Study of the double mathematical pendulum - I. Numerical investigation of homoclinic transversal intersections

A.V.Ivanov
Physics Department, St.-Petersburg State University, Ulyanov str. 1, kor.1, Petrodvorec, St.-Peterburg, 198904, *Russia.*
E-mail address: alex@ai2634.spb.edu

AMS classification scheme numbers: 34C15, 58C25, 58F22

Abstract

We study numerically the double mathematical pendulum in terms of the Poincaré section. Approximate positions of some hyperbolic periodic points are obtained visually from the pictures of the phase portrait, and their precise coordinates are calculated by use of the Newton method. The pictures of the corresponding stable and unstable manifolds are drawn, the positions of some homoclinic points are found numerically, and their homoclinic invariants are calculated. This is done for 3 chosen sets of system parameters and values of the energy. The nonnullity of the mentioned homoclinic invariants implies the nonintegrability of the system for these values of the parameters and energy. It means that the related Poincaré map has no a first integral on that energy level.

1 Introduction

The mathematical double pendulum consists of two masses m_1 and m_2 attached to consequently joined arms of lengths l_1 and l_2, as it is shown on Fig.1, the upper end of the first arm being fixed, and the whole system being subjected to the action of the constant gravity acceleration, g.

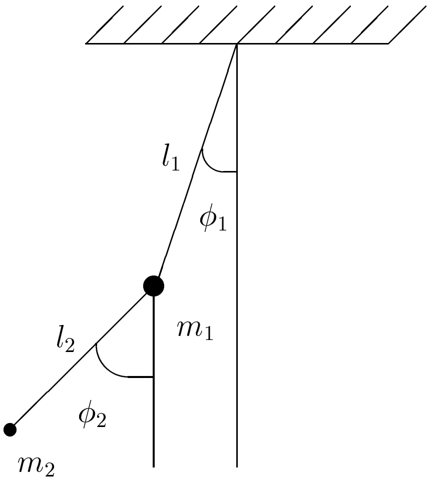

Fig.1

Denote by ϕ_1 and ϕ_2 the angles of deviation of the arms from the vertical axis. Then the Hamiltonian, H, of the system reads:

$$H = \frac{1}{(m_1 + m_2 \sin^2(\phi_1 - \phi_2))} \left(\frac{p_1^2}{l_1^2} - \frac{p_1 p_2 \cos(\phi_1 - \phi_2)}{l_1 l_2} + \frac{p_2^2 (1 + \frac{m_1}{m_2})}{l_2^2} \right) +$$

$$+ (m_1 + m_2) g l_1 (1 - \cos\phi_1) + m_2 g l_2 (1 - \cos\phi_2), \tag{1.1}$$

where p_1 and p_2 are generalized momenta corresponding to the coordinates ϕ_1, ϕ_2.

We will study the motion of the system (1.1) on the energy surface $H = E$, the value of the energy, E, being considered as an additional parameter of the system.

The number of parameters can be reduced, if we set

$$\varepsilon = \frac{m_2}{m_1},\ \mu = \frac{l_2}{l_1},\ \nu = \frac{E}{m_1 g l_1},\ \tilde{H} = \frac{H}{m_1 g l_1}.$$

Then the expression (1.1) and the equation of energy surface can be rewritten as

$$\tilde{H} = \frac{1}{(1 + \varepsilon \sin^2(\phi_1 - \phi_2))} \left(p_1^2 - \frac{p_1 p_2 \cos(\phi_1 - \phi_2)}{\mu} + \frac{p_2^2 (1 + \varepsilon)}{\varepsilon \mu^2} \right) +$$

$$+ (1 + \varepsilon)(1 - \cos\phi_1) + \varepsilon\mu(1 - \cos\phi_2) \tag{1.2a}$$

$$\tilde{H}(\phi_1, \phi_2, p_1, p_2) = \nu. \tag{1.2b}$$

Numerical simulations (see e.g. [1]) show that the system (1.2) exhibits chaotic dynamics, and hence one can expect it not to be integrable. Let us precise the definition of integrability.

Consider a Hamiltonian system defined on a symplectic manifold X, $\dim X = 2n$, with Hamiltonian $\tilde{H}$. We say the system is *integrable* on an open subset $G \subset X$ if there exist n functions I_1, I_2,..., I_n defined on G such that

1. $\{I_k, \tilde{H}\} = 0$, $k = 1, ..., n$

2. $\{I_k, I_j\} = 0,\ k, j = 1, ..., n$

3. $\{\nabla I_k(x)\}_{k=1}^n$ are linearly independent for all $x \in G$,

where $\{\cdot, \cdot\}$ denotes the Poisson bracket.

Given $\nu \in \mathbf{R}$, consider the energy surface $\tilde{H} = \nu$. We say that the system is integrable on the level of energy ν, if there exist an open neighborhood G of $H^{-1}(\nu)$ such that the system is integrable on G. Without loss of generality we can choose G in the form $G = \bigcup_{\nu_1 < \nu < \nu_2} H^{-1}(\nu)$, where ν_1, ν_2 are constants.

It follows from the Liouville-Arnold theorem [1] that if the subset $H^{-1}([\nu - \varepsilon_0, \nu + \varepsilon_0])$ is compact for some positive ε_0 and the system is integrable in a domain $H^{-1}(]\nu - \varepsilon, \nu + \varepsilon[)$, $\varepsilon < \varepsilon_0$ then the system is isomorphic to an integrable model system on $B \times T^n = \{(I, \phi)\}$ with a Hamiltonian $H_{integrable}$ depending only on I. If there is no such a neighborhood G, we say that the system is *nonintegrable* at the energy level ν.

Now we formulate our main conjecture:

For all nonzero values of the parameters ε, μ, ν, the system (1.2) is nonintegrable.

In this first part we develop a numerical method which enables us to prove the nonintegrability of the system for fixed values of the parameters.

As an example we choose three cases:

1. $\varepsilon = 0.01,\ \mu = 5,\ \nu = 1.16;$
2. $\varepsilon = 0.01,\ \mu = 10,\ \nu = 0.80;$
3. $\varepsilon = 0.01,\ \mu = 7,\ \nu = 0.16.$

Our method consists of finding numerically transversal homoclinic trajectories. The latters cannot exist in the integrable case, as it follows from the above described model.

2 Poincaré section. Poincaré Map. Phase portraits

Let $f^t : X \to X$, $X = T^*(\mathbf{T^2}) = \mathbf{T^2} \times \mathbf{R^2}$ be the Hamiltonian flow, defined by the Hamiltonian function (1.2a). The flow f^t preserves a symplectic form $\Omega = \sum_{i=1}^{2} d\phi_i \wedge dp_i$, defined on X.

