Stability of driven systems with growing gaps, Quantum rings and Wannier ladders

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Abstract

We consider a quantum particle in a periodic structure submitted to a constant external electromotive force. The periodic background is given by a smooth potential plus singular point interactions and has the property that the gaps between its bands are growing with the band index. We prove that the spectrum is pure point–i.e. trajectories of wave packets lie in compact sets in Hilbert space– if the Bloch frequency is non-resonant with the frequency of the system and satisfies a Diophantine type estimate, or if it is resonant. Furthermore it is shown that the KAM method employed in the non-resonant case produces uniform bounds on the growth of energy for driven systems.

1 Introduction

We study stability of the dynamics of one electron in a 1d periodic structure with infinitely many open gaps driven by a constant electromotive force. To be specific we consider two realizations: the Stark-Wannier problem for a periodic background interaction $V(x) = V(x+L)$ defined by the Hamiltonian

$$H_S = -\Delta + V(x) + Fx \qquad \text{on } L^2(\mathbb{R})$$

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and an electron on a conducting ring in the plane threaded by a linearly increasing magnetic flux line $\Phi(t) = Ft$ whose dynamics is defined by

$$H_R(t) = (-i\partial_x - Ft)^2 + V(x) \qquad \text{on } L^2(S^1).$$

The stability of a general time dependent system was addressed for example in [17, 10, 13, 26, 22, 12, 5]. The discussion of stability may be summarized in the question whether a wave packet can get delocalized during its time evolution. To answer this one may study the time behavior of expectations of observables; if the system is periodic in time the spectral properties of the Floquet operator–i.e.: the evolution over one period– can provide precise information on the stability. To mention one example: if the periodically time dependent system is confined and unbounded and the spectrum of its Floquet operator is absolutely continuous then the energy expectation grows in time for any initial state, so the system is unstable.

The special case considered here was intensively studied since Wannier conjectured existence of ladders of eigenvalues; see [19, 25] for background on this story. In [7] it was proven that for smooth background potential $V \in C^2(\mathbb{R})$ –in fact $C^{1+\epsilon}$– the spectrum of H_S is absolutely continuous which leads to unbounded growth of the energy for $H_R(t)$, see [4]. On the other hand in [3], [18] a comb of δ' point interactions was considered. It was shown that this model is physically important, in particular it describes idealized geometric scatterers. It was proven that the spectrum has no absolutely continuous component leaving the possibility of eigenstates, singular continuous spectrum and unbounded energy growth. Furthermore a conjecture on the essential spectrum was made. See also [23] for the geometric scatterer aspect and [24] for a second proof of absence of absolute continuity.

It was argued by Ao [1] that the spectral nature depends on the gap structure of the periodic background. He conjectured that for gap behavior $\Delta E_n = \mathcal{O}(1/n^\alpha)$ one has point spectrum for $\alpha < 0$ at least for "non-resonant" F and continuous spectrum for $\alpha > 0$. For $\alpha = 0$ a phase transition from pure point to continuous spectrum with growing F is expected –see also [9], [8]; furthermore the spectral nature seems to depend also on number theoretical properties of the driving frequency FL. This critical case corresponds to the driven Kronig-Penney model; another realization of constant gaps is the –explicitly solvable– forced harmonic oscillator considered by [17, 20, 11, 10].

Our contribution, here, is to show stability for V a comb of δ' interactions plus a smooth bounded background. We prove that the spectrum of H_S is pure point; H_R is periodic up to a gauge transformation, the Floquet

Hamiltonian of the transformed problem is unitarily equivalent to H_S, its spectrum is also pure point. We are able to prove this in two different settings. Firstly for a large subset of frequencies FL which do not resonate with the frequencies of the background we use a KAM algorithm in order to treat the δ' interaction as perturbation of the decoupled problem where δ' is replaced by a Neumann boundary condition. This algorithm needs as input a matrix which has sufficient off diagonal decay. Because of the singularity of the interaction it is not evident that the Floquet Hamiltonian considered here has such a matrix representation. A detailed spectral analysis is necessary to show that this is the case. Technically the basic observation is that the eigenfunctions do not concentrate at the singularity if the band index grows. A consequence is that the gaps are increasing and the transition matrix has the required properties. This result was announced in [2]. In the second case for the countable set of resonant frequencies we prove the conjecture of [3, 18] concerning the location of the essential spectrum; a general argument based again on the off-diagonal decay of matrix elements allows to conclude that also in this case the spectrum is pure point.

These results strongly suggest that in the models considered here, in fact for $\alpha < 0$ in the Ao language and for reasonable boundedness of the transition matrix, pure pointness of the spectrum should not depend on number theoretical properties of the frequency!

The fact that the spectrum of the Floquet operator is pure point– in the H_R picture– does not imply on general grounds that the energy expectation is bounded in time as the example in [14] shows. We prove here that applicability of the KAM method provides a uniform bound on the energy growth, so this applies here to the non-resonant case and holds true for a subclass of general time dependent Hamiltonians studied in [26, 22] which complements their results.

The organization of the paper is as follows. In section 2 we fix notations, define the problem in detail and provide a regularization necessary for the methods in the following sections to work. Section 3 is devoted to the study of the non-resonant case; the result on pure pointness is Corollary 3.3, the bounds on energy growth are Corollary 3.4 and Theorem 3.5. In section 4 we determine the essential spectrum and prove pure pointness in the resonant case.

2 The problem and its Matrix Representation

We consider the class of potentials

$$V(x) = \sum_{n\in\mathbb{Z}} \beta\delta'(x - nL) + W(x)$$

with $W(x) = W(x+L)$ a differentiable multiplication operator on $L^2(\mathbb{R})$ and δ' defined by the expressions (1) below. We refer to [3] and references therein for background material on this model.

