An Elementary Approach to Finite Size Conditions for the Exponential Decay of Covariances in Lattice Spin Models

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Abstract. We consider lattice spin models with finite range interaction and compact spin space. Using the Markov property we derive in an elementary way, very much close to Lieb–Simon type of argument, finite size conditions for the exponential decay of finite volume covariances. Our method covers at the same time the case of completely analytical Gibbsian random fields introduced by Dobrushin and Shlosman for which covariances between local observables decay exponentially fast in any volume, and the case of Gibbsian random fields for which the decay of covariances is proved only on regular (e.g. cubic) volumes.

Key Words: Lattice spin models, decay of correlations, finite size conditions, complete analyticity.

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1. INTRODUCTION

The aim of this note is to present in a unified way some simple observations on the relationship between the Markov property and finite size conditions for the exponential decay of covariances for classical lattice models of compact spins with finite range interaction (see section 3 of [M] for previous results in the context of ± 1 spins).

Finite size conditions, e.g. decay assumption on covariances or truncated correlations in subsets of some given cube of side l uniformly in the boundary conditions or, what is the same, assumption of very weak dependence on the boundary conditions of Gibbs measures restricted to subsets of some given cube of side l, have been introduced long ago (see e.g. [DP], [Lie], [DS2], [DS3], [O], [OP], [BM], [Yo] and [MO1], [MO2] for a detailed review) in order to analyze the system in the so called one phase region. More precisely they have been introduced to explicitely show that, in the pure phase region, the model enjoys all the nice properties of the high temperature phase (e.g. exponential tree decay of semiinvariants, analyticity and existence of a convergent cluster expansion) provided it is analyzed on the correct length scale, typically somewhat larger than the correlation length.

Such a philosophy is quite explicit in the celebrated series of papers by Dobrushin and Shlosman (see [DS1], [DS2], [DS3]) on completely analytical Gibbsian fields. There (see the introduction of [DS3]), the authors compare, by quoting the beginning of "Anna Karenin", each Gibbsian field outside the phase transition region to an "happy family" with basically the same behaviour of all the others "happy families". In particular, among the others, they prove the following important result (see condition $\mathrm{III}^{\mathrm{constr}}\mathrm{b}$ and theorem 4.1 of [DS3]).

Theorem [Dobrushin–Shlosman]. *Given a decreasing function $\varphi(l)$ with $\lim_{l\to\infty}\varphi(l)\, l^{2(d-1)} = 0$ there exists $l = l(\varphi) < \infty$ such that, if for all volumes $V \in \mathbb{Z}^d$ with diameter smaller than l, all subsets Λ of V, all $x \in \partial^+ V$, all possible boundary conditions τ and τ' that differ only at the site x and all functions f which depend only on the spins inside Λ the following holds*

$$|\,\mu_\Lambda^\tau(f) - \mu_\Lambda^{\tau'}(f)\,| \le \|f\|_\infty\, \varphi\big(dist(\Lambda, x)\big) \tag{1.1}$$

then the interaction under consideration is completely analytical and covariances decay exponentially fast in any finite or infinite volume. In the above formula μ_Λ^τ denotes the Gibbs measure relative to the set Λ with boundary condition τ.

Unfortunately, and contrary to the original conjecture of Dobrushin and Shlosman, the above finite volume condition cannot hold, in general, in the whole one phase region, the reason being that the requirement "for all V with diameter smaller than l and all $\Lambda \subset V$" is too strong (see [MO1], [MO2] for explicit examples and critical analysis and [EFS], [MO3], [MO4] for connections with renormalization group pathologies).

Later Olivieri [O] and Olivieri and Picco [OP] relaxed considerably the above finite size condition by roughly requiring (1.1) only for regular sets V that are the disjoint union of large enough cubes. Using their condition they were able to prove the convergence of a suitable cluster expansion and thus all the nice properties of a completely analytical interaction but only on regular volumes.

As an example of a model which outside the phase coexistence region is completely analytical on regular volumes but not completely analytical in the original sense of [DS1] we should mention the two dimensional Ising model (see [SchSh]). We warn however the reader that in the beautiful result of [SchSh] the dimensionality plays a key role. There is in fact good evidence that the three dimensional Ising model at very low temperature and small positive magnetic field is not even completely analytical on regular volumes (see the introduction of [CM]).

Unfortunately both the method of [DS3], based on coupling techniques, and that of [O] and [OP], based on cluster expansion, looks quite involved if one is just interested in deriving the exponential decay of covariances (and not analiticity or other more delicate properties) starting from a finite size condition similar to that described above. This is an important point since in many applications, for example in the analysis of the relaxational properties of Glauber or Kawasaki dynamics in the one phase region (see e.g. [M] and references therein), not all the strength of the notion of completely analytical (on all or only on regular volumes) is needed, but just the exponential decay of covariances.

On the other hand, the old results by Simon [Si2] and Lieb [Lie] on the decay of covariances for ferromagnetic Ising spins suggest that there should be an elementary way to derive the exponential decay of covariances simply from a finite size condition like the one above plus the Markov property of the random field.

In this paper we describe how to implement such a simple idea for classical lattice models of compact spins with finite range interaction. In particular we prove the following result (see theorem 3.3 for an expanded version).

Let $h_x^s(\sigma) = \exp\Big[\sum_{\Lambda\ni x}\big[\Phi_\Lambda(\sigma^{x,s}) - \Phi_\Lambda(\sigma)\big]\Big]$, *where* $\sigma^{x,s}$ *denotes the configuration obtained from* σ *by replacing the spin* $\sigma(x)$ *with an admissible value* s *and* $\{\Phi_\Lambda\}$ *is the interaction. Let*

$$\psi(l) = \sup_{s,t,\tau}\ \sup_{V\subset B_{2l-1}}\ \sum_{2l-1\le|x|\le 2l-1+r} |\mu_V^\tau(h_0^s, h_x^t)|$$

where r *is the range of the interaction,* B_{2l-1} *denotes the ball of radius* $2l-1$ *in* Z^d *and* μ_V^τ *is the Gibbs measure in* V *with boundary condition* τ. *Then there exists a positive constant* $\varepsilon = \varepsilon(d, \|\Phi\|)$ *such that, if* $l^{d-1}\psi(l) \le \varepsilon$, *then any finite volume covariance between two local functions* f *and* g *decays exponentially fast with the distance between their supports, with a rate which is uniform in the volume, in the boundary condition and in the choice of* f *and* g.

Our proof, which covers at the same time the case of complete analiticity on arbitrary volumes or only on regular ones, is based on the following sequence of three steps:

step 1) we show that the exponential decay of covariances follows from the exponential decay of a particular type of covariance (see proposition 3.1 below).
step 2) we then show, using the Markov property, that the special covariance of step 1) satisfies a kind of Lieb–Simon inequality (see lemma 3.2 below).
step 3) we finally prove, using step 2), that the special covariance of step 1) decays exponentially fast provided a natural finite size condition is satisfied (see theorem 3.3 below).

The method described here is confined to the problem of showing exponential decay of covariances and it does not add any new contribution to the original and more complex question of complete analiticity of the interaction to which Dobrushin contributed so much. Nevertheless I would have enjoyed very much discussing it with him and hear his opinion. As everybody else I really miss his scientific and human presence very much.

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2. THE MODEL

Here we describe the models that will be considered in this note and we fix the notation. For a general reference see e.g. [Si1] or [Geo].

2.1 Notation.

The lattice. We consider the d dimensional lattice $\mathbb{Z}^d$ with sites $x = \{x_1, \ldots, x_d\}$ and norm $|x| = \max_{i\in\{1,\ldots,d\}} |x_i|$. The associated distance function is denoted by $d(\cdot,\cdot)$. By Q_l we denote the cube of all $x = (x_1, \ldots, x_d) \in \mathbb{Z}^d$ such that $x_i \in \{0, \ldots, l-1\}$. If $x \in \mathbb{Z}^d$, $Q_l(x)$ stands for $Q_l + x$. We also let B_l be the ball of radius l centered at the origin, *i.e.* $B_l = Q_{2l+1}((-l, \ldots, -l))$.
If Λ is a finite subset of $\mathbb{Z}^d$ we write $\Lambda \subset\subset \mathbb{Z}^d$. The cardinality of Λ is denoted by $|\Lambda|$. $\mathbb{F}$ is the set of all nonempty finite subsets of $\mathbb{Z}^d$.
We finally define the exterior *n–boundary* as $\partial_n^+\Lambda = \{x \in \Lambda^c : d(x,\Lambda) \le n\}$. Given $r \in \mathbb{Z}_+$, we say that a subset V of $\mathbb{Z}^d$ is *r-connected* if for any two sites , y in V there exists $\{x^1, \ldots, x^n\} \subset V$ such that $x^1 = y$, $x^n = z$ and $|x^{i+1} - x^i| \le r$ for $i = 2, \ldots, n$.

Regular sets. A finite subset Λ of $\mathbb{Z}^d$ is said to be a *l–regular*, $l \in \mathbb{Z}_+$, if there exists $x \in \mathbb{Z}^d$ such that Λ is the union of a finite number of cubes $Q_l(x^i + x)$ where $x^i \in l\mathbb{Z}^d$. We denote the class of all such sets by $\mathbb{F}_l$. Notice that any set is 1–regular *i.e.* $\mathbb{F}_{l=1} = \mathbb{F}$.

The configuration space. Our *configuration space* is $\Omega = S^{\mathbb{Z}^d}$, where S is a compact metric space, or $\Omega_V = S^V$ for some $V \subset \mathbb{Z}^d$. The single spin space S is endowed with its natural topology and Ω with the corresponding product topology. Given $\sigma \in \Omega$ and $\Lambda \subset \mathbb{Z}^d$ we denote by σ_Λ the natural projection over Ω_Λ. If U, V are disjoint, $\sigma_U\eta_V$ is the configuration on $U \cup V$ which is equal to σ on U and η on V.

Local functions. If f is a measurable function on Ω, the support of f, denoted by Λ_f, is the smallest subset of $\mathbb{Z}^d$ such that $f(\sigma)$ depends only on σ_{Λ_f}. f is called *local* if Λ_f is finite. $\mathcal{F}_\Lambda$ stands for the σ–algebra generated by the set of projections $\{\pi_x\}$, $x \in \Lambda$, from Ω to S, where $\pi_x : \sigma \mapsto \sigma(x)$. When $\Lambda = \mathbb{Z}^d$ we set $\mathcal{F} \equiv \mathcal{F}_{\mathbb{Z}^d}$ and $\mathcal{F}$ coincides with the Borel σ–algebra on Ω with respect to the topology introduced above. By $\|f\|_\infty$ we mean the supremum norm of f.

2.2 Gibbs Measures

Definition 2.1. *A finite range, translation–invariant potential $\{\Phi_\Lambda\}_{\Lambda\in\mathbb{F}}$ is a collection of real, local continuous functions on Ω with the following properties*

(1) $\Phi_\Lambda = \Phi_{\Lambda+x}$ for all $\Lambda \in \mathbb{F}$ and all $x \in \mathbb{Z}^d$

(2) *For each* Λ *the support of* Φ_Λ *coincides with* Λ

(3) *There exists* $r > 0$ *such that* $\Phi_\Lambda = 0$ *if* $\operatorname{diam}\Lambda > r$. r *is called the range of the interaction.*

(4) $\|\Phi\| \equiv \sum_{\Lambda \ni 0} \|\Phi_\Lambda\|_\infty < \infty$

Given a *potential* or *interaction* Φ with the above four properties and $V \in \mathbb{F}$, we define the Hamiltonian $H_V^\Phi : \Omega \mapsto \mathbb{R}$ by

$$H_V^\Phi(\sigma) = -\sum_{\Lambda:\, \Lambda \cap V \neq \emptyset} \Phi_\Lambda(\sigma)$$

For $\sigma, \tau \in \Omega$ we also let $H_V^{\Phi,\tau}(\sigma) = H_V^\Phi(\sigma_V \tau_{V^c})$ and τ is called the *boundary condition*. For each $V \in \mathbb{F}$, $\tau \in \Omega$ the (finite volume) conditional Gibbs measure on $(\Omega, \mathcal{F})$, are given by

$$d\mu_V^{\Phi,\tau}(\sigma) = \begin{cases} \left(Z_V^{\Phi,\tau}\right)^{-1} \exp[-H_V^{\Phi,\tau}(\sigma)] \prod_{x \in \Lambda} d\mu_0(\sigma(x)) & \text{if } \sigma(x) = \tau(x) \text{ for all } x \in V^c \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

where $Z_V^{\Phi,\tau}$ is the proper normalization factor called partition function and $\mu_0(\cdot)$ is some apriori probability measure on S. Notice that in (2.1) we have adsorbed in the interaction Φ the usual inverse temperature factor β in front of the Hamiltonian. In most notation we will drop the superscript Φ if that does not generate confusion.

Given a measurable bounded function f on Ω, $\mu_V(f)$ denotes the *function* $\sigma \mapsto \mu_V^\sigma(f)$ where $\mu_V^\sigma(f)$ is just the average of f w.r.t. μ_V^σ. Analogously, if $X \in \mathcal{F}$, $\mu_V^\tau(X) \equiv \mu_V^\tau(\mathbb{1}_X)$, where $\mathbb{1}_X$ is the characteristic function on X. $\mu_V^\tau(f,g)$ stands for the covariance or *truncated correlation* (with respect to μ_V^τ) of f and g. The set of measures (2.1) satisfies the DLR compatibility conditions

$$\mu_\Lambda^\tau(\mu_V(X)) = \mu_\Lambda^\tau(X) \qquad \forall\, X \in \mathcal{F} \qquad \forall\, V \subset \Lambda \subset\subset \mathbb{Z}^d \tag{2.2}$$

Definition 2.2. *A probability measure* μ *on* $(\Omega, \mathcal{F})$ *is called a Gibbs measure for* Φ *if*

$$\mu(\mu_V(X)) = \mu(X) \qquad \forall\, X \in \mathcal{F} \qquad \forall\, V \in \mathbb{F} \tag{2.3}$$

Remark. In the above definition we could have replaced the $\sigma-$algebra $\mathcal{F}$ with $\mathcal{F}_V$ (see section 2.3.2 in [EFS]).

3. MAIN RESULTS

In order to state our results on the decay of covariances, we first need few additional definition. Given a site $x \in \mathbb{Z}^d$, a spin value $s \in S$ and a configuration $\sigma \in \Omega$, we denote by $\sigma^{x,s}$ the new configuration

$$\sigma^{x,s}(y) = \begin{cases} \sigma(y) & \text{if } y \neq x \\ s & \text{if } y = x \end{cases}$$

and by h_x^s the function

$$h_x^s(\sigma) = \exp\Big[\sum_{\Lambda \ni x}\big[\Phi_\Lambda(\sigma^{x,s}) - \Phi_\Lambda(\sigma)\big]\Big]$$

Notice that

$$e^{-2\|\Phi\|} \leq \inf_{s\in S} \|h_x^s\|_\infty \leq \sup_{s\in S} \|h_x^s\|_\infty \leq e^{2\|\Phi\|}$$

The main reason for introducing h_x^s is that it represents, apart from a normalization factor, the relative density of $\mu_V^{\tau^{x,s}}$ w.r.t. μ_V^τ. In particular the following key identity holds

$$\mu_V^{\tau^{x,s}}(f) - \mu_V^\tau(f) = \frac{\mu_V^\tau(h_x^s, f)}{\mu_V^\tau(h_x^s)} \tag{3.1}$$

With these notation fixed we define, for any positive integer l, the non–negative, bounded, translation invariant function $\Gamma_l(x-y)$ on $\mathbb{Z}^d$ by

$$\Gamma_l(x-y) = \begin{cases} \sup_{V\in\mathbb{F}_l:\, \partial_r^+ V \ni x,y} \sup_{\tau\in\Omega} \sup_{s,t\in S} |\mu_V^\tau(h_x^s, h_y^t)| & \text{if } \exists\, V \in \mathbb{F}_l :\ \partial_r^+ V \ni x, y \\ 0 & \text{otherwise} \end{cases} \tag{3.2}$$