Given a value $\nu > 0$ of the rescaled energy, consider the surface $\tilde{S} \subset \tilde{H}^{-1}(\nu)$ defined by the equality

$$\tilde{S} = \tilde{H}^{-1}(\nu) \cap \Phi_0,$$

where Φ_0 is the hyperplane $\{\phi_1 = 0\}$.

Let S be a submanifold of the space X. Denote by $i_S : S \to X$ the inclusion map. We call S a Poincaré section, if the following is true:

1. $S \subset \tilde{H}^{-1}(\nu)$ for some ν;
2. S is symplectic, i.e. $\Omega_S = i_S^*\Omega$ is a symplectic form on S;
3. the vector field of the Hamiltonian $\tilde{H}$, $\sum_{i=1}^{2}(-\frac{\partial \tilde{H}}{\partial \phi_i}\frac{\partial}{\partial \phi_i} + \frac{\partial \tilde{H}}{\partial p_i}\frac{\partial}{\partial p_i})$, is transversal in $\tilde{H}^{-1}(\nu)$ to S at each point of S.

It is not difficult to see that in our case the surface $\tilde{S}$ satisfies the first two conditions and $\Omega_S = d\phi_2 \wedge dp_2$. Consider the third condition. It is equivalent to $\dot{\phi}_1|_{\Phi_0} \neq 0$. Thus it fails on the line defined by the formula:

$$\frac{p_2^2}{\mu^2}\left(\frac{\cos^2\phi_2}{4}-\frac{(1+\varepsilon)}{\varepsilon}\right)+(1+\varepsilon\sin^2\phi_2)(\nu-\varepsilon\mu(1-\cos\phi_2))=0. \tag{2.1}$$

The surface $\tilde{S}$ is divided by the line (2.1) onto two components $S_\pm$, corresponding to the left hand part of (2.1) greater (resp. less) then zero.

In the present work we put the Poincaré section $S=S_+$. Let $z\in S$. We call $\tau(z)=\min\{t>0: f^t(i_S(z))\in S\}$ the first-return time. Then the Poincaré map $F: S\to S$ is defined by the formula:

$$F(z)=i_S^{-1}(f^{\tau(z)}(i_S(z))). \tag{2.2}$$

It is known (see e.g.[7]) that the Poincaré map preserves the form Ω_S. Besides, as it follows from the invariance of the dynamical system with respect to reversing the time and due to a special choice of the Poincaré section, the Poincaré map satisfies the following condition: $F(z_1,z_2)=F(-z_1,z_2)$. To illustrate the behavior of the trajectories of the system, we consider the Poincaré sections for $\varepsilon=0.01, \mu=10$, the third parameter, energy ν, varying in the interval $[0.09, 11.31]$. The series consists of 12 phase portraits (fig. 2.1 - 2.12). The first one describes the limit case, when the system energy close to the minimum value $\nu=0$. For $\nu=0$ the system is quadratic and therefore integrable. All its orbits are quasiperiodic (or with very big period). Besides the regular component of the motion, a stochastic one appears with energy increasing. While the perturbation is not big, the majority of the invariant tori continues to exist, being deformed a little. However a part of them is destroyed and new tori, elliptical and hyperbolic points, stochastic layers appear on their place (fig. 2.2). It is to be noted that size of the Poincaré section S depends on the system energy. As a consequence there are empty (white) regions on fig. 2.1, 2.2. Under further energy increasing the stochastic motion begins to dominate (fig. 2.3-2.6). But when the energy becomes sufficiently high, an inverse process occurs (fig. 2.7-2.11) and we approach to the second limit case $\nu=\infty$, when the system becomes integrable (integrability follows from presence of second integral of motion - the full momentum). As in the first case, all orbits are quasiperiodic here (fig.2.12). It is not difficult to see that all figures have periodic trajectory in the center, which goes through the point $(0,0)$. It is described by a following equation

$$\left(p_2-\frac{\varepsilon\mu\nu^{\frac{1}{2}}}{2(1+\varepsilon)}\right)^2+\frac{\mu\phi_2^2}{2(1+\varepsilon)}=\frac{\varepsilon^2\mu^2\nu}{4(1+\varepsilon)^2}.$$

The results of this computer experiment are in a good agreement with data obtained in the paper [11].

3 Periodic hyperbolic points of the Poincaré map

In this section we describe our method of finding periodic hyperbolic points.

By the definition, a periodic hyperbolic point z_0 of the map F satisfies the equation

$$F^p(z_0)=z_0 \text{ for some integer } p$$

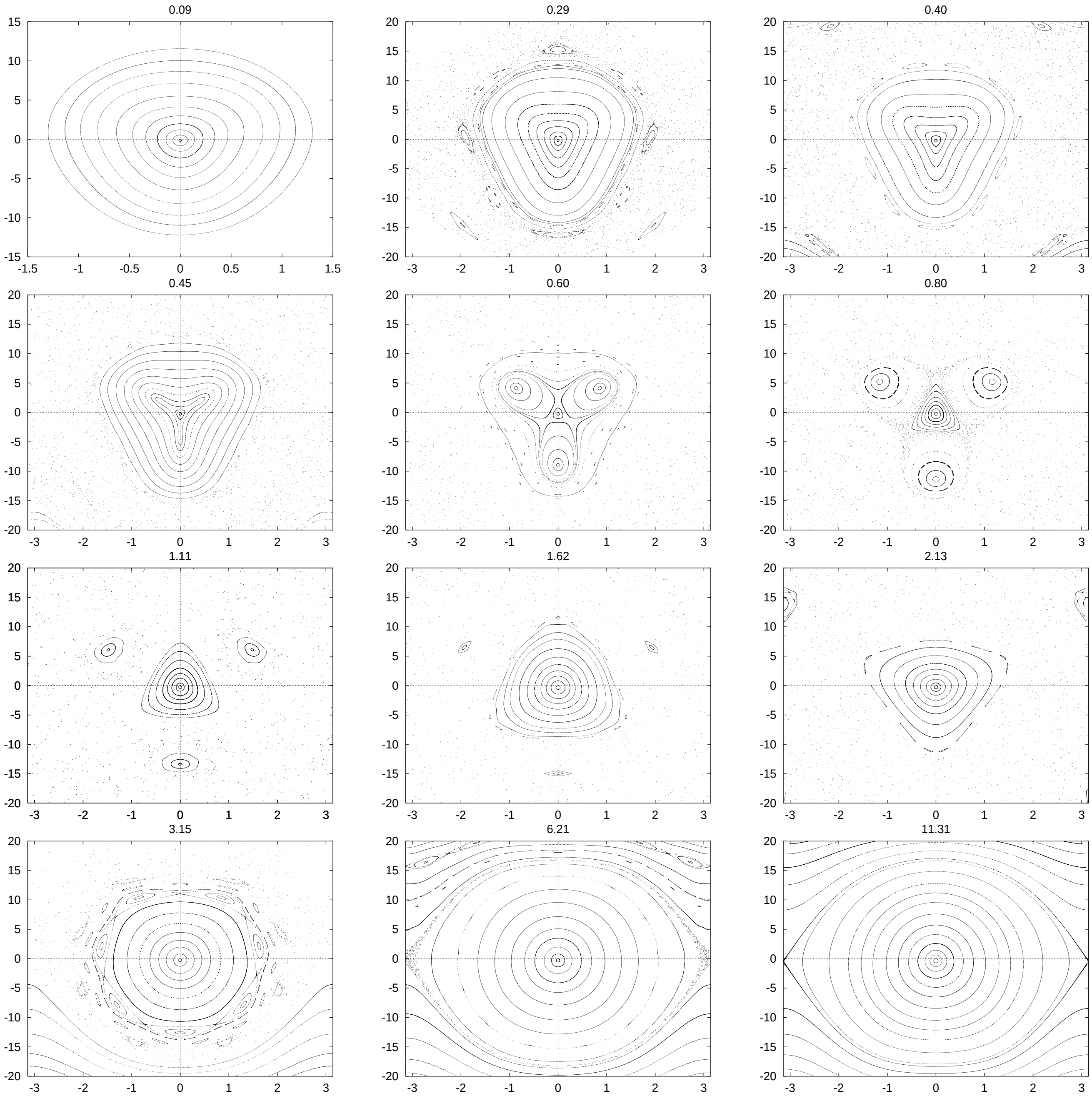

Fig.2.1-2.12 A series of Poincaré sections ϕ_2 versus p_2, corresponding to $\varepsilon = 0.01, \mu = 10$, which shows dynamic changes of the double mathematical pendulum with respect to energy increasing. (The value of the energy, ν, is written at the top of each figure).

and the matrix of the derivative of the map $T = F^p$ at this point has real eigenvalues, which are not equal to ± 1. Since the Poincaré map preserves the form Ω_S, the eigenvalues λ_1, λ_2 of the matrix $T'(z_0)$ satisfy the following condition:

$$\lambda_1 \lambda_2 = 1. \tag{3.1}$$

As far as hyperbolic objects are stable with respect to small perturbations, it is possible to use Newton method to find periodic hyperbolic points.

Assume that for some point z_1 the following condition is fulfilled:

there is a neighborhood of point z_1, $U(z_1) = \{z : ||z - z_1|| \leq C_1\}$, $C_1 > 0$ such that:

1)$||(T'(z) - I)^{-1}|| \leq C_2$, if $z \in U(z_1)$,

2)$||T''(z)|| \leq C_3$, if $z \in U(z_1)$,

3)$z_k \in U(z_1), k \geq 1$,

4)$C_2 \sqrt{C_3} e_1 < 1$, where $e_1 = ||\varepsilon_1||$.

Let us build a sequence $\{z_k\}_{k=1}^{\infty}$ by the rule:

$$z_{k+1} = z_k - (T'(z_k) - I)^{-1} \varepsilon_k, \tag{3.2}$$

where $\varepsilon_k = F(z_k) - z_k$.

As it follows from [8], there is a unique fixed hyperbolic point z_0 of the map T, such that the sequence (3.2) tends to it.

Thus knowing a first approximation z_1, we can find a periodic hyperbolic point of the Poincaré map with arbitrary accuracy. To calculate z_{k+1}, it is necessary to know the value $T'(z_k)$. To do that we consider Hamilton equations presented in the form:

$$\dot{x}(t) = f(x), \tag{3.3}$$

where $x_1 = \phi_1$, $x_2 = p_1$, $x_3 = \phi_2$, $x_4 = p_2$ and

$f(x)$ - a vector of right part of Hamilton equations.

If we take an initial point z on the Poincaré section, the solution of the system (3.3) will satisfy the following initial condition:

$$x(t, z)|_{t=0} = g(z), \tag{3.4}$$

where $g(z) = i_S(z)$.

Thus we can write

$$x(t, z) = g(z) + \int_0^t f(x(s, z)) ds. \tag{3.5}$$

The components of the map T can be expressed as:

$$T_j(z) = x_{2j}(\tau_p(z), z), \ j = 1, 2, \tag{3.6}$$

where $\tau_p(z)$ is the first-return time for T.

Applying the chain rule and taking into account specific feature of our choice of the Poincaré section, we get

$$\frac{\partial T_i}{\partial z_j}(z) = \frac{\partial x_{2i}}{\partial z_j}(\tau_p(z), z) - \frac{f_{2i}(x(\tau_p(z), z))}{f_1(x(\tau_p(z), z))} \frac{\partial x_1}{\partial z_j}(\tau_p(z), z), \ i, j = 1, 2, \tag{3.7}$$

where f_i are the components of the function in (3.3)

In this work periodic hyperbolic points of the Poincaré map are calculated for the three different sets of parameters given in the introduction.

The Poincaré sections, corresponding to these parameters, are presented on fig. 3.1, 3.2 and 3.3.

The initial approximations of the hyperbolic points are chosen experimentally. For this purpose, the phase portraits are analyzed, taking into account the locations of regular islands and stochastic layers, which appear on the region of resonance. As it is shown in [9], sequences of alternating elliptic and hyperbolic points appear in the place of the destroyed invariant tori. Elliptic points are the centres of regular islands, hyperbolic ones are located between these islands, separating them. Neighborhoods, where initial approximations of the hyperbolic points are chosen, for the first set of the parameters are presented on the fig. 3.1. Neighborhoods, corresponding to other cases, are chosen in the same way. The results of calculations of the periodic hyperbolic points are presented in tables 1a,1b,1c, accordingly. The obtained hyperbolic points are marked at the phase portraits on fig.3.1-3.3.

table 1a

№	value ϕ_2	value p_2	period
1	1,0488473161408	-0.47666956084763	4
2	0	2,55056306336664	4
3	-1,0488473161408	-0,47666956084763	4
4	0	-2,59872272125134	4

table 1b

№	value ϕ_2	value p_2	period
1	0,52047771645691	-2,85080318620684	3
2	0	5,02214211436370	3
3	-0,52047771645691	-2,85080318620684	3

table 1c

№	value ϕ_2	value p_2	period
1	1,07955070161988	-3,10327802883132	3
2	0	6,45619087194293	3
3	-1,07955070161988	-3,10327802883132	3

It is necessary to note that in the first case the presented hyperbolic points are the ones of period 4 (i.e. they are fixed points for map F^4), but in the second and the third cases they are points of period 3. Thus in each case the calculated hyperbolic points belong to one hyperbolic periodic trajectory. Projections of these hyperbolic periodic trajectories on the torus of angular coordinates are presented on fig. 3.4, 3.5, 3.6. It is not difficult to see that the considered trajectories are homotopic to zero.

It is discovered experimentally that the hyperbolic points, z_k, $k = 1, ..., p$, in each case satisfy the following equation

$$F(z_0^{k\ (mod\ p)}) = z_0^{k+1\ (mod\ p)},$$

where k is the number of the hyperbolic point, according to the tables 1a,b,c, p is their period.

4 Construction of separatrices of the Poincaré map

Let z_k, $1 \leq k \leq p$ be hyperbolic points of period p, belonging to a periodic hyperbolic trajectory γ_{hyp}. As it is shown in [10], γ_{hyp} has two invariant curves: the stable W^s and the unstable W^u separatrices, defined as:

$$W^u(\gamma_{hyp}) = \{z \in S : F^n(z) \to \gamma_{hyp}\ as\ n \to \infty\},$$

$$W^s(\gamma_{hyp}) = \{z \in S : F^n(z) \to \gamma_{hyp}\ as\ n \to -\infty\},$$

It is not difficult to prove that there are injective maps $\Phi_k^{u,s} : R^1 \to S$, $1 \leq k \leq p$ for the periodic hyperbolic trajectory γ_{hyp} such that

$$\bigcup_{k=1}^{p} \Phi_k^{u,s}(\mathbf{R^1}) = W^{u,s}(\gamma_{hyp}),$$

$$\Phi_{k+1\ (mod\ p)}^{u,s}(\lambda_1^{(\pm 1)}\xi) = F \circ \Phi_{k\ (mod\ p)}^{u,s}(\xi),\ 1 \leq k \leq p,$$

$$\Phi_k^{u,s}(0) = z_k,\ 1 \leq k \leq p, \tag{4.1}$$

where $\lambda_1, \lambda_1^{-1} = \lambda_2$ are the eigenvalues of the linear part of the map F at the point z_k, $|\lambda_1| > 1$, the sign $(+)$ corresponds to W^u and the sign $(-)$ corresponds to W^s ([12]).

Thus the separatrices $W^{u,s}(\gamma_{hyp})$ consist of p parts $W_k^{u,s} = \Phi_k^{u,s}(R^1)$, $k = 1, ..., p$, each of them corresponds to the periodic hyperbolic point z_k.

Represent the expansion of $W_k^{u,s}$ in a neighborhood of the periodic hyperbolic point z_k in the form:

$$z_k^{u,s}(\xi) = \sum_{m=0}^{\infty} z_{k,m}^{u,s}\xi^m. \tag{4.2}$$

Substitute (4.2) in the second equality (4.1). Expanding it on powers ξ and comparing coefficients under same power ξ, we get a system of linear equations for the coefficients $z_{k,m}^{u,s}$:

$$F(z_{k,0}^{u,s}) = z_{k+1},$$

$$F'(z_{k,0}^{u,s})z_{k,1}^{u,s} = \lambda_1^{(\pm 1)} z_{k+1,1}^{u,s},$$

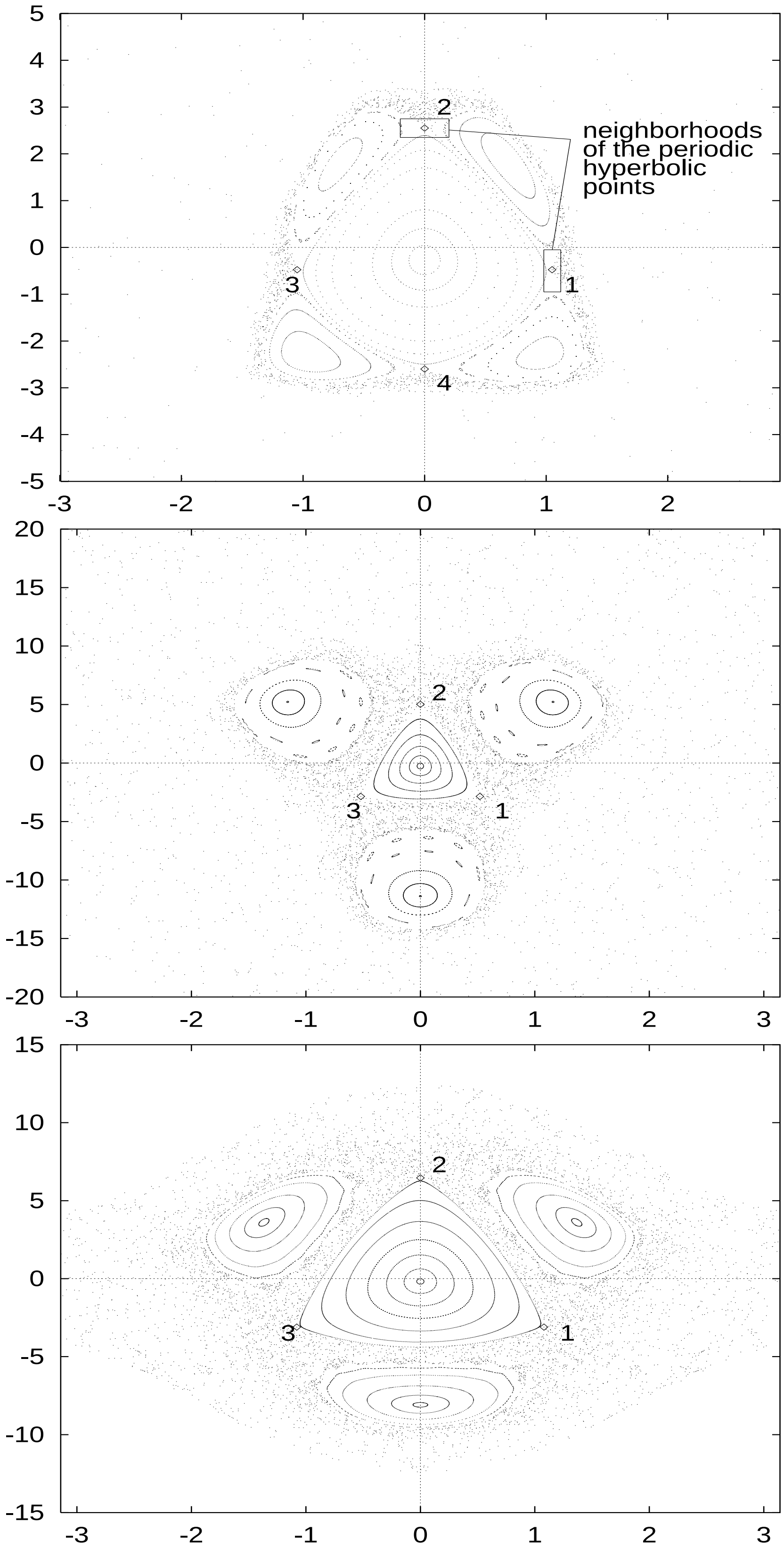

Fig.3.1-3.3 Poincaré sections ϕ_2 versus p_2 corresponding to $(\varepsilon = 0.01,\ \mu = 5,\ \nu = 1.16)$**,** $(\varepsilon = 0.01,\ \mu = 10,\ \nu = 0.80)$**,** $(\varepsilon = 0.01,\ \mu = 7,\ \nu = 0.16)$**, accordingly.**

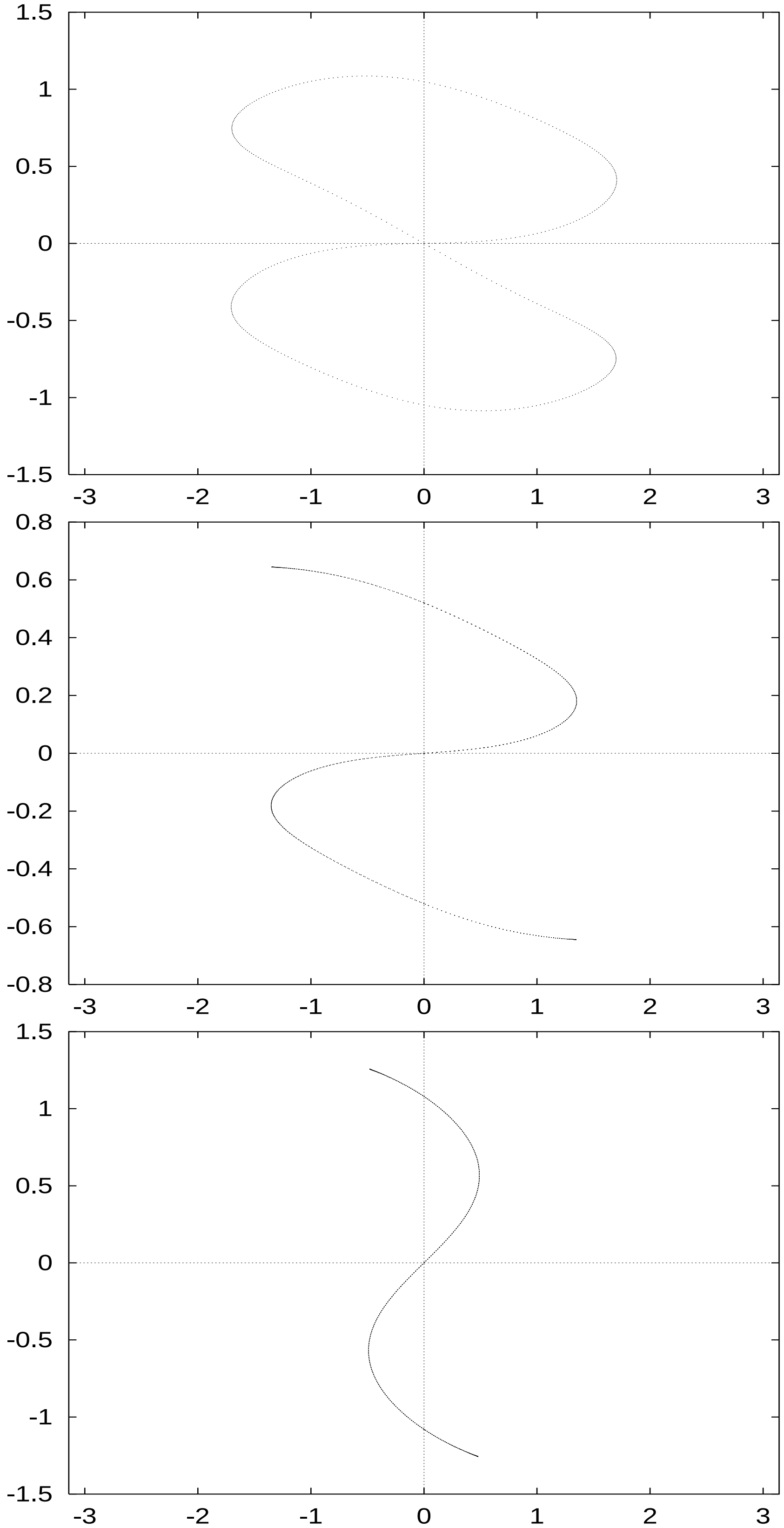

Fig.3.4-3.6 Projections of periodic hyperbolic trajectories on the torus ϕ_1 versus ϕ_2 corresponding to $(\varepsilon = 0.01,\ \mu = 5,\ \nu = 1.16)$, $(\varepsilon = 0.01,\ \mu = 10,\ \nu = 0.80)$, $(\varepsilon = 0.01,\ \mu = 7,\ \nu = 0.16)$, accordingly.

$$F'(z_{k,0}^{u,s})z_{k,2}^{u,s} - (\lambda_1^{(\pm 1)})^2 z_{k+1,2}^{u,s} = Q_2(z_{k+1,0}^{u,s}, z_{k+1,1}^{u,s}),$$

....

$$F'(z_0^{u,s})z_{k,m}^{u,s} - (\lambda_1^{(\pm 1)})^m z_{k+1,m}^{u,s} = Q_m(z_{k+1,0}^{u,s}, z_{k+1,1}^{u,s}, .., z_{k+1,m-1}^{u,s}), \qquad (4.3)$$

where Q_m are polinomial vector-functions.

Notice that it is necessary to know values of the derivatives of the map F at the point z_k to find the coefficients $z_{k,m}^{u,s}$, since they enter the expressions for the functions Q_m. Taking this into account, we consider formula (3.7). Differentiating this expression with respect to z_j m times and putting in this formula $p = 1$, we obtain the formula for m-th derivative of the map F. It is easy to see that the expression contains derivatives of the type

$$\frac{\partial^l x_s}{\partial z_{j_1} \partial z_{j_2} ... \partial z_{j_l}}, s = 1, 2, 4; \; l = 1, .., k. \qquad (4.4)$$

Derivatives of the type (4.4) satisfy the following Cauchy problems:

$$\frac{\partial}{\partial t}(\frac{\partial^l x_i}{\partial z_{j_1} \partial z_{j_2} ... \partial z_{j_l}})(t, z) = \sum_{k=1}^{4} \frac{\partial f_j}{\partial x_k}(x(t, z)) \frac{\partial^l x_k}{\partial z_{j_1} \partial z_{j_2} ... \partial z_{j_l}}(t, z) + R_i$$

$$\frac{\partial^l x_i}{\partial z_{j_1} \partial z_{j_2} ... \partial z_{j_l}}(t, z)|_{t=0} = \frac{\partial^l g_i}{\partial z_{j_1} \partial z_{j_2} ... \partial z_{j_l}}(z), i = 1, .., 4; j = 1, 2; s = 1, .., l, \qquad (4.5)$$

where functions R_i depend on the same derivatives, their orders do not exceed $l - 1$. This statement is proven by direct differentiating of (3.11).