The limit $\beta \to \infty$ represents decoupling of the cells by a Neumann boundary condition. For β large the problem is a perturbation of the decoupled case, but only in the quadratic form sense. We shall show in this chapter that in spite of the singular character of the perturbation the problem can be represented by a matrix operator with polynomial off-diagonal decay. Let us fix

Notations. L, F, β are positive numbers. We shall, however, mostly work with the parameters $\omega := FL, g := 1/\beta$ and employ the symbols $T := \frac{2\pi}{FL}$ for the Bloch period; $D := -i\partial$ where ∂ denotes a partial derivative; $\mathbf{D}M(\mathbf{O}M)$ for the diagonal (off-diagonal) part of a matrix M; χ for the binary code defined by $\chi(True) := 1, \chi(False) := 0$; *cte* for a generic constant, independent of the parameters, which may change from line to line. H^n denotes a Sobolev space of order n. We shall try to avoid to note the dependence of parameters of a quantity if we feel that this is possible while keeping clarity.

The Stark-Wannier Hamiltonian is

$$H_S = D_x^2 + Fx + W(x)$$

defined on

$$\begin{aligned} &D(H_S) = \{\psi \in H^2(\mathbb{R} \setminus (\mathbb{Z}L)), H_S\psi \in L^2(\mathbb{R}), \\ &\psi'(nL+) - \psi'(nL-) = 0, \psi(nL+) - \psi(nL-) = \beta\psi'(nL)\}. \end{aligned} \tag{1}$$

The Hamiltonian for the driven ring is informally :

$$H_R(t) = \nabla^2 + W(x) + (\nabla\delta) \qquad \text{on } L^2(S^1),$$

where $\nabla := D_x - Ft$; we use, however, the time dependent gauge transformation $\exp(-iFtx)$ to transform the propagator into the periodic one of period

T generated by the Hamiltonian

$$\begin{aligned} H(t) &= H(t,\omega,g) := D_x^2 + \frac{\omega}{L}x + W(x) \\ D(H(t)) &:= \{\psi \in H^2((0,L); e^{i\omega t}\psi'(L) - \psi'(0) = 0, \\ &\quad g(e^{i\omega t}\psi(L) - \psi(0)) = -\psi'(0)\}. \end{aligned} \tag{2}$$

Notice that because the domain is t dependent an argument for existence of the propagator is needed. This will be shown by mapping the problem to one whose propagator is known to exist, see remark 2.3 at the end of this section.

We shall henceforth study the point spectrum of the Floquet Hamiltonian

$$K = K(\omega, g) = D_t^{per} + H(t,\omega,g) \tag{3}$$

acting in $L^2((0,T), dt; L^2((0,L), dx))$ on the domain

$$D(K) = \{\psi \in H^1((0,T), D(H(t,\omega,g)), \psi(T,x) = \psi(0,x))\}.$$

An eigenvector ϕ of K with eigenvalue ϵ will provide us with a Bloch-Floquet solution of the Schrödinger equation

$$D_t\psi(t,x) + H(t)\psi(t,x) = 0$$

which is of the form

$$\psi(t,x) = e^{-i\epsilon t}\phi(t,x)$$

with ϕ periodic in t.

A second reason to introduce K is the unitary equivalence of H_S and K, see [4]:

$$U_B H_S = K U_B$$

where U_B is the Bloch transformation

$$\begin{aligned} U_B &: L^2(\mathbb{R}) \to L^2((0,T)\times(0,L), dt\ dx) \\ (U_B\psi)(t,x) &= \frac{1}{\sqrt{T}}\sum_{\gamma\in\mathbb{Z}} e^{i\gamma\omega t}\psi(x+\gamma L). \end{aligned}$$

The matrix representation M of K is constructed as follows: let $\{\psi_n(t)\}_{n\in\mathbb{N}}$ be a periodic orthonormal eigenbasis of $H(t)$: $\psi_n(t+T)=\psi_n(t)$.

$$\{\phi_j\}_{j\in\mathbb{Z}\times\mathbb{N}}, \qquad \phi_j(t,x)=\phi_{j_1,j_2}(t,x):=\frac{e^{i\omega t j_1}}{\sqrt{T}}\psi_{j_2}(t,x)$$

then is a basis of $L^2((0,T),dt;L^2((0,L),dx))$. Using $\langle .,.\rangle$ for the scalar product in x space we define

$$\begin{aligned} M_{jk} &= \langle\langle \phi_j, K\phi_k\rangle\rangle := \int_0^T \langle \phi_j, K\phi_k\rangle(t)\ dt \qquad (4)\\ &= \frac{1}{T}\int_0^T e^{i(j_1-k_1)\omega t}\left((k_1\omega+E_{k_2}(t))\delta_{j_2k_2}+\langle\psi_{j_2}, D_t\psi_{k_2}\rangle(t)\right)\ dt. \end{aligned}$$

In the rest of this section we shall study the properties of the eigenvalues $E_n(t,\omega,g)$ of H and the coupling matrix $\langle\psi_n, D_t\psi_m\rangle$.

For $\psi\in H^1((0,L)),\phi\in D(H(t))$ we find by integration by parts:

$$\begin{aligned} \langle\psi,H\phi\rangle &= \langle\psi',\phi'\rangle+\langle\psi,(\frac{\omega}{L}x+W)\phi\rangle \\ &\quad +g\overline{(e^{i\omega t}\psi(L)-\psi(0))}(e^{i\omega t}\phi(L)-\phi(0)). \end{aligned}$$

So denoting the Neumann decoupled operator ($g=0$) by

$$H_0=D_x^2+\frac{\omega}{L}x+W(x) \qquad \text{with } \psi'(0)=\psi'(L)=0$$

we have the representation

$$H=H_0+g|f(t,\omega)\rangle\langle f(t,\omega)|$$

where

$$f(t,\omega)=e^{-i\omega t}\delta_L-\delta_0.$$

f is in $H^{-1}((0,L))$ so we are in the framework of generalized rank-one perturbations; we shall use the results of [27]. $H(t,\omega,g)$ is an analytic family with constant form domain $H^1(0,L)$ for $(t,\omega,g)\in S_{\alpha_t}\times S_{\alpha_\omega}\times\mathbb{C}$ for some $\alpha_t,\alpha_\omega>0$ where $S_\alpha:=\{z\in\mathbb{C};|\mathrm{Im}z|<\alpha\}$. For the resolvent $R(z)=(H-z)^{-1}$ it holds:

$$R(z)-R_0(z)=-\frac{g}{1+gG(z)}|R_0(z)f\rangle\langle R_0(\bar{z})f| \qquad (5)$$

with $G(z)=G(z,t,\omega):=\langle f,R_0(z)f\rangle$.

In the sequel we make statements for g small enough. This could be circumvented by the use of an adiabatic technique. We shall not do so as in section 3 the smallness of g will be essential anyhow. We obtain

Theorem 2.1 *For g sufficiently small, ω in a given interval $[\omega_-, \omega_+] \subset (0, \infty)$, $T = \frac{2\pi}{\omega}$, $t \in [0, T]$ the operator $H(t, \omega, g)$ as defined in equation (2) has simple discrete spectrum. For its eigenvalues $E_n = E_n(t, \omega, g)$ it holds uniformly in t, ω, g, n:*

(i) $\frac{E_{n+1} - E_n}{n} \geq cte > 0$,

(ii) $0 \leq \partial_\omega E_n \leq cte < 1$;

furthermore there exists a basis $\{\psi_n\}$ of eigenfunctions of H with $\psi_n(t+T) = \psi_n(t)$, $\psi_n \in C^\infty([0, T] \times [\omega_-, \omega_+] \times [0, g_{max}])$ such that in the C^∞ topology and uniformly in t, ω, n:

(iii) $\langle \psi_n, D_t \psi_m \rangle = \frac{O(g)}{|n^2 - m^2|}$ *for* $n \neq m$,

(iv) $\langle \psi_n, D_t \psi_n \rangle = O(g/n)$.

Proof. The behavior for large n has to be controlled. We compare H to the Neumann Laplacian $-\Delta^N$ on $H^2(0, L)$ with boundary conditions $\psi'(0) = \psi'(L) = 0$, whose eigenvalues are $(\pi n/L)^2$. This is done in two steps: first we compare $H(t, \omega, 0)$ –which is actually time independent– to $-\Delta^N$ using regular perturbation theory; secondly the difference $H(g) - H(g = 0)$ is treated using formula (5).

By a Wronskian argument the eigenvalues of $H(t, \omega, 0)$ are simple and for n large enough it holds:

$$|E_n(t, \omega, 0) - (\frac{\pi n}{L})^2| \leq cte\|\frac{\omega}{L}x + W\|.$$

The reason why the transition matrix decays and eigenvalues stay nearby upon switching on g is contained in the following auxiliary result.

Lemma 2.2 *For $\phi \in L^2((0, L))$ and the eigenprojections P_n of H it holds in the C^∞ topology and uniformly in n:*

$$|(P_n(t, \omega, g)\phi)(L)| + |(P_n(t, \omega, g)\phi)(0)| \leq cte\|\phi\|$$

Proof.(Of the Lemma) By Riesz's formula we have with a circle Γ_n of length $|\Gamma_n|$ centered at $(\pi n/L)^2$

$$P_n = -\frac{1}{2\pi i} \oint_{\Gamma_n} R(z)\, dz.$$

Denote be a, b indices which take the values 0 and L. In order to prove the estimate on $P_n\phi(a) = \langle\delta_a, P_n\phi\rangle$ by Krein's formula–equation (5)– it is sufficient to show

$$\sup_{z\in\Gamma_n}(|\langle\delta_a, R_0(z)\phi\rangle| + |\langle\delta_a, R_0(z)\delta_b\rangle|) = O(\frac{1}{|\Gamma_n|}).$$

To do this we use regular perturbation theory and the fact that the "gaps" of the Neumann operator are growing. Denote $d_n := \frac{\pi^2}{L^2}((n)^2 - (n-1)^2)$ and choose n_0 such that $\sup_\omega \|\frac{\omega}{L}x + W\| < \frac{d_{n_0}}{2}$; for $n > n_0$ choose a suitable number M and and $|\Gamma_n| := \frac{d_n}{M}$. Then it holds for the resolvent R^N of the Neumann Laplacian

$$\|(\frac{\omega}{L}x + W)R^N(z)\| \leq \frac{cte}{n} \leq 1 \text{ and } \|R^N(z)\| = O(\frac{1}{n})$$

uniformly for $z \in \Gamma_n$ and

$$\begin{aligned} R_0 &= R^N(1 + (\frac{\omega}{L}x + W)R^N)^{-1} \\ &= R^N - R^N(1 + (\frac{\omega}{L}x + W)R^N)^{-1}(\frac{\omega}{L}x + W)R^N. \end{aligned}$$

So the question is reduced to the explicit calculation of $\|\delta_a R^N\|$ and $\langle\delta_a, R^N\delta_b\rangle$. $R^N(z)$ is given by its kernel $R^N(z)\phi(x) = \int R^N(x, y; z)\phi(y)\, dy$

$$R^N(x, y; z) := -\frac{1}{\sqrt{z}\sin\sqrt{z}L}\cos(\sqrt{z}(x \wedge y))\cos(\sqrt{z}((y \vee x) - L))$$

with the notation $x \wedge y$ $(x \vee y)$for the minimum (maximum) of x and y. One finds for example

$$\|\delta_0 R^N(z)\| \leq \|\frac{1}{\sqrt{z}\sin\sqrt{z}L}\cos\sqrt{z}(y - L)\| = O(\frac{1}{n})$$

and similarly $\|\delta_L R^N(z)\| = O(\frac{1}{n}), |\langle\delta_a, R^N(z)\delta_b\rangle| = O(\frac{1}{n})$ for a, b in $\{0, L\}$ uniformly for $z \in \Gamma_n$. These estimates are preserved upon differentiation with respect to t, ω, g. □

To continue with the proof of the theorem for the eigenvalues we show

$$E_n(t, \omega, g) = (\frac{n\pi}{L})^2 + \frac{\omega}{2} + \langle W\rangle + \frac{4g}{L}(1 - (-1)^n\cos\omega t) + O(1/n) \quad (6)$$

in the C^∞ topology. $\langle W\rangle$ denotes the mean value $(1/L)\int_0^L W$. Indeed by regular perturbation theory with the notation $V := \frac{\omega x}{L} + W + g|f\rangle\langle f|$ it is a corollary of the previous lemma that

$$
\begin{aligned}
& E_n - (\frac{n\pi}{L})^2 - tr(P_n^N V) \\
& = -\frac{1}{2\pi i} tr(\oint_{\Gamma_n} (z - (\frac{n\pi}{L})^2) R(z) V R^N(z) V R^N(z)\, dz) = O(1/n)
\end{aligned}
$$

With $P_n^N = \frac{2}{L}|\cos\frac{n\pi}{L}x\rangle\langle\cos\frac{n\pi}{L}x| \quad (n \geq 1)$ the explicit term of the approximation follows. From this we infer the assertions concerning the eigenvalues for n large enough. For the lowest finitely many n we employ continuity of $\partial_\omega E_n$ and compactness of $[\omega_-, \omega_+]$ to deduce (ii).

The eigenfunctions are now constructed as

$$
\psi_n(t,\omega,g) := \frac{P_n(t,\omega,g)\psi_n^0(\omega)}{\|P_n(t,\omega,g)\psi_n^0(\omega)\|}
$$

for any time independent choice of eigenfunctions ψ_n^0 of H_0. We differentiate $HP = EP$ in the quadratic form sense to get $P_m\partial_t P_n = \frac{P_m \partial_t H P_n}{E_n - E_m}$. It follows from Lemma 2.2 for the off-diagonal part:

$$
\langle\psi_m, D_t\psi_n\rangle = \frac{\langle\psi_m, D_t H\psi_n\rangle}{E_n - E_m} = \frac{O(g)}{|n^2 - m^2|}.
$$

For the diagonal a calculation using Lemma 2.2 yields

$$
\langle\psi_n, D_t\psi_n\rangle = \frac{1}{\|\psi_n\|^2}\frac{1}{2}\langle\psi_n^0, [P_n, D_t P_n]\psi_n^0\rangle = O(\frac{g}{n}).
$$

$\square$

Remark 2.3 *The existence of the propagator $U(t,s)$ of $H(t)$ is a corollary of the preceding theorem: denote by $J(t)$ the unitary between $l^2(\mathbb{N})$ and $L^2((0,L))$ which maps the n'th canonical base vector to $\psi_n(t)$. Then $J^{-1}(t)(D_t + H(t))J(t) = D_t + h(t)$ where the matrix operator h is defined by*

$$
h_{nm} = E_n\delta_{nm} + \langle\psi_n, D_t\psi_m\rangle.
$$

h is analytic with constant domain so its propagator $u(t,s)$ exists. U is then given by

$$
U(t,s) = J(t)u(t,s)J^{-1}(s).
$$

3 Stability for non-resonant frequencies

In this section we shall employ the KAM algorithm to diagonalize the matrix M of the Floquet operator K. M is considered as a perturbation of its diagonal $\mathbf{D}M$. For generic values of the frequency ω the eigenvalues of $\mathbf{D}M$ form a dense subset of the real line [16]. We shall show in Corollary 3.3 that for a large set of "non-resonant" ω the spectrum of K is pure point, in Corollary 3.4 and in Theorem 3.5 that the energy of the system stays bounded.

In order to measure the decay of matrix elements consider the following Banach algebras (see [15]): let $r, \delta \geq 0, \Omega \subset (0,\infty), \langle x \rangle := (1+x^2)^{1/2}$,

$$\mathcal{B}(\Omega, r, \delta) := \{\ \omega \mapsto M(\omega) \in \mathcal{B}(l^2(\mathbb{Z}\times\mathbb{N})); \infty > \|M\|_{\Omega,r,\delta} :=$$
$$\sum_{d\in\mathbb{Z}^2} e^{|d|r}\langle |d|\rangle^\delta \sup_{\omega\neq\omega', i-j=d} \left(|M_{ij}(\omega)| + |\frac{M_{ij}(\omega)-M_{ij}(\omega')}{\omega-\omega'}| \right) \Bigg\}.$$

The result of the KAM algorithm we need here is:

Theorem 3.1 *Let $\tau \in (0,\infty)$ be large enough, $\Omega = [\omega_-, \omega_+] \subset (0,\infty)$, $M = M(\omega) = M^*(\omega)$ a family of matrix operators in $l^2(\mathbb{Z}\times\mathbb{N})$ such that*

$\|\mathbf{O}M\|_{\Omega,0,\tau} < \infty$, $M_{jj} = \omega j_1 + e_{j_2}(\omega) \quad (j = (j_1, j_2) \in \mathbb{Z}\times\mathbb{N})$ *with*

$\inf_{\omega,n} \frac{e_{n+1}-e_n}{n} \geq cte > 0$,

$|||e||| := \sup_{\omega\neq\omega',n,m} \left| \frac{(e_n-e_m)(\omega)-(e_n-e_m)(\omega')}{\omega-\omega'} \right| < 1.$

Then there is a $\delta \in (0,\tau)$ such that for γ small enough and $\|\mathbf{O}M\|_{\Omega,0,\tau} \leq \gamma^2$ there is a set of good frequencies $\Omega_\infty \subset \Omega$ with measure

$$|\Omega \setminus \Omega_\infty| = O(\gamma)$$

and a unitary family $U_\infty(\omega)$ with $\|U_\infty\|_{\Omega,0,\delta} < \infty$ such that

$$U_\infty M U_\infty^{-1}(\omega) = M_\infty(\omega), \quad \mathbf{O}M_\infty = 0 \qquad (\omega \in \Omega_\infty).$$

Remarks 3.2

(i) In particular $M(\omega)$ has a basis of eigenfunctions f_j which decay polynomially: $f_j(k) = (U_\infty^{-1})_{kj} = O(|k-j|^{-\delta})$;

(ii) Actually $\delta = \tau - cte$ so δ can be chosen arbitrarily large if τ is arbitrarily large.

Outline of the Proof. The KAM method in its quantum guise first introduced by [6] is by now quite standard. We shall, however, give only a descriptive proof and refer to [15] and references therein for analytic details.

The idea is to successively diminish the size of small off-diagonal elements by unitary transformations. We use the function $\chi(True) := 1, \chi(False) := 0$. Denote the matrices

$$\mathbf{D}M_{ij} := M_{ij}\chi(i = j), \quad \mathbf{O}M := M - \mathbf{D}M,$$
$$\mathbf{D}_n M_{ij} := M_{ij}\chi(|i-j| = n), \quad \mathbf{B}_n M := \sum_{j=0}^{n} \mathbf{D}_n M,$$

and define recursively

$$\begin{aligned} &M_1 := \mathbf{B}_1 M, \quad U_0 := \mathbb{I}, \\ &W_{n;ij} := \frac{M_{n;ij}}{M_{n;ii} - M_{n;jj}}\chi(i \neq j), \quad U_n := e^{W_n}U_{n-1}, \\ &M_{n+1} := U_n(\mathbf{B}_{n+1}M)U_n^{-1} = e^{W_n}M_n e^{-W_n} + U_n(\mathbf{D}_{n+1}M)U_n^{-1}. \end{aligned}$$

The idea of this is based on the identity

$$ad_{W_n}(\mathbf{D}M_n) := [W_n, \mathbf{D}M_n] = -\mathbf{O}M_n$$

which by the Lie Schwinger formula

$$e^W M e^{-W} = \sum_{k=0}^{\infty} \frac{ad_W^k(M)}{k!}$$

leads to

$$M_{n+1} = \mathbf{D}M_n + \sum_{k=1}^{\infty} \frac{k}{(k+1)!} ad_{W_n}^k(\mathbf{O}M_n) + U_n \mathbf{D}_{n+1} M U_n^{-1}.$$

W_n is to be estimated by $\mathbf{O}M_n$ so the second term in $\mathbf{O}M_{n+1}$ will be quadratic in $\|\mathbf{O}M_n\|$. It is in this estimate where one looses the resonant ω giving rise to small divisors. For each step one proves that for $\sigma > 1$ and γ_n small enough there is an open set $\Omega_{n+1} \subset \Omega_n$ with $|\Omega_n \setminus \Omega_{n+1}| \leq cte\ \frac{\gamma_n}{1-|||e|||}$ such that for $r_{n+1} < r_n$ it holds:

$$\|W_n\|_{\Omega_{n+1}, r_{n+1}, \delta} \leq \frac{cte}{\gamma_n^2 (r_n - r_{n+1})^{2\sigma+1}} \|\mathbf{O}M\|_{\Omega_n, r_n, \delta}\ ; \tag{7}$$

here $|||e|||$ estimates the space part of $\mathbf{D}M_n$. The bad frequencies are controlled by a diophantine estimate

$$\Omega_n \setminus \Omega_{n+1} = \bigcup_{k,m,n\in\mathbb{N},n>m} \{\omega; |\omega k + e_m - e_n| < \gamma(k+n-m)^{-\sigma}\}.$$

The growing gap property is then used to show that the contributions to the measure are summable. Estimating now $\|ad_W^k M\| \leq cte^k \|W\|^k \|M\|$ it follows for $r_{n+1} < r_n$ with the shorthand $\|\cdot\|_n := \|\cdot\|_{\Omega_n, r_n, \delta}$

$$\begin{aligned} &\|\mathbf{O}M_{n+1}\|_{n+1} \leq \\ &cte\|W_n\|_{n+1} e^{cte\|W_n\|_{n+1}} \|\mathbf{O}M_n\|_n + e^{2\sum_j \|W_j\|_{n+1}} \|\mathbf{D}_{n+1}M\|_n, \end{aligned} \tag{8}$$

$$\begin{aligned} &\|\mathbf{D}M_{n+1} - \mathbf{D}M_n\|_{n+1} \leq \\ &cte\|W_n\|_{n+1} e^{cte\|W_n\|_{n+1}} \|\mathbf{O}M_n\|_n + e^{2\sum_j \|W_j\|_{n+1}} \|\mathbf{D}_{n+1}M\|_n, \end{aligned}$$

$$\|U_n^{\pm}\|_{n+1} \leq e^{\|W_n\|_{n+1}} \|U_{n-1}^{\pm}\|_n.$$

The choice $\gamma_n = O(1/n^\mu), r_n = O(1/n^{\nu-1})$ for suitable μ, ν in estimate (8) then leads to a quadratic estimate for $\|W_n\|_{n+1}$:

$$\|W_n\|_{n+1} \leq c_1 \|W_n\|_{n+1}^2 e^{cte\|W_n\|_{n+1}} + c_2 \langle n \rangle^\beta$$

where the constant c_2 is proportional to $\|\mathbf{O}M\|_{\Omega,0,\tau}$, $\beta = 2\mu + \nu(2\sigma+1) - \tau$. If $\|\mathbf{O}M\|_{\Omega,0,\tau}$ is small enough and τ large enough this implies that $\|W_n\|$ is summable, and that $\mathbf{D}M_n$ and U_n are convergent. $\square$

As a consequence of this and the analysis in section 2 we obtain that the spectrum is pure point:

Corollary 3.3 *For the Floquet Hamiltonian defined in equation (3) it holds:*

$$K(\omega, g) \text{ has a basis of eigenvectors in } L^2((0,T), dt; L^2((0,L), dx))$$

provided g is small enough and $\omega \in \Omega_\infty \subset [\omega_-, \omega_+]$ the set constructed in Theorem (3.1) with measure $|[\omega_-, \omega_+] \setminus \Omega_\infty| = O(\sqrt{g})$.

Proof. Identifying $L^2((0,L),dx)$ with $l^2(\mathbb{N})$ via the eigenbasis $\{\psi_n\}$ of H constructed in Theorem 2.1 we find that K is unitarily equivalent to

$$D_t + E_n(t)\delta_{nm} + \langle \psi_n, D_t\psi_m\rangle$$

on $L^2((0,T),dt;l^2(\mathbb{N}))$, which is of the form

$$D_t + h_0(\omega,g) + gV(t,\omega,g)$$

where V is C^∞ bounded and $O(1)$ in g, and $h_0(\omega,g)_{ij} := \langle E_j\rangle\delta_{ij}$. Applying a version of the superadiabatic regularization as in [21],[15, Thm.3.6] we get that K is unitarily equivalent to an operator of the same form whose fluctuating part has the property that $(m^2-n^2)^\tau V_{mn}$ is C^∞ bounded in l^2 uniformly in the parameters for any $\tau > 0$. Furthermore, by Theorem 2 the diagonal elements still satisfy $\frac{e_{n+1}-e_n}{n} \geq cte > 0$ and $0 \leq \partial_\omega e_n < 1$ for g small enough.

Finally going to the Fourier representation K turns out to be unitarily equivalent to a matrix M on $l^2(\mathbb{Z}\times\mathbb{N})$ which has the properties required in Theorem 3.1. $\square$

By pure pointness of $\sigma(K)$ every trajectory $\{t\in\mathbb{R}; U(t)\psi\}$ generated by H is a precompact set. It follows that

$$\lim_{r\to\infty}\sup_{t\in\mathbb{R}}\|\chi(H(t)>r)U(t)\psi\| = 0.$$

This does, however, not imply that the energy expectation $|\langle\psi, H(t)\psi\rangle|$ is bounded, as the example given in [14] shows.

We shall show now that boundedness of the energy is in fact always ensured in cases where the KAM algorithm used above applies.

Corollary 3.4 *For g small enough there exists a set of frequencies $\Omega_\infty \subset [\omega_-,\omega_+] \subset (0,\infty)$ with $|[\omega_-,\omega_+]\setminus\Omega_\infty| = O(\sqrt{g})$ such that for $\omega\in\Omega_\infty$ and for the propagator U of $H(t,\omega,g)$ it holds*

(1) $U(t) = U_p(t)e^{-iGt}U_p^{-1}(0)$

where G commutes and is relatively bounded with respect to $H(0)$ and U_p is T periodic and C^∞ as a bounded operator in $L^2((0,L),dx)$;

(2) $|\langle U(t)\psi, H(t)U(t)\psi\rangle| \leq cte$

for $\psi\in Q(H(0))$, uniformly for $t\in\mathbb{R}$.

Proof. Application of the KAM algorithm to the matrix M of K gave $U_\infty M U_\infty^{-1} = M_\infty$. We transform back to the space of (t,x) functions using the basis $\{\phi_j\}$ whose space part is $\{\psi_{j_2}\}$ as constructed in Theorem 2.1. Denote

$$\mathcal{T}_\infty := \sum_{j,k\in\mathbb{Z}\times\mathbb{N}} (U_\infty)_{jk} |\phi_j\rangle\langle\phi_k|.$$

By construction U_∞ is a Toeplitz matrix in the indices corresponding to the Fourier variable, i.e. $(U_\infty)_{jk}$ is of the form $(U_\infty)_{j_1-k_1,j_2,k_2}$. Consequently $\mathcal{T}_\infty$ is fibered, i.e.: $\mathcal{T}_\infty\psi(t,x) = T_\infty(t)\psi(t,x)$ for $T_\infty(t)$ unitary, periodic and C^∞ bounded in $L^2((0,L),dx)$; $(M_\infty)_{jj} = \omega j_1 + e^\infty_{j_2}$ with $e^\infty_{j_2} - \langle E_{j_2}\rangle = O(g)$ uniformly in n,ω. The reader may consult [15] for a more detailed discussion of this point. By Theorem 3.1 this results in

$$\mathcal{T}_\infty K \mathcal{T}_\infty^{-1} = \sum (M_\infty)_{jj}|\phi_j\rangle\langle\phi_j| = D_t + H_\infty(t)$$

where H_∞ is defined by $(D_t + H_\infty(t))\psi_m(t) = e^\infty_m \psi_m(t)$. Denote

$$U_A(t) := \sum_m |\psi_m(t)\rangle\langle\psi_m(0)|$$

then the relation

$$(D_t + H_\infty(t))U_A(t) = U_A(t)\sum e^\infty_m P_m(0)$$

holds on $D(H(0))$ so with the definition $G := \sum e^\infty_m P_m(0)$ we obtain

$$(U_A^{-1}T_\infty)H(T_\infty^{-1}U_A) + \big((U_A^{-1}T_\infty)D_t(T_\infty^{-1}U_A)\big) = G.$$

This formula implies the asserted form for the propagator with the definition $U_p(t) := T_\infty^{-1}U_A(t)$ and the fact that U_A is C^∞ bounded by Theorem 2.1, furthermore it shows that U_p preserves domains: $U_p(t)D(H(0)) \subset D(H(t))$.

For ψ in the form domain of $H(0)$ it holds with $\varphi := U_p^{-1}(0)\psi$

$$\begin{aligned}
&\langle U(t)\psi, H(t)U(t)\psi\rangle = -\langle U(t)\psi, D_t U(t)\psi\rangle \\
&= \langle e^{-iGt}\varphi, (G - (U_p^{-1}(D_t U_p))(t)e^{-iGt}\varphi\rangle \\
&= \langle\varphi, G\varphi\rangle - \langle\varphi, e^{iGt}(U_p^{-1}(D_t U_p))(t)e^{-iGt}\varphi\rangle
\end{aligned}$$

which is bounded uniformly in time as $(U_p^{-1}(D_t U_p))$ is periodic. □

By the same method we complement now the results of [26] which were much extended in [22, 5]. These authors estimate the propagator by time dependent methods to discuss stability of the energy expectations. The spectral method used here is better suited to provide bounds valid on an infinite time scale. The differentiability properties on the potential could be relaxed. However, we do not make an effort, here, to do so.

Theorem 3.5 *Let $T, g > 0$, W a T periodic C^∞ function with values in the bounded operators on a Hilbert space. Consider $H_0 = \sum_n E_n P_n$ with growing gaps: $\frac{E_{n+1}-E_n}{n^\alpha} \geq cte > 0$ for an $\alpha > 0$.*

Then it holds for the propagator U of $H(t) := H_0 + gW(t)$, $\psi \in Q(H_0)$:

$$|\langle U(t)\psi, H(t)U(t)\psi\rangle| \leq cte$$

provided g is small enough and $\omega \in \Omega_\infty \subset [\omega_-, \omega_+] \subset (0, \infty)$, the set constructed as in Theorem (3.1) with measure $|[\omega_-, \omega_+] \setminus \Omega_\infty| = O(\sqrt{g})$.

Proof. By [15] the KAM algorithm is applicable, so we can proceed as in the previous corollary and find U_p, G with $U(t) = U_p(t)e^{-iGt}U_p^{-1}(0)$. □

4 Stability for resonant frequencies

In this section we shall show that for resonant frequencies $\omega \in \mathbb{Q}(\frac{L}{\pi})^2$ the spectrum of K is still pure point.

It is equivalent to show that the time T map $U(T)$ of H has pure point spectrum. We go to the matrix representation of section 2. Let $\{\psi_n\}$ be the basis found in Theorem 2.1, $\{e_n\}$ the standard basis of l^2 and $J = J(t, \omega, g)$ the unitary operator $J := \sum_0^\infty |e_n\rangle\langle\psi_n|$. We have

$$J(t)(D_t + H(t))J^{-1}(t) = D_t + h(t)$$
$$h_{nm} := E_n\delta_{nm} + \langle\psi_n, D_t\psi_m\rangle.$$

An approximation of E_n was worked out in equation (6). Let $g_n(t) = g_n(t+T) := \int_0^t \frac{4g}{L}(-1)^{n+1}\cos\omega s \; ds$, G the gauge transformation defined by $G_{nm}(t) = \exp(ig_n(t))\delta_{nm}$. Now

$$G(D_t + h)G^{-1} - \left(D_t + ((\frac{n\pi}{L})^2 + \frac{\omega}{2} + \langle W\rangle + \frac{4g}{L})\delta_{nm}\right)$$
$$= O(\frac{1}{n})\delta_{nm} + e^{i(g_n - g_m)}\langle\psi_n, D_t\psi_m\rangle;$$

by Theorem 2.1 the matrix function on the right hand side is a C^∞ function in the Hilbert Schmidt norm $\|a\|_{HS} := (\sum_{nm} |a_{nm}|^2)^{1/2}$ and a forteriori in the compact operators on $l^2(\mathbb{N})$. We now make use of the argument of Enß and Veselić to conclude:

Theorem 4.1 *For $\omega \in \mathbb{Q}(\frac{L}{\pi})^2, g$ small enough, $T = \frac{2\pi}{\omega}$ it holds*

(i) $\sigma_{ess}(U(T,\omega,g)) = \left\{\exp(-i((\frac{\pi n}{L})^2 + \frac{\omega}{2} + \langle W\rangle + \frac{4g}{L})T); n \in \mathbb{N}\right\}$,

(ii) $L^2((0,L))$ *has a basis of eigenvectors of* $U(T,\omega,g)$.

Proof. Denote $\tilde{h} := GhG^{-1} + G(D_t G^{-1})$, $\tilde{h}_0 := ((\frac{n\pi}{L})^2 + \frac{\omega}{2} + \langle W\rangle + \frac{4g}{L})\chi(n = m)$ and by $\tilde{U}, \tilde{U}_0$ their propagators. The spectrum

$$\sigma(\tilde{U}_0(T,\omega,g)) = \left\{\exp(-i((\frac{\pi n}{L})^2 + \frac{\omega}{2} + \langle W\rangle + \frac{4g}{L})T), n \in \mathbb{N}\right\}$$

is a discrete set. Furthermore

$$\int_s^t \tilde{U}_0^{-1}(\tilde{h} - \tilde{h}_0)\tilde{U}_0$$

is compact for every $s, t \in \mathbb{R}$. By Theorem 5.2 of [17] $\tilde{U}(T) - \tilde{U}_0(T)$ is compact. So $\sigma_{ess}(\tilde{U}(T)) = \sigma_{ess}(\tilde{U}_0(T))$, which cannot contain continuous spectrum so $\sigma(\tilde{U}(T))$ is pure point. $G(T) = \mathbb{I}$, from the unitary equivalence

$$U(T) = J^{-1}(T)\tilde{U}(T)J(T).$$

we conclude that the spectrum of $U(T)$ is pure point $\square$

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References

[1] Ping Ao. Absence of localization in energy space of a Bloch electron driven by a constant electric force. *Phys. Rev. B*, 41:3998–4001, 1990.

[2] J. Asch, P. Duclos, and P. Exner. Stark-Wannier hamiltonians with pure point spectrum. In M. Demuth and B.-W. , Schultze, editors, *Partial Differential Operators and Mathematical Physics*. Birkhäuser, 1997.

[3] J.E. Avron, P. Exner, and Y. Last. Periodic Schrödinger operators with large gaps and Wannier-Stark ladders. *Phys. Rev. Lett.*, 72:896–899, 1994.

[4] J.E. Avron and J. Nemirovsky. Quasienergies, stark hamiltonians and energy growth for driven

quantum rings. *Phys. Rev. Lett.*, 68:2212–2215, 1992.

[5] J. Barbaroux and A. Joye. Expectation values of observables in time-dependent quantum mechanics. *J.Stat.Phys., to appear*, 1998.

[6] J. Bellissard. Stability and instability in quantum mechanics. In S. Albeverio and P. Blanchard, editors, *Trends and Developments in the Eighties*. Word Scientific, 1985.

[7] F. Bentosela, R. Carmona, P. Duclos, B. Simon, B. Souillard, and R. Weder. Schrödinger operators with an electric field and random or deterministic potentials. *Commun. Math. Phys.*, 88:387–397, 1983.

[8] F. Bentosela, V. Grecchi, and F. Zironi. Approximate ladder of resonances in a semiinfinite crystal. *J.Phys. C*, 15:7119–7131, 1982.

[9] A.M. Berezhkovski and A.A. Ovchinnikov. True width of electron levels in a crystal in a static electric field. *Sov. Phys.: Solid State*, 18:1908–1911, 1976.

[10] L. Bunimovich, H.R. Jauslin, J.L. Lebowitz, A. Pellegrinotti, and P. Nielaba. Diffusive energy growth in classical and quantum driven oscillators. *J.Stat.Phys.*, 62:793–817, 1991.

[11] M. Combescure. The quantum stability problem for time-periodic perturbations of the harmonic oscillator. *Ann. Inst. H. Poincaré*, 47:62–82, 451–454, 1987.

[12] S. De Bièvre and G. Forni. Transport properties of kicked and quasi-periodic hamiltonians. *Comm. Math. Phys.*, 1998, to appear.

[13] C.R. de Oliveira. Some remarks concerning stability for nonstationary quantum systems. *J.Stat.Phys.*, 78:1055–1066, 1995.

[14] R. del Rio, S. Jitomirskaya, Y. Last, and B. Simon. Operators with singular continuous spectrum, iv. Hausdorff dimensions, rank one perturbations, and localization. *J. d'Analyse Math.*, 69:153–200, 1996.

[15] P. Duclos and P. Št'ovíček. Floquet hamiltonians with pure point spectrum. *Commun. Math. Phys.*, 177:327–374, 1996.

[16] P. Duclos, P. Št'ovíček, and M. Vittot. Pertubation of an Eigen-value from a dense Point Spectrum: a general Floquet hamiltonian. *Preprint CPT*, (P.3559), 1997.

[17] V. Enss and K. Veselić. Bound states and propagating states for time-dependent hamiltonians. *Ann.Inst.H.Poincaré, sect. A*, 39:159–191, 1983.

[18] P. Exner. The absence of the absolutely continuous spectrum for δ' Wannier-Stark ladders. *J. Math. Phys.*, 36:4561–4570, 1995.

[19] V. Grecchi, , and A. Sacchetti. Lifetime of the Wannier-Stark resonances and perturbation theory. *Comm. Math. Phys.*, 185:359–378, 1993.

[20] G. Hagedorn, M. Loss, and J. Slawny. Non stochasticity of time-dependent quadratic hamiltonians and

spectra of canonical transformations. *J.Phys.A.*, 19:521–531, 1986.

[21] J. Howland. Floquet operator with singular spectrum i,ii. *Ann.Inst.H.Poincaré, sect. A*, 49:309–334, 1989.

[22] A. Joye. Upper bounds for the energy expectation in time-dependent quantum mechanics. *J. Stat. Phys.*, 85:575–606, 1996.

[23] A. Kiselev. Some examples in one-dimensional geometric scattering on manifolds. *J. Math.Anal.Appl.*, 212:263 ff., 1997.

[24] M. Maioli and A. Sacchetti. Absence of the absolutely continuous spectrum for Stark-Bloch operators with strongly singular periodic potentials, erratum. *J. Phys. A*, 31:1115–1119, 1998.

[25] E.E. Mendez and G. Bastard. Wannier Stark resonances. *Physics Today*, 46(6):34–42, 1993.

[26] G. Nenciu. Adiabatic theory: Stability of systems with increasing gaps. *Ann. Inst.H.P.*, 67(4):411–424, 1997.

[27] B. Simon. Spectral analysis of rank one perturbations and applications. In J. Feldman, R. Froese, and L. Rosen, editors, *CRM Lecture Notes*, volume 8. AMS, 1995.