Our first result says that, in order to prove decay of the covariance $\mu_V^\tau(f,g)$ between any two bounded local functions f and g uniformly in $V \in \mathbb{F}_l$ and in the boundary condition τ, it is sufficient to show decay of $\Gamma_l(x-y)$ as $|x-y| \to \infty$. In particular, if $\Gamma_l(x-y)$ decays exponentially fast then $\mu_V^\tau(f,g)$ will also decay exponentially fast with the distance between the supports of f and g. Notice that, since $\mathbb{F}_{l=1}$ coincides with $\mathbb{F}$, the exponential decay of $\Gamma_1(x-y)$ will then imply the exponential decay of covariances in any finite volume *i.e.* the Dobrushin–Shlosman complete analyticity of the model [DS1]. However it is important to remember that it is possible to have exponential decay of $\Gamma_l(x-y)$ for some large enough l and not of $\Gamma_1(x-y)$ (see e.g. section 2 of [MO2]).

Proposition 3.1 . *There exists a positive finite constant C depending on r and $\|\Phi\|$ such that for any pair of bounded local functions f and g, any $V \in \mathbb{F}_l$ and any $\tau \in \Omega$*

$$|\mu_V^\tau(f,g)| \leq C\|\bar f\|_\infty \|\bar g\|_\infty \sum_{x\in\partial_r^+(V_f)^c} \sum_{y\in\partial_r^+(V_g)^c} \Gamma_l(x-y) \qquad \textit{if} \quad d(V_f, V_g) \geq 6l + 2r$$

Here $\bar f = f - \mu_V^\tau(f)$ and, if $V = \cup_{i\in I} Q_l(x^i + x)$, V_f (V_g) is the union of those cubes $Q_l(x^i + x)$, $i \in I$, that intersect the support of f (g).

Proof.

We fix a set $V \in \mathbb{F}_l$, a boundary condition τ, two bounded local functions f and g with zero

μ_V^τ–mean and we assume that $d(\Lambda_f, \Lambda_g) \ge 6l + 2r$. Then we write

$$\begin{aligned} |\mu_V^\tau(f\,g)| &= |\mu_V^\tau(f\,\mu_{V\setminus V_f}(g))| \\ &\le \|f\|_\infty \, \|\mu_{V\setminus V_f}(g)\|_\infty^\tau \end{aligned} \tag{3.3}$$

where in $\|\mu_{V\setminus V_f}(g)\|_\infty^\tau$ the supremum over the boundary condition in $\partial_r^+(V\setminus V_f)$ has to be taken only over configurations that agree with τ in $\partial_r^+ V$. In order to bound from above $\|\mu_{V\setminus V_f}(g)\|_\infty^\tau$ we use the following simple formula valid for any pairs of sets $\Lambda_1 \subset \Lambda_2$ and any function f that depends only on the spins in Λ_1

$$\begin{aligned} \sup_{\substack{\sigma,\tau;\\ \sigma_{\Lambda_2^c}=\tau_{\Lambda_2^c}}} |\mu_{\Lambda_1}^\sigma(f) - \mu_{\Lambda_2}^\tau(f)| &= \sup_{\substack{\sigma,\tau;\\ \sigma_{\Lambda_2^c}=\tau_{\Lambda_2^c}}} \left| \int d\mu_{\Lambda_2}^\tau(\eta)[\mu_{\Lambda_1}^\sigma(f) - \mu_{\Lambda_1}^\eta(f)] \right| \\ &\le \sup_{\substack{\sigma,\eta;\\ \sigma_{\Lambda_2^c}=\eta_{\Lambda_2^c}}} |\mu_{\Lambda_1}^\sigma(f) - \mu_{\Lambda_1}^\eta(f)| \\ &\le \sum_{x\in(\partial_r^+\Lambda_1)\cap\Lambda_2} \sup_\tau \sup_{s\in S} |\mu_{\Lambda_1}^{\tau^{x,s}}(f) - \mu_{\Lambda_1}^\tau(f)| \\ &= \sum_{x\in(\partial_r^+\Lambda_1)\cap\Lambda_2} \sup_\tau \sup_{s\in S} \frac{|\mu_{\Lambda_1}^\tau(h_x^s, f)|}{\mu_{\Lambda_1}^\tau(h_x^s)} \\ &\le e^{2\|\Phi\|} \sum_{x\in(\partial_r^+\Lambda_1)\cap\Lambda_2} \sup_\tau \sup_{s\in S} |\mu_{\Lambda_1}^\tau(h_x^s, f)| \end{aligned} \tag{3.4}$$

where we have used a simple telescoping argument to derive the second inequality and (3.1) for the forth equality.
We can now apply (3.4) to g with $\Lambda_2 = V$, $\Lambda_1 = V\setminus V_f$ (remember that $\mu_V^\tau(g) = 0$) and get that the r.h.s. of (3.3) is bounded from above by

$$|\mu_V^\tau(f\,g)| \le e^{2\|\Phi\|}\|f\|_\infty \sum_{x\in\partial_r^+(V_f)^c} \sup_\tau \sup_{s\in S} |\mu_{V\setminus V_f}^\tau(h_x^s, g)| \tag{3.5}$$

We can now repeat the previous steps for each term $\mu_{V\setminus V_f}^\tau(h_x^s, g)$ in the above sum and write

$$\begin{aligned} |\mu_{V\setminus V_f}^\tau(h_x^s, g)| &= |\mu_{V\setminus V_f}^\tau(g\,\mu_{V\setminus V_0}(\bar h_x^s))| \\ &\le \|g\|_\infty \sum_{y\in\partial_r^+(V_g)^c} \sup_\tau \sup_{s\in S} |\mu_{V\setminus V_0}^\tau(h_x^s, h_y^t)| \end{aligned} \tag{3.6}$$

where $\bar h_x^s \equiv h_x^s - \mu_{V\setminus V_f}^\tau(h_x^s)$ and $V_0 = V_f \cup V_g$.
Notice that, by construction, $V\setminus V_0$ is still a l–regular set. Therefore we can estimate from above $\sup_\tau \sup_{t,s\in S} \sum_{y\in V_g} |\mu_{V\setminus V_0}^\tau(h_x^s, h_y^t)|$ with $\Gamma_l(x-y)$ and get the statement of the proposition. $\square$

In order to analyze the decay properties of the function $\Gamma_l(x-y)$ we first show that $\Gamma_l(x-y)$ satisfies the following recursive inequalities which are reminiscent of the well known Lieb–Simon inequalities

([Si2],[Lie]). In order to formulate them, for any integer l we first denote by $\mathcal{F}_l^0$ the class of those sets $V \in \mathcal{F}_l$ that are union of cubes $Q_l(x^i)$, $x^i \in l\mathbb{Z}^d$, that are contained in B_{2l-1} *i.e.* with $|x^i| \le l$. Then we set

$$\begin{aligned} \varphi(l) =: & \sup_{x\in\partial_r^+ Q_l} \sup_{\tau\in\Omega} \sup_{s,t\in S} \sup_{\substack{V\in\mathcal{F}_l^0: \\ x\in\partial_r^+ V \\ Q_l \subset V}} \sum_{z\in\partial_r^+ B_{2l-1}} |\mu_V^\tau(h_x^s, h_z^t)| \\ \psi(l) =: & \sup_{x\in\partial_r^+ Q_l} \sup_{\tau\in\Omega} \sup_{s,t\in S} \sup_{\substack{V\subset B_{2l-1} \\ \partial_r^+ V \ni x}} \sum_{z\in\partial_r^+ B_{2l-1}} |\mu_V^\tau(h_x^s, h_z^t)| \end{aligned} \tag{3.7}$$

Remark. It is important to observe that in the definition of $\varphi(l)$ and $\psi(l)$ appear only covariances on subsets of B_{2l-1}. However, and this is also the main difference between the two quantities, in the definition of $\varphi(l)$ covariances are computed in l–regular volumes while in the definition of $\psi(l)$ subsets of B_{2l-1} of arbitrary shape appear. This fact implies, in particular, that $\psi(l) \ge \varphi(l)$. It is also clear that both $\varphi(l)$ and $\psi(l)$ can be bounded from above in terms of $\Gamma_l(l)$ and $\Gamma_1(l)$. However that is not so interesting since the latter require the computation of covariances on scale l but in arbitrarily large sets and not just on sets of side $\approx l$.

Lemma 3.2. *Take $l > 2r$. Then for any $x, y \in \mathbb{Z}^d$ such that $|x-y| \ge 8l$ and $x \in \partial_r^+ Q_l$*
(a)

$$\Gamma_l(x-y) \le e^{4\|\Phi\|}\,\varphi(l) \sum_{z\in\partial_r^+ B_{3l-1}^c} \Gamma_l(z-y) \tag{3.8}$$

and

$$\Gamma_1(x-y) \le e^{4\|\Phi\|}\,\psi(l) \sum_{z\in\partial_r^+ B_{3l-1}^c} \Gamma_1(z-y) \tag{3.9}$$

(b) Fix an integer $n \ge 4$ and let $x^1 \dots x^N$ be some ordering of all sites $x \in l\mathbb{Z}^d$ with $|x| = nl$. Then

$$\Gamma_l(x-y) \le \sum_{j=1}^N \sum_{z,z'\in\partial_r^+ Q_l(x^j)^c} \Gamma_l(z-x)\,\Gamma_l(z'-y) \tag{3.10}$$

and

$$\Gamma_1(x-y) \le \sum_{z:\,|z|=nl} \Gamma_1(x-z)\,\Gamma_1(z-y) \tag{3.11}$$

Proof.
(a). Let us just prove (3.8) since the proof of (3.9) is exactly the same once one replaces the class $\mathbb{F}_l$ with $\mathbb{F}$.
We fix two sites x, y with $|x-y| \ge 8l$, two spin values $s, t \in S$, a set $V \in \mathbb{F}_l$ such that x and y both belong to $\partial_r^+ V$ and a boundary configuration τ. Without loss of generality we may assume that $V = \cup_{i=1}^k Q_l(x^i)$, $x^i \in l\mathbb{Z}^d$ with $x^1 = 0$ and $x \in \partial_r^+ Q_l$. Next we set

$$\begin{aligned} A &= V \cap B_{2l-1} \\ B &= B_{3l-1} \setminus B_{2l-1} \\ C &= V \cap B_{3l-1}^c \end{aligned}$$

and we observe that, by construction, $x \in A$, $y \in C$ and that neither h^s_x nor h^t_y depend on the spins in B since $l > 2r$. Therefore, using the DLR equations together with the Markov property of our Gibbsian field and by denoting with $\bar h^s_x$ the function $h^s_x - \mu^\tau_V(h^s_x)$ and similarly for $\bar h^t_y$, we can write

$$\begin{aligned} |\mu^\tau_V(h^s_x, h^t_y)| &= |\mu^\tau_V(\bar h^s_x\, \bar h^t_y)| \\ &= |\mu^\tau_V\big(\mu_C(\bar h^t_y)\,\mu_A(\bar h^s_x)\big)| \\ &\le \|\mu_A(\bar h^s_x)\|^\tau_\infty\, \|\mu_C(\bar h^t_y)\|^\tau_\infty \end{aligned} \tag{3.12}$$

where, as in (3.3), the superscript τ in $\|\cdot\|^\tau_\infty$ norms means that the supremum is taken only over those boundary conditions that agree with τ on $\partial^+_r V$.

We claim at this point that

$$\begin{aligned} &\|\mu_A(\bar h^s_x)\|^\tau_\infty \le e^{2\|\Phi\|}\varphi(l) \\ &\|\mu_C(\bar h^t_y)\|^\tau_\infty \le e^{2\|\Phi\|} \sum_{z\in(\partial^+_r C)\cap B} \Gamma_l(z-y) \end{aligned} \tag{3.13}$$

so that (3.8) follows from (3.12), the observation that $(\partial^+_r C)\cap B \subset \partial^+_r B^c_{3l-1}$ and the arbitrariness of V and τ.

In turn the first inequality in (3.13) follows immediately if we use (3.4) with $\Lambda_2 = V$, $\Lambda_1 = A$ and $f = \bar h^s_x$. To prove the second one we use (3.4) with $\Lambda_2 = V$, $\Lambda_1 = C$ and $f = \bar h^t_y$.

(b). Again we prove only (3.10). The key tool is the formula for conditional covariance which, for Gibbsian random fields, reads

$$\mu^\tau_V(f,g) = \mu^\tau_V\big(\mu_{V\setminus\Lambda}(f,g)\big) + \mu^\tau_V\big(\mu_{V\setminus\Lambda}(f), \mu_{V\setminus\Lambda}(g)\big) \tag{3.14}$$

where V and Λ are any pairs of sets in $\mathbb{F}$.

As before we fix two sites x, y with $|x-y| \ge 2nl$, two spin values $s,t \in S$, a set $V \in \mathbb{F}_l$ such that x and y both belong to $\partial^+_r V$ and a boundary configuration τ. Without loss of generality we may assume that $V = \cup_{i=1}^k Q_l(x^i)$, $x^i \in l\mathbb{Z}^d$ with $x^1 = 0$ and $x \in \partial^+_r Q_l$. Clearly, in order to get a non–zero covariance $\mu^\tau_V(h^s_x, h^t_y)$ and because of the condition $|x-y| \ge 2nl$, some of the x^i's will satisfy $|x^i| = nl$. Let us denote by $Q^1 \dots Q^m$ the corresponding cubes ordered in some way and let us define inductively the sets $V_0 \dots V_m$ as follows

$$\begin{aligned} V_0 &= V \\ V_{i+1} &= V_i \setminus Q_{i+1} \qquad i = 0\dots m-1 \end{aligned}$$

Obviously each V_i belongs to $\mathbb{F}_l$. Then, using (3.14), we start by writing

$$\mu^\tau_V(h^s_x, h^t_y) = \mu^\tau_V\big(\mu_{V_1}(h^s_x, h^t_y)\big) + \mu^\tau_V\big(\mu_{V_1}(h^s_x), \mu_{V_1}(h^t_y)\big) \tag{3.15}$$

The absolute value of the second term in the r.h.s. of (3.15), thanks to (3.4), is bounded from above by

$$\sup_{\tau,\tau'}\ \sup_{s,s',t,t'} \sum_{z,z'\in\partial^+_r(Q^1)^c} |\mu^\tau_{V_1}(h^s_x, h^{s'}_z)|\,|\mu^{\tau'}_{V_1}(h^t_y, h^{t'}_{z'})| \le \sum_{z,z'\in\partial^+_r(Q^1)^c} \Gamma_l(x-z)\,\Gamma_l(y-z')$$

We can now reapply the above scheme to the term $\mu_{V_1}(h^s_x, h^t_y)$ in (3.15) and so on. The expansion we get for the starting covariance looks as follows

$$|\mu^\tau_V(h^s_x, h^t_y)| \le |\mu^\tau_V\big(\mu_{V_m}(h^s_x, h^t_y)\big)| + \sum_{i=1}^m \sum_{z,z' \in \partial^+_r (Q^i)^c} \Gamma_l(x-z)\,\Gamma_l(y-z') \tag{3.16}$$

where we have used DLR equations to write

$$\mu^\tau_V\Big[\mu_{V_1}[\ldots\mu_{V_{m-1}}[\mu_{V_m}(h^s_x, h^t_y)]]\Big] = \mu^\tau_V\big(\mu_{V_m}(h^s_x, h^t_y)\big)$$

The proof is now complete once we observe that, by construction, $\mu^\tau_{V_m}(h^s_x, h^t_y) = 0$ for any τ since x and y belong to two different connected components of V_m at relative distance greater than r and that the second term in the r.h.s. of (3.16) is increased if we extend the sum over all cubes $Q_l(x^i)$ with $|x^i| = nl$. $\square$

Let us finally derive some consequences of the above simple key lemma.
We let $c = c(d,r) = \sup_l |\partial^+_r B^c_{3l-1}|\, l^{-(d-1)}$. Then we have

Theorem 3.3. *Let $l \ge 2r$. Then, with the same notation of lemma 3.2, let $m = -\log\big(c\,e^{4\|\Phi\|}\, l^{d-1}\,\varphi(l)\big)$ and $m' = -\log\big(c\,e^{4\|\Phi\|}\, l^{d-1}\,\psi(l)\big)$*

(a) if $m > 0$ then

$$\Gamma_l(x-y) \le e^{4\|\Phi\|} e^{-m\lfloor \frac{|x-y|-8l}{4l}\rfloor}$$

if $m' > 0$ then

$$\Gamma_1(x-y) \le e^{4\|\Phi\|} e^{-m'\lfloor \frac{|x-y|-8l}{4l}\rfloor}$$

(b) if $\liminf_{|x|\to\infty} |x|^{d-1}\,\Gamma_l(x) = 0$ then $\Gamma_l(x)$ decays exponentially fast. Similarily for $\Gamma_1(x)$.