In particular under $m = 2$ we can write:

$$\frac{\partial^2 F_i}{\partial z_{j_1} \partial z_{j_2}}(z) = \frac{\partial^2 x_{2i}}{\partial z_{j_1} \partial z_{j_2}}(\tau(z), z) +$$

$$+ \sum_{k=1}^{4} (A_{i,k}(x(\tau(z), z)) \frac{\partial x_k}{\partial z_{j_1}}(\tau(z), z) \frac{\partial x_1}{\partial z_{j_2}}(\tau(z), z) +$$

$$+ B_{i,k}(x(\tau(z), z)) \frac{\partial x_k}{\partial z_{j_2}}(\tau(z), z) \frac{\partial x_1}{\partial z_{j_1}}(\tau(z), z) +$$

$$+ C_i(x(\tau(z), z)) \frac{\partial x_1}{\partial z_{j_1}}(\tau(z), z) \frac{\partial x_1}{\partial z_{j_2}}(\tau(z), z) + D_i(x(\tau(z), z)) \frac{\partial^2 x_1}{\partial z_{j_1} \partial z_{j_2}}(\tau(z), z),$$

$$A_{i,k} = \frac{\partial f_{2i}}{\partial x_k}(x(\tau(z), z)) + \frac{\partial f_1}{\partial x_k}(x(\tau(z), z)) \frac{f_{2i}(x(\tau(z), z))}{f_1^2(x(\tau(z), z))},$$

$$B_{i,k} = -\frac{\partial}{\partial x_k}(\frac{f_{2i}}{f_1})(x(\tau(z), z)),$$

$$C_i = -\sum_{k=1}^{4} \frac{f_k(x(\tau(z),z))}{f_1(x(\tau(z),z))} \frac{\partial}{\partial x_k}(\frac{f_{2i}}{f_1})(x(\tau(z),z)),$$

$$D_i = -\frac{f_{2i}(x(\tau(z),z))}{f_1(x(\tau(z),z))},$$

$$\frac{\partial}{\partial t}(\frac{\partial^2 x_i}{\partial z_{j_1}\partial z_{j_2}})(t,z) = \sum_{k=1}^{4} \frac{\partial f_j}{\partial x_k}(x(t,z)) \frac{\partial^2 x_k}{\partial z_{j_1}\partial z_{j_2}}(t,z)+$$

$$+\sum_{k,r=1}^{4} \frac{\partial^2 f_i}{\partial x_k \partial x_r}(x(\tau(z),z)) \frac{\partial x_k}{\partial z_{j_1}}(\tau(z),z) \frac{\partial x_r}{\partial z_{j_2}}(\tau(z),z),$$

$$\frac{\partial^2 x_i}{\partial z_{j_1}\partial z_{j_2}}(t,z)|_{t=0} = \frac{\partial^2 g_i}{\partial z_{j_1}\partial z_{j_2}}(z),\ i=1,..,4;\ j_1,j_2=1,2.$$

It is necessary to solve m Cauchy problems of the type (4.5), to find m-th derivative of the map F.

The coefficients of the separatrices expansion are obtained for the same values of the parameters of the double mathematical pendulum as in previous sections. This expansion takes into account the terms up to the third order. The results of the calculations are presented in the tables 2a, 2b, 2c.

table 2a

coefficient	point 1	point 2	point 3	point 4
$(z^u_{k,1})_1$	1	-1	-1	1
$(z^u_{k,1})_2$	7,0742	1,9053	-3,887	-0,8511
$(z^u_{k,2})_1$	10,1331	-0,631	-2,07	0,6232
$(z^u_{k,2})_2$	18,2885	5,1954	4,1938	-2,3059
$(z^u_{k,3})_1$	6,0748	-3,7842	-4,3841	3,216
$(z^u_{k,3})_2$	9,3541	-1,108	-2,1146	4,5824
$(z^s_{k,1})_1$	1	1	-1	-1
$(z^s_{k,1})_2$	-3,887	1,8905	7,0742	-0,8235
$(z^s_{k,2})_1$	-2,07	-1,0204	10,1331	0,3106
$(z^s_{k,2})_2$	4,1938	-4,9342	18,2885	0,9077
$(z^s_{k,3})_1$	4,3841	3,7602	-6,0748	-0,1063
$(z^s_{k,3})_2$	-2,1146	-4,3112	9,3541	1,3547

Note that the coefficients for the points 2, 3, 4 can be obtained from those of the point 1 by applying the corresponding iterate of the Poincaré map to the 3-jet of the separatrix. This is also true for the other considered cases. We put all these data in the tables 2a,b,c for the sake of completeness.

table 2b

coefficient	point 1	point 2	point 3
$(z^u_{k,1})_1$	1	-1	-1
$(z^u_{k,1})_2$	1,5227	10,216	-21,1991
$(z^u_{k,2})_1$	-1,488	2,1311	1,8408
$(z^u_{k,2})_2$	-3,3983	-20,1850	33,1721
$(z^u_{k,3})_1$	-2,6571	-5,6122	3,5613
$(z^u_{k,3})_2$	1,0249	7,803	-9,8302
$(z^s_{k,1})_1$	1	1	-1
$(z^s_{k,1})_2$	-21,1991	10,0638	1,5227
$(z^s_{k,2})_1$	1,8408	-1,5096	-1,488
$(z^s_{k,2})_2$	33,1721	6,6696	-3,3983
$(z^s_{k,3})_1$	-3,5613	-2,4598	2,6571
$(z^s_{k,3})_2$	-9,8302	8,3155	1,0249

table 2c

coefficient	point 1	point 2	point 3
$(z^u_{k,1})_1$	1	-1	-1
$(z^u_{k,1})_2$	2,6715	19,2623	-21,7869
$(z^u_{k,2})_1$	-0,0701	4,0208	4,6203
$(z^u_{k,2})_2$	-0,2637	67,1454	-12,4454
$(z^u_{k,3})_1$	3,1386	2,1976	-5,3051
$(z^u_{k,3})_2$	1,2544	-26,3199	-7,201
$(z^s_{k,1})_1$	1	1	-1
$(z^s_{k,1})_2$	-21,7869	-1,677	2,6715
$(z^s_{k,2})_1$	4,6203	0,2345	-0,0701
$(z^s_{k,2})_2$	-12,4454	-3,2262	-0,2637
$(z^s_{k,3})_1$	5,3051	-6,7135	-3,1386
$(z^s_{k,3})_2$	-7,201	4,7833	1,2544

Once the expansions of $W_k^{u,s}$, $k = 1, ..., p$ are found, it is possible to continue them out of the neighborhood by using the Poincaré map.

We consider a fundamental area $\Gamma^{u,s} = [\xi, \lambda_1 \xi]$. As it follows from the second equality of (4.1), the parts of separatrices, $W_k^{u,s}$, can be represented as $W_k^{u,s} = \bigcup\limits_{n \in \mathbf{Z}} F^n \circ \Phi^{u,s}_{p+1-n \ (mod\ p)}(\Gamma^{u,s})$. Chosing ξ sufficiently small so to be able to use the expansion (4.2), we can construct the separatrices of the trajectory γ_{hyp}.

