}\, d^m x$.

By m and $e = -|e|$ we denote the mass and the charge of the electron, respectively. The fine structure constant is $\alpha = e^2 \approx \frac{1}{137}$. By $A = (A^0, \mathbf{A})$ we denote the electromagnetic vector potential, which is a given real valued vector function. Here $A^0 \equiv \Phi$ is the electric potential, and $\mathbf{A}$ is the magnetic vector potential. We denote $\nabla_\mu := \partial/\partial x^\mu$ and $P := i\nabla - eA$.

Our system of units is such that the speed of light c and Planck's constant $\hbar$ are both 1, whereas $m \sim 1$ (equivalently, Compton's wavelength ~ 1).

By $e^{\lambda\mu\nu}$ and $e^{\varkappa\lambda\mu\nu}$ we denote the totally antisymmetric pseudotensor in $\mathbb{M}^3$ and $\mathbb{M}^4$, respectively; we take $e^{012} = e^{0123} = +1$. In $\mathbb{M}^3$ we define the dual of an antisymmetric tensor T as $T^\star{}_\lambda := \frac{1}{2}e_{\lambda\mu\nu}T^{\mu\nu}$, and in $\mathbb{M}^4$ as $T^\star{}_{\varkappa\lambda} := \frac{1}{2}e_{\varkappa\lambda\mu\nu}T^{\mu\nu}$. In $\mathbb{M}^3$ we define the vector product as $[v, w]^\lambda := e^{\lambda\mu\nu}v_\mu w_\nu$; accordingly, $\mathrm{curl} := [\nabla,\ \cdot\ ]$, and $\mathrm{curl}_A := [\nabla + ieA,\ \cdot\ ] = -i[P,\ \cdot\ ]$. We define divergence on vectors and tensors as $\mathrm{div}\, u := \nabla_\mu u^\mu$ and $(\mathrm{div}\, T)^\nu := \nabla_\mu T^{\mu\nu}$, respectively.

"Pseudo" refers to quantities which behave in the "correct" way (as scalars, vectors, or tensors) under changes of coordinates preserving orientation, and which get the "wrong" sign under change of orientation.

By $\boldsymbol{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$ we denote the "vector" of Pauli matrices,

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and by $\gamma = (\gamma^0, \gamma^1, \gamma^2, \gamma^3)$ the "vector" of Dirac matrices. In Sections 3 and 4 we use the spinor representation of bispinors as opposed to the more common standard representation (see Section 21 of [LL4] for details); consequently the Dirac matrices appearing in these sections are

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \qquad \gamma^\mu = \begin{pmatrix} 0 & -\sigma^\mu \\ \sigma^\mu & 0 \end{pmatrix}, \quad \mu = 1, 2, 3.$$

3. A tensor interpretation of bispinors

A bispinor in $\mathbb{M}^4$ is a set of four complex numbers

$$\psi = \begin{pmatrix}\xi\\ \eta\end{pmatrix} = \begin{pmatrix}\xi^1\\ \xi^2\\ \eta_{\dot 1}\\ \eta_{\dot 2}\end{pmatrix} \tag{3.1}$$

which change under passive Lorentz transformations in the following way. Under proper Lorentz transformations $\xi' = B\xi$ and $\eta' = (B^*)^{-1}\eta$, where B is a complex 2×2 matrix given by

$$B = \exp\left(-\tfrac{1}{2}\phi\,\mathbf{n}\cdot\boldsymbol{\sigma}\right) = \cosh(\tfrac{1}{2}\phi) - \mathbf{n}\cdot\boldsymbol{\sigma}\sinh(\tfrac{1}{2}\phi)$$

for a boost of speed $V = \tanh(\phi)$ along the unit vector $\mathbf{n} \in \mathbb{R}^3$, and by

$$B = \exp\left(\tfrac{1}{2}i\vartheta\,\mathbf{n}\cdot\boldsymbol{\sigma}\right) = \cos(\tfrac{1}{2}\vartheta) + i\mathbf{n}\cdot\boldsymbol{\sigma}\sin(\tfrac{1}{2}\vartheta)$$

for a rotation by angle ϑ around $\mathbf{n}$. Here "passive" means that we transform the coordinate system and not the the bispinors themselves, and the prime refers to the representation of the bispinor in the new coordinate system. Under space inversion $\xi' = i\eta$ and $\eta' = i\xi$. Under time inversion $\xi' = \sigma^2\overline{\xi}$, $\eta' = \sigma^2\overline{\eta}$.

Remark 3.1. A full rotation of the coordinate system ($\vartheta = 2\pi$) changes the sign of a bispinor, which means that bispinors are defined up to a choice of sign.

Throughout this section we assume that we are dealing with a bispinor satisfying the technical condition $\eta^*\xi \neq 0$.

Let $\mathfrak{t}$ be a fixed real time–like vector and let $\{e^{(0)}, e^{(1)}, e^{(2)}, e^{(3)}\}$, $\langle e^{(k)}, e^{(l)}\rangle = g^{kl}$, be some coordinate basis. Set $\tau := \operatorname{sign}\langle \mathfrak{t}, e^{(0)}\rangle$. In other words, to every coordinate system we assign a number $\tau = +1$ or $\tau = -1$. We assume that for the original coordinate system (and, consequently, for all coordinate systems obtained from the original one by proper Lorentz transformations and space inversion) $\tau = +1$. Thus, the vector $\mathfrak{t}$ fixes the positive direction of time.

Put

$$\rho := |2\eta^*\xi|, \qquad \theta := \arg(2\eta^*\xi), \tag{3.2}$$

$$j^\mu := \psi^*\gamma^0\gamma^\mu\psi, \qquad u^\mu := -i\psi^T\gamma^0\gamma^2\gamma^\mu\psi, \tag{3.3}$$

$$f^{(0)} := \frac{\tau j}{\rho}, \qquad f^{(1)} := \frac{\operatorname{Re} u}{\rho}, \qquad f^{(2)} := \frac{\tau \operatorname{Im} u}{\rho}. \tag{3.4}$$

Theorem 3.2. *The quantities*

$$\{\rho,\, \theta;\, f^{(k)},\ k = 0, 1, 2\} \tag{3.5}$$

defined in accordance with (3.2)–(3.4) have the following properties:

$\rho \in \mathbb{R}_+$ *is a scalar;*

$\theta \in \mathbb{S}^1$ *is a pseudoscalar;*

the $f^{(k)}$ *are real vectors forming an orthonormal triad, i.e.,* $\langle f^{(k)}, f^{(l)} \rangle = g^{kl}$.

Conversely, given a set (3.5) with the above three properties there is a unique bispinor (3.1) satisfying (3.2)–(3.4).

Of course, "uniqueness" of a bispinor is understood as "uniqueness up to the choice of sign"; see Remark 3.1.

P r o o f of Theorem 3.2. Clearly $f^{(1)}$ and $f^{(2)}$ are real whilst $\left(\gamma^0\gamma^\mu\right)^* = \gamma^{\mu *}\gamma^0 = \gamma^0\gamma^\mu$ from which it follows that j and hence $f^{(0)}$ are also real.

Given a Lorentz transformation Λ of the coordinate system let $\Lambda^\mu{}_\nu$, $\mu, \nu = 0,1,2,3$, be the real 4×4 matrix in terms of which the vector a transforms to the vector a' with components given by

$$a'^\mu = \Lambda^\mu{}_\nu a^\nu .$$

For a boost of speed $V = \tanh(\phi)$ along the unit vector $\mathbf{n} \in \mathbb{R}^3$

$$\Lambda = \exp\left(-\phi M_{\mathbf{n}}\right) = I + \left(\cosh(\phi) - 1\right) M_{\mathbf{n}}^2 - \sinh(\phi) M_{\mathbf{n}},$$

where

$$M_{\mathbf{n}} = \begin{pmatrix} 0 & n^1 & n^2 & n^3 \\ n^1 & 0 & 0 & 0 \\ n^2 & 0 & 0 & 0 \\ n^3 & 0 & 0 & 0 \end{pmatrix},$$

whilst, for a rotation by angle ϑ around $\mathbf{n}$,

$$\Lambda = \exp\left(-\vartheta N_{\mathbf{n}}\right) = I + \left(1 - \cos(\vartheta)\right) N_{\mathbf{n}}^2 - \sin(\vartheta) N_{\mathbf{n}},$$

where

$$N_{\mathbf{n}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -n^3 & n^2 \\ 0 & n^3 & 0 & -n^1 \\ 0 & -n^2 & n^1 & 0 \end{pmatrix}.$$

