NON-GAUSSIAN SURFACE PINNED BY A WEAK POTENTIAL

J.-D. DEUSCHEL AND YVAN VELENIK

Abstract. We consider a model of a two-dimensional interface of the SOS type, with finite-range, even, strictly convex, twice continuously differentiable interactions. We prove that, under an arbitrarily weak potential favouring zero-height, the surface has finite mean square heights. We consider the cases of both square well and δ potentials. These results extend previous results for the case of nearest-neighbours Gaussian interactions in [DMRR] and [BB]. We also obtain estimates on the tail of the height distribution implying, for example, existence of exponential moments. In the case of the δ potential, we prove a spectral gap estimate for linear functionals. We finally prove exponential decay of the two-point function (1) for strong δ-pinning and the above interactions, and (2) for arbitrarily weak δ-pinning, but with finite-range Gaussian interactions.

1. Introduction

Even though the understanding of phase separation and related interfacial phenomena for two-dimensional systems such as the Ising model, has greatly improved recently, the situation for three-dimensional systems remains quite unsatisfactory. For example, even in the three-dimensional Ising model, several basic questions remain open: Existence of a roughening transition, proof that the wetting transition occurs at a non-trivial value of the boundary magnetic field (or proof of the contrary), or even instability of the $(1,1,1)$-interface. To gain some insights in these problems, it is very useful to consider simpler SOS-type, effective models for interfaces. In these models, the interface is modelized as a function $\underline{h}$ from a subset of $\mathbb{Z}^d$ to $\mathbb{R}$ (or $\mathbb{Z}$, but we restrict our attention to the former case) where $h_i \equiv \underline{h}(i)$ represents the height of the surface above, or below, the site i; the energy associated to this surface is specified by some function of these heights, $H(\underline{h}) = \sum_{i,j} \Psi_{ij}(h_i - h_j)$. Unfortunately, even these much simplified models remain rather difficult to handle, and most of the results which have been obtained are restricted to the Harmonic case, where $\Psi_{ij}(h_i - h_j) = \frac{1}{2}(h_i - h_j)^2$. It is therefore valuable to find ways to extend such results to a larger class of models, by providing arguments which are less sensitive to the particular features of the underlying interaction.

The aim of the present work is to extend to a large class of interactions results about the pinning of an interface by a weak potential. We emphasize that these results do not follow from perturbation around the Gaussian case. Let $\Lambda_L = [-L, L]^2 \cap \mathbb{Z}^2$. It is well known that the mean square height at any fixed site $i \in \Lambda_L$ w.r.t. the Gaussian measure with 0-boundary condition has a logarithmic divergence in the thermodynamic limit. However, it was shown in [DMRR] that the addition of an arbitrarily weak self-potential favouring height 0 would localize the surface, in the sense that this mean square height remains bounded uniformly in L. We show that this holds for a large class of finite-range non-Gaussian interactions.

As a byproduct of our technique, we also obtain an improved version of a result of [BB] proving the existence of a massgap for the model with nearest-neighbours, Gaussian

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interactions, in the presence of an arbitrarily weak pinning potential. Even though we are not able to extend their result to non-Gaussian interactions, we show that it holds in the following situations: (1) For a large class of finite-range non-Gaussian interactions but with a sufficiently strong δ-pinning, and (2) for a large class of finite-range Gaussian interactions with arbitrarily weak δ-pinning.

We finally prove the following new result (as far as we know) on the tail of the distribution, valid for the same class of finite-range non-Gaussian interactions submitted either to square-well or δ potentials: the probability that the height of the interface above some site i is larger than T (large) is bounded from above by $\exp(-\mathcal{O}(T^2/\log T))$; this implies of course existence of all moments, including exponential ones. If the interaction has bounded second derivatives, then we also prove the corresponding lower bound.

We restrict our attention to dimension two since it is the relevant case to describe an interface in a three-dimensional medium. It is also the most interesting one as far as pinning is concerned. Indeed, in dimensions greater than two, the situation is completely different: The mean square of the height of the interface is already finite *without* a pinning potential. The behaviour of the two-point function is also different: without pinning potential, it has a power-law decay. However, the addition of such a potential would make this decay exponential.

In Section 2, we define the models and state the main results of this paper. Proofs of these statements are given in Sections 3 and 4. Our main estimate, Proposition 5.1, is proved in Section 5. Some technical estimates are given in the appendix.

2. Models and results

Let r be some strictly positive integer, the *range* of the interaction. The interaction between sites i and j, $\Psi_{i,j}(h_j - h_i)$ is supposed to satisfy the following conditions:

- **Translation invariance:** $\Psi_{i,j} = \Psi_{0,j-i} \equiv \Psi_{j-i}$.
- **Finite range:** $\Psi_k \equiv 0$ if $\|k\|_1 > r$.
- **Symmetry:** $\Psi_k = \Psi_{-k}$ and $\Psi_k(x) = \Psi_k(-x)$.
- **Smoothness:** Ψ_k is twice continuously differentiable.
- **Irreducibility:** Ψ_k is convex, i.e. $\Psi_k''(x) \geqslant 0$. Moreover, there exists $c > 0$ such that the random walk on $\mathbb{Z}^2$ with transition rates $P_c(0,j)$ given by 1 if $\Psi_k''(h) \geqslant c\ \forall h$, and 0 otherwise, is irreducible.

All these conditions are natural, and standard in this kind of problem, except for the last one, which happens to be necessary to be able to use standard result about the random walk appearing in the random walk representation (described below) of a related Gaussian model obtained using Brascamp-Lieb inequality.

Remark: The hypothese on translation invariance could be removed easily. We only left it for notational convenience.

Let $b \in \mathbb{R}$ and let $\Lambda \Subset \mathbb{Z}^2$. The Gibbs measure with b-b.c. in Λ is the probability measure on $\mathbb{R}^{\mathbb{Z}^2}$ given by

$$\mu_\Lambda^b(\,\mathrm{d}\underline{h}) \doteqdot \frac{1}{Z_\Lambda^b} \exp\{-\sum_{\langle ij \rangle_r \cap \Lambda \neq \varnothing} \Psi_{j-i}(h_i - h_j)\} \prod_{i\in\Lambda} \mathrm{d}h_i \prod_{i \notin \Lambda} \delta_b(\,\mathrm{d}h_i)\,, \tag{1}$$

where $\langle ij\rangle_r$ is any pair of distinct sites i and j such that $\|j-i\|_\infty \leqslant r$ and δ_b is the point-mass at b. Expectation value and variance with respect to μ^b_Λ are denoted by $\langle\,\cdot\,\rangle^b_\Lambda$ and $\mathrm{var}^b_\Lambda(\,\cdot\,)$.

Let ε and a be two strictly positive real numbers; the potential $V:\mathbb{R}\to\mathbb{R}$ is defined by

$$V(h) \doteqdot -\,\varepsilon\chi(|h|\leqslant a)\,, \tag{2}$$

where $\chi(\mathcal{A})$ is the indicator function of the event $\mathcal{A}$. The Gibbs measure with 0-b.c. on Λ and potential V is the probability measure defined by

$$\mu^V_\Lambda(\,\mathrm{d}\underline{h}) \doteqdot \frac{1}{Z^V_\Lambda}\exp\{-\sum_{i\in\Lambda}V(h_i)\}\;\mu^0_\Lambda(\,\mathrm{d}\underline{h})\,. \tag{3}$$

Expectation value with respect to this measure is written $\langle\,\cdot\,\rangle^V_\Lambda$.

Let $\Lambda_L \doteqdot [-L,L]^2\cap\mathbb{Z}^2$. Our first result shows that, for any ε and a, the mean square height of the field is finite, uniformly in L. This generalizes the corresponding result of [DMRR] valid for nearest-neighbors Gaussian interactions.

Theorem 2.1. *There exists a constant $C_1 = C_1(a(e^\varepsilon-1),c,r)<\infty$ such that, for any $i\in\Lambda_L$ and $\forall L$,*

$$\langle h_i^2\rangle^V_{\Lambda_L}\leqslant C_1\,.$$

Moreover, if $a(e^\varepsilon-1)\sqrt{c}$ is small, then there exists $C_2=C_2(r)>0$ such that $C_1\leqslant 4a^2+\frac{C_2}{c}|\log(a(e^\varepsilon-1)\sqrt{c})|$.

Note that the fact that C_1 depends on a only through the product $a\sqrt{c}$ is natural, since otherwise we could improve the result by rescaling the field $\underline{h}$.

In fact, using the same techniques, it is possible to obtain a much stronger statement about pinning of the field, namely existence of exponential moments. Indeed, this is a consequence of the following estimates on the tail of the height distribution.