Proof.
(a) Let

$$\gamma(n) \equiv \sup_{\substack{x,y;\\ |x-y|\ge n}} \Gamma_l(x-y)$$

Let $n \ge 8l$. If we use translation invariance together with lemma 3.2, we immediately get

$$\begin{aligned}\gamma(n) &\le e^{4\|\Phi\|}\varphi(l) \sup_{\substack{x,y;\\ x\in\partial^+_r Q_l\\ |x-y|\ge n}} \sum_{z\in\partial^+_r B^c_{3l-1}} \Gamma_l(z-y)\\ &\le e^{4\|\Phi\|} c(d,r)\,\varphi(l)\, l^{d-1}\gamma(n-4l)\\ &\le e^{-m}\gamma(n-4l)\end{aligned} \tag{3.17}$$

Part (a) of the theorem now follows from a simple iteration of (3.17) together with the observation that $\Gamma_l(x-y) \le e^{4\|\Phi\|}$.

(b). Using the hypothesis together with (3.10) we can conclude that there exists an integer L such that

$$\gamma(nL) \le \frac{1}{2}\gamma((n-1)L) \qquad \forall\, n \ge L$$

i.e.

$$\gamma((n+1)L) \le (\frac{1}{2})^n \gamma(L) \le e^{4\|\Phi\|} (\frac{1}{2})^n$$

Similarily for $\Gamma_1(n)$. $\square$

Remark.

i) Thus theorem 3.3 is exactly the analogue of the constructive, finite size conditions of Dobrushin and Shlosman ([DS3]) or of Olivieri [O], depending whether one accepts to impose decay of covariances in regions of arbitrary shape ($l^{d-1}\,\psi(l)$ is small) or only on l–regular regions ($l^{d-1}\,\varphi(l)$ is small). If we recall the very definition (3.7) of $\varphi(l)$ or of $\psi(l)$, then we easily recognize that in both cases the required decay in $|x-y|$ for finite volume covariances $\mu_V^\tau(h_x^s, h_y^t)$ is at least a polynomial decay with a power not smaller than $2(d-1)$. This is so because the decay of the covariance on scale l has to beat the *square* of the surface of a cube of side $\approx l$. We will see below that for ferromagnetic ± 1 spins the power $2(d-1)$ can be improved to $d-1$.

ii) With some extra work one can prove that, if for some given C, $\alpha > 0$ and some large enough l_0

$$\sup_{\substack{x,y\in\partial_r^+ Q_{l_0} \\ |x-y|\ge \frac{l_0}{4}}} \sup_\tau \sup_{s,t\in S} |\mu^\tau_{Q_{l_0}}(h_x^s, h_y^)| \le C e^{-\alpha l_0}$$

then it is possible to find a constant $k = k(d, r, \|\Phi\|)$ and $m = m(d, r, \alpha)$ such that assumption (a) of the theorem holds for $l = kl_0$. In other words, in order to check that $l^{d-1}\,\varphi(l)$ is small, one has just to check that covariances are sufficiently small in a fixed, large enough cube and not for all subsets $V \in \mathcal{F}_l^0$ such that $Q_l \subset V$.

iii) Part (b) of the theorem gives, in particular, exponential decay as soon as $\Gamma_l(x-y)$ decays like $\frac{1}{|x-y|^\alpha}$ with $\alpha > d-1$. It is important to underline however that such result is *not* constructive in the sense that in order to check the hypothesis behind it one has to control the decay of covariances for *all* sets in $\mathbb{F}_l$ and not just for all sets in $\mathcal{F}_l^0$.

iv) It is quite clear that all the above analysis can be extended, with the obvious modification, to non–translation invariant interactions (e.g random one). In particular one gets exponential decay if $\sup_x l^{d-1}\,\varphi_x(l)$ is small where $\varphi_x(l)$ is as in (3.7) but with the cubes Q_l and B_{2l-1} shifted by x.

4. ATTRACTIVE SYSTEMS

In this section we consider ferromagnetic discrete spin models and we improve slightly our results. Although our discussion basically follows that of appendix 1 of [MO2], we decided to include it for completeness.

We take as single spin space the set $S = \{0, 1\}$ and we assume that the potential Φ satisfies, besides (1)...(4) of section 2, the following extra condition (ferromagnetic interaction)

(5) $\Phi_\Lambda(\sigma) = J_\Lambda \prod_{x\in\Lambda} \sigma(x)$ *where* $J_\Lambda \geq 0$ *if* $|\Lambda| \geq 2$.

Let us introduce a partial order in Ω by letting $\sigma \leq \eta$ if $\sigma(x) \leq \eta(x)$ for any $x \in \mathbb{Z}^d$. Then it is known that, under assumption (5), the Gibbs measures μ_V^τ enjoy the FKG property. In particular, if $\tau' \geq \tau$ there exists a joint representation (or coupling) $\nu_V^{\tau',\tau}(\cdot,\cdot)$ of $\mu_V^{\tau'}$ and μ_V^τ which is above the diagonal.