We prolong the separatrices by given $\xi = 10^{-4}$ and making 100 iterations. The parts of them which correspond to the first set of the system parameters are presented on fig. 4.1-4.3. In fig.4.2 we mark the homoclinic points, where homoclinic invariant is calculated. These figures illustrate the motion in a stochastic layer. In particular, it is possible to observe the effect of separatrix splitting.

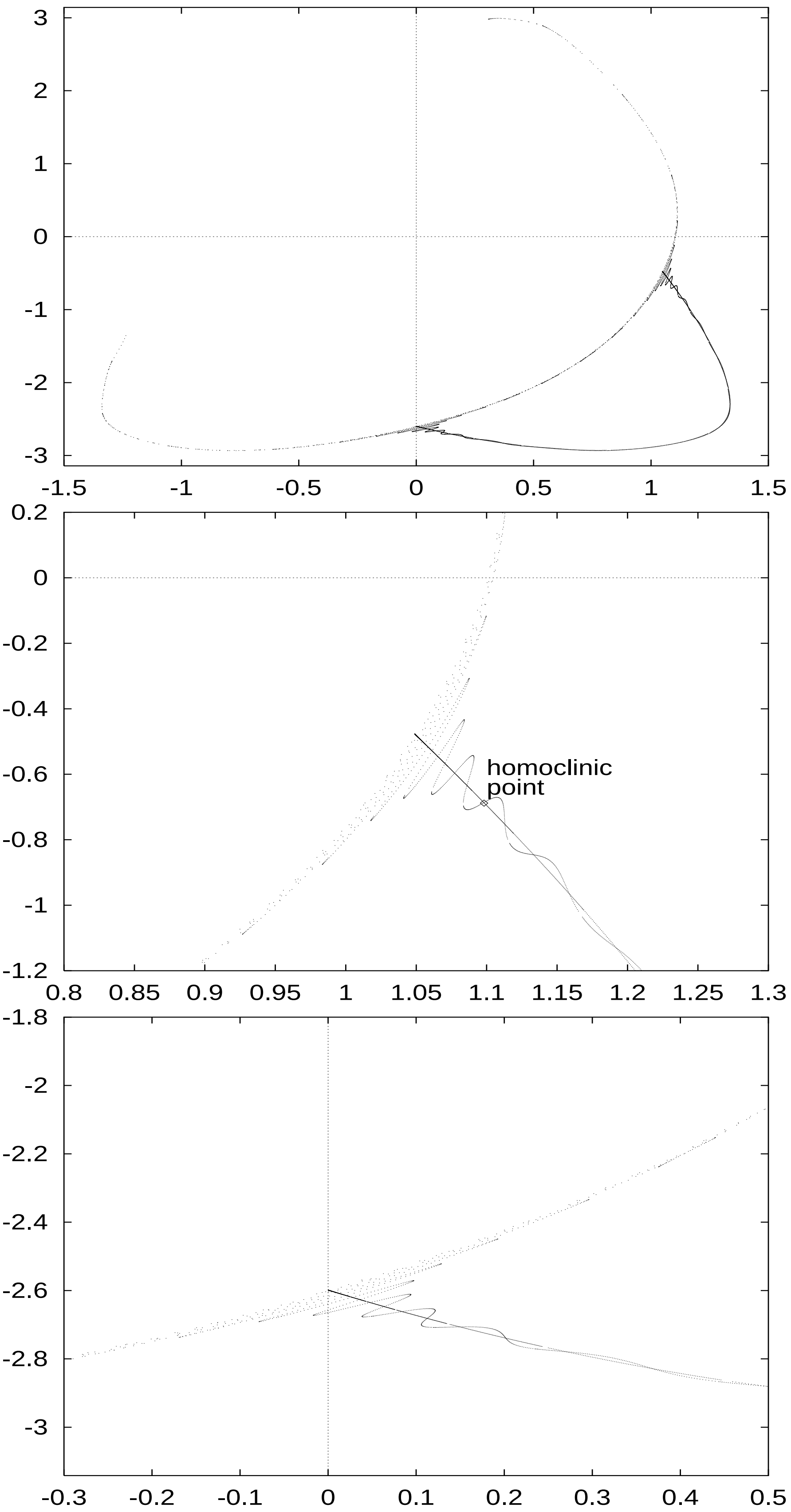

Fig.4.1-4.3 The splitting of W_4^u and W_1^s corresponding to $(\varepsilon = 0.01,\ \mu = 5,\ \nu = 1.16)$**.**

5 Homoclinic invariant

Assume that z_k is a hyperbolic point of period p of the Poincaré map. Then the stable $W_k^s(z_k)$ and the unstable $W_k^u(z_k)$ parts of separatrices are divided by the point z_k into two branches. Denote by $W_{k,\pm}^{u,s}(z_k)$ the branch of $W_k^{u,s}(z_k)$ corresponding to the positive (resp. negative) values of the variable ξ in the expansion (4.2). Consider $W_{k,+}^{u,s}(z_k)$. It is possible to pass to a new variable y by the formula:

$$\xi = e^y.$$

In terms of the new variable the restriction of F on $W_{k,+}^u(z_k)$ looks like a shift

$$y \to y + h, \tag{5.1}$$

where $h = \log(\lambda_1)$.

The same statement is true for the part $W_{k,+}^s(z_k)$ of the stable separatrix. We keep the same expression (5.1) in this case by use of the change

$$\xi = e^{-y}.$$

The variable y is defined uniquely up to an additive constant. The following expression for the tangent vector to $W_{k,+}^{u,s}$ at a point $z \in W_{k,+}^{u,s}$ is invariant with respect to the choice of the parametrization:

$$\vec{e}_k^{u,s}(z) = \frac{d}{dy}\Phi_k^{u,s}(e^{\pm y}). \tag{5.2}$$

We assume that z is a homoclinic point and $z \in W_{k,+}^u(z_k) \cap W_{l,+}^s(z_l)$ (where z_l belongs to the orbit of z_k). Denote by γ the trajectory of point z:

$$\gamma = \{F^n(z) : n \in \mathbf{Z}\}.$$

Then the homoclinic invariant of the homoclinic trajectory γ is defined as (see [5]):

$$w(\gamma) = \Omega_S(\vec{e}_k^u(z), \vec{e}_l^s(z)). \tag{5.3}$$

The right hand part of (5.3) is invariant with respect to F, so the homoclinic invariant is constant on the homoclinic trajectories.

In the present work the homoclinic invariants are calculated for the sets of the system parameters, considered in sections 3 and 4. For this purpose we analize the splitting of the separatrices branches $W^u_{k\ (mod\ p),+}$ and $W^s_{k+1\ (mod\ p),+}$, constructed in the section 4 for each set of the system parameters and find numerically homoclinic points and the tangent vectors (5.2) at these points. The results of the calculations are presented in tables 3a, 3b, 3c, accordingly.

table 3a

index (k,l)	homoclinic point	tangent vector $\vec{e}_k^u$	tangent vector $\vec{e}_l^s$	homoclinic invariant
(1,2)	(0.0396,2.6254)	(-4.9077,13.8745)	(0.9998,1.8892)	-23.1433
(2,3)	(-0.0396,2.6254)	(-0.9998,1.8892)	(4.9077,13.8745)	-23.1433
(3,4)	(-1.0982,-0.6883)	(-1.0002,-3.8893)	(1.1881,-23.2412)	27.8667
(4,1)	(1.0982,-0.6883)	(-1.1881,-23.2412)	(1.0002,-3.8893)	27.8667

table 3b

index (k,l)	homoclinic point	tangent vector $\vec{e}_k^u$	tangent vector $\vec{e}_l^s$	homoclinic invariant
(1,2)	(0.0514,5.5394)	(-1.3642,12.5071)	(0.9999,10.0612)	-26.2313
(2,3)	(-0.0514,5.5394)	(-0.9999,10.0612)	(1.3642,12.5071)	-26.2313
(3,1)	(0.5621,-3.7333)	(-4.1560,-9.3515)	(1.0002,-21.2015)	97.4668

table 3c

index (k,l)	homoclinic point	tangent vector $\vec{e}_k^u$	tangent vector $\vec{e}_l^s$	homoclinic invariant
(1,2)	(0.0419,-3.1735)	(0.0012,0.0093)	(1.0001,-1.6625)	-0.0113
(2,3)	(-0.0419,-3.1735)	(-1.0001,-1.6625)	(-0.0012,0.0093)	-0.0113
(3,1)	(1.1294,-4.1883)	(-0.0036,-0.0051)	(1.0001,-21.7893)	0.0835

The index (k,l) refers to the homoclinic point, belonging to $W^u_{k,+} \cap W^s_{l,+}$.

The coincidence of the values of the homoclinic invariant in some cases follows from a symmetry of the Poincaré map and that of the homoclinic points.

As it shown in [5], the nonnullity of the homoclinic invariant implies the nonintegrability of the dynamic system. Since the property of nonintegrability is invariant with respect to small perturbations [7], the following theorem is true:

Theorem 1 *There are neighborhoods of points* $\vec{a}_1 = (0.01, 5, 1.16)$, $\vec{a}_2 = (0.01, 10, 0.80)$, $\vec{a}_3 = (0.01, 7, 0.16)$ *in the space of parameters* $A = \{\vec{a} = (\varepsilon, \mu, \nu)\}$ *of the double mathematical pendulum such that the system corresponding to these parameters is nonintegrable.*

However the sizes of these neighborhoods are not yet obtained.

Proofs of the nonintegrability the double mathematical pendulum for some values of the parameters are presented in the papers [3], [11]. Note that the method developed in our work can be applied for arbitrary values of the parameters, which is not the case in the mentioned papers [3] and [11].

6 Accuracy of numerical calculations

As it follows from sections 3 and 4, our main numerical calculations deal with integrating of differential equations. They were performed by using of Runge-Kutta methods of 7 and 8 orders with the automatic step control. During the calculation of fixed hyperbolic points of the Poincaré map, the criteria of accuracy were the values $\varepsilon = F(z_0) - z_0$, $\delta_k = ||z_{k+1} - z_k||$ and the condition (4.1). The accuracy of these calculations is of the order 10^{-15} and 10^{-14}, accordingly. If we take into account the other operations, involved into our calculations, the accuracy of the construction of the separatrices and the calculation of the homoclinic invariant can be upper estimated by 10^{-5}. This bound is very rough, however, it is sufficient to claim the nonintegrability of the system.

7 Conclusions

In the present paper a procedure of investigation of a dynamical system depending on parameters is described. The scheme consists of finding of periodic hyperbolic points of Poincaré map corresponding to the dynamic system and calculation of their homoclinic invariants. This procedure is applied to the case of the double mathematical pendulum at some points in the space of parameters. We prove nonintegrability of the dynamical system in some neighborhoods of these points. However the bounds for the neighborhoods are not yet calculated. To obtain these bounds will be the purpose of future studies.

Investigations are performed under supervision of professor V.F.Lazutkin. Doctor M.B.Tabanov, the former advicer of the author, who died in 1996, made an effective contribution in understanding of the problem. The author thanks doctor V.G.Gelfreich for his assistance in the realization of the program for the procedure mentioned above and professor C.Simó for his attention to the work. This work was supported by INTAS grant 93-339-ext, CRDF grant RMI-227, RFFI grant 97-01-00612 and a grant of the State Higher Educational Committe in Russia. The author thanks the University of Barcelona for the hospitality during his stay in Barcelona in july 1998.

References

[1] V.I.Arnold, V.V.Kozlov, A.I.Neishtadt, Encyclopedia of Mathematical Science. Dynamical Systems III, Springer, Berlin, 1988.

[2] V.I.Arnold, Mathematical methods of classical mechanics, Springer, Heidelberg, 1978.

[3] S. Bolotin, P. Negrini, A variational criterion for nonintegrability, Universita delgi Studi di Roma "La Sapirenza", 1996.

[4] A.A. Burov, Nonexistence of an additional integral of the problem of a planar heavy double pendulum, Prikl. Matem. i Mekhan. 50, 168- 171, 1986.

[5] V.G. Gelfreich, V.F. Lazutkin, N.V. Svanidze, A refined formula for the separatrix splitting for the standard map, Physica D 71 (1994) 82-101.

[6] L.D.Landau, E.M.Lifshic, Theoretical physics, vol.1, Nauka, Moscow, 1988 (in russian).

[7] V.F. Lazutkin, KAM Theory and Semiclassical Approximations of Eigenfunctions, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3.Folge, Band 24, A Series of Modern Surveys in Mathematics.

[8] L.V.Kantorovich, V.I.Krylov, Numerical methods in analysis, Phizmatgiz, Moscow, 1962 (in russian).

[9] J. Moser, Stable and random motions in dynamical systems. With special emphasis on celestial mechanics, Princeton, N.Y. Princeton unvi. press, 1973.

[10] H.Poincaré, Les Methodes Nouvelles de la Mechanique Celeste, vols. 1-3, Gauthier-Villars, Paris, 1892; (Dover, New York, 1957), reprint.

[11] P.H. Richter and H.-J. Scholz, Chaos in Classical Mechanics: The Double Pendulum: In: Stochastic Phenomena and Chaotic Behavior in Complex Systems. Schuster, P.(ed.), Berlin, Springer 1984.

[12] M.B. Tabanov, Splitting of separatrices for Birkhoff's billiard under some symmetrical perturbation of an ellipse, Universite Paris 7 - U.F.R de Mathematiques C.N.R.S.