If the Lorentz transformation Λ is proper the corresponding transformation of the bispinor ψ is linear and so can be represented by a complex 4×4 matrix $S(\Lambda)$; that is $\psi' = S(\Lambda)\psi$. Clearly

$$S(\Lambda) = \begin{pmatrix} B(\Lambda) & 0 \\ 0 & \left(B^*(\Lambda)\right)^{-1} \end{pmatrix},$$

where $B(\Lambda)$ is the complex 2×2 matrix defined above. A straightforward check using the explicit forms of the matrices $S(\Lambda)$ and $B(\Lambda)$ for boosts and spatial rotations gives us

$$\Lambda^\mu{}_\nu \gamma^\nu = S^{-1}(\Lambda)\gamma^\mu S(\Lambda) \tag{3.6}$$

in these cases (see Section 7.2 of [S] for more details). Formula (3.6) immediately extends to all proper Lorentz transformations since any such transformation can be written as a product of boosts and rotations. On the other hand det $\big(B(\Lambda)\big) = 1$ from which we get $\sigma^2 B^{-1}(\Lambda) = B^T(\Lambda)\sigma^2$ and $\sigma^2 B^*(\Lambda) = \overline{B(\Lambda)}^{\,-1}\sigma^2$. Hence

$$S^*(\Lambda)\gamma^0 = \gamma^0 S^{-1}(\Lambda), \tag{3.7}$$

$$\overline{S(\Lambda)}^{\,-1}\gamma^2 = \gamma^2 S^{-1}(\Lambda). \tag{3.8}$$

Now, for proper Lorentz transformations,

$$\eta'^*\xi' = \big((B^*(\Lambda))^{-1}\eta\big)^* B(\Lambda)\xi = \eta^* B^{-1}(\Lambda)B(\Lambda)\xi = \eta^*\xi,$$

so $\rho' = \rho$ and $\theta' = \theta$. Also, by (3.6) and (3.7),

$$\begin{aligned} j'^{\mu} &= \psi'^*\gamma^0\gamma^\mu\psi' = \psi^* S^*(\Lambda)\gamma^0\gamma^\mu S(\Lambda)\psi \\ &= \psi^*\gamma^0 S^{-1}(\Lambda)\gamma^\mu S(\Lambda)\psi = \psi^*\gamma^0\Lambda^\mu{}_\nu\gamma^\nu\psi = \Lambda^\mu{}_\nu j^\nu, \end{aligned}$$

whilst a similar argument using (3.8) as well gives

$$\begin{aligned} u'^{\mu} &= -i\psi^T S^T(\Lambda)\gamma^0\gamma^2\gamma^\mu S(\Lambda)\psi \\ &= -i\psi^T\gamma^0\overline{S(\Lambda)}^{\,-1}\gamma^2\gamma^\mu S(\Lambda)\psi = -i\psi^T\gamma^0\gamma^2 S^{-1}(\Lambda)\gamma^\mu S(\Lambda)\psi = \Lambda^\mu{}_\nu u^\nu. \end{aligned}$$

Under space inversion $\xi' = i\eta$ and $\eta' = i\xi$ so $\eta'^*\xi' = \xi^*\eta = \overline{\eta^*\xi}$ and $\psi' = S(\Lambda)\psi$ where $S(\Lambda) = \begin{pmatrix} 0 & iI \\ iI & 0 \end{pmatrix}$. Thus $\rho' = \rho$ and $\theta' = -\theta$. On the other hand $\Lambda^\mu{}_\nu =$ diag$(+1,-1,-1,-1)$ which can be used to directly check that (3.6) to (3.8) are still valid; it follows that $j'^\mu = \Lambda^\mu{}_\nu j^\nu$ and $u'^\mu = \Lambda^\mu{}_\nu u^\nu$.

Under time inversion $\xi' = \sigma^2\overline{\xi}$ and $\eta' = \sigma^2\overline{\eta}$ so $\eta'^*\xi' = \overline{\eta}^*(\sigma^2)^*\sigma^2\overline{\xi} = \overline{\eta^*\xi}$, giving $\rho' = \rho$ and $\theta' = -\theta$ once again. Now $\Lambda^\mu{}_\nu = $ diag$(-1,+1,+1,+1)$. We cannot extend the definition of $S(\Lambda)$ to cover the present case; however if we set $S_t = \begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}$ then

$$\psi' = S_t\overline{\psi}, \quad \Lambda^\mu{}_\nu\overline{\gamma^\nu} = -S_t^{-1}\gamma^\mu S_t, \quad S_t^*\gamma^0 = \gamma^0 S_t^{-1} \quad \text{and} \quad \overline{S_t}^{\,-1}\gamma^2 = \overline{\gamma^2}S_t^{-1}.$$

Therefore

$$j'^\mu = \psi^T S_t^*\gamma^0\gamma^\mu S_t\overline{\psi} = -\psi^T\gamma^0\Lambda^\mu{}_\nu\overline{\gamma^\nu}\,\overline{\psi} = -\Lambda^\mu{}_\nu\overline{j^\nu} = -\Lambda^\mu{}_\nu j^\nu,$$

(n.b. γ^0 and j are real) and

$$\begin{aligned} u'^\mu &= -i\psi^* S_t^T\gamma^0\gamma^2\gamma^\mu S_t\overline{\psi} \\ &= -i\psi^*\gamma^0\overline{S_t}^{\,-1}\gamma^2\gamma^\mu S_t\overline{\psi} = i\psi^*\gamma^0\overline{\gamma^2}\Lambda^\mu{}_\nu\overline{\gamma^\nu}\,\overline{\psi} = \Lambda^\mu{}_\nu\overline{u^\nu}. \end{aligned}$$

However $\tau' = -\tau$ so ${f^{(k)}}'^{\mu} = \Lambda^{\mu}{}_{\nu} {f^{(k)}}^{\nu}$ for $k = 0, 1, 2$.

Since any Lorentz transformation can be written as a combination of a proper transformation, space inversion and time inversion, the above calculations show that ρ is a scalar, θ is a pseudoscalar and $f^{(k)}$ is a vector for $k = 0, 1, 2$.

The fact that $\{f^{(k)},\ k = 0, 1, 2\}$ is an orthonormal triad can be checked directly by a somewhat lengthy calculation. However, since we have established that the $f^{(k)}$'s are vectors, it suffices to prove they are orthonormal for one particular choice of the coordinate system.

Claim: there exists a unique proper coordinate system in which

$$\psi = \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \pm\sqrt{\frac{\rho}{2}} \begin{pmatrix} e^{i\theta/2} \\ 0 \\ e^{-i\theta/2} \\ 0 \end{pmatrix} \tag{3.9}$$

for some $\rho \in \mathbb{R}_+$ and $\theta \in \mathbb{S}^1$. Of course, these ρ and θ must be given by (3.2).

Indeed, we have

$$\begin{aligned} j^{\mu} j_{\mu} &= \left(\xi^* \xi + \eta^* \eta\right)^2 - \sum_{\nu=1}^{3} \left(\xi^* \sigma^{\nu} \xi - \eta^* \sigma^{\nu} \eta\right)^2 \\ &= 4\left(\overline{\xi^1} \eta_{\dot{1}} + \overline{\xi^2} \eta_{\dot{2}}\right)\left(\xi^1 \overline{\eta_{\dot{1}}} + \xi^2 \overline{\eta_{\dot{2}}}\right) = \left|2\eta^* \xi\right|^2 = \rho^2 > 0. \end{aligned}$$

Therefore the real vector j is time-like so we can choose a proper coordinate system, given by an appropriate boost, in which $\mathbf{j} = \mathbf{0}$ and $j^0 = \rho$; n.b. $j^0 = \xi^* \xi + \eta^* \eta > 0$. Now suppose we rotate the coordinate system by an angle $\vartheta \in \mathbb{R}$ about the unit vector $\mathbf{n} = (n^1, n^2, 0) \in \mathbb{R}^3$. From above we have $\xi' = B\xi$ where

$$B = \begin{pmatrix} \cos(\frac{1}{2}\vartheta) & (n^2 + in^1)\sin(\frac{1}{2}\vartheta) \\ (-n^2 + in^1)\sin(\frac{1}{2}\vartheta) & \cos(\frac{1}{2}\vartheta) \end{pmatrix}.$$

Choosing n^1, n^2 and ϑ so that $(-n^2 + in^1)\sin(\frac{1}{2}\vartheta)\xi^1 + \cos(\frac{1}{2}\vartheta)\xi^2 = 0$ we thus have ${\xi'}^2 = 0$. Since we are only rotating the coordinate system we must have $\mathbf{j}' = \mathbf{0}$ and ${j'}^0 = \rho$. An elementary analysis of these formulae gives us ${\eta'}_{\dot{2}} = 0$ and $|{\eta'}_{\dot{1}}| = \left|{\xi'}^1\right| = \sqrt{\frac{\rho}{2}}$.

Suppose now that we are in the rotated coordinate system (and drop the primes). Rotate this coordinate system by an angle $\vartheta \in \mathbb{R}$ about $\mathbf{n} = (0, 0, 1)$. From above we have $\xi' = B\xi$ and $\eta' = (B^*)^{-1}\eta$ where

$$B = \begin{pmatrix} \alpha & 0 \\ 0 & \overline{\alpha} \end{pmatrix}, \qquad \alpha = e^{i\vartheta/2}.$$

Therefore ${\xi'}^2 = {\eta'}_{\dot{2}} = 0$, ${\xi'}^1 = \alpha\xi^1$ and ${\eta'}_{\dot{1}} = \alpha\eta_{\dot{1}}$. Now, $|\overline{\eta_{\dot{1}}}| = |\xi^1|$ so we can choose ϑ such that ${\xi'}^1 = \overline{{\eta'}_{\dot{1}}}$. However ${\eta'}^* \xi' = \eta^* \xi$ (since we have only changed the coordinate system by proper Lorentz transformations) so $2{\eta'}^* \xi' = 2\left({\xi'}^1\right)^2 =$

$2\left(\overline{\eta'_1}\right)^2 = \rho e^{i\theta}$, giving us the required representation (3.9). In this coordinate system it is straightforward to check that

$$f^{(0)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad f^{(1)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad f^{(2)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \tag{3.10}$$

(contravariant representation). Clearly, (3.10) can be true only in one proper coordinate system. This completes the proof of the claim.

Formula (3.10) shows that $\{f^{(k)},\ k = 0,1,2\}$ is an orthonormal triad. The proof of the direct statement of Theorem 3.2 is complete.

Conversely, suppose we have a set (3.5) with the required properties. Choose the proper coordinate system in which (3.10) holds and define the bispinor ψ by (3.9). It follows that this ψ satisfies (3.2)–(3.4). Now, let $\widetilde{\psi}$ be another bispinor satisfying (3.2)–(3.4). The above claim associates with $\widetilde{\psi}$ a special proper coordinate system in which $\widetilde{\psi}$ is given by the right–hand side of (3.9). But (3.10) can be true only in one proper coordinate system, so the special coordinate system for the bispinor $\widetilde{\psi}$ is the one we are already working in. This implies $\widetilde{\psi} = \psi$. □

Set ${f^{(3)}}^{\nu} := e^{\varkappa\lambda\mu\nu} {f^{(0)}}_{\varkappa} {f^{(1)}}_{\lambda} {f^{(2)}}_{\mu}$. We shall call the pseudovector $f^{(3)}$ *spin*. Clearly, $\{f^{(k)},\ k = 0,1,2,3\}$ is an orthonormal tetrad (orthonormal frame).

4. The Dirac equation in tensor form

From now on our bispinor ψ is a function of the point x in space–time. We assume that ψ is defined on some open set $O \subset \mathbb{M}^4$, and that ψ is smooth and satisfies the technical condition $\eta^*\xi \neq 0$. Our objective is to rewrite the Dirac equation (1.1) in terms of the tensor quantities (3.5).

It will be convenient for us to deal with Euler–Lagrange functionals rather than with the corresponding differential equations. It is well known that the Euler–Lagrange functional for (1.1) is

$$\int \psi^* \gamma^0 \left(\gamma^{\varkappa} (i\nabla - \tau e A)_{\varkappa} - m \right) \psi \, d^4x \tag{4.1}$$

(we included the factor τ because the Dirac equation is CT–invariant, but not T–invariant). Integration in (4.1) is carried out over O. We do not assume, however, that the integral converges and treat (4.1) as a formal expression. This is acceptable for our purposes because we are interested not in (4.1) itself, but only in its variation generated by a variation of the bispinor $\delta\psi \in C_0^\infty(O)$. The same applies to all subsequent functionals.

Further on we use

$$\int \mathrm{Re}\Big(\psi^*\gamma^0\big(\gamma^\varkappa(i\nabla-\tau eA)_\varkappa - m\big)\psi\Big)\,d^4x \tag{4.2}$$

instead of (4.1) as the Euler–Lagrange functional for (1.1). This is possible because the variation of (4.1) is real. Incidentally, without the Re the integrand is not a scalar (time inversion leads to complex conjugation).

Let us now examine the rotation of the orthonormal frame $\{f^{(k)},\ k=0,1,2,3\}$ as we move from one point x to another. Let

$$\delta {f^{(k)}}_\mu = (\nabla_\nu {f^{(k)}}_\mu)\,\delta x^\nu \tag{4.3}$$

be the increment of the vector function $f^{(k)}$ when we move from x to a close point $x+\delta x$. We define the (antisymmetric) tensor of infinitesimal rotations R as the solution of the linear system

$$\delta {f^{(k)}}_\mu = R_{\mu\lambda}{f^{(k)}}^\lambda, \qquad k=0,1,2,3. \tag{4.4}$$

The explicit formula for the solution of (4.4) is

$$R_{\mu\lambda} = \sum_{j,l=0}^{3} g_{jl}\,(\delta {f^{(j)}}_\mu)\,{f^{(l)}}_\lambda\,. \tag{4.5}$$

In particular, formulae (4.5), (4.3) imply

$$\big(\mathrm{div}(R^\star)\big)^\varkappa = \frac{1}{2}\sum_{j,l=0}^{3} g_{jl}\,e^{\epsilon\varkappa\mu\lambda}\,(\nabla_\epsilon {f^{(j)}}_\mu)\,{f^{(l)}}_\lambda\,. \tag{4.6}$$

As $R^\star$ is a pseudotensor, $\mathrm{div}(R^\star)$ is a pseudovector.

Theorem 4.1. *We have the identity*

$$\begin{aligned}\mathrm{Re}\Big(\psi^*\gamma^0\big(\gamma^\varkappa(i\nabla-\tau eA)_\varkappa - m\big)\psi\Big) &= \\ -\left[\frac{1}{2}\big\langle f^{(3)},\,\mathrm{div}(R^\star)+\mathrm{grad}\,\theta\big\rangle \;+\; e\big\langle f^{(0)},A\big\rangle \;+\; m\cos\theta\right]\rho\,. &\end{aligned} \tag{4.7}$$

P r o o f of Theorem 4.1. Since both sides of (4.7) are scalars it suffices to check the identity at each point $x\in O$ in only one coordinate system. Choosing the coordinate system given by the claim in the proof of Theorem 3.2 (based at the point x) we have

$${f^{(k)}}_\mu = g_{k\mu}, \qquad k,\mu = 0,1,2,3;$$

n.b. this, and other expressions to follow, are *not* tensor identities but rather identities which hold at the point x for our special choice of coordinates. Using this expression and (4.6) we have

$$\big(\mathrm{div}(R^\star)\big)_3 = \nabla_0 {f^{(1)}}_2 - \nabla_1 {f^{(0)}}_2 + \nabla_2 {f^{(0)}}_1\,,$$

and hence

$$\rho\big\langle f^{(3)},\,\mathrm{div}(R^\star)+\mathrm{grad}\,\theta\big\rangle=\rho\big(\nabla_0 f^{(1)}{}_2-\nabla_1 f^{(0)}{}_2+\nabla_2 f^{(0)}{}_1+\nabla_3\theta\big)$$
$$=\nabla_0\big(\rho f^{(1)}{}_2\big)-\nabla_1\big(\rho f^{(0)}{}_2\big)+\nabla_2\big(\rho f^{(0)}{}_1\big)-ie^{-i\theta}\nabla_3\big(\rho e^{i\theta}\big)+i\nabla_3\rho\,.$$