Theorem 2.2. *There exist $C_3=C_3(a(e^\varepsilon-1),c,r)$ and $T_0=T_0(a(e^\varepsilon-1),c,r)a$ such that, for all $T>T_0$ with $T\gg a$, and all L,*

$$\mu^V_{\Lambda_L}(h_i\geqslant T)\leqslant e^{-C_3\,T^2/\log T}\,.$$

Moreover, if $\Psi''_k(h)\leqslant\frac{1}{c}$, for all k and h, then there exists a constant $C_4=C_4(a(e^\varepsilon-1),c,r)<\infty$ such that, for all $T>1$ and all L,

$$\mu^V_{\Lambda_L}(h_i\geqslant t)\geqslant e^{-C_4\,T^2/\log T}\,.$$

In [BB], a statement analogous to Theorem 2.1 was proved, together with the exponential decay of the 2-point function, for a slightly different measure to which we will refer as the δ-pinning. Their measure corresponds to

$$\mu^J_{\Lambda_L}(\,\mathrm{d}\underline{h}) = \frac{1}{Z^J_{\Lambda_L}}\exp[-\sum_{\langle ij\rangle_r\subseteq\Lambda_L}\Psi(h_i-h_j)-\sum_{\substack{\langle ij\rangle_r\\ i\in\Lambda_L,\,j\notin\Lambda_L}}\Psi(h_i)]\prod_{i\in\Lambda_L}(\,\mathrm{d}h_i+e^J\delta_0(\,\mathrm{d}h_i))\,, \tag{4}$$

where J is some real parameter. (In fact, they considered the Gaussian case, with periodic boundary conditions and nearest-neighbors interaction). Expectation value and variance with respect to $\mu^J_{\Lambda_L}$ are written $\langle\,\cdot\,\rangle^J_{\Lambda_L}$ and $\mathrm{var}^J_{\Lambda_L}(\,\cdot\,)$.

The measure in (4) can be seen as the limit of the measure $\mu^V_{\Lambda_L}$, when $\varepsilon \to \infty$ with $2(e^\varepsilon - 1)a = e^J$ (using, for example, Lebesgue's Theorem). Since the bounds in Theorems 2.1 and 2.2 only depend on the product $(e^\varepsilon - 1)a$, they readily give the following[1]

Corollary 2.1. *There exists a constant $C_1' = C_1'(J, c, r) < \infty$ such that, for any $i \in \Lambda_L$ and for all L,*

$$\langle h_i^2 \rangle^J_{\Lambda_L} \leqslant C_1' .$$

Moreover, if $e^J\sqrt{c}$ is small, then there exists $C_2' = C_2'(r) > 0$ such that $C_1' \leqslant \frac{C_2'}{c}|\log(e^J\sqrt{c})|$.

Corollary 2.2. *There exist $C_3' = C_3'(J, c, r)$ and $T_0 = T_0(J, c, r)$ such that, for all $T > T_0$ and all L,*

$$\mu^J_{\Lambda_L}(h_i \geqslant T) \leqslant e^{-C_3'\, T^2/\log T} .$$

Moreover, if $\Psi_k''(h) \leqslant \frac{1}{c}$, for all k and h, then there exists a constant $C_4' = C_4'(J, c, r) < \infty$ such that, for all $T > 1$ and all L,

$$\mu^J_{\Lambda_L}(h_i \geqslant t) \geqslant e^{-C_4'\, T^2/\log T} .$$

It is in fact possible to obtain bounds on more general quantities. Indeed, we have the following

Theorem 2.3. *There exists a constant $C_5 = C_5(J, c, r) < \infty$ such that, for any $\alpha : \Lambda_L \to \mathbb{R}$ and all L,*

$$\mathrm{var}^J_{\Lambda_L}((\alpha, h)) \leqslant C_5(\alpha, \alpha) ,$$

where $(\alpha, h) = \sum_{i\in\Lambda} \alpha(i) h_i$.

Remark: This result implies in particular that, for any $\Lambda' \subseteq \Lambda_L$,

$$\mathrm{var}^J_{\Lambda_L}\Big(\frac{1}{|\Lambda'|}\sum_{i\in\Lambda'} h_i\Big) = \mathcal{O}(|\Lambda'|^{-1}) . \tag{5}$$

It does not seem possible with our techniques to prove the same kind of result for arbitrary functions. However, it is still possible to get the following

Corollary 2.3. *Let $C_{\mathrm{b}}^{1,\mathrm{o}}$ be the set of functions $F \in C^1(\mathbb{R}^{\Lambda_L})$ such that $\|f_i\|_\infty \equiv \|\frac{\partial}{\partial h_i} F\|_\infty < \infty$, $i \in \Lambda_L$, and F is odd in each coordinate, i.e. $F(T_i h) = -F(h)$ where $(T_i h)_j = (1 - 2\delta_{i,j})h_j$. Then there exists a constant $C_5' = C_5'(J, c, r) < \infty$ such that*

$$\mathrm{var}^J_{\Lambda_L}(F) \leqslant C_5' \sum_{i\in\Lambda_L} \|f_i\|^2_\infty .$$

The results we are able to obtain about exponential decay of the two-point function are less satisfactory. Since it is not clear how the technique used in this paper (and taken from [BB]) should be used to prove exponential decay of the 2-point function in the non-Gaussian case (the corresponding random walk representation being much more complicated, see below), we have to restrict our attention to Gaussian interactions. The result we obtain in this case reads

[1] Notice that the result on pinning in [BB] cannot be deduced from the corresponding statement in [DMRR] since the bound given in this last work diverges in the limit $\varepsilon \to \infty$, $2(e^\varepsilon - 1)a = e^J$.

Theorem 2.4. *Suppose that, in addition to the above hypotheses, the interaction is Gaussian, i.e. $\Psi_k(x) = c_k x^2$. Then there exists $C_6 = C_6(J,c,r) > 0$ such that, for all L,*

$$\langle h_i h_j \rangle_{\Lambda_L}^J \leqslant \tfrac{1}{C_6} e^{-C_6 \|j-i\|_1} .$$

Moreover, if $e^J\sqrt{c}$ is small, then there exists $C_7 = C_7(r) > 0$ such that $C_6 \geqslant C_7 |\log(e^J\sqrt{c})|$.

This improves the result of [BB] since it holds for 0-b.c. and finite-range interactions. In the non-Gaussian case, we are only able to prove exponential decay in the strong pinning regime, i.e. when the parameter J is sufficiently large. In this case, the following can be proved,

Theorem 2.5. *There exist J_0 and $C_8 = C_8(J,c,r) > 0$ such that, for all $J \geqslant J_0$, and all L,*

$$\langle h_i h_j \rangle_{\Lambda_L}^J \leqslant \tfrac{1}{C_8} e^{-C_8 \|j-i\|_1} .$$

The basic strategy to obtain these exponential decay results is taken from [BB]. Our main contribution is Proposition 5.1 which replaces an estimate in [BB] the proof of which, based on reflection positivity and an entropy estimate, limited their analysis to nearest-neighbors Gaussian interactions. Our method, more robust, is inspired in part by their entropy estimate.

A basic tool for our analysis is the following *random walk representation* of two-point functions (see [DGI]). For any $\Lambda \Subset \mathbb{Z}^2$ and any $i,j \in \Lambda$ (it is possible that $i=j$), the following holds

$$\langle h_i h_j \rangle_\Lambda^0 = \mathbb{E}_i^{\Lambda,0}[\int_0^{\tau_\Lambda} \chi(\eta_s = j)\, \mathrm{d}s] , \tag{6}$$

where η is a random walk on $\mathbb{Z}^2$ starting at i, η_s its position at time s and $\tau_\Lambda = \inf\{s \geqslant 0 : \eta_s \notin \Lambda\}$. Expectation of an event $\mathcal{A}$ depending only on η is given by

$$\mathbb{E}_i^{\Lambda,0}[\mathcal{A}] = \langle \mathbb{E}_{i,\cdot}[\mathcal{A}] \rangle_\Lambda^0 , \tag{7}$$

with $\mathbb{E}_{i,\underline{h}}[\,\cdot\,]$ denoting joint expectation w.r.t. the symmetric diffusion $\underline{h}(s)$ and the random walk in $\mathbb{Z}^2$, starting at i, with jump-rate at time s given by

$$p_{\underline{h}(s)}(i,j) = 2\Psi''_{j-i}(h_j(s) - h_i(s)) . \tag{8}$$

Observe that in the Gaussian case $\Psi''_k = c_k$ is independent of $\underline{h}$ and therefore $\mathbb{E}_i^{\Lambda,0}[\mathcal{A}] \equiv \mathbb{E}_i[\mathcal{A}]$, the expectation w.r.t. the random walk in the plane starting at i, with jump-rate $p(i,j) = 2c_{j-i}$.

We also use Brascamp-Lieb inequality in the following formulation. Let us introduce the following measure,

$$\mu_\Lambda^{0,t}(\,\mathrm{d}\underline{h}) = \frac{1}{Z_\Lambda^{0,t}} e^{t(\alpha,h)} \mu_\Lambda^0(\,\mathrm{d}\underline{h}) . \tag{9}$$

Expectation value and variance w.r.t. $\mu_{\Lambda_L}^{0,t}$ are written $\langle\,\cdot\,\rangle_{\Lambda_L}^{0,t}$ and $\mathrm{var}_{\Lambda_L}^{0,t}$. Then for any $\alpha : \Lambda \to \mathbb{R}$; then

$$\mathrm{var}_\Lambda^{0,t}(\sum_{i\in\Lambda} (\alpha,h)) \leqslant \frac{1}{c} \mathrm{var}_\Lambda^{0,\mathrm{G}}(\sum_{i\in\Lambda} (\alpha,h)) , \tag{10}$$

where $\mathrm{var}_{\Lambda}^{0,\mathrm{G}}$ is the variance w.r.t. the Gaussian measure with 0-b.c. in Λ, obtained by setting $\Psi_k(x) = x^2/2$ if $P_c(0,k) = 1$ and 0 otherwise (see beginning of the section).

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3. Mean square height of the pinned field and tail estimates

This section is devoted to the proof of Theorems 2.1 and 2.2.

3.1. Mean square.

We prove now Theorem 2.1.

Expectation value with respect to $\mu_{\Lambda_L}^V$ has the following convenient representation, close to the one used in [BI] and [BB] in the case of the δ-pinning,

$$\begin{aligned}\langle\,\cdot\,\rangle_{\Lambda_L}^V &= \frac{1}{Z_{\Lambda_L}^V}\int \mathrm{d}\underline{h}\,\cdot\, e^{-\sum_{\langle ij\rangle_r\cap\Lambda\neq\varnothing}\Psi_{j-i}(h_i-h_j)} \prod_{j\in\Lambda_L}\Big(1+(e^{\varepsilon}-1)\chi(|h_j|\leqslant a)\Big)\\ &= \sum_{A\subseteq\Lambda_L}(e^{\varepsilon}-1)^{|A|}\frac{Z_{\Lambda_L}(A)}{Z_{\Lambda_L}^V}\,\langle\,\cdot\,|\,|h_j|\leqslant a,\,\forall j\in A\rangle_{\Lambda_L}^0\,,\end{aligned} \tag{11}$$

where $Z_{\Lambda_L}(A) \doteqdot \int \mathrm{d}\underline{h}\; e^{-\sum_{\langle ij\rangle_r\cap\Lambda\neq\varnothing}\Psi_{j-i}(h_i-h_j)}\prod_{j\in A}\chi(|h_j|\leqslant a)$.

An upper bound on the mean square height of the field is easily obtained using (11). Indeed, we can write

$$\langle h_i^2\rangle_{\Lambda_L}^V = \sum_{A\subseteq\Lambda_L}(e^{\varepsilon}-1)^{|A|}\frac{Z_{\Lambda_L}(A)}{Z_{\Lambda_L}^V}\,\langle h_i^2\,|\,|h_j|\leqslant a,\,\forall j\in A\rangle_{\Lambda_L}^0\,. \tag{12}$$

Using Lemma 6.3 and (10), we get

$$\langle h_i^2\,|\,|h_j|\leqslant a,\,\forall j\in A\rangle_{\Lambda_L}^0 \leqslant 4a^2 + 4\langle h_i^2\rangle_{A^{\mathrm{c}}}^0 \leqslant 4a^2 + \frac{4}{c}\langle h_i^2\rangle_{A^{\mathrm{c}}}^{0,\mathrm{G}}\,, \tag{13}$$

where $A^{\mathrm{c}} \doteqdot \Lambda_L\setminus A$.

Observe that the random-walk representation of Section 2 gives

$$\langle h_i^2\rangle_{A^{\mathrm{c}}}^{0,\mathrm{G}} = \mathbb{E}_i\Big[\int_0^\infty \chi(\eta_s=i)\chi(T_A>s)\,\mathrm{d}s\Big]\,, \tag{14}$$

where $\mathbb{E}_i[\,\cdot\,]$ denotes expectation with respect to the random walk starting at the site i, η_s is the position of the RW at time s, and $T_A \doteqdot \inf_{s\geqslant 0}\{\eta_s\in A\}$. This last expression can be easily bounded using a well-known result about symmetric, irreducible random walks (see e.g. **P**12.3 in [S]); we obtain

$$\langle h_i^2\rangle_{A^{\mathrm{c}}}^{0,\mathrm{G}} \leqslant \frac{\widetilde{C}}{c}\log\,\mathrm{d}(i,A)\,, \tag{15}$$

for some absolute constant $\widetilde{C}$. Let $R_{\min}$ be the smallest value of the diameter of sets B for which Proposition 5.1 applies. Since the range of the random-walk is r-connected, we can use our main estimate, Proposition 5.1, which shows that there exists $K>0$ such that

($B_R(i)$ is the ball with radius R and center i)

$$\begin{aligned}\langle h_i^2\rangle_{\Lambda_L}^V &\leqslant 4a^2+\frac{C}{c}\log R_{\min}+\sum_{R\geqslant R_{\min}}\sum_{\substack{A\subseteq\Lambda_L\\ A\cap B_R(i)=\varnothing\\ A\cap B_{R+1}(i)\neq\varnothing}}\frac{C}{c}\log R\\ &\leqslant 4a^2+\frac{C}{c}\log R_{\min}+\sum_{R\geqslant R_{\min}} e^{-K\,R^2}\,\frac{C}{c}\log R\,. \end{aligned}\tag{16}$$

This completes the proof of Theorem 2.1; the estimate on C_3 follows by taking the optimal $R_{\min}$ above.

3.2. **Tail estimate.** We prove now Theorem 2.2. This proof is close to the previous one. Let us first prove the upper bound. Using the representation (11), we can write

$$\begin{aligned}\langle\chi(h_i>T)\rangle_{\Lambda_L}^V &= \sum_{A\subseteq\Lambda_L}\nu(A)\,\langle\chi(h_i>T)\,\big|\,|h_j|\leqslant a,\,\forall j\in A\rangle_{\Lambda_L}^0\\ &=\sum_{R\geqslant 1}\sum_{\substack{A\subseteq\Lambda_L\\ A\cap B_R(i)=\varnothing\\ A\cap B_{R+1}(i)\neq\varnothing}}\nu(A)\langle\chi(h_i>T)\,\big|\,|h_j|\leqslant a,\,\forall j\in A\rangle_{\Lambda_L}^0\,,\end{aligned}\tag{17}$$

where $\nu(A)=(e^{\varepsilon}-1)^{|A|}Z_{\Lambda_L}(A)/Z_{\Lambda_L}^V$. Lemma 6.2 gives

$$\langle\chi(h_i>T)\,\big|\,|h_j|\leqslant a,\,\forall j\in A\rangle_{\Lambda_L}^0\leqslant\langle\chi(h_i>T-a)\rangle_{A^c}^0\,.\tag{18}$$

Now this probability is easily evaluated: There exists $\underline{C}>0$ such that

$$\langle\chi(h_i>T-a)\rangle_{A^c}^0\leqslant\exp(-\underline{C}\,T^2/\log R)\,.\tag{19}$$

Indeed, this follows from Chebyshev's inequality, Brascamp-Lieb inequality (10) and the variance estimate (15).

Let $R_{\min}$ be large enough so that we can apply our main estimate to sets B with $\mathrm{diam}B>R_{\min}$. We get

$$\langle\chi(h_i>T)\rangle_{\Lambda_L}^V\leqslant e^{-\mathcal{O}(T^2)}+\sum_{R\geqslant R_{\min}}e^{-\underline{K}R^2-\underline{C}\frac{T^2}{\log R}}\,.\tag{20}$$

We now have to find the asymptotic behaviour in T of this sum. Observe that the function $KR^2+\underline{C}\frac{T^2}{\log R}$ is convex, with a unique minimum at R_0 solution of

$$R_0\log R_0=\sqrt{\underline{C}/2K}\;T\,.\tag{21}$$

We cannot solve this equation, however we can easily find a lower bound on R_0:

$$R_0>\sqrt{\underline{C}/2K}\;\frac{T}{\log T}\equiv\overline{R}\,.\tag{22}$$

Observe that

$$1\geqslant\frac{\overline{R}}{R_0}\geqslant 1-\mathcal{O}(\frac{\log\log T}{\log T})\,.\tag{23}$$

The required upper bound is obtained by splitting the sum in the following way:

$$\sum_{R\geqslant 1}e^{-\underline{K}R^2-\underline{C}\frac{T^2}{\log R}}=\sum_{R=1}^{T}e^{-\underline{K}R^2-\underline{C}\frac{T^2}{\log R}}+\sum_{R>T}e^{-\underline{K}R^2-\underline{C}\frac{T^2}{\log R}}\,.\tag{24}$$

The exponential in the first sum is maximum when $R = R_0$. Therefore,

$$\sum_{R=1}^{T} e^{-\underline{K}R^2 - \underline{C}\frac{T^2}{\log R}} \leqslant T\, e^{-\underline{K}R_0^2 - \underline{C}\frac{T^2}{\log R_0}} \leqslant T\, e^{-\underline{C}\,\frac{T^2}{\log T}(1+\mathcal{O}(\frac{\log\log T}{\log T}))}. \tag{25}$$

The other part of the sum is easily taken care of by using the bound

$$e^{-\underline{K}R^2 - \underline{C}\frac{T^2}{\log R}} \leqslant e^{-\underline{K}R^2}, \tag{26}$$

and estimating the corresponding sum. This finally proves that

$$\sum_{R\geqslant 1} e^{-\underline{K}R^2 - \underline{C}\frac{T^2}{\log R}} \leqslant e^{-\underline{C}\,\frac{T^2}{\log T}(1+\mathcal{O}(\frac{\log\log T}{\log T})}. \tag{27}$$

The proof of the lower bound is very similar. The main change is that we have to use some kind of reverse Chebyshev's inequality to bound $\langle\chi(h_i > T+a)\rangle^0_{A^c}$. This can be done by using the following well-known inequality (see [DS] for example),

$$\log\frac{\mu^0_{A^c}(h_i > T+a)}{\mu^{0,\alpha}_{A^c}(h_i > T+a)} \geqslant -\frac{H(\mu^{0,\alpha}_{A^c}\,|\,\mu^0_{A^c}) + e^{-1}}{\mu^{0,\alpha}_{A^c}(h_i > T+a)}, \tag{28}$$

where $H(\mu\,|\,\nu)$ is the relative entropy of μ w.r.t. ν, and

$$\mu^{0,\alpha}_{A^c}(\,\mathrm{d}\underline{h}) = \frac{1}{Z^{0,\alpha}_{A^c}} e^{\alpha h_i} \mu^0_{A^c}(\,\mathrm{d}\underline{h}). \tag{29}$$

Differentiating $\langle h_i\rangle^{0,\alpha}_{A^c}$ and using the reverse Brascamp-Lieb inequality [DGI] to bound the resulting variance in terms of the corresponding Gaussian quantity, we easily get

$$\langle h_i\rangle^{0,\alpha}_{A^c} \geqslant c\,\alpha\,\mathrm{var}^{\mathrm{G}}_{A^c}(h_i) \geqslant Cc\,\alpha\,\log R, \tag{30}$$

for some $C > 0$; the last inequality follows from well-known result on symmetric, irreducible random walks, as above. This implies that

$$H(\mu^{0,\alpha}_{A^c}\,|\,\mu^0_{A^c}) \leqslant \alpha\langle h_i\rangle^{0,\alpha}_{A^c} \leqslant Cc\alpha^2\log R. \tag{31}$$

We also have

$$\mu^{0,\alpha}_{A^c}(h_i > T+a) \geqslant 1 - e^{\alpha(T+a)}e^{-Cc\alpha^2\log R} = 1 - e^{-\alpha(\alpha Cc\log R - T - a)}. \tag{32}$$

Choosing $\alpha = 2(T+a)/Cc\log R$, this yields

$$\mu^0_{A^c}(h_i > T+a) \geqslant (1 - e^{-\frac{2(T+a)^2}{Cc\log R}})\exp\{-\frac{4(T+a)^2}{Cc\log R}\Big/(1 - e^{-\frac{2(T+a)^2}{Cc\log R}})\}. \tag{33}$$

To obtain the desired lower bound, it suffices to restrict the sum over R in (17) to the single term $R = \overline{R}$, apply Lemma 6.2 to replace $\langle\chi(h_i > T)\,\big|\,|h_j| \leqslant a,\,\forall j\in A\rangle^0_{\Lambda_L}$ by $\langle\chi(h_i > T+a)\rangle^0_{A^c}$, and finally use (33).

4. Results for the δ-pinning

In this section, we prove Theorems 2.3, 2.4 and 2.5, and Corollary 2.3.

4.1. **Estimate of variances.** We first prove Theorem 2.3 and its corollary. The proof is quite similar to those given in the previous section. We introduce the symmetric operator

$$\mathfrak{K}_N(i,j) = \mathbb{E}_i\,[\int_0^{\tau_{B_N(i)}} \chi(\eta_s = j)\,\mathrm{d}s]\,, \tag{34}$$

where $B_N(i) = \{j \in \mathbb{Z}^2 : \|j-i\|_1 < N\}$ and $\tau_{B_N(i)} = \inf\{s \geqslant 0 : \eta_s \notin B_N(i)\}$. From an easy adaptation of (1.21) in [L], we know that there exists a constant C such that

$$\sup_i \sum_j \mathfrak{K}_N(i,j) = \sup_i \mathbb{E}_i\,[\tau_{B_N(i)}] \leqslant CN^2\,. \tag{35}$$

Therefore, using (10) and our main estimate,

$$\begin{aligned}
\mathrm{var}^J_{\Lambda_L}(\sum_i \alpha(i)h_i) &= \sum_{A\subseteq\Lambda_L} e^{J|A|}\frac{Z^0_{A^{\mathrm{c}}}}{Z^J_{\Lambda_L}}\mathrm{var}^0_{A^{\mathrm{c}}}(\sum_i \alpha(i)h_i) \\
&\leqslant \frac{1}{c}\sum_{A\subseteq\Lambda_L} e^{J|A|}\frac{Z^0_{A^{\mathrm{c}}}}{Z^J_{\Lambda_L}}\mathrm{var}^{0,\mathrm{G}}_{A^{\mathrm{c}}}(\sum_i \alpha(i)h_i) \\
&\leqslant \frac{1}{c}\sum_{i\in\Lambda_L}|\alpha(i)|\mathbb{E}_i\,[\int_0^{\tau_{B_M(i)}}|\alpha(\eta_s)|\,\mathrm{d}s] \\
&\qquad + \frac{1}{c}\sum_{i\in\Lambda_L}|\alpha(i)|\sum_{N\geqslant M}\mathbb{E}_i\,[\int_{\tau_{B_N(i)}}^{\tau_{B_{N+1}(i)}}|\alpha(\eta_s)|\,\mathrm{d}s]e^{-KN} \\
&\leqslant \frac{1}{c}(|\alpha|,\mathfrak{K}_M|\alpha|) + \frac{1}{c}\sum_{N\geqslant M}(|\alpha|,\mathfrak{K}_{N+1}|\alpha|)e^{-KN} \\
&\leqslant \overline{C}\sum_{i\in\Lambda_L}\alpha(i)^2\,.
\end{aligned} \tag{36}$$

provided we choose M large enough. This proves Theorem 2.3.

Let us prove now Corollary 2.3. We have, similarly as above,

$$\begin{aligned}
\mathrm{var}^J_{\Lambda_L}(F) &= \sum_{A\subseteq\Lambda_L}\nu(A)\langle F^2\rangle^0_{A^{\mathrm{c}}} - \Big\{\sum_{A\subseteq\Lambda_L}\nu(A)\langle F\rangle^0_{A^{\mathrm{c}}}\Big\}^2 \\
&= \sum_{A\subseteq\Lambda_L}\nu(A)\mathrm{var}^0_{A^{\mathrm{c}}}(F) + \sum_{A\subseteq\Lambda_L}\nu(A)\big(\langle F\rangle^0_{A^{\mathrm{c}}}\big)^2 - \Big\{\sum_{A\subseteq\Lambda_L}\nu(A)\langle F\rangle^0_{A^{\mathrm{c}}}\Big\}^2.
\end{aligned} \tag{37}$$

Since F is odd in each coordinate, we know that

$$\langle F\rangle^0_{A^{\mathrm{c}}} = 0 \qquad \forall A \subseteq \Lambda_L\,. \tag{38}$$

Also, by a version of Brascamp-Lieb inequality proved in [DGI],

$$\mathrm{var}^0_{A^{\mathrm{c}}}(F) \leqslant \frac{1}{c}\mathrm{var}^{0,\mathrm{G}}_{A^{\mathrm{c}}}(\widehat{F})\,, \tag{39}$$

where $\widehat{F}(h) \equiv \sum_i \|f_i\|_\infty h_i$. Now the result follows from Theorem 2.3.

4.2. **Mass generation.** The proofs of the last two theorems follow closely the approach of [BB]; they are based on the representation (11), which yields the following expression for the 2-point function (valid for any interactions),

$$\begin{aligned}\langle h_i h_j\rangle^J_{\Lambda_L} &= \sum_{A\subseteq\Lambda_L} e^{J|A|}\frac{Z^0_{A^c}}{Z^J_{\Lambda_L}}\,\langle h_i h_j\rangle^0_{A^c} \\ &= \sum_{A\subseteq\Lambda_L} e^{J|A|}\frac{Z^0_{A^c}}{Z^J_{\Lambda_L}}\,\mathbb{E}^A_i[\int_0^\infty \chi(\eta_s=j)\chi(T_A>s)\,\mathrm{d}s]\,, \end{aligned}\tag{40}$$

where $\mathbb{E}^A_i[\,\cdot\,]$ denotes expectation value w.r.t. the random walk in random environment described at the end of Section 2. Let us first prove Theorem 2.4. In this case, the expectation with respect to the random walk is independent of A and therefore can be permuted with the sum over A, as was done in Section 3. Without loss of generality, we restrict our attention to sites i and j which are sufficiently far from one another so that we can use our main estimate, Proposition 5.1, to get

$$\langle h_i h_j\rangle^J_{\Lambda_L} = \int_0^\infty \mathbb{E}_i[\chi(\eta_s=j)\sum_{\substack{A\subseteq\Lambda_L\\ A\cap\eta_{[0,s]}=\varnothing}} e^{J|A|}\frac{Z^0_{A^c}}{Z^J_{\Lambda_L}}]\,\mathrm{d}s \leqslant \int_0^\infty \mathbb{E}_i[\chi(\eta_s=j)e^{-K|\eta_{[0,s]}|}]\,\mathrm{d}s\,. \tag{41}$$

This can be estimated as in [BB]. We give here a proof for completeness. Writing

$$G_N(i,j) \doteqdot \int_0^\infty \mathbb{E}_i[\chi(\eta_s=j)\chi(\tau^i_N>s)]\,\mathrm{d}s\,, \tag{42}$$

with $\tau^i_N \doteqdot \inf_{s\geqslant 0}\{\|\eta_s-i\|_\infty \geqslant N\}$, we get

$$\begin{aligned}\langle h_i h_j\rangle^J_{\Lambda_L} &\leqslant \mathbb{E}_i[\int_0^\infty \chi(\eta_s=j)e^{-K|\eta_{[0,s]}|}\,\mathrm{d}s] \\ &\leqslant \sum_{N\geqslant\|j-i\|_1} (G_{N+1}(i,j)-G_N(i,j))e^{-KN} \\ &\leqslant (1-e^{-K})\sum_{N\geqslant\|j-i\|_1} G_N(i,j)e^{-K(N-1)} \\ &\leqslant (1-e^{-K})\sum_{N\geqslant\|j-i\|_1} G_N(i,i)e^{-K(N-1)} \\ &\leqslant e^{-C_6\|i-j\|_1}\,. \end{aligned}\tag{43}$$

This concludes the proof of Theorem 2.4. We don't know how to make the corresponding proof in the non-Gaussian case, since in that case the expectation with respect to the random walk *does* depend on the set A, so it is not possible to permute it with the sum over sets A. Moreover, we cannot use the trick of Section 3 to recover the Gaussian case, since Brascamp-Lieb inequality does not apply to $\langle h_i h_j\rangle^0_{A^c}$. It is however possible to do something when J is large enough, as shown now. Let us write $i\overset{A^c}{\longleftrightarrow}j$ if i and j belong to the same r-connected (see Section 5) component of A^c. Since $\langle h_i h_j\rangle^0_{A^c}\neq 0$ only if $i\overset{A^c}{\longleftrightarrow}j$, we can write

$$\langle h_i h_j \rangle_{\Lambda_L}^J = \sum_{\substack{A \subseteq \Lambda_L \\ i \overset{A^c}{\longleftrightarrow} j}} e^{J|A|} \frac{Z^0_{A^c}}{Z^J_{\Lambda_L}} \langle h_i h_j \rangle^0_{A^c} .$$

$$\leqslant \tfrac{1}{2} \sum_{\substack{A \subseteq \Lambda_L \\ i \overset{A^c}{\longleftrightarrow} j}} e^{J|A|} \frac{Z^0_{A^c}}{Z^J_{\Lambda_L}} \left(\langle h_i^2 \rangle^0_{A^c} + \langle h_j^2 \rangle^0_{A^c} \right)$$

$$\leqslant \tfrac{1}{2} \sum_{\substack{A \subseteq \Lambda_L \\ i \overset{A^c}{\longleftrightarrow} j}} e^{J|A|} \frac{Z^0_{A^c}}{Z^J_{\Lambda_L}} \left(\langle h_i^2 \rangle^{0,\mathrm{G}}_{A^c} + \langle h_j^2 \rangle^{0,\mathrm{G}}_{A^c} \right) . \tag{44}$$

Each of these two terms can be decomposed in the following way.

$$\sum_{\substack{A \subseteq \Lambda_L \\ i \overset{A^c}{\longleftrightarrow} j}} e^{J|A|} \frac{Z^0_{A^c}}{Z^J_{\Lambda_L}} \langle h_i^2 \rangle^{0,\mathrm{G}}_{A^c} \leqslant \int_0^\infty \mathbb{E}_i [\chi(\eta_s = i) \chi(|\eta_{[0,s]}| \geqslant \|i-j\|_1) \sum_{\substack{A \subseteq \Lambda_L \\ A \cap \eta_{[0,s]} = \varnothing}} e^{J|A|} \frac{Z^0_{A^c}}{Z^J_{\Lambda_L}} \mathrm{d}s]$$

$$+ \int_0^\infty \mathbb{E}_i [\chi(\eta_s = i) \chi(|\eta_{[0,s]}| < \|i-j\|_1) \sum_{\substack{A \subseteq \Lambda_L \\ i \overset{A^c}{\longleftrightarrow} j}} e^{J|A|} \frac{Z^0_{A^c}}{Z^J_{\Lambda_L}} \mathrm{d}s] . \tag{45}$$

The first of these integral can be dealt with as before. To control the second one, observe that our main estimate Proposition 5.1 implies the existence of a constant $\widehat{K} > 0$ such that and

$$\sum_{\substack{A \subseteq \Lambda_L \\ i \overset{A^c}{\longleftrightarrow} j}} e^{J|A|} \frac{Z^0_{A^c}}{Z^J_{\Lambda_L}} \leqslant \sum_{\substack{B \subseteq \Lambda_L \\ i \overset{B}{\longleftrightarrow} j}} \sum_{\substack{A \subseteq \Lambda_L \\ A \cap B = \varnothing}} e^{J|A|} \frac{Z^0_{A^c}}{Z^J_{\Lambda_L}} \leqslant \sum_{\substack{B \subseteq \Lambda_L \\ i \overset{B}{\longleftrightarrow} j}} e^{-K|B|} \leqslant e^{-\widehat{K}\|j-i\|_1} , \tag{46}$$

as soon as K is large enough (which is true if J is sufficiently large); here the sum is over sets B which are r-connected and contain i and j. Using this, the second integral in (45) can easily be seen to decay exponentially with $\|i-j\|_1$; Theorem 2.5 follows.

5. Proof of the main estimate

This section is devoted to the proof of Proposition 5.1, which is the main estimate of this paper. The most important is the first statement, but the other also appear to be useful. This proposition roughly states that an arbitrarily weak pinning potential is sufficient to decrease (strictly) the free energy; its power, however, lies in the fact that it is not restricted to well-behaved subsets (in the sense of Van Hove for example), but even applies to "one-dimensional" ones.

We say that a set $D \subseteq \mathbb{Z}^2$ is *M-connected*, if, for any $x, y \in D$, there exists an ordered sequence $(t_0 \equiv x, t_1, \dots, t_n \equiv y)$ of sites of D such that $\|t_k - t_{k-1}\|_1 \leqslant M$, for all $k = 1, \dots, n$. The *diameter* of a set D is defined by $\mathrm{diam} D = \max_{x,y \in D} \|x-y\|_1$.

Proposition 5.1.
1. Let $B \subseteq \Lambda_L$ be M-connected and such that $\mathrm{diam} B \geqslant \big(a(e^\varepsilon - 1)\sqrt{c}\big)^{-C(M)}$ *for some $C(M)$ large enough. Then, there exists $K = K(a(e^\varepsilon - 1)\sqrt{c}, M)$, independent of B, such*

that

$$\sum_{\substack{A\subset\Lambda_L\\ A\cap B=\varnothing}} (e^{\varepsilon}-1)^{|A|}\ \frac{Z_{\Lambda_L}(A)}{Z^V_{\Lambda_L}} \leqslant \exp\{-K\,|B|\}\,.$$

Moreover, if $a(e^{\varepsilon}-1)\sqrt{c}$ *is small enough, then there exists* $C_8 = C_8(M)$ *such that* $K > (a(e^{\varepsilon}-1)\sqrt{c})^{C_8}$.
2. For any $B \subseteq \Lambda_L$,

$$\sum_{\substack{A\subset\Lambda_L\\ A\cap B=\varnothing}} (e^{\varepsilon}-1)^{|A|}\ \frac{Z_{\Lambda_L}(A)}{Z^V_{\Lambda_L}} \geqslant \exp\{-\varepsilon\,|B|\}\,.$$

3. For all $\xi < 1$, *there exists* $C_9 = C_9(a(e^{\varepsilon}-1)\sqrt{c}, M)$ *such that, for all* M*-connected* $B \subseteq \Lambda_L$ *and all* L,

$$\sum_{\substack{A\subset\Lambda_L\\ A\supset B}} (e^{\varepsilon}-1)^{|A|}\frac{Z_{\Lambda_L}(A)}{Z^V_{\Lambda_L}} \geqslant (1-\xi)\bigl(1+(C_9 a(e^{\varepsilon}-1)\sqrt{c})^{-1}\bigr)^{-|B|}\,.$$

Remark: In the case of the δ potential, a look at the proof shows that the statement corresponding to point 2. takes the form

$$\sum_{\substack{A\subset\Lambda_L\\ A\cap B=\varnothing}} e^{J|A|}\ \frac{Z^0_{A^{\mathrm{c}}}}{Z^J_{\Lambda_L}} \geqslant (1+e^J)^{-|B|}\,. \tag{47}$$

Proof. Let us first 1. We introduce the following notations:

$$D^k \doteqdot \{t\in\Lambda_L\ :\ \mathrm{d}_1(t,D)\leqslant k\}\,, \tag{48}$$
$$\partial^{\mathrm{ext}} D \doteqdot D^1\setminus D\,. \tag{49}$$

The weights $\overline{\rho}(A) \doteqdot (e^{\varepsilon}-1)^{|A|}\ \frac{Z_{\Lambda_L}(A)}{Z^V_{\Lambda_L}}$ define a probability measure on $\{A\subseteq\Lambda_L\}$, which we denote by Prob. We also use the notation $\overline{A} \doteqdot A\cup\partial^{\mathrm{ext}}\Lambda_L$. What we want to obtain is a upper bound on

$$\begin{aligned}\mathrm{Prob}[A\cap B=\varnothing] &= \sum_{k\,\geqslant\,0}\mathrm{Prob}[A\cap B^k=\varnothing \text{ and } \overline{A}\cap\partial^{\mathrm{ext}}B^k\neq\varnothing]\\ &\leqslant \sum_{k\,\geqslant\,0}\mathrm{Prob}[A\cap B^k=\varnothing\,|\,\overline{A}\cap\partial^{\mathrm{ext}}B^k\neq\varnothing]\,.\end{aligned} \tag{50}$$

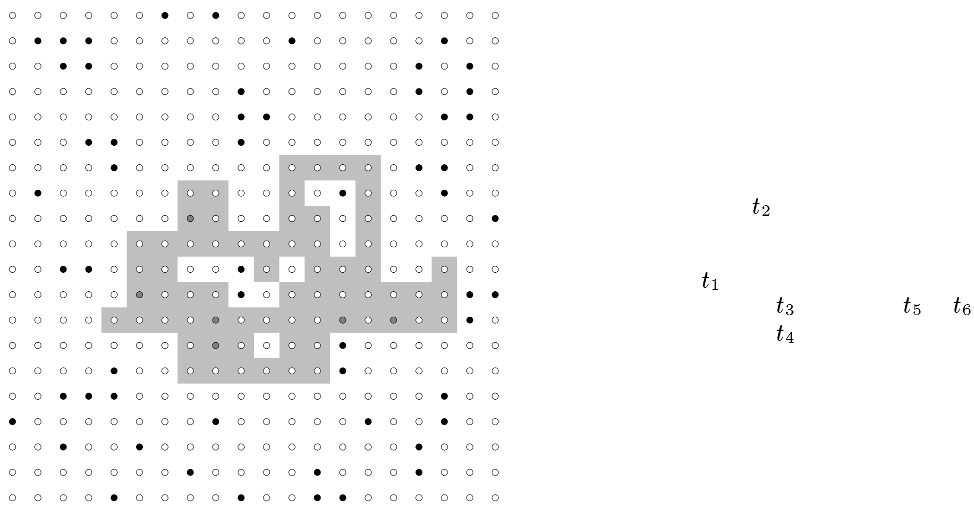

FIGURE 1. The set A is represented by the black dots and the set $C = \{t_1, \ldots, t_6\}$ by the grey ones; the set B^k (connected, here) is composed of all the sites centered on a shaded plaquette. The set A_k is composed of the union of A and the sites $t_1, \ldots, t_k$.

Observe that B^k is also M-connected and $\mathrm{diam} B^k > \mathrm{diam} B$. We can write

$$
\begin{aligned}
\mathrm{Prob}[A \cap B^k = \varnothing \,|\, \overline{A} \cap \partial^{\mathrm{ext}} B^k \neq \varnothing] &= \frac{\displaystyle\sum_{\substack{A \subseteq \Lambda_L \setminus B^k \\ \overline{A} \cap \partial^{\mathrm{ext}} B^k \neq \varnothing}} (e^{\varepsilon} - 1)^{|A|} Z_{\Lambda_L}(A)}{\displaystyle\sum_{\substack{A \subseteq \Lambda_L \\ \overline{A} \cap \partial^{\mathrm{ext}} B^k \neq \varnothing}} (e^{\varepsilon} - 1)^{|A|} Z_{\Lambda_L}(A)} \\
&= \frac{\displaystyle\sum_{\substack{A \subseteq \Lambda_L \setminus B^k \\ \overline{A} \cap \partial^{\mathrm{ext}} B^k \neq \varnothing}} (e^{\varepsilon} - 1)^{|A|} Z_{\Lambda_L}(A)}{\displaystyle\sum_{C \subseteq B^k} (e^{\varepsilon} - 1)^{|C|} \sum_{\substack{A \subseteq \Lambda_L \setminus B^k \\ \overline{A} \cap \partial^{\mathrm{ext}} B^k \neq \varnothing}} (e^{\varepsilon} - 1)^{|A|} Z_{\Lambda_L}(A \cup C)} \\
&= \left\{ \sum_{C \subseteq B^k} (e^{\varepsilon} - 1)^{|C|} \sum_{\substack{A \subseteq \Lambda_L \setminus B^k \\ \overline{A} \cap \partial^{\mathrm{ext}} B^k \neq \varnothing}} \rho(A) \frac{Z_{\Lambda_L}(A \cup C)}{Z_{\Lambda_L}(A)} \right\}^{-1} \\
&\leqslant \left\{ \sum_{C \subseteq B^k} (e^{\varepsilon} - 1)^{|C|} \inf_{\substack{A \subseteq \Lambda_L \setminus B^k \\ \overline{A} \cap \partial^{\mathrm{ext}} B^k \neq \varnothing}} \frac{Z_{\Lambda_L}(A \cup C)}{Z_{\Lambda_L}(A)} \right\}^{-1},
\end{aligned}
\tag{51}
$$

where $\rho(A) \doteqdot (e^{\varepsilon} - 1)^{|A|} Z_{\Lambda_L}(A) \Big/ \sum_{\substack{A \subseteq \Lambda_L \setminus B^k \\ \overline{A} \cap \partial^{\mathrm{ext}} B^k \neq \varnothing}} (e^{\varepsilon} - 1)^{|A|} Z_{\Lambda_L}(A)$.

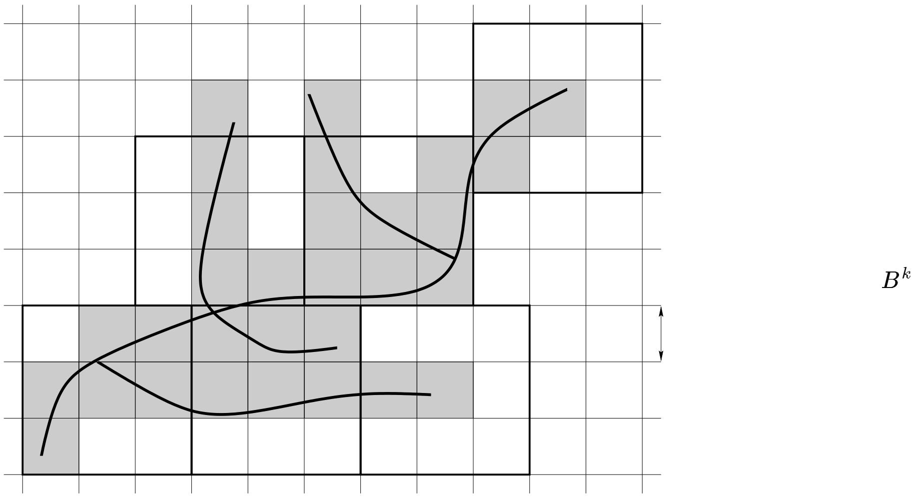

FIGURE 2. The shaded cells represents the set $\{\mathcal{C}_j,\, j \in \mathbb{J}\}$; the six large 3×3 squares represents the cells $\{\mathcal{D}_i,\, i = 1, \ldots, N_D\}$. The summation will be done on all sets $C \subseteq B^k$ containing exactly one site in each of the cells $\mathcal{C}_j,\, j \in \mathbb{J}$.

One has therefore to bound the ratio of partition functions. If we enumerate the elements of C, say $C = \{t_1, \ldots, t_{|C|}\}$, and define $A_k \doteqdot A \cup \{t_1, \ldots, t_k\}$, we get

$$\frac{Z_{\Lambda_L}(A \cup C)}{Z_{\Lambda_L}(A)} = \frac{Z_{\Lambda_L}(A_1)}{Z_{\Lambda_L}(A_0)} \frac{Z_{\Lambda_L}(A_2)}{Z_{\Lambda_L}(A_1)} \cdots \frac{Z_{\Lambda_L}(A_{|C|})}{Z_{\Lambda_L}(A_{|C|-1})}\,. \tag{52}$$

But, using Lemmas 6.4 and 6.5,

$$\frac{Z_{\Lambda_L}(A_k)}{Z_{\Lambda_L}(A_{k-1})} = \mu^0_{\Lambda_L}(|h_{t_k}| \leqslant a \,\big|\, |h_j| \leqslant a,\, \forall j \in A_{k-1}) \geqslant \tfrac{1}{2}\mu^0_{A^c_{k-1}}(|h_{t_k}| \leqslant a) \geqslant \frac{a}{8\langle |h_{t_k}| \rangle^0_{A^c_{k-1}}}\,. \tag{53}$$

Therefore,

$$\frac{Z_{\Lambda_L}(A \cup C)}{Z_{\Lambda_L}(A)} \geqslant \prod_{k=1}^{|C|} \frac{a}{8\langle |h_{t_k}| \rangle^0_{A^c_{t_{k-1}}}}\,. \tag{54}$$

To go further, we need to use the properties that B^k inherited from B. Let $l \in \mathbb{N}$ large enough (in particular $l \gg M$), but small compared to $\operatorname{diam} B$; we consider a grid of spacing l in Λ, with cells $\mathcal{C}_i$. Observe that there exists two numbers $\nu \in (0,1]$ and $\rho \in (0,1]$, independent of the set B^k and of l, such that the following properties hold:

- $B^k \subseteq \bigcup_{j \in \mathbb{J}} \mathcal{C}_j$,
- $B^k \cap \mathcal{C}_j \neq \varnothing$, for all $j \in \mathbb{J}$,
- $|B^k \cap \mathcal{C}_j| > \frac{\nu}{M} l$, for all $j \in \tilde{\mathbb{J}}$,

where $\{\mathcal{C}_j,\, j \in \mathbb{J}\}$ is a connected set of cells, and $\tilde{\mathbb{J}} \subseteq \mathbb{J}$ with $|\tilde{\mathbb{J}}| \geqslant \rho |\mathbb{J}|$. Indeed, the first statements are a simple consequence of the M-connectedness of B^k, and the last one is proven in the following way. Let $\{\mathcal{D}_i,\, i = 1, \ldots, N_D\}$ be a set of disjoint square boxes in $\mathbb{Z}^2$, build with exactly 9 cells of the grid defined above, and such that the middle-cell of each such box belongs to $\{\mathcal{C}_j,\, j \in \mathbb{J}\}$. We suppose that these boxes are chosen in such a way as to maximize N_D under these constraints. Then

- At most $12N_D$ cells of $\{\mathcal{C}_j,\, j\in\mathbb{J}\}$ are outside every $\mathcal{D}_i$.
- Each $\mathcal{D}_i$ contains at most 9 cells of $\{\mathcal{C}_j,\, j\in\mathbb{J}\}$.

Therefore, $9N_D+12N_D\geqslant|\mathbb{J}|$, i.e.

$$N_D\geqslant\tfrac{1}{21}|\mathbb{J}|\,. \tag{55}$$

Now, $|B^k\cap\mathcal{D}_i|>l/r$, for all $i=1,\ldots,N_D$. Consequently, each box $\mathcal{D}_i$ contains at least one cell $\mathcal{C}$ with $|B^k\cap\mathcal{C}|\geqslant\frac{1}{9M}l$. Choosing $\nu=\frac{1}{9}$, this implies that

$$|\tilde{\mathbb{J}}|\geqslant\tfrac{1}{9}N_D\geqslant\tfrac{1}{189}|\mathbb{J}|. \tag{56}$$

We can therefore take $\rho=\frac{1}{189}$.

We'll restrict the summation in (51) on sets $C\subseteq B^k$ which satisfy

$$|C\cap\mathcal{C}_i|=1\,,\quad \forall i\in\mathbb{J}\,. \tag{57}$$

We number the elements of C as above, but in such a way as to ensure that

$$\{\mathcal{C}\,:\,\mathcal{C}\ni t_i\,,1\leqslant i\leqslant k\} \tag{58}$$

is connected for all $1\leqslant k\leqslant|C|=|\mathbb{J}|$. Then $\mathrm{d}_1(t_k,A_{k-1})\leqslant\sqrt{5}l$, for all $k>1$. We further ask that $\mathrm{d}_1(t_1,\overline{A})\leqslant\sqrt{5}l$, which is always possible. Using this, we obtain

$$\frac{Z_{\Lambda_L}(A\cup C)}{Z_{\Lambda_L}(A)}\geqslant\left(\frac{a\overline{K}\sqrt{c}}{\sqrt{\log l}}\right)^{|\mathbb{J}|}. \tag{59}$$

Indeed, (10) implies

$$\begin{aligned}\langle|h_{t_k}|\rangle^0_{A^c_{k-1}}&\leqslant\big[\langle h^2_{t_k}\rangle^0_{A^c_{k-1}}\big]^{\frac{1}{2}}\\&\leqslant\big[\frac{1}{c}\langle h^2_{t_k}\rangle^{0,\mathrm{G}}_{A^c_{k-1}}\big]^{\frac{1}{2}}\\&\leqslant\frac{\overline{K}}{\sqrt{c}}\sqrt{\log l}\,,\end{aligned} \tag{60}$$

since, by construction, $\mathrm{d}_1(t_k,A_{t_{k-1}})\leqslant\sqrt{5}l$, and the expectation value can be estimated using the random walk representation and standard results about irreducible, symmetric random walk, see **P**11.6 and **P**12.3 in [S] for example. Therefore,

$$\begin{aligned}\sum_{C\subseteq B^k}(e^\varepsilon-1)^{|C|}\inf_{\substack{A\subseteq\Lambda_L\setminus B^k\\ \overline{A}\cap\partial^{\mathrm{ext}}B^k\neq\varnothing}}\frac{Z_{\Lambda_L}(A\cup C)}{Z_{\Lambda_L}(A)}&\geqslant\sum_{\substack{C\subseteq B^k\\|C\cap\mathcal{C}_i|=1,\forall i\in\mathbb{J}}}(e^\varepsilon-1)^{|\mathbb{J}|}\left(\frac{a\overline{K}\sqrt{c}}{\sqrt{\log l}}\right)^{|\mathbb{J}|}\\&\geqslant(\nu l)^{\rho|\mathbb{J}|}\left(\frac{(e^\varepsilon-1)a\sqrt{c}\overline{K}}{\sqrt{\log l}}\right)^{|\mathbb{J}|}\\&=\left(\frac{(e^\varepsilon-1)a\sqrt{c}\overline{K}(\nu l)^\rho}{\sqrt{\log l}}\right)^{|\mathbb{J}|}\\&\geqslant e^{|\mathbb{J}|}\geqslant e^{|B|/l^2}\,,\end{aligned} \tag{61}$$

which implies

$$\mathrm{Prob}[A\cap B^k=\varnothing\,|\,\overline{A}\cap\partial^{\mathrm{ext}}B^k\neq\varnothing]\leqslant\exp\{-\tilde{K}_l\,|B^k|\}\,, \tag{62}$$

for some $\tilde{K}_l > 0$ independent of B^k (provided $\mathrm{diam} B \gg l \geqslant l_0(a(e^\varepsilon - 1)\sqrt{c}, M))$. Therefore, we finally have

$$\mathrm{Prob}[A \cap B = \varnothing] \leqslant \exp\{-K\,|B|\}\,, \tag{63}$$

for some $K > 0$. The explicit bound on K follows by optimizing over l_0 above; this also explains the constraint on $\mathrm{diam} B$.

Let us prove 2. Proceeding as in (51), we can write

$$\begin{aligned}\sum_{\substack{A \subseteq \Lambda_L \\ A \cap B = \varnothing}} (e^\varepsilon - 1)^{|A|}\, \frac{Z_{\Lambda_L}(A)}{Z^V_{\Lambda_L}} &\geqslant \Big\{ \sum_{C \subseteq B} (e^\varepsilon - 1)^{|C|} \sup_{A \subseteq \Lambda_L \setminus B} \frac{Z_{\Lambda_L}(A \cup C)}{Z_{\Lambda_L}(A)} \Big\}^{-1} \\ &\geqslant \Big\{ \sum_{C \subseteq B} (e^\varepsilon - 1)^{|C|} \Big\}^{-1} \\ &= e^{-\varepsilon |B|}\,. \end{aligned} \tag{64}$$

We finally prove 3. Let R be a large enough integer; we write

$$\begin{aligned}\mathrm{Prob}[A \supset B] &\geqslant \mathrm{Prob}[A \supset B,\, \overline{A} \cap (B^R \setminus B) \neq \varnothing] \\ &= \mathrm{Prob}[A \supset B \,|\, \overline{A} \cap (B^R \setminus B) \neq \varnothing](1 - \mathrm{Prob}[\overline{A} \cap (B^R \setminus B) = \varnothing]) \\ &\geqslant (1 - \xi)\mathrm{Prob}[A \supset B \,|\, \overline{A} \cap (B^R \setminus B) \neq \varnothing]\,, \end{aligned} \tag{65}$$

provided R is large enough, by part 1. of the proposition. Now, similarly as before,

$$\begin{aligned}&\mathrm{Prob}[A \supset B \,|\, \overline{A} \cap (B^R \setminus B) \neq \varnothing] \\ &\qquad = \frac{\displaystyle\sum_{\substack{A \supset B \\ \overline{A} \cap (B^R \setminus B) \neq \varnothing}} (e^\varepsilon - 1)^{|A|} Z_{\Lambda_L}(A)}{\displaystyle\sum_{\substack{A \supset B \\ \overline{A} \cap (B^R \setminus B) \neq \varnothing}} (e^\varepsilon - 1)^{|A|} Z_{\Lambda_L}(A) \Big(\sum_{C \subseteq B} (e^\varepsilon - 1)^{-|C|} \frac{Z_{\Lambda_L}(A \setminus C)}{Z_{\Lambda_L}(A)} \Big)}\,. \end{aligned} \tag{66}$$

The conclusion follows from (53) and the construction, which imply that

$$\frac{Z^0(A)}{Z^0(A \setminus C)} \geqslant \Big(\frac{\sqrt{c} a \overline{K}}{\sqrt{\log R}} \Big)^{|C|}\,. \tag{67}$$

Indeed this gives

$$\begin{aligned}\mathrm{Prob}[A \supset B \,|\, \overline{A} \cap (B^R \setminus B) \neq \varnothing] &\geqslant \Big\{ \sum_{C \subseteq B} \big(\frac{\sqrt{\log R}}{a(e^\varepsilon - 1)\sqrt{c} \overline{K}} \big)^{|C|} \Big\}^{-1} \\ &= \Big\{ 1 + \frac{\sqrt{\log R}}{a(e^\varepsilon - 1)\sqrt{c}\overline{K}} \Big\}^{-|B|}\,. \end{aligned} \tag{68}$$

□

6. Appendix: Proofs of some technical estimates

In this section, we give the proofs of several technical statements used in the previous ones. Since FKG inequality is used several times, we recall that, as a consequence of Corollary 1.7 in [HP], measures of the form

$$\frac{\mu^b_\Lambda(\,\cdot\,\prod_{i\in\Lambda} f_i(h_i))}{\mu^b_\Lambda(\prod_{i\in\Lambda} f_i(h_i))} \tag{69}$$

are FKG.

Lemma 6.1. *Let g be a positive, even function which is increasing on $\mathbb{R}^+$ and such that $g(0)=0$. Then, for any $\Lambda \Subset \mathbb{Z}^2$, any $A \subseteq \Lambda$ and any $j \in \Lambda \setminus A$,*

$$\langle g(h_j) \,\big|\, |h_k| \leqslant a \;\forall k \in A\rangle^0_\Lambda \leqslant \langle g(h_j+a) \,\big|\, h_j \geqslant -a\rangle^0_{\Lambda\setminus A}\,.$$

Proof. We introduce $g_\nearrow(h_j) = g(h_j \vee 0)$. Using symmetry, FKG twice and translation invariance, we can write

$$\begin{aligned}
\langle g(h_j) \,\big|\, |h_k| \leqslant a \;\forall k \in A\rangle^0_\Lambda &= \langle g_\nearrow(h_j) \,\big|\, |h_k| \leqslant a \;\forall k \in A,\, h_j \geqslant 0\rangle^0_\Lambda\\
&\leqslant \langle g_\nearrow(h_j) \,\big|\, h_k = a \;\forall k \in A,\, h_j \geqslant 0\rangle^0_\Lambda\\
&\leqslant \langle g_\nearrow(h_j) \,\big|\, h_k = a \;\forall k \in A,\, h_j \geqslant 0\rangle^a_\Lambda\\
&= \langle g_\nearrow(h_j+a) \,\big|\, h_k = 0 \;\forall k \in A,\, h_j \geqslant -a\rangle^0_\Lambda\\
&= \langle g(h_j+a) \,\big|\, h_j \geqslant -a\rangle^0_{\Lambda\setminus A}\,.
\end{aligned} \tag{70}$$

Let us explain how the two inequalities are obtained. Let $\lambda > 0$. Since $\prod_{k\in A}\chi(h_k > a-\lambda)$ and $g_\nearrow$ are increasing, and the measure

$$\chi(h_j \geqslant 0)\prod_{k\in A}\chi(|h_k| \leqslant a)\,\mathrm{d}\mu^0_\Lambda$$

is FKG,

$$\langle g_\nearrow(h_j) \,|\, |h_k| \leqslant a,\, \forall k \in A, h_j \geqslant 0\rangle^0_\Lambda \leqslant \langle g_\nearrow(h_j) \,|\, |h_k| \in (a-\lambda, a],\, \forall k \in A, h_j \geqslant a\rangle^0_\Lambda\,. \tag{71}$$

Letting λ go to zero gives the first inequality. The second follows from the observation that $\mu^0_\Lambda(\,\mathrm{d}\underline{h}) = \phi_b(\underline{h})\mu^b_\Lambda(\,\mathrm{d}\underline{h})$, with $\phi_b(\underline{h}) = \prod_{\substack{\langle ik\rangle_r\\ i\in\Lambda,\, k\notin\Lambda}} \exp\{\Psi_{k-i}(h_i - b) - \Psi_{k-i}(h_i)\}$ decreasing if $b > 0$. Indeed,

$$\frac{\mathrm{d}}{\mathrm{d}h_i}(\Psi_{k-i}(h_i-b) - \Psi_{k-i}(h_i)) = -\int_{h_i-b}^{h_i} \Psi''_{k-i}(x)\,\mathrm{d}x < 0\,, \forall i \in \Lambda\,. \tag{72}$$

□

Lemma 6.2. *Let $T > 0$. Then, for all $\Lambda \Subset \mathbb{Z}^2$, $A \subseteq \Lambda$ and $i \in \Lambda \setminus A$,*

$$\mu^0_{A^c}(h_i > T + a) \leqslant \mu^0_\Lambda(h_i > T \,\big|\, |h_j| \leqslant a,\, \forall j \in A) \leqslant \mu^0_{A^c}(h_i > T - a)\,.$$

Proof. The proof is completely similar to the previous one. We have

$$\begin{aligned}
\langle \chi(h_i > T) \,\big|\, |h_j| \leqslant a \;\forall j \in A\rangle^0_{\Lambda_L} &\leqslant \langle \chi(h_i > T) \,\big|\, |h_j| = a \;\forall j \in A\rangle^0_{\Lambda_L}\\
&\leqslant \langle \chi(h_i > T) \,\big|\, |h_j| = a \;\forall j \in A\rangle^a_{\Lambda_L}\\
&\leqslant \langle \chi(h_i > T - a)\rangle^0_{A^c}\,,
\end{aligned} \tag{73}$$

and

$$\begin{aligned}\langle \chi(h_i > T) \,\big|\, |h_j| \leqslant a \;\forall j \in A \rangle^0_{\Lambda_L} &\geqslant \langle \chi(h_i > T) \,\big|\, |h_j| = -a \;\forall j \in A \rangle^0_{\Lambda_L} \\ &\geqslant \langle \chi(h_i > T) \,\big|\, |h_j| = -a \;\forall j \in A \rangle^{-a}_{\Lambda_L} \\ &\geqslant \langle \chi(h_i > T + a) \rangle^0_{A^{\mathrm{c}}} \,. \end{aligned} \tag{74}$$

□

Lemma 6.3. *For any $\Lambda \Subset \mathbb{Z}^2$ and $A \subseteq \Lambda$,*

$$\langle h_i^2 \,|\, |h_j| \leqslant a, \forall j \in A \rangle^0_\Lambda \leqslant 4a^2 + 4 \langle h_i^2 \rangle^0_{\Lambda \setminus A} \,.$$

Proof. This follows easily from Lemma 6.1 with $g(x) = x^2$, and FKG inequality which yields

$$\langle ((h_i + a) \vee 0)^2 \,\big|\, h_i \geqslant -a \rangle^0_{\Lambda\setminus A} \leqslant \langle ((h_i + a) \vee 0)^2 \,\big|\, h_i \geqslant 0 \rangle^0_{\Lambda\setminus A} \,. \tag{75}$$

□

Lemma 6.4. *For any $\Lambda \Subset \mathbb{Z}^2$ and $A \subseteq \Lambda$,*

$$\mu^0_\Lambda(|h_i| \leqslant a \,|\, |h_j| \leqslant a, \forall j \in A)^0_\Lambda \geqslant \tfrac{1}{2} \mu^0_{\Lambda\setminus A}(|h_i| \leqslant a) \,.$$

Proof. Lemma 6.1 with $g(x) = \chi(|x| > a)$ implies that

$$\begin{aligned}\mu^0_\Lambda(|h_i| \leqslant a \,|\, |h_j| \leqslant a, \forall j \in A)^0_\Lambda &\geqslant \mu^0_{\Lambda\setminus A}(|h_i + a| \leqslant a \,\big|\, h_i \geqslant -a) \\ &= \mu^0_{\Lambda\setminus A}(h_i \leqslant 0 \,\big|\, h_i \geqslant -a) \\ &\geqslant \mu^0_{\Lambda\setminus A}(-a \leqslant h_i \leqslant 0) \\ &= \tfrac{1}{2} \mu^0_{\Lambda\setminus A}(|h_i| \leqslant a) \,. \end{aligned} \tag{76}$$

□

Lemma 6.5. *For any $\Lambda \Subset \mathbb{Z}^2$ and $i \in \Lambda$,*

$$\mu^0_\Lambda(|h_i| \leqslant a) \geqslant \frac{a}{4\langle |h_i| \rangle^0_\Lambda} \wedge \tfrac{1}{2} \,.$$

Proof. The proof follows from the following elementary result, which is proved in [DMRR]: *Let X be a random variable whose density under $\mathbb{P}$ is even and decreasing on $\mathbb{R}^+$. Then $\mathbb{P}[|X| \leqslant a] \geqslant \frac{a}{4\mathbb{E}[|X|]} \wedge \frac{1}{2}$, where $\mathbb{E}[\,\cdot\,]$ is the expectation value with respect to $\mathbb{P}$.*

Let F_j be the density of h_j under μ^0_Λ. The evenness is obvious; let us check the monotonicity.

$$F_j(x) = \frac{1}{Z^0_\Lambda} \int \prod_{t \in \mathbb{R}^{\Lambda\setminus\{j\}}} \mathrm{d}h_t \prod_{\substack{\langle kl \rangle_r \\ k,l \in \Lambda\setminus\{j\}}} \mathrm{e}^{-\Psi_{l-k}(h_k - h_l)} \prod_{\substack{\langle jk \rangle_r \\ k \in \Lambda}} \mathrm{e}^{-\Psi_{k-j}(h_k - x)} \prod_{\substack{\langle jk \rangle_r \\ k \notin \Lambda}} \mathrm{e}^{-\Psi_{k-j}(x)} \prod_{\substack{\langle kl \rangle_r \\ k \in \Lambda\setminus\{j\} \\ l \notin \Lambda}} \mathrm{e}^{-\Psi_{l-k}(h_k)} \,,$$

$$F'_j(x) = \tilde{C}_1 \sum_{\substack{\langle jk \rangle_r \\ k \notin \Lambda}} \langle -\Psi'_{k-j}(x) \,\big|\, h_j = x \rangle^0_\Lambda + \tilde{C}_2 \sum_{\substack{\langle jk \rangle_r \\ k \in \Lambda}} \langle \Psi'_{k-j}(h_k - x) \,\big|\, h_j = x \rangle^0_\Lambda \,, \tag{77}$$

where $\tilde{C}_1$ and $\tilde{C}_2$ are two positive constants. Now, for $x \geqslant 0$, $\Psi'_{k-j}(x) \geqslant 0$ and therefore the first term is negative. By FKG,

$$\langle \Psi'_{k-j}(h_k - x) \,\big|\, h_j = x \rangle^0_\Lambda = \langle \Psi'_{k-j}(h_k) \,\big|\, h_j = 0 \rangle^{-x}_\Lambda \leqslant \langle \Psi'_{k-j}(h_k) \rangle^0_{\Lambda\setminus\{j\}} = 0 \,, \tag{78}$$

since Ψ'_{k-j} is increasing and odd. Consequently, $F_j'(x) \leqslant 0$ for $x \geqslant 0$. $\square$

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Fachbereich Mathematik, Sekr. MA 7-4, TU-Berlin, Strasse des 17. Juni 136, D-10623 Berlin, Germany

E-mail address: `deuschel@math.tu-berlin.de`

Fachbereich Mathematik, Sekr. MA 7-4, TU-Berlin, Strasse des 17. Juni 136, D-10623 Berlin, Germany

E-mail address: `velenik@math.tu-berlin.de`