In analogy with (3.2) we define the translation invariant, bounded, non–negative function $\alpha_l(x-y)$ by

$$\alpha_l(x-y) = \sup_{V\in\mathbb{F}_l} \sup_{\tau\in\Omega} |\mu_V^{\tau^y}(\sigma(x)) - \mu_V^\tau(\sigma(x))| \tag{4.1}$$

where τ^y simply denotes the configuration whose spin at y has been flipped. The reason behind the above definition is that, for ferromagnetic $\{0, 1\}$ spins, it is slightly easier to work with $\alpha_l(x-y)$ than with $\Gamma_l(x-y)$. The two are however simply related. Notice in fact that

$$|\mu_V^{\tau^y}(\sigma(x)) - \mu_V^\tau(\sigma(x))| = \nu_V^{\tau^y,\tau}(\sigma(x) \neq \eta(x)) \tag{4.2}$$

since $\nu_V^{\tau^y,\tau}(\sigma(x) \geq \eta(x)\ \forall\, x \in V) = 1$ if $\tau^y \geq \tau$. As a consequence we get that, for any V, any $x, y \in \partial_r^+ V$ and any $s, t \in \{0, 1\}$

$$\begin{aligned} |\mu_V^\tau(h_x^s, h_y^t)| &= |\mu_V^\tau(h_y^t)|\,|\mu_V^{\tau^{y,t}}(h_x^s) - \mu_V^\tau(h_x^s)| \\ &\leq e^{2\|\Phi\|}|\,\nu_V^{\tau^{y,t},\tau}\big(h_x^s(\sigma) - h_x^s(\eta)\big)\,| \\ &\leq e^{4\|\Phi\|}\nu_V^{\tau^y,\tau}\big(\exists\, z \text{ with } |z-x| \leq r:\ \sigma(z) \neq \eta(z)\big) \\ &\leq e^{4\|\Phi\|} \sum_{\substack{z\in V\\ d(x,z)\leq r}} \sup_\tau \nu_V^{\tau^y,\tau}\big(\sigma(z) \neq \eta(z)\big) \end{aligned} \tag{4.3}$$

where we have used (3.1) to derive the first equality.

Thus if the function $\alpha_l(x-y)$ decays exponentially fast then so does $\Gamma_l(x-y)$. Of course also the converse holds because of proposition 3.1.

In complete analogy with Lemma 3.2 we now have

Lemma 4.1. *If* $|x-y| \geq 8l$ *and* $x \in Q_l$*, then*

$$\alpha_l(x-y) \leq \sum_{z\in\partial_r^+ B_{2l-1}} \varphi(x,z)\, \alpha_l(z-y)$$

where

$$\varphi(x,z) = \sup_\tau \sup_{\substack{V\in\mathbb{F}_l^0\\ Q_l\subset V}} \nu_V^{\tau^z,\tau}\big(\sigma(x) \neq \eta(x)\big)$$

A similar statement holds with $\alpha_l(x-y)$ replaced by $\alpha_1(x-y)$ and $\varphi(x,z)$ by

$$\psi(x,z)=\sup_{\tau}\sup_{V\subset B_{2l-1}}\nu_V^{\tau^z,\tau}\big(\sigma(x)\neq\eta(x)\big)$$

.

Remark. Using (4.2) together with (3.4) one immediately checks that

$$\varphi(x,z)\leq e^{2\|\Phi\|}\sup_{\tau}\sup_{\substack{V\in\mathbb{F}_l\\ V\subset B_{2l-1}}}\sup_{t\in\{0,1\}}|\mu_V^{\tau}(\sigma(x),h_y^t)|$$

Furthermore, as in theorem 3.3, if for some positive m

$$\sup_{x\in Q_l}\sum_{z\in\partial_r^+B_{2l-1}}\varphi(x,z)\leq e^{-m} \tag{4.4}$$

then

$$\alpha_l(x-y)\leq e^{2\|\Phi\|}e^{-m\lfloor\frac{|x-y|-8l}{4l}\rfloor}$$

Notice that the finite size condition (4.4) only requires a polynomial decay of finite volume covariances with a power just greater than $d-1$ and not $2(d-1)$ as in the more general, non attractive case.

Proof.
Without loss of generality let x,y,V,A,B,C be as in the proof of part (a) of lemma 3.2. Then, using DLR, a standard telescope argument together with (4.2), we write

$$\begin{aligned}
\mu_V^{\tau^y}(\sigma(x))-\mu_V^{\tau}(\sigma(x)) &= \int d\nu_V^{\tau^y,\tau}(\eta',\eta)\Big(\mu_A^{\eta'}(\sigma(x))-\mu_A^{\eta}(\sigma(x))\Big)\\
&\leq \int d\nu_V^{\tau^y,\tau}(\eta',\eta)\Big(\sum_{z\in\partial_r^+A}\chi\big(\eta'(z)\neq\eta(z)\big)\sup_{\alpha}|\mu_A^{\alpha^z}(\sigma(x))-\mu_A^{\alpha}(\sigma(x))|\Big)\\
&\leq \sum_{z\in\partial_r^+A}\sup_{\alpha}\nu_A^{\alpha^z,\alpha}\big(\sigma'(x)\neq\sigma(x)\big)\,\nu_V^{\tau^y,\tau}\big(\eta'(z)\neq\eta(z)\big)\\
&\leq \sum_{z\in\partial_r^+A}\varphi(x,z)\,\alpha_l(z-y)
\end{aligned}$$

□

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