Combining the general definitions (3.2)–(3.4) with the specific value of the bispinor $\psi(x)$ in our chosen coordinate frame (formula (3.9)) we get

$$\begin{aligned}
\nabla_0\big(\rho f^{(1)}{}_2\big) &= i\Big(-\overline{\xi^1}\nabla_0\xi^1+\xi^1\nabla_0\overline{\xi^1}-\overline{\eta_{\dot 1}}\nabla_0\eta_{\dot 1}+\eta_{\dot 1}\nabla_0\overline{\eta_{\dot 1}}\Big)\,,\\
\nabla_1\big(\rho f^{(0)}{}_2\big) &= i\Big(\overline{\xi^1}\nabla_1\xi^2-\xi^1\nabla_1\overline{\xi^2}-\overline{\eta_{\dot 1}}\nabla_1\eta_{\dot 2}+\eta_{\dot 1}\nabla_1\overline{\eta_{\dot 2}}\Big)\,,\\
\nabla_2\big(\rho f^{(0)}{}_1\big) &= -\overline{\xi^1}\nabla_2\xi^2-\xi^1\nabla_2\overline{\xi^2}+\overline{\eta_{\dot 1}}\nabla_2\eta_{\dot 2}+\eta_{\dot 1}\nabla_2\overline{\eta_{\dot 2}}\,,\\
e^{-i\theta}\nabla_3\big(\rho e^{i\theta}\big) &= 2\overline{\xi^1}\nabla_3\xi^1+2\eta_{\dot 1}\nabla_3\overline{\eta_{\dot 1}}\,,\\
\nabla_3\rho &= \overline{\xi^1}\nabla_3\xi^1+\xi^1\nabla_3\overline{\xi^1}+\overline{\eta_{\dot 1}}\nabla_3\eta_{\dot 1}+\eta_{\dot 1}\nabla_3\overline{\eta_{\dot 1}}\,.
\end{aligned}$$

The above expressions can then be combined producing

$$\begin{aligned}
&\rho\big\langle f^{(3)},\,\mathrm{div}(R^\star)+\mathrm{grad}\,\theta\big\rangle\\
&\quad= -2\,\mathrm{Re}\Big\{\overline{\xi^1}\big(i\nabla_0\xi^1+i\nabla_1\xi^2+\nabla_2\xi^2+i\nabla_3\xi^1\big)\\
&\qquad\qquad +\overline{\eta_{\dot 1}}\big(i\nabla_0\eta_{\dot 1}-i\nabla_1\eta_{\dot 2}-\nabla_2\eta_{\dot 2}-i\nabla_3\eta_{\dot 1}\big)\Big\}\\
&\quad= -2\,\mathrm{Re}\left\{\begin{pmatrix}\xi^* & \eta^*\end{pmatrix}\left[\begin{pmatrix}I & 0\\ 0 & I\end{pmatrix}i\nabla_0+\sum_{\nu=1}^{3}\begin{pmatrix}\sigma^\nu & 0\\ 0 & -\sigma^\nu\end{pmatrix}i\nabla_\nu\right]\begin{pmatrix}\xi\\ \eta\end{pmatrix}\right\}\\
&\quad= -2\,\mathrm{Re}\,\big(\psi^*\gamma^0\gamma^\mu i\nabla_\mu\psi\big)\,.
\end{aligned}$$

From the definition of $f^{(0)}$ we have

$$\rho e\langle f^{(0)},A\rangle=\rho f^{(0)\,\mu}eA_\mu=\psi^*\gamma^0\gamma^\mu\tau eA_\mu\psi,$$

whilst the definition of ρ and θ gives

$$\rho m\cos\theta=m\,\mathrm{Re}(2\eta^*\xi)=m\begin{pmatrix}\xi^* & \eta^*\end{pmatrix}\begin{pmatrix}0 & I\\ I & 0\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix}=m\psi^*\gamma^0\psi.$$

The result now follows from the fact that both of these expressions are real. □

Formulae (4.2) and (4.7) imply

Corollary 4.2. *The Euler–Lagrange functional for the Dirac equation can be written as*

$$\int\left[\frac{1}{2}\big\langle f^{(3)},\,\mathrm{div}(R^\star)+\mathrm{grad}\,\theta\big\rangle\;+\;e\big\langle f^{(0)},A\big\rangle\;+\;m\cos\theta\right]\rho\,d^4x\,. \tag{4.8}$$

Variation of (4.8) with respect to the scalar ρ, pseudoscalar θ and the moving frame $\{f^{(k)},\ k=0,1,2,3\}$ produces a (nonlinear) system of tensor differential equations equivalent to the Dirac equation (1.1). Of course, in performing this variation one has to remember the constraints: all the quantities are real, ρ is positive, and $\langle f^{(k)}, f^{(l)}\rangle = g^{kl}$.

5. Simplifications in the case of $\mathbb{M}^3$

Let us return to the Dirac equation (1.1) and consider the case when A and ψ do not depend on x^3. Then (1.1) separates into two systems of two equations :

$$\begin{pmatrix} P_0 & P_\mp \\ -P_\pm & -P_0 \end{pmatrix}\begin{pmatrix}\varphi_\pm \\ \chi_\pm\end{pmatrix} = m\begin{pmatrix}\varphi_\pm \\ \chi_\pm\end{pmatrix}, \tag{5.1}$$

where $P_\pm := P_1 \pm iP_2$ and

$$\begin{pmatrix}\varphi_+ \\ \varphi_-\end{pmatrix} = \frac{\xi+\eta}{\sqrt{2}}, \qquad \begin{pmatrix}\chi_- \\ \chi_+\end{pmatrix} = \frac{\xi-\eta}{\sqrt{2}}.$$

(The relation with standard notation is $\varphi_+ = \varphi_1,\ \varphi_- = \varphi_2,\ \chi_- = \chi_1,\ \chi_+ = \chi_2$; see formula (21.17) in [LL4].) Accordingly, simplifications occur in the tensor functional (4.8). We give the final result omitting intermediate calculations.

We are now working in $\mathbb{M}^3$ and the particle is described by the set of quantities

$$\{\rho;\ f^{(k)},\ k=1,2\} \tag{5.2}$$

where $\rho\in\mathbb{R}_+$ is a scalar and the $f^{(k)}$ are real vectors forming an orthonormal dyad, i.e., $\langle f^{(k)}, f^{(l)}\rangle = g^{kl}$. As we are in a 3–space it is convenient to use the notion of a vector product. Put $f^{(0)} := [f^{(1)}, f^{(2)}]$, and define the pseudovector of infinitesimal rotations r as the solution of the linear system $\delta f^{(k)} = [r, f^{(k)}]$, $k=0,1,2$. Then $r=-R^\star$ where

$$R_{\mu\lambda} = \sum_{j,l=0}^{2} g_{jl}\,(\delta f^{(j)}{}_\mu)\, f^{(l)}{}_\lambda \tag{5.3}$$

(cf. (4.5)), and

$$\operatorname{div} r = -\frac{1}{2}\sum_{j,l=0}^{2} g_{jl}\langle f^{(j)}, \operatorname{curl} f^{(l)}\rangle \tag{5.4}$$

(cf. (4.6)). The functional (4.8) turns into

$$\int\left[\frac{1}{2}\operatorname{div} r + e\big\langle f^{(0)}, A\big\rangle \pm m\right]\rho\, d^3x\,, \tag{5.5}$$

where the sign corresponds to that in (5.1). The functional(s) (5.5) should be varied with respect to ρ and the moving frame $\{f^{(k)},\ k=0,1,2\}$.

Remark 5.1. The $f^{(0)}$ from this section is a pseudovector in $\mathbb{M}^3$, and it coincides up to sign with the corresponding part of the 4–vector $f^{(0)}$ from Sections 3 and 4.

Remark 5.2. The set (5.2) is equivalent to a complex valued vector function u satisfying the constraint $\langle u,u\rangle=0$; the equivalence is established by the formula $u=\rho(f^{(1)}+if^{(2)})$, cf. (3.4). This is not surprising: Cartan originally defined spinors as complex vectors u satisfying $\langle u,u\rangle=0$, see Section 52 in [C].

6. A model equation in $\mathbb{M}^3$

The arguments in this section are not mathematically rigorous, and are needed to motivate the introduction of the equation (1.2).

Let $\alpha\to+0$ be an asymptotic parameter. Assume that the external electromagnetic field A is smooth and satisfies

$$eA\sim\alpha^2,\qquad \partial_x^\beta A\sim\alpha^{|\beta|}A\,, \tag{6.1}$$

where $\beta=(\beta_0,\dots,\beta_3)$ is an arbitrary multiindex, $|\beta|=\beta_0+\ldots+\beta_3$, $\partial_x^\beta A=\partial_x^\beta A^\mu=(\nabla_0)^{\beta_0}\ldots(\nabla_3)^{\beta_3}A^\mu$, and "$\sim$" stands for "asymptotically of the order of". In other words, we assume that the field is weak (potential energy of the electron $\sim\alpha^2$) and slowly varying (each differentiation gives an additional α).

Our scaling assumptions (6.1) are meant to model the situation which occurs in the hydrogen or positronium atoms. Indeed for the hydrogen atom

$$eA^0=e\Phi=-\alpha r^{-1},\qquad \mathbf{A}\equiv\mathbf{0}, \tag{6.2}$$

where $r=\sqrt{(x^1)^2+(x^2)^2+(x^3)^2}$. But the characteristic length associated with the wave functions of bound states is

$$r\sim\alpha^{-1}. \tag{6.3}$$

The latter is established by elementary analysis of the corresponding Schrödinger equation; say, the wave function of the ground state is $\Psi=e^{-\alpha mr}$. Formulae (6.2), (6.3) imply (6.1).

Of course, such arguments should be treated with a fair degree of caution as the Coulomb potential has a singularity at the origin. However, in theoretical physics it is common to disregard this technical difficulty, and it is known (see, e.g., Sections 33 and 34 in [LL4]) that one can get very sharp results on the basis of formal asymptotic calculations of the type (6.1)–(6.3).

The question we address now is whether it is possible to suggest simple tensor equations which would be asymptotically equivalent (up to a certain accuracy in powers of the small parameter α) to the Dirac equation. In our search we accept equations whose algebraic structure may be totally different from that of the Dirac equation, as long as they have (asymptotically) the required spectral properties.

Examination of (5.5), (5.4) and Remark 5.2 suggests (1.2) as the natural candidate in $\mathbb{M}^3$. Let us rewrite (1.2) as

$$-i[P,u]=\pm mu\,, \tag{6.4}$$

and formally analyse the properties of this equation. Rigorous mathematical analysis is deferred till the next section.

Let us first make some general observations.

Observation 1: the equation (6.4) is not algebraically equivalent to the Dirac equation (5.1). This is clear from the fact that the number of equations in (6.4) (3 equations) and (5.1) (2 equations) is different. Also, there are no spinors in (6.4).

Observation 2: the equation (6.4) is Lorentz invariant. In fact, (6.4) is probably "more invariant" than the Dirac equation because it can be used in curved space–time: the notions of vector product and curl are defined on any pseudo–Riemannian 3–manifold.

Observation 3: the equation (6.4) is formally self–adjoint:

$$\int \langle \overline{v}\,, -i[P,u]\mp mu\rangle\, d^3x \;=\; \int \langle\, \overline{-i[P,v]\mp mv}\,, u\rangle\, d^3x\,.$$

Moreover, it has an Euler–Lagrange functional which can be written as

$$\int \langle \overline{u}\,, -i[P,u]\mp mu\rangle\, d^3x \qquad \text{or} \qquad \int \mathrm{Re}\,\langle \overline{u}\,, -i[P,u]\mp mu\rangle\, d^3x\,.$$

Equation (6.4) can be presented in matrix form as

$$\begin{pmatrix} 0 & P_2 & -P_1 \\ P_2 & 0 & P_0 \\ -P_1 & -P_0 & 0 \end{pmatrix} \begin{pmatrix} u^0 \\ u^1 \\ u^2 \end{pmatrix} = \pm im \begin{pmatrix} u^0 \\ u^1 \\ u^2 \end{pmatrix}.$$

The first row gives $u^0=\mp i\big(P_2u^1-P_1u^2\big)/m$ which can then be used to eliminate u^0 from the remaining two rows. This results in a 2×2 second order system of equations which is equivalent to equation (6.4):

$$\begin{pmatrix} m^2+{P_2}^2 & \pm imP_0-P_2P_1 \\ \mp imP_0-P_1P_2 & m^2+{P_1}^2 \end{pmatrix} \begin{pmatrix} u^1 \\ u^2 \end{pmatrix} = 0\,. \tag{6.5}$$

Now, suppose we are looking for bound state solutions; that is, assume that A does not depend on x^0 and u is of the form $u(x^1,x^2)e^{-i\varepsilon x^0}$. For such vector functions we have $P_0=\varepsilon-e\Phi$, and (6.5) reduces to

$$\mathcal{A}\mathbf{u}=\varepsilon\mathcal{B}\mathbf{u} \tag{6.6}$$

where

$$\mathcal{A} := \begin{pmatrix} m^2 + {P_2}^2 & \mp ime\Phi - P_2P_1 \\ \pm ime\Phi - P_1P_2 & m^2 + {P_1}^2 \end{pmatrix}, \qquad \mathcal{B} := \begin{pmatrix} 0 & \mp im \\ \pm im & 0 \end{pmatrix},$$

$$\mathbf{u} = \begin{pmatrix} u^1 \\ u^2 \end{pmatrix} : \mathbb{R}^2 \longrightarrow \mathbb{C}^2,$$

and ε is the spectral parameter. Note that $\mathcal{A}$ is not elliptic.

Observation 4: equation (6.6) asymptotically reduces to the Pauli equation. Indeed, the unitary transformation

$$\begin{pmatrix} u^1 \\ u^2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} \varphi_+ \\ \varphi_- \end{pmatrix}$$

turns (6.6) into

$$\begin{pmatrix} m^2 + \frac{P_-P_+}{2} \pm me\Phi \mp \varepsilon m & -\frac{{P_-}^2}{2} \\ -\frac{{P_+}^2}{2} & m^2 + \frac{P_+P_-}{2} \mp me\Phi \pm \varepsilon m \end{pmatrix} \begin{pmatrix} \varphi_+ \\ \varphi_- \end{pmatrix} = 0\,. \tag{6.7}$$

The latter system formally reduces to the scalar equation

$$\left(\frac{P_\mp P_\pm}{2m} + e\Phi - \frac{1}{4m^2} {P_\mp}^2 \left(m + \varepsilon + \frac{P_\pm P_\mp}{2m} - e\Phi \right)^{-1} {P_\pm}^2 \right) \varphi_\pm \;=\; (\varepsilon - m)\varphi_\pm\,.$$

Suppose we are looking for the bound states of the electron, so that $\varepsilon \approx +m$, and suppose that our eigenfunction inherits the slow variation property of the potential, ${P_1}^{\beta_1}{P_2}^{\beta_2}\varphi_\pm \sim \alpha^{\beta_1+\beta_2}\varphi_\pm$. Then the above scalar equation can be rewritten as

$$\left(\frac{P_\mp P_\pm}{2m} + e\Phi + O(\alpha^4) \right) \varphi_\pm \;=\; (\varepsilon - m)\varphi_\pm\,. \tag{6.8}$$

But $P_\mp P_\pm = \mathbf{P}^2 \mp eH^3$, where $\mathbf{P}^2 := {P_1}^2 + {P_2}^2$, and $H^3 = ie^{-1}(P_2P_1 - P_1P_2) = \nabla_2 A_1 - \nabla_1 A_2$ is the intensity of the magnetic field, see formula (23.5) and list of notation in [LL2]. Therefore, (6.8) takes the form

$$\left(\frac{\mathbf{P}^2 \mp eH^3}{2m} + e\Phi + O(\alpha^4) \right) \varphi_\pm \;=\; (\varepsilon - m)\varphi_\pm\,, \tag{6.9}$$

which is the Pauli equation perturbed by the $O(\alpha^4)$ term; note that our scaling assumptions (6.1) imply $eH^3 \sim \alpha^3$, so the magnetic term cannot be included in $O(\alpha^4)$. Similar arguments (see also Section 33 in [LL4], as well as Theorem 6.8 in [T]) reduce (5.1) to (6.9). Thus, we have (formally) shown that the energy levels of our model equation (6.4) and those of the 2–dimensional Dirac equation (5.1) are related as

$$\varepsilon_{\mathrm{model}} = \varepsilon_{\mathrm{Dirac}} + O(\alpha^4)\,. \tag{6.10}$$

Normally one subtracts the rest mass from ε and deals with $E := \varepsilon - m \sim \alpha^2$, so (6.10) means that the relative accuracy in the determination of E is $\sim \alpha^2$.

7. Spectral properties of the model equation

Throughout this section we assume that the electromagnetic vector potential A does not depend on x^0, is smooth and vanishes at infinity. In addition we assume

$$\|e\Phi\|_{L^\infty} < m\,, \tag{7.1}$$

and that the first derivatives of $\mathbf{A}$ vanish at infinity.

Define the operator $\mathcal{P}$ formally by

$$\mathcal{P} = \begin{pmatrix} {P_2}^2 & -P_2P_1 \\ -P_1P_2 & {P_1}^2 \end{pmatrix} = \begin{pmatrix} -P_2 \\ P_1 \end{pmatrix} \begin{pmatrix} -P_2 & P_1 \end{pmatrix} = \begin{pmatrix} -P_2 & P_1 \end{pmatrix}^* \begin{pmatrix} -P_2 & P_1 \end{pmatrix}.$$

More precisely, we consider first the nonnegative symmetric operator $C_0^\infty(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ given by the above expression, and define $\mathcal{P}$ as the Friedrichs extension of the latter; see Theorem 4.4.5 from [D]. Thus, $\mathcal{P} : D(\mathcal{P}) \to L^2(\mathbb{R}^2)$ is a nonnegative self–adjoint operator defined on some $D(\mathcal{P}) \supset \mathcal{S}(\mathbb{R}^2) \supset C_0^\infty(\mathbb{R}^2)$ (here $\mathcal{S}$ stands for the Schwartz class).

We have $\mathcal{A} = \mathcal{P} + m^2 I + e\Phi\mathcal{B}$, and the operator $\mathcal{A} : D(\mathcal{A}) \to L^2(\mathbb{R}^2)$, $D(\mathcal{A}) = D(\mathcal{P})$, is self–adjoint. Furthermore, condition (7.1) means that $\|e\Phi\mathcal{B}\| < m^2$, so the operator $\mathcal{A}$ is positive definite: $a := \inf \sigma(\mathcal{A}) > 0$. Therefore $\mathcal{A}^{-\frac{1}{2}}$ is a bounded self–adjoint operator in $L^2(\mathbb{R}^2)$. We can now rewrite (6.6) as

$$\mathcal{A}^{-\frac{1}{2}}\mathcal{B}\mathcal{A}^{-\frac{1}{2}}\mathbf{v} = \frac{1}{\varepsilon}\mathbf{v}. \tag{7.2}$$

The latter spectral problem is well–posed because $\mathcal{A}^{-\frac{1}{2}}\mathcal{B}\mathcal{A}^{-\frac{1}{2}}$ is a bounded self-adjoint operator in $L^2(\mathbb{R}^2)$.

Theorem 7.1. *The essential spectrum of the operator $\mathcal{A}^{-\frac{1}{2}}\mathcal{B}\mathcal{A}^{-\frac{1}{2}}$ is the interval $[-m^{-1}, m^{-1}]$.*

Theorem 7.1 says, in effect, that for potentials which are sufficiently weak, smooth and well behaved at infinity, the basic structure of the spectrum of our model equation (6.4) is the same as that of the Dirac equation (5.1).

P r o o f of Theorem 7.1.

Part 1 of the proof. Let us prove

$$\left[-m^{-1}, m^{-1}\right] \subset \sigma_{\mathrm{ess}}\left(\mathcal{A}^{-\frac{1}{2}}\mathcal{B}\mathcal{A}^{-\frac{1}{2}}\right). \tag{7.3}$$

Let ε be an arbitrary real number such that $|\varepsilon| > m$. Suppose we have a sequence $\{\mathbf{u}^{(n)}\}$ such that

$$\mathbf{u}^{(n)} \in C_0^\infty(\mathbb{R}^2), \tag{7.4}$$

$$\left\|\mathbf{u}^{(n)}\right\| = 1, \tag{7.5}$$

$$\|(\mathcal{A}-\varepsilon\mathcal{B})\mathbf{u}^{(n)}\|\to 0 \tag{7.6}$$

(a sequence of approximate eigenfunctions of problem (6.6)). Put $\mathbf{v}^{(n)}=\mathcal{A}^{\frac12}\mathbf{u}^{(n)}\in D(\mathcal{A}^{\frac12})\subset L^2(\mathbb{R}^2)$. We have $\|\mathbf{v}^{(n)}\|\ge a^{\frac12}$ and

$$\|(\varepsilon^{-1}-\mathcal{A}^{-\frac12}\mathcal{B}\mathcal{A}^{-\frac12})\mathbf{v}^{(n)}\|=|\varepsilon|^{-1}\|\mathcal{A}^{-\frac12}(\mathcal{A}-\varepsilon\mathcal{B})\mathbf{u}^{(n)}\|\le|\varepsilon|^{-1}a^{-\frac12}\|(\mathcal{A}-\varepsilon\mathcal{B})\mathbf{u}^{(n)}\|,$$

so $\|(\varepsilon^{-1}-\mathcal{A}^{-\frac12}\mathcal{B}\mathcal{A}^{-\frac12})\mathbf{v}^{(n)}\|\to 0$. This implies $\varepsilon^{-1}\in\sigma(\mathcal{A}^{-\frac12}\mathcal{B}\mathcal{A}^{-\frac12})$. As ε^{-1} is an arbitrary number in $(-m^{-1},m^{-1})\setminus\{0\}$ we arrive at (7.3). Thus, we have reduced the proof of (7.3) to the construction of a sequence with properties (7.4)–(7.6).

Let us denote by $\mathcal{P}_0$ the operator $\mathcal{P}$ in the case $\mathbf{A}\equiv\mathbf{0}$, and by $\mathcal{A}_0$ the operator $\mathcal{A}$ in the case $A\equiv 0$; of course, $\mathcal{A}_0=\mathcal{P}_0+m^2I$.

Suppose we have a sequence $\{\mathbf{u}^{(n)}\}$ with properties (7.4), (7.5) and

$$\|(\mathcal{A}_0-\varepsilon\mathcal{B})\mathbf{u}^{(n)}\|\to 0\,. \tag{7.7}$$

Let us modify this sequence by translating the functions, that is, by replacing each $\mathbf{u}^{(n)}(x)$ by $\mathbf{u}^{(n)}(x-x^{(n)})$, where $x^{(n)}\in\mathbb{R}^2$. Clearly, if the sequence of points $\{x^{(n)}\}$ is chosen to tend to infinity sufficiently quickly, then $\|(\mathcal{A}-\mathcal{A}_0)\mathbf{u}^{(n)}\|\to 0$ and (7.7) will imply (7.6). The proof of (7.3) has been reduced to the construction of a sequence with properties (7.4), (7.5), (7.7).

As functions of Schwartz class can be approximated by C_0^∞ functions, we can relax the condition (7.4) by replacing it with

$$\mathbf{u}^{(n)}\in\mathcal{S}(\mathbb{R}^2). \tag{7.8}$$

The differential operator $\mathcal{A}_0-\varepsilon\mathcal{B}$ has constant coefficients, so it is natural to switch from the $\mathbf{u}^{(n)}(x)$ to their Fourier transforms

$$\widehat{\mathbf{u}}^{(n)}(\xi):=\frac{1}{2\pi}\int e^{-i\langle x,\xi\rangle}\mathbf{u}^{(n)}(x)\,d^2x\,,\qquad \langle x,\xi\rangle=x^1\xi_1+x^2\xi_2.$$

We now have to construct a sequence $\{\widehat{\mathbf{u}}^{(n)}\}$ such that

$$\widehat{\mathbf{u}}^{(n)}\in\mathcal{S}(\mathbb{R}^2), \tag{7.9}$$

$$\|\widehat{\mathbf{u}}^{(n)}\|=1, \tag{7.10}$$

$$\|(\widehat{\mathcal{A}}_0-\varepsilon\mathcal{B})\widehat{\mathbf{u}}^{(n)}\|\to 0\,, \tag{7.11}$$

where

$$\widehat{\mathcal{A}}_0-\varepsilon\mathcal{B}\equiv\widehat{\mathcal{A}}_0(\xi)-\varepsilon\mathcal{B}=\begin{pmatrix}\xi_2^2+m^2 & -\xi_2\xi_1\pm i\varepsilon m\\ -\xi_1\xi_2\mp i\varepsilon m & \xi_1^2+m^2\end{pmatrix}$$

is the (full) symbol of $\mathcal{A}_0-\varepsilon\mathcal{B}$. We have $\det(\widehat{\mathcal{A}}_0(\xi)-\varepsilon\mathcal{B})=m^2(\xi_1^2+\xi_2^2+m^2-\varepsilon^2)$ and $|\varepsilon|>m$, so we can choose an η to give $\det(\widehat{\mathcal{A}}_0(\eta)-\varepsilon\mathcal{B})=0$. Let $\widehat{\mathbf{u}}$ be a normalised (constant) vector in the null space of $\widehat{\mathcal{A}}_0(\eta) - \varepsilon\mathcal{B}$, and let $\{\phi^{(n)}\}$ be a sequence of scalar functions such that $\phi^{(n)} \in C_0^\infty(\mathbb{R}^2)$, $\|\phi^{(n)}\| = 1$, and $\operatorname{supp}\phi^{(n)} \to \{\eta\}$. It is easy to see that the vector functions $\widehat{\mathbf{u}}^{(n)}(\xi) := \widehat{\mathbf{u}}\,\phi^{(n)}(\xi)$ have the required properties (7.9)–(7.11).

Part 2 of the proof. Let us prove $\sigma_{\mathrm{ess}}(\mathcal{A}^{-\frac{1}{2}}\mathcal{B}\mathcal{A}^{-\frac{1}{2}}) \subset [-m^{-1}, m^{-1}]$.

Since $\|\mathcal{B}\| = m$ it is sufficient to show $\sigma_{\mathrm{ess}}(\mathcal{A}) \subset [m^2, +\infty)$. However, $\mathcal{A} = \mathcal{P} + m^2 I + e\Phi\mathcal{B}$, so, in turn, it is sufficient to show

$$\sigma_{\mathrm{ess}}(\mathcal{P} + \Psi\mathcal{B}) \subset [0, +\infty)\,, \tag{7.12}$$

where $\Psi = e\Phi$ is an arbitrary smooth real valued function vanishing at infinity.

Claim: for any $\Psi \in C_0^\infty(\mathbb{R}^2)$ we have

$$\sigma_{\mathrm{ess}}\big(\mathcal{P}_0 + I + \Psi\mathcal{B}\big) \subset [0, +\infty)\,. \tag{7.13}$$

Indeed, let us define the operators $\mathcal{Q}_0 := \mathcal{P}_0 + I$, $\mathcal{R} := \mathcal{Q}_0^{-\frac{1}{2}}\Psi\mathcal{B}\mathcal{Q}_0^{-\frac{1}{2}}$. The operator $\mathcal{Q}_0$ is a positive definite differential operator with constant coefficients, and its symbol is

$$\mathcal{Q}_0(\xi) = \begin{pmatrix} \xi_2^2 + 1 & -\xi_2\xi_1 \\ -\xi_1\xi_2 & \xi_1^2 + 1 \end{pmatrix}.$$

Therefore $\mathcal{Q}_0^{-\frac{1}{2}}$ is a pseudodifferential operator with symbol

$$\mathcal{Q}_0^{-\frac{1}{2}}(\xi) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{1}{\Xi(1+\Xi)}\begin{pmatrix} \xi_2^2 & -\xi_2\xi_1 \\ -\xi_1\xi_2 & \xi_1^2 \end{pmatrix}, \quad \Xi = \big(1 + \xi_1^2 + \xi_2^2\big)^{\frac{1}{2}}.$$

The principal symbol of the operator $\mathcal{Q}_0^{-\frac{1}{2}}$ is

$$M(\xi) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{1}{\xi_1^2 + \xi_2^2}\begin{pmatrix} \xi_2^2 & -\xi_2\xi_1 \\ -\xi_1\xi_2 & \xi_1^2 \end{pmatrix},$$

so $\mathcal{Q}_0^{-\frac{1}{2}}$ is a pseudodifferential operator of order 0. It follows that $\mathcal{R}$ is a pseudodifferential operator of order 0. The principal symbol of $\mathcal{R}$ is

$$\pm\, im\, M(\xi)\begin{pmatrix} 0 & -\Psi(x) \\ \Psi(x) & 0 \end{pmatrix} M(\xi) \,=\, 0\,,$$

so $\mathcal{R}$ is in fact a pseudodifferential operator of order -1. Now, the Schwartz kernel $K_{\mathcal{R}}(x,y)$ of the operator $\mathcal{R}$ is given by the oscillatory integral

$$\pm\frac{im}{(2\pi)^2}\int \mathcal{Q}_0^{-\frac{1}{2}}(\xi)\, e^{i\langle x-z,\xi\rangle}\begin{pmatrix} 0 & -\Psi(z) \\ \Psi(z) & 0 \end{pmatrix}\mathcal{Q}_0^{-\frac{1}{2}}(\eta)\, e^{i\langle z-y,\eta\rangle} d^2\xi\, d^2z\, d^2\eta\,.$$

Let $\Omega \subset \mathbb{R}^2$ be a bounded open set with $\operatorname{supp}\Psi \subset \Omega$. A standard calculation shows that $K_{\mathcal{R}}(x,y)$ is smooth and rapidly decreasing in both x and y outside

$\Omega\times\Omega$ (n.b. this calculation relies on the fact that the matrix function $\mathcal{Q}_0^{-\frac{1}{2}}(\xi)$ is smooth on the whole of $\mathbb{R}^2$, including the point $\xi = 0$). Since $K_{\mathcal{R}}(x,y)$ is the integral kernel of a pseudodifferential operator of order -1 inside Ω, it follows that the operator $\mathcal{R}$ is compact. Hence the operator $I+\mathcal{R}$ is nonnegative on a subspace of finite co-dimension. Consequently, the quadratic form

$$\int \mathbf{v}^*(I+\mathcal{R})\mathbf{v}\,d^2x\,,\qquad \mathbf{v}\in L^2(\mathbb{R}^2)\,,$$

is nonnegative on a subspace of finite co-dimension. Equivalently,

$$\int \left(\mathcal{Q}_0^{\frac{1}{2}}\mathbf{u}\right)^*(I+\mathcal{R})\mathcal{Q}_0^{\frac{1}{2}}\mathbf{u}\,d^2x\,,\qquad \mathbf{u}\in D(\mathcal{P}_0^{\frac{1}{2}})\,,$$

is nonnegative on a subspace of finite co-dimension. But the latter is the quadratic form associated with the operator $\mathcal{P}_0+I+\Psi\mathcal{B}$. Formula (7.13) now follows.

Let us now remove the restriction $\Psi\in C_0^\infty(\mathbb{R}^2)$ and show that (7.13) still holds. Given an arbitrary $\epsilon>0$ we can decompose Ψ as $\Psi=\Psi_0+\Psi_1$ where $\Psi_0\in C_0^\infty$ and $\|\Psi_1\|_{L^\infty}\le m^{-1}\epsilon$. We have

$$\mathcal{P}_0+I+\Psi\mathcal{B}=(\mathcal{P}_0+I+\Psi_0\mathcal{B})+\Psi_1\mathcal{B}.$$

According to (7.13) $\sigma_{\mathrm{ess}}(\mathcal{P}_0+I+\Psi_0\mathcal{B})\subset[0,+\infty)$, whereas $\Psi_1\mathcal{B}$ is a bounded operator with $\|\Psi_1\mathcal{B}\|\le\epsilon$. This implies $\sigma_{\mathrm{ess}}(\mathcal{P}_0+I+\Psi\mathcal{B})\subset[-\epsilon,+\infty)$. As $\epsilon>0$ is arbitrary we arrive at (7.13) for general Ψ.

Now, given an arbitrary $\delta\in(0,1)$ we have

$$\mathcal{P}_0+\Psi\mathcal{B}\ge\delta\mathcal{P}_0+\Psi\mathcal{B}=\delta(\mathcal{P}_0+I+\delta^{-1}\Psi\mathcal{B})-\delta I.$$

According to (7.13) $\sigma_{\mathrm{ess}}(\mathcal{P}_0+I+\delta^{-1}\Psi\mathcal{B})\subset[0,+\infty)$, so $\sigma_{\mathrm{ess}}(\mathcal{P}_0+\Psi\mathcal{B})\subset[-\delta,+\infty)$. As $\delta\in(0,1)$ is arbitrary we conclude that

$$\sigma_{\mathrm{ess}}(\mathcal{P}_0+\Psi\mathcal{B})\subset[0,+\infty)\,. \tag{7.14}$$

Thus, we have proved (7.12) in the case $\mathbf{A}\equiv\mathbf{0}$. Let us now remove this restriction. For any $\nu>0$ and any smooth vector function $\mathbf{u}$ we have pointwise

$$\left|P_1u^2-P_2u^1\right|^2=\frac{\nu}{1+\nu}\left|\nabla_1u^2-\nabla_2u^1\right|^2-\nu e^2\left|A_1u^2-A_2u^1\right|^2+\left|\widetilde{P}_1u^2-\widetilde{P}_2u^1\right|^2,$$

where $\widetilde{P}_\varkappa:=i(1+\nu)^{-\frac{1}{2}}\nabla_\varkappa-e(1+\nu)^{\frac{1}{2}}A_\varkappa\,,\ \varkappa=1,2$. This implies

$$\int\mathbf{u}^*\mathcal{P}\mathbf{u}\,d^2x\ge\frac{\nu}{1+\nu}\int\mathbf{u}^*\mathcal{P}_0\mathbf{u}\,d^2x\,-\,C\,\nu\,\|\mathbf{u}\|^2\,,\qquad\mathbf{u}\in C_0^\infty(\mathbb{R}^2),$$

with $C=e^2\left\|{A_1}^2+{A_2}^2\right\|_{L^\infty}$. Consequently,

$$\int\mathbf{u}^*(\mathcal{P}+\Psi\mathcal{B})\mathbf{u}\,d^2x\ge\frac{\nu}{1+\nu}\int\mathbf{u}^*(\mathcal{P}_0+\widetilde{\Psi}\mathcal{B})\mathbf{u}\,d^2x-C\,\nu\,\|\mathbf{u}\|^2\,,\qquad\mathbf{u}\in C_0^\infty(\mathbb{R}^2)\,,$$

with $\widetilde{\Psi} = \nu^{-1}(1+\nu)\Psi$. The latter formula and (7.14) imply $\sigma_{\mathrm{ess}}(\mathcal{P}+\Psi\mathcal{B}) \subset [-C\nu, +\infty)$. As $\nu > 0$ is arbitrary we arrive at (7.12). $\square$

Comprehensive analysis of the discrete spectrum of the operator $\mathcal{A}^{-\frac{1}{2}}\mathcal{B}\mathcal{A}^{-\frac{1}{2}}$ is a non–trivial task which lies outside the scope of this paper. We shall, however, briefly deal with the most basic situation.

Suppose $\Phi \in \mathcal{S}(\mathbb{R}^2)$ and satisfies (7.1), and suppose $\mathbf{A} \equiv 0$. Let ε^{-1}, $|\varepsilon| < m$, be an eigenvalue of (7.2). Then, using arguments similar to those in the second part of the proof of Theorem 7.1, one can show that $\mathbf{u} \in \mathcal{S}(\mathbb{R}^2)$. Therefore, in studying the discrete spectrum we can work with (6.6) or (6.7) rather than with (7.2).

Suppose now Φ is radially symmetric. Let us introduce polar coordinates $x^1 = r\cos\vartheta$, $x^2 = r\sin\vartheta$, and expand the $\varphi_\pm$ as

$$\varphi_\pm = \sum_{k\in\mathbb{Z}} e^{i(k\mp 1)\vartheta}\varphi_\pm^{(k)}(r)\,. \tag{7.15}$$

Substituting these expansions into (6.7) we see that the latter separates into systems of ordinary differential equations

$$\begin{pmatrix} H^\pm_{k-1} & G_k \\ G_{-k} & H^\mp_{k+1} \end{pmatrix} \begin{pmatrix} \varphi_+^{(k)} \\ \varphi_-^{(k)} \end{pmatrix} = 0\,, \tag{7.16}$$

where k runs through $\mathbb{Z}$, and

$$H_l^\pm := m - \frac{1}{2m}\left(\frac{1}{r}\frac{d}{dr}r\frac{d}{dr} - \frac{l^2}{r^2}\right) \mp (\varepsilon - e\Phi)\,,$$

$$G_l := \frac{1}{2m}\left(\frac{1}{r}\frac{d}{dr}r\frac{d}{dr} + \frac{2l}{r}\frac{d}{dr} + \frac{l^2-1}{r^2}\right).$$

The orthogonal transformation

$$\begin{pmatrix} \varphi_+^{(k)} \\ \varphi_-^{(k)} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix}$$

turns (7.16) into

$$\frac{k}{r}g' + \frac{k}{r^2}g - \frac{k^2}{r^2}f - m^2 f \pm m(\varepsilon - e\Phi)g = 0\,, \tag{7.17}$$

$$g'' + \frac{1}{r}g' - \frac{1}{r^2}g - \frac{k}{r}f' + \frac{k}{r^2}f - m^2 g \pm m(\varepsilon - e\Phi)f = 0\,, \tag{7.18}$$

where the prime stands for the derivative in r. Resolving (7.17) with respect to f and substituting the resulting expression into (7.18), we reduce our problem to a single second order equation

$$g'' + \frac{3k^2 + m^2r^2}{(k^2+m^2r^2)r}\, g' + \left(\frac{k^2 - m^2r^2}{(k^2+m^2r^2)r^2} - \frac{k^2}{r^2} \pm \frac{2km(\varepsilon - e\Phi)}{k^2+m^2r^2} \pm \frac{ke\Phi'}{mr} + (\varepsilon - e\Phi)^2 - m^2 \right) g = 0\,. \tag{7.19}$$

Let us compare our model equation (7.19) with the Klein–Gordon equation

$$g'' + \frac{1}{r}g' + \left(-\frac{n^2}{r^2} + (\varepsilon - e\Phi)^2 - m^2 \right) g = 0\,, \tag{7.20}$$

$n \in \mathbb{Z}$. In view of (6.7), (7.15) it is natural to compare the two equations taking $|n| = |k \mp 1|$ when we are looking for the bound states of the electron ($\varepsilon \approx +m$), and $|n| = |k \pm 1|$ when we are looking for the bound states of the positron ($\varepsilon \approx -m$). Clearly, for $k = 0$ (7.19) and (7.20) coincide. For $k \neq 0$ these equations differ and we shall compare their spectra asymptotically, assuming that $e\Phi$ is of the form $\alpha^2\Psi(\alpha r)$, $\Psi \in \mathcal{S}(\mathbb{R}^2)$, $\alpha \to +0$; this is a particular case of the situation (6.1). Asymptotic analysis gives $\varepsilon_{\text{model}} = \varepsilon_{\text{KG}} + O(\alpha^4)$, which in turn implies $\varepsilon_{\text{model}} = \varepsilon_{\text{Dirac}} + O(\alpha^4)$.

We hope to justify a version of (6.10) in the case of more general Φ and $\mathbf{A}$ by means of asymptotic perturbation techniques (viz. Chapter VIII of [K]). Here the technical difficulty is that in contrast to the Dirac operator which can be viewed as an analytic perturbation of the Pauli operator (see Chapter 6 of [T] for more details) the perturbation in (6.9) is not analytic.

8. Conclusion

The results of Sections 6 and 7 seem to indicate that the basic effects attributed to spinors can be explained (at least in Minkowski 3–space) using simple tensor models. This observation may be useful in relation to attempts at modelling the electron as a soliton–like solution of some nonlinear system of partial differential equations. The usual approach, see, e.g., [W] and [EGS], involves the so–called Maxwell–Dirac equation. In our view, it might make sense looking also at other nonlinear systems which do not necessarily have spinors occurring explicitly but may still produce spinor effects. With this goal in mind, let us compare our model equation (1.2) with the Maxwell system.

Define the operator of exterior differentiation d mapping vectors to antisymmetric tensors as $(du)_{\mu\nu} := i\nabla_\mu u_\nu - i\nabla_\nu u_\mu$, and its dual δ as $(\delta T)^\nu := i\nabla_\mu T^{\mu\nu}$; we also define the action of δ on vectors as $\delta u := i\nabla_\mu u^\mu$. By analogy, define $(d_A u)_{\mu\nu} := P_\mu u_\nu - P_\nu u_\mu$, $(\delta_A T)^\nu := P_\mu T^{\mu\nu}$. Now, squaring (1.2) gives

${\mathrm{curl}_A}^2 u = m^2 u$. But ${\mathrm{curl}_A}^2 = \delta_A d_A$, so our model equation (1.2) becomes

$$\delta_A d_A u = m^2 u\,. \tag{8.1}$$

According to formula (30.2) of [LL2] the Maxwell system can be written as

$$\delta dA = -4\pi j\,, \qquad \delta A = 0\,, \tag{8.2}$$

where j is the (given) current. The similarity between (8.1) and (8.2) is remarkable, and we hope to build further mathematical models on the basis of this similarity.

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