DISTRIBUTION OF ZEROS OF RANDOM AND QUANTUM CHAOTIC SECTIONS OF POSITIVE LINE BUNDLES

BERNARD SHIFFMAN AND STEVE ZELDITCH

Abstract. We study the limit distribution of zeros of certain sequences of holomorphic sections of high powers L^N of a positive holomorphic Hermitian line bundle L over a compact complex manifold M. Our first result concerns 'random' sequences of sections. Using the natural probability measure on the space of sequences of orthonormal bases $\{S_j^N\}$ of $H^0(M,L^N)$, we show that for almost every sequence $\{S_j^N\}$, the associated sequence of zero currents $\frac{1}{N}Z_{S_j^N}$ tends to the curvature form ω of L. Thus, the zeros of a sequence of sections $s_N \in H^0(M,L^N)$ chosen independently and at random become uniformly distributed. Our second result concerns the zeros of quantum ergodic eigenfunctions, where the relevant orthonormal bases $\{S_j^N\}$ of $H^0(M,L^N)$ consist of eigensections of a quantum ergodic map. We show that also in this case the zeros become uniformly distributed.

1. Introduction

This paper is concerned with the limit distribution of zeros of 'random' holomorphic sections and of 'quantum ergodic' eigensections of powers of a positive holomorphic line bundle L over a compact complex manifold M. To introduce our subject, let us consider the simplest case where $M = \mathbb{CP}^m$ and where L is the hyperplane section bundle. As is well-known, sections of L^N are given by homogeneous polynomials $p_N(z_0, z_1, \dots, z_m)$ of degree N on $\mathbb{C}^{m+1}$; these polynomials are called $\mathrm{SU}(m+1)$ polynomials when we consider them as elements of a measure space with an $\mathrm{SU}(m+1)$-invariant Gaussian measure (see §4). We are concerned with the question: what is the limit distribution of zeros $Z_N = \{p_N = 0\} \subset M$ of a sequence $\{p_N\}$ of such polynomials as the degree $N \to \infty$? Of course, if we consider all possible sequences, then little can be said. However, if we consider only the typical behavior, then there is a simple answer: if the sequence $\{p_N\}$ is chosen independently and at random from the ensembles of homogeneous polynomials of degree N and $\mathcal{L}^2$-norm one, then the zero sets of $\{p_N\}$ almost surely become uniformly distributed with respect to the volume form induced by ω.

The same conclusion is true for any positive Hermitian holomorphic line bundle (L,h) over any compact complex manifold M. In place of homogeneous polynomials of degree N, one now considers holomorphic sections $s_N \in H^0(M,L^N)$. The curvature form $\omega = c_1(h)$ of h defines a Kähler structure on M, and the metrics h, ω provide a Hermitian inner product on $H^0(M,L^N)$. (See equations (1)–(2) in §2.) We then have the notion of a 'random' sequence of $\mathcal{L}^2$-normalized sections of $H^0(M,L^N)$. Namely, we consider the probability space $(\mathcal{S}, d\mu)$, where $\mathcal{S}$ equals the product $\prod_{N=1}^{\infty} SH^0(M,L^N)$ of the unit spheres $SH^0(M,L^N)$ in

Date: March 6, 1998.

Research of the first author partially supported by NSF grant #DMS-9500491; research of the second author partially supported by NSF grant #DMS-9703775.

$H^0(M, L^N)$ and μ is the product of Haar measures on these spheres. Given a sequence $\mathbf{s} = \{s_N\} \in \mathcal{S}$, we associate the currents of integration Z_{s_N} over the zero divisors of the sections s_N. In complex dimension 1, Z_{s_N} is simply the sum of delta functions at the zeros of s_N. Our first result states that for a random (i.e., for almost all) $\mathbf{s} \in \mathcal{S}$, the sequence of zeros of the sections s_N are asymptotically uniformly distributed:

THEOREM **1.1.** *For μ-almost all $\mathbf{s} = \{s_N\} \in \mathcal{S}$, $\frac{1}{N} Z_{s_N} \to \omega$ weakly in the sense of measures; in other words,*

$$\lim_{N\to\infty} \left(\frac{1}{N} Z_{s_N}, \varphi\right) = \int_M \omega \wedge \varphi$$

for all continuous $(m-1, m-1)$ forms φ. In particular,

$$\lim_{N\to\infty} \frac{1}{N} \mathrm{Vol}_{2m-2}\{z \in U : s_N(z) = 0\} = m \mathrm{Vol}_{2n} U \ ,$$

for U open in M (where Vol_k denotes the Riemannian k-volume in (M, ω)).

The key ideas in the proof of Theorem 1.1 (as well as Theorem 1.2 below) are Tian's theorem [T, Z4] on approximating the metric ω using the sections of $H^0(M, L^N)$ (see Theorem 2.1) and an asymptotic estimate of the variances of Z_{s_N}, regarded as a current-valued random variable (Lemma 3.3).

A closely related issue is the distribution of zeros of sections $\{S_j^N\}$ forming random orthonormal bases of $H^0(M, L^N)$. Such bases are increasingly used to model orthonormal bases of quantum chaotic eigenfunctions; e.g., see [BBL, Ha, LS, NV]. The properties of these bases are very similar to those of random orthonormal bases of spherical harmonics studied in [Z1] and [V]. To study the zeros of random orthonormal bases, we introduce the probability space $(\mathcal{ONB}, d\nu)$, where $\mathcal{ONB}$ is the infinite product of the sets $\mathcal{ONB}_N$ of orthonormal bases of the spaces $H^0(M, L^N)$, and $\nu = \prod_{N=1}^{\infty} \nu_N$, where ν_N is Haar probability measure on $\mathcal{ONB}_N$. A point of $\mathcal{ONB}$ is thus a sequence $\mathbf{S} = \{(S_1^N, \dots, S_{d_N}^N)\}_{N\geq 1}$ of orthonormal bases (where $d_N = \dim H^0(M, L^N)$), and we may ask whether all of the zero sets $Z_{S_j^N}$ are tending simultaneously to the uniform distribution. The answer is still essentially yes, but for technical reasons we have to delete a subsequence of relative density zero of the sections.

THEOREM **1.2.** *For ν-almost all $\mathbf{S} = \{(S_1^N, \dots, S_{d_N}^N)\} \in \mathcal{ONB}$, we have*

$$\frac{1}{d_N} \sum_{j=1}^{d_N} \left(\frac{1}{N} Z_{S_j^N} - \omega, \varphi\right)^2 \to 0$$

for all continuous $(m-1, m-1)$ forms φ. Equivalently, for each N there exists a subset $\Lambda_N \subset \{1, \dots, d_N\}$ such that $\frac{\#\Lambda_N}{d_N} \to 1$ and

$$\lim_{N\to\infty, j\in\Lambda_N} \frac{1}{N} Z_{S_j^N} = \omega$$

weakly in the sense of measures.

Our final result pertains to actual quantum ergodic eigenfunctions rather than to random sections and shows that their zero divisors also become uniformly distributed in the high power limit. Recall that a quantum map is a unitary operator which 'quantizes' a symplectic map on a symplectic manifold. In our setting, the symplectic manifold is the Kähler manifold

(M, ω) and the map is a symplectic transformation $\chi : (M, \omega) \to (M, \omega)$. Under certain conditions, χ may be quantized as a sequence of unitary operators $U_{\chi,N}$ on $H^0(M, L^N)$. The sequence defines a semiclassical Fourier integral operator of Hermite type (or equivalently a semiclassical Toeplitz operator). For the precise definitions and conditions, we refer to [Z3]. We call $U_{\chi,N}$ a 'quantum ergodic map' if χ is also an ergodic transformation of (M, ω).

THEOREM **1.3.** *Let $(L, h) \to (M, \omega)$ be a positive Hermitian line bundle over a Kähler manifold with $c_1(h) = \omega$ and let $U_{\chi,N} : H^0(M, L^N) \mapsto H^0(M, L^N)$ be a quantum ergodic map. Further, let $\{S_1^N, \dots, S_{d_N}^N\}$ be an orthonormal basis of eigensections of $U_{\chi,N}$. Then there exists a subsequence $\Lambda \subset \{(N, j) : N = 1, 2, 3, \dots, j \in \{1, \dots, d_N\}\}$ of density one such that*

$$\lim_{N\to\infty,(N,j)\in\Lambda} \frac{1}{N} Z_{S_j^N} = \omega$$

weakly in the sense of measures.

This result was proved independently by Nonnenmacher-Voros [NV] in the case of the theta bundle over an elliptic curve $\mathbb{C}/\mathbb{Z}^2$. The main step is to establish the following result:

LEMMA **1.4.** *Let $(L, h) \to (M, \omega)$ be a positive Hermitian holomorphic line bundle over a Kähler manifold M with $c_1(h) = \omega$. Let $s_N \in H^0(M, L^N)$, $N = 1, 2, \dots,$ be a sequence of sections with the property that $\|s_N(z)\|^2 \to 1$ in the weak* sense as $N \to \infty$. Then $\frac{1}{N} Z_N \to \omega$ weakly in the sense of measures.*

The convergence hypothesis means that $\int_M \varphi(z)\|s_N(z)\|^2 dz \to \int_M \varphi(z) dz$ for all $\varphi \in \mathcal{C}^0(M)$. Our proof of Lemma 1.4 is somewhat different and more general than that of [NV], but both are based on potential theory. The lemma was motivated by an analogous result of Sodin [So] on the asymptotic equidistribution of zero sets of sequences of rational functions in one variable (see also [RSh, RSo] for the higher dimensional case); Sodin's result in turn arose from the Brolin-Lyubich Theorem in complex dynamics (cf., [FS]). The connection between Lemma 1.4 and Theorems 1.2, 1.3 will be established in §5, the main point being that both random orthonormal bases and orthonormal bases of chaotic eigenfunctions satisfy the hypothesis of the lemma (Theorems 5.1, 5.2).

We end this introduction with a brief discussion of related results. There is an extensive literature on the distribution of zeros of random polynomials, beginning with the classical papers of Bloch-Polya [BP], Littlewood-Offord [LO], Kac [Ka] and Erdos-Turan [ET] on polynomials in one variable. The articles of Bleher-Di [BD] and Shepp-Vanderbei [SV] contain recent results and further references. In addition to the mathematical literature there is a growing physics literature on zeros of random polynomials and chaotic quantum eigenfunctions, see in particular [BD, BBL, Ha, LS, NV]. As in this paper, these articles are largely concerned with the distribution of zeros in the semiclassical limit. The main theme is that the distribution of zeros of eigenfunctions of quantum maps should reflect the signature of the dynamics of the underlying classical system: in the case of ergodic quantum maps, the zeros should be uniformly distributed in the semiclassical limit while in the completely integrable case they should concentrate in a singular way. Random polynomials (or more generally sections) are believed to provide an accurate model for quantum chaotic eigenfunctions and hence there is interest in understanding how their zeros are distributed and how the zeros are correlated.

To our knowledge, the prior results on distribution of zeros of random holomorphic sections only go as far as determining the average distribution. In the special case of SU(2) polynomials it is shown in [BBL] that the average distribution is uniform. Our result that the expected distribution is achieved asymptotically by almost every sequence of sections appears to be new even in that case. Regarding zeros of quantum ergodic eigenfunctions, the only prior rigorous result appears to be that of [NV] mentioned above. We should also mention the study of the zeros of certain sections of positive line bundles in the almost complex setting which has recently been made by Donaldson [D]; the relevant zero sets were also shown to be uniformly distributed in the high power limit.

Acknowledgments: We would like to thank S. Nonnenmacher and A. Voros for sending us a copy of their paper [NV] prior to publication and to acknowledge their priority on the overlapping result. We would also like to thank W. Minicozzi for discussions of Donaldson's paper at the outset of this work and for suggesting that we study random sequences of sections.

2. Background

We begin by introducing some terminology and basic properties of orthonormal bases of holomorphic sections of powers of a positive line bundle.

2.1. **Notation.** Throughout this paper, we let L denote an ample holomorphic line bundle over an m-dimensional compact complex (projective) manifold M. We denote the space of global holomorphic sections of L by $H^0(M,L)$. We let $\mathcal{D}^{p,q}(M)$ denote the space of $\mathcal{C}^\infty$ (p,q)-forms on M, and we let $\mathcal{D}'^{p,q}(M) = \mathcal{D}^{m-p,m-q}(M)'$ denote the space of (p,q)-currents on M; $(T,\varphi) = T(\varphi)$ denotes the pairing of $T \in \mathcal{D}'^{p,q}(M)$ and $\varphi \in \mathcal{D}^{m-p,m-q}(M)$. If L has a smooth Hermitian metric h, its *curvature form* $c_1(h) \in \mathcal{D}^{1,1}(M)$ is given locally by

$$c_1(h) = -\frac{\sqrt{-1}}{\pi}\partial\bar\partial \log \|e_L\|_h ,$$

where e_L is a nonvanishing local holomorphic section of L, and $\|e_L\|_h = h(e_L,e_L)^{1/2}$ denotes the h-norm of e_L. The curvature form $c_1(h)$ is a de Rham representative of the Chern class $c_1(L) \in H^2(M,\mathbb{R})$; see [GH, SS]. Since L is ample, we can give L a metric h with strictly positive curvature form, and we give M the Kähler metric $\omega = c_1(h)$. Then $\int_M \omega^m = c_1(L)^m \in \mathbb{Z}^+$. Finally, we give M the volume form

$$dV = \frac{1}{c_1(L)^m}\omega^m , \tag{1}$$

so that M has unit volume: $\int_M dV = 1$.

This paper is concerned with the spaces $H^0(M,L^N)$ of sections of $L^N = L \otimes \cdots \otimes L$. The metric h induces Hermitian metrics h_N on L^N given by $\|s^{\otimes N}\|_{h_N} = \|s\|_h^N$. We give $H^0(M,L^N)$ the inner product structure

$$\langle s_1, s_2\rangle = \int_M h_N(s_1,s_2)dV \qquad (s_1,s_2 \in H^0(M,L^N)) , \tag{2}$$

and we write $|s| = \langle s,s\rangle^{1/2}$. We let $d_N = \dim H^0(M,L^N)$. It is well known that for N sufficiently large, d_N is given by the Hilbert polynomial of L, whose leading term is $\frac{c_1(L)^m}{m!}N^m$ (see, for example [SS, Chapter 7]).

For a holomorphic section $s \in H^0(M, L^N)$, we let Z_s denote the current of integration over the zero divisor of s. In a local frame e_L^N for L^N, we can write $s = \psi e_L^N$, where ψ is a holomorphic function. We recall the Poincaré-Lelong formula

$$Z_s = \frac{\sqrt{-1}}{\pi}\partial\bar\partial \log|\psi| = \frac{\sqrt{-1}}{\pi}\partial\bar\partial \log\|s\|_{h_n} + N\omega \ . \tag{3}$$

We also consider the normalized zero divisor

$$\widetilde{Z}_s^N = \frac{1}{N} Z_s \ ,$$

so that the currents $\widetilde{Z}_s^N$ are de Rham representatives of $c_1(L)$, and thus

$$\left(\widetilde{Z}_s^N, \omega^{m-1}\right) = \frac{c_1(L)^m}{m!} \ . \tag{4}$$

Equation (4) says that the currents $\widetilde{Z}_s^N$ all have the same mass.

For example, we consider the hyperplane section bundle, denoted $\mathcal{O}(1)$, over $\mathbb{CP}^m$. Sections $s \in H^0(\mathbb{CP}^m, \mathcal{O}(1))$ are linear functions on $\mathbb{C}^{m+1}$; The zero divisors Z_s are projective hyperplanes. The line bundle $\mathcal{O}(1)$ carries a natural metric h_{FS} given by

$$\|s\|_{h_{\mathrm{FS}}}([w]) = \frac{|(s,w)|}{|w|} \ , \qquad w = (w_0, \dots, w_m) \in \mathbb{C}^{m+1} \ ,$$

for $s \in \mathbb{C}^{m+1*} \equiv H^0(\mathbb{CP}^m, \mathcal{O}(1))$, where $|w|^2 = \sum_{j=0}^m |w_j|^2$ and $[w] \in \mathbb{CP}^m$ is the complex line through w. The curvature form of h_{FS} is given by

$$c_1(h_{\mathrm{FS}}) = \omega_{\mathrm{FS}} = \frac{\sqrt{-1}}{2\pi}\partial\bar\partial \log|w|^2 \ , \tag{5}$$

where ω_{FS} is the Fubini-Study Kähler form on $\mathbb{CP}^m$. Here, ω_{FS} is normalized so that it represents the generator of $H^2(\mathbb{CP}^m, \mathbb{Z})$. The N-th tensor power of $\mathcal{O}(1)$ is denoted $\mathcal{O}(N)$. Elements of $H^0(\mathbb{CP}^m, \mathcal{O}(N))$ are homogeneous polynomials on $\mathbb{C}^{m+1}$ of degree N; hence, $\dim H^0(\mathbb{CP}^m, \mathcal{O}(N)) = \binom{N+m}{m} = \frac{1}{m!}N^m + \cdots$.

2.2. **Holomorphic sections and CR holomorphic functions.** The setting for our analysis is the Hardy space $H^2(X) \subset \mathcal{L}^2(X)$ where $X \to M$ is the principal S^1 bundle associated to L. To be precise, let L^* be the dual line bundle to L and let $D = \{v \in L^* : h(v,v) < 1\}$ be its unit disc bundle relative to the metric induced by h and let $X = \partial D = \{v \in L^* : h(v,v) = 1\}$. The positivity of $c_1(h)$ is equivalent to the disc bundle D being strictly pseudoconvex in L^* (see [Gr]).

We let $r_\theta x = e^{i\theta}x$ ($x \in X$) denote the S^1 action on X and denote its infinitesimal generator by $\frac{\partial}{\partial\theta}$. As the boundary of a strictly pseudoconvex domain, X is a CR manifold, and the Hardy space $H^2(X)$ mentioned above is by definition the space of square integrable CR functions on X. Equivalently, it is the space of boundary values of holomorphic functions on D which are in $\mathcal{L}^2(X)$. The S^1 action on X commutes with the Cauchy-Riemann operator $\bar\partial_b$; hence $H^2(X) = \bigoplus_{N=0}^\infty H^2_N(X)$ where $H^2_N(X) = \{f \in H^2(X) : f(r_\theta x) = e^{iN\theta} f(x)\}$. A section s of L determines an equivariant function $\hat{s}$ on L^* by the rule: $\hat{s}(z,\lambda) = (\lambda, s(z))$ ($z \in M, \lambda \in L^*_z$). It is clear that if $\tau \in \mathbb{C}$ then $\hat{s}(z, \tau\lambda) = \tau\hat{s}$. We will usually restrict $\hat{s}$ to X and then the equivariance property takes the form: $\hat{s}(r_\theta x) = e^{i\theta}\hat{s}(x)$. Similarly, a section s_N of L^N determines an equivariant function $\hat{s}_N$ on L^*: put $\hat{s}_N(z,\lambda) = \left(\lambda^N, s_N(z)\right)$ where

$\lambda^N = \lambda \otimes \cdots \otimes \lambda$; then $\hat{s}_N(r_\theta x) = e^{iN\theta} s_N(x)$. The map $s \mapsto \hat{s}$ is a unitary equivalence between $H^0(M, L^N)$ and $H^2_N(X)$.

We now recall the strong form of Tian's theorem [T] given in [Z4]:

THEOREM **2.1.** [Z4] *Let M be a compact complex manifold of dimension m (over $\mathbb{C}$) and let $(L, h) \to M$ be a positive Hermitian holomorphic line bundle. Let $\{S_1^N, \dots, S_{d_N}^N\}$ be any orthonormal basis of $H^0(M, L^N)$ (with respect to the inner product defined above). Then there exists a complete asymptotic expansion*

$$\sum_{j=1}^{d_N} \|S_j^N(z)\|^2_{h_N} = a_0 N^m + a_1(z) N^{m-1} + a_2(z) N^{m-2} + \dots$$

with $a_0 = \frac{c_1(L)^m}{m!}$ and with the lower coefficients $a_j(z)$ given by invariant polynomials in the higher derivatives of h. More precisely, for any $k \geq 0$,

$$\left\| \sum_{i=0}^{d_N} \|S_i^N\|^2_{h_N} - \sum_{j<R} a_j N^{m-j} \right\|_{\mathcal{C}^k} \leq C_{R,k} N^{m-R}.$$

Note that since the S_j^N have unit length (as elements of $H^0(M, L^N)$), if we integrate the above asymptotic expansion over M (with respect to the volume dV, we get simply d_N. Thus the integrals of the a_j are the coefficients of the Hilbert polynomial of L. (The constant a_0 differs from that of [T] and [Z4], since we use here the normalized volume dV on M.)

The canonical map

$$\Phi_N : M \to \mathbb{P}H^0(M, L^{\otimes N})^*, \qquad z \mapsto \{s \in H^0(M, L^{\otimes N}) : s(z) = 0\} \tag{6}$$

can be described in terms of an orthonormal basis $\mathbf{S} = \{S_1^N, \dots, S_{d_N}^N\}$ by the map

$$\Phi_N^{\mathbf{S}} : M \to \mathbb{CP}^{d_N - 1}, \qquad z \mapsto [S_1^N(z), \dots, S_{d_N}^N(z)] \,. \tag{7}$$

We shall drop the $\mathbf{S}$ and denote the map given in (7) simply by Φ_N. For N sufficiently large, the sections $\{S_1^N, \dots, S_{d_N}^N\}$ do not have common zeros and (7) gives a holomorphic embedding, by the Kodaira embedding theorem; see [GH, SS].

Theorem 2.1 can be regarded as an asymptotic formula for the distortion function between the metrics h_N and $\Phi_N^* h_{\rm FS}$ on the line bundle L^N. It also gives the following asymptotic estimate of the Riemannian distortion of the maps Φ_N:

COROLLARY **2.2.** [Z4] *Let $\omega_{\rm FS}$ denote the Fubini-Study form on $\mathbb{CP}^{d_N-1}$. Then for any $k \geq 0$,*

$$\left\| \frac{1}{N} \Phi_N^*(\omega_{\rm FS}) - \omega \right\|_{\mathcal{C}^k} = O(\frac{1}{N}) \,.$$

3. ZEROS OF RANDOM SECTIONS

Our first aim is to determine the expected value of the normalized zero divisor $\widetilde{Z}_s$ as s is chosen at random from the unit sphere

$$SH^0(M, L^N) := \{s \in H^0(M, L^N) : |s| = 1\}$$

(or equivalently as $[s] \in \mathbb{P}H^0(M, L^N)$ is chosen at random with respect to the Fubini-Study volume). As above, we fix one orthonormal basis $\{S_j^N\}$ of $H^0(M, L^N)$ and write $S_j^N = f_j e_L^N$

relative to a holomorphic frame (= nonvanishing section) e_L^N over an open set $U \subset M$. Any section in $SH^0(M, L^N)$ may then be written as $s = \sum_{j=1}^{d_N} a_j f_j e_L^N$ with $\sum_{j=1}^{d_N} |a_j|^2 = 1$. To simplify the notation we let $f = (f_1, \dots, f_{d_N}) : U \to \mathbb{C}^{d_N}$ (which is a local representation of Φ_N) and we put

$$\sum_{j=1}^{d_N} a_j f_j = \langle a, f \rangle.$$

Hence

$$\widetilde{Z}_s^N = \frac{\sqrt{-1}}{N\pi} \partial\bar{\partial} \log |\langle a, f\rangle| \,. \tag{8}$$

3.1. **Expected distribution of zeros.** We shall frequently use the notation $E(Y)$ for the expected value of a random variable Y on a probability space $(\Omega, d\mu)$, i.e. $E(Y) = \int_\Omega Y d\mu$.

We view $\widetilde{Z}_s^N$ as a $\mathcal{D}'^{1,1}(M)$-valued random variable (which we call simply a 'random current') as s varies over $SH^0(M, L^N)$ regarded as a probability space with the standard measure, which we denote by μ_N. The expected distribution of zeros of the random section s is the current $E(\widetilde{Z}_s^N) \in \mathcal{D}'^{1,1}(M)$ given by

$$\left(E(\widetilde{Z}_s^N), \varphi\right) = \int_{S^{2d_N-1}} \left(\widetilde{Z}_s^N, \varphi\right) d\mu_N, \quad \varphi \in \mathcal{D}^{m-1,m-1}(M), \tag{9}$$

where we identify $SH^0(M, L^N)$ with the unit $(2d_N+1)$-sphere $S^{2d_N-1} \subset \mathbb{C}^{d_N}$. In fact, we have the following simple formula for the expected zero-distribution in terms of the map Φ_N given by equation (7):

Lemma **3.1.** *For N sufficiently large so that Φ_N is defined, we have:*

$$E(\widetilde{Z}_s^N) = \frac{1}{N} \Phi_N^* \omega_{\mathrm{FS}}$$

Lemma 3.1 is a special case of Lemma 4.3 below. We give here a short alternate proof of Lemma 3.1 which will serve as an introduction to our estimate on the variance (Lemma 3.3) to be given below. We write

$$\omega_N = \frac{1}{N} \Phi_N^* \omega_{\mathrm{FS}} \,. \tag{10}$$

In terms of our fixed orthonormal basis, we have:

$$\omega_N = \frac{\sqrt{-1}}{2\pi N} \partial\bar{\partial} \log \sum_{j=1}^{d_N} |f_j^N|^2 = \frac{\sqrt{-1}}{2\pi N} \partial\bar{\partial} \log |f|^2, \tag{11}$$

where $f = (f_0, \dots, f_{d_N})$ is a local representation of Φ_N as defined above. Let φ be a smooth $(m-1, m-1)$ form, which we shall refer to as a 'test form'. We may assume that we have a coordinate frame for L on Support φ. By (8), we must show that

$$\frac{\sqrt{-1}}{\pi N} \int_{S^{2d_N-1}} \int_M \partial\bar{\partial} \log |\langle a, f\rangle| \wedge \varphi d\mu_N(a) = (\omega_N, \varphi) \,. \tag{12}$$

To compute the integral, we write $f = |f|u$ where $|u| \equiv 1$. Evidently, $\log|\langle a, f\rangle| = \log|f| + \log|\langle a, u\rangle|$. The first term gives

$$\frac{\sqrt{-1}}{\pi N}\int_M \partial\bar\partial \log|f| \wedge \varphi = \int_M \omega_N \wedge \varphi. \tag{13}$$

We now look at the second term. We have

$$\frac{\sqrt{-1}}{\pi}\int_{S^{2d_N-1}}\int_M \partial\bar\partial \log|\langle a, u\rangle| \wedge \varphi d\mu_N(a) = \frac{\sqrt{-1}}{\pi}\int_M \partial\bar\partial\left[\int_{S^{2d_N-1}} \log|\langle a, u\rangle| d\mu_N(a)\right] \wedge \varphi = 0, \tag{14}$$

since the average $\int \log|\langle a, \omega\rangle| d\mu_N(a)$ is a constant independent of u for $|u| = 1$, and thus the operator $\partial\bar\partial$ kills it. □

Combining Corollary 2.2 and Lemma 3.1, we obtain:

Proposition **3.2.** *$E(\widetilde{Z}_s^N) = \omega + O(\frac{1}{N})$; i.e., for each smooth test form φ, we have*

$$E(\widetilde{Z}_s^N, \varphi) = \int_M \omega \wedge \varphi + O(\frac{1}{N}) .$$

3.2. **Variance estimate.** The purpose of this section is to obtain the variance estimate we need to obtain Theorem 1.2. Let φ be a test form. It follows from our formula for the expectation (Lemma 3.1) that the variance of $(\widetilde{Z}_s^N, \varphi)$ is given by

$$E\left((\widetilde{Z}_s^N - \omega_N, \varphi)^2\right) = E\left(|(\widetilde{Z}_s^N, \varphi) - (\omega_N, \varphi)|^2\right) = E\left((\widetilde{Z}_s^N, \varphi)^2\right) - (\omega_N, \varphi)^2 . \tag{15}$$

We have the following estimate of the variance:

Lemma **3.3.** *Let φ be any smooth test form. Then*

$$E\left(|(\widetilde{Z}_s^N, \varphi) - (\omega_N, \varphi)|^2\right) = O(\frac{1}{N^2}).$$

Proof: We again let f be a local representation of Φ_N. Using (8) we easily obtain

$$E\left((\widetilde{Z}_s^N, \varphi)^2\right) = \frac{-1}{\pi^2 N^2}\int_M\int_M (\partial\bar\partial\varphi(z))(\partial\bar\partial\varphi(w)) \int_{S^{2d_N-1}} \log|\langle f(z), a\rangle| \log|\langle f(w), a\rangle| d\mu_N(a) \tag{16}$$

As in the previous lemma we write $f = |f|u$ with $|u| \equiv 1$. Then

$$\begin{aligned}\log|\langle f(z), a\rangle| \log|\langle f(w), a\rangle| \;&=\; \log|f(z)|\log|f(w)| + \log|f(z)|\log|\langle u(w), a\rangle| \\ &\quad + \log|f(w)|\log|\langle u(z), a\rangle| + \log|\langle u(w), a\rangle|\log|\langle u(z), a\rangle|.\end{aligned}$$

The first term contributes

$$\frac{-1}{\pi^2 N^2}\int_M\int_M (\partial\bar\partial\varphi(z))(\partial\bar\partial\varphi(w)) \log|f(z)|\log|f(w)| = \frac{1}{N^2}(\varphi, \Phi_N^*\omega_{\rm FS})^2 = (\varphi, \omega_N)^2. \tag{17}$$

The middle two terms contribute zero to the integral by (14). The lemma at hand thus comes down to the following claim:

$$\left|\int_M\int_M (\partial\bar\partial\varphi(z))(\partial\bar\partial\varphi(w)) \int_{S^{2d_N-1}} \log|\langle u(z), a\rangle| \log|\langle u(w), a\rangle| d\mu_N(a)\right| = O(1). \tag{18}$$

It suffices to show that

$$G_N(x,y) := \int_{S^{2d_N-1}} \log|\langle x,a\rangle| \log|\langle y,a\rangle| d\mu_N(a) = C_N + O(1) \quad (x,y\in S^{2d_N-1}), \tag{19}$$

where C_N is a constant and the $O(1)$ term is uniformly bounded on $S^{2d_N-1}\times S^{2d_N-1}$. To verify (19), we consider the Gaussian integral

$$\widetilde{G}_N(x,y) := \int_{\mathbb{C}^{d_N}} e^{-|a|^2} \log|\langle x,a\rangle| \log|\langle y,a\rangle| da. \tag{20}$$

We evaluate (20) in two different ways. First, we use spherical coordinates $a=\rho\sigma$ with $\sigma\in S^{2d_N-1}$. We have

$$\widetilde{G}_N(x,y) = \int_0^\infty \int_{S^{2d_N-1}} e^{-\rho^2}\rho^{2d_N-1}\left(\log\rho+\log|\langle x,\sigma\rangle|\right)\left(\log\rho+\log|\langle y,\sigma\rangle|\right) d\rho d\sigma \tag{21}$$

where $d\sigma$ denotes the (non-normalized) volume element on the unit sphere. Multiplying out we get four terms. The only term that is non-constant is the term containing both x and y. We then have

$$\begin{aligned}\widetilde{G}_N(x,y) &= C_N + \left[\int_0^\infty e^{-\rho^2}\rho^{2d_N-1}d\rho\right]\int_{S^{2d_N-1}} \log|\langle x,\sigma\rangle|\log|\langle y,\sigma\rangle| d\sigma \\ &= C_N + \frac{(d_N-1)!}{2}\int_{S^{2d_N-1}} \log|\langle x,\sigma\rangle|\log|\langle y,\sigma\rangle| d\sigma.\end{aligned}$$

We now evaluate $\widetilde{G}_N(x,y)$ a second way by noting that coordinates in $\mathbb{C}^{d_N}$ may be chosen so that $x=(1,0,\dots,0)$, $y=(\zeta_1,\zeta_2,0,\dots,0)$. Write $a'=(a_1,a_2)$, $\tilde{a}=(a_3,\dots,a_{d_N})$, $\zeta'=(\zeta_1,\zeta_2)$. Then the integral becomes

$$\widetilde{G}_N(x,y) = \left[\int_{\mathbb{C}^{d_N-2}} e^{-|\tilde{a}|^2} d\tilde{a}\right]\psi(\zeta') = \pi^{d_N-2}\psi(\zeta') \tag{22}$$

where

$$\psi(\zeta') = \int_{\mathbb{C}^2} e^{-|a'|^2}\log|a_1|\log|\langle a',\zeta'\rangle| da' \quad (\zeta'\in S^3\subset\mathbb{C}^2)\ . \tag{23}$$

(To be precise, we have a well-defined continuous map $\zeta: S^{2d_N-1}\times S^{2d_N-1}\to S^3/S^1=\mathbb{CP}^1$ and $\psi(\zeta')=\psi(\zeta(x,y))$.) By the Cauchy-Schwartz inequality, we have

$$\begin{aligned}|\psi(\zeta')| &\le \left[\int_{\mathbb{C}^2} e^{-|a'|^2}(\log|a_1|)^2 da'\right]^{1/2}\left[\int_{\mathbb{C}^2} e^{-|a'|^2}(\log|\langle a',\zeta'\rangle|)^2 da'\right]^{1/2} \\ &= \int_{\mathbb{C}^2} e^{-|a'|^2}(\log|a_1|)^2 da' \ = \ C \ < \ +\infty\ ,\end{aligned}$$

for all $\zeta'\in S^3$. Since

$$d\sigma = \sigma(S^{2d_N-1})d\mu_N = \frac{2\pi^{d_N}}{(d_N-1)!}d\mu_N\ ,$$

we have

$$G_N(x,y) = \frac{1}{\pi^{d_N}}\left(\widetilde{G}_N(x,y)-C_N\right) = \frac{1}{\pi^2}\psi(\zeta') + C'_N\ . \tag{24}$$

Thus

$$E\left(|(\widetilde{Z}_s^N,\varphi)-(\omega_N,\varphi)|^2\right) \le \frac{C}{\pi^4N^2}\sup\|\partial\bar\partial\varphi\|^2 \tag{25}$$

□

3.3. **Almost everywhere convergence.** We can now complete the proof of Theorem 1.1 on the convergence of the zero sets for a random sequence of sections of increasing degree, viewed as an element of the probability space $\mathcal{S} = \prod_{N=1}^{\infty} SH^0(M, L^N)$ with the measure $\mu = \prod_{N=1}^{\infty} \mu_N$. Recall that we identify the unit sphere $SH^0(M, L^N) \subset H^0(M, L^N)$ with the $(2d_N - 1)$-sphere $S^{2d_N-1} \subset \mathbb{C}^{d_N}$ (using the Hermitian inner product described in §2.1); the measure μ_N is Haar probability measure on S^{2d_N-1}.

An element in $\mathcal{S}$ will be denoted $\mathbf{s} = \{s_N\}$. Since

$$|(\widetilde{Z}_{s_N}, \varphi)| \leq (\widetilde{Z}_{s_N}, \omega^{m-1})\|\varphi\|_{\mathcal{C}^0} = c_1(L)^m \|\varphi\|_{\mathcal{C}^0} ,$$

by considering a countable $\mathcal{C}^0$-dense family of test forms, we need only consider one test form φ. By Lemma 2.2, it suffices to show that

$$(\widetilde{Z}_{s_N} - \omega_N, \varphi) \to 0 \qquad \text{almost surely} .$$

Consider the random variables

$$Y_N(\mathbf{s}) = (\widetilde{Z}_{s_N} - \omega_N, \varphi)^2 \geq 0 . \tag{26}$$

By Lemma 3.3,

$$\int_{\mathcal{S}} Y_N(\mathbf{s}) d\mu(\mathbf{s}) = O(\frac{1}{N^2}) .$$

Therefore

$$\int_{\mathcal{S}} \sum_{N=1}^{\infty} Y_N d\mu = \sum_{N=1}^{\infty} \int_{\mathcal{S}} Y_N d\mu < +\infty,$$

and hence $Y_N \to 0$ almost surely. □

Remark: Since Lemma 3.3 gives an $O\left(\frac{1}{N^2}\right)$ bound on $E(Y_N)$, we have for any $\epsilon > 0$, $\int_{\mathcal{S}} N^{1-2\epsilon} Y_N d\mu = O\left(\frac{1}{N^{1+2\epsilon}}\right)$. Thus the above proof actually shows that

$$\left|\left(\widetilde{Z}_N, \varphi\right) - (\omega, \varphi)\right| \leq O\left(\frac{1}{N^{\frac{1}{2}-\epsilon}}\right) , \quad \text{almost surely.}$$

3.4. **Zeros of random orthonormal bases.** We now switch our attention to sequences of orthonormal bases and prove Theorem 1.2. We let $\mathcal{ONB} = \prod_{N=1}^{\infty} \mathcal{ONB}_N$ denote the space of sequences $\{(S_1^N, \dots, S_{d_N}^N) : N = 1, 2, \dots\}$, where $(S_1^N, \dots, S_{d_N}^N)$ is an element of the space $\mathcal{ONB}_N$ of orthonormal bases for $H^0(M, L^N)$. Choosing a fixed

$$\mathbf{e} = \{e_j^N : j = 0, \dots, d_N,\ N = 1, 2, \dots\} \in \mathcal{ONB}$$

gives the identifications $\mathcal{ONB}_N \equiv \mathrm{U}(d_N)$ (the unitary group of rank d_N) and

$$\mathcal{ONB} \equiv \prod_{N=1}^{\infty} \mathrm{U}(d_N). \tag{27}$$

We give $\mathcal{ONB}$ the measure

$$\nu := \prod_{N=1}^{\infty} \nu_N , \tag{28}$$

where ν_N is the unit-mass Haar measure on $\mathrm{U}(d_N)$.

The variance analogous of Lemma 3.3 carries over to orthonormal bases:

LEMMA **3.4.** *For a smooth test form φ, we have*

$$E\Big((\widetilde{Z}^N_{S^N_j}-\omega_N,\varphi)^2\Big)=O(\frac{1}{N^2})$$

Proof: Let $\pi^N_j:\mathcal{ONB}_N\to SH^0(M,L^N)$ denote the projection to the j-th factor. Since $\pi^N_{j*}\nu_N=\mu_N$, we see that

$$E_{\mathrm{U}(d_N)}\Big((\widetilde{Z}^N_{S^N_j}-\omega_N,\varphi)^2\Big)=E_{S^{2d_N-1}}\Big((\widetilde{Z}^N_s-\omega_N,\varphi)^2\Big)\ ,$$

and thus Lemmas 3.3 and 3.4 are equivalent. □

The proof of Theorem 1.2 follows easily from Lemma 3.4 exactly as in the proof of Theorem 1.1. (The equivalence of the second conclusion follows from [Z2, §1.3].)

4. Zeros of SU(k) polynomials

As an example, we apply Lemma 3.1 to the case $M=\mathbb{CP}^m$, $L=\mathcal{O}(1)$, where we give L the standard Hermitian metric h_{FS}, whose curvature is the Fubini-Study Kähler form $\omega=\omega_{\mathrm{FS}}$ on $\mathbb{CP}^m$. We also extend Lemma 3.1 to the case of simultaneous zeros.

4.1. SU(2) **polynomials.** First consider $m=1$. Elements of $H^0(M,L^N)=H^0(\mathbb{CP}^1,\mathcal{O}(N))$ are homogeneous polynomials in two variables of degree N, or equivalently, polynomials in one variable of degree $\le N$. A basis is given by $\sigma_j=z^j,\ j=0,\dots,N$. The inner product in $H^0(M,L^N)$ is given by

$$\langle\sigma_j,\sigma_k\rangle=\int_{\mathbb{C}}\frac{z^j\,\bar z^k}{(1+|z|^2)^N}\omega=\frac{1}{\pi}\int_{\mathbb{C}}\frac{z^j\,\bar z^k}{(1+|z|^2)^{N+2}}dxdy\ .$$

Writing the integral in polar coordinates, we see that the σ_j are orthogonal, and

$$|\sigma_j|^2=2\int_0^\infty\frac{r^{2j+1}}{(1+r^2)^{N+2}}dr=\frac{1}{(N+1)\binom{N}{j}}\ . \tag{29}$$

We thus can choose an orthonormal basis

$$S^N_j=(N+1)^{\frac12}\binom{N}{j}^{\frac12}z^j,\qquad j=0,\dots,N\ .$$

Next, we note that

$$\sum_{j=1}^N\|S^N_j\|^2=(1+|z|^2)^{-N}\sum_{j=1}^N(N+1)\binom{N}{j}|z^{2j}|\equiv N+1\ ,$$

and thus $\omega_N=\frac{1}{N}\Phi_N^*\omega_{\mathrm{FS}}=\omega$. We thus recover the following result of [BBL, Appendix C] on 'random SU(2) polynomials':

THEOREM **4.1.** [BBL] *Suppose we have a random polynomial*

$$P(z)=c_0+c_1z+\cdots+c_Nz^N\ ,$$

where $\mathrm{Re}\,c_0,\ \mathrm{Im}\,c_0,\ \dots,\ \mathrm{Re}\,c_N,\ \mathrm{Im}\,c_N$ *are independent Gaussian random variables with mean 0 and variances*

$$E\left((\mathrm{Re}\,c_j)^2\right)=E\left((\mathrm{Im}\,c_j)^2\right)=\binom{N}{j}\ .$$

Then the expected distribution of zeros of P is uniform over $\mathbb{CP}^1\approx S^2$.

In fact, Theorem 1.1 tells us that for a random sequence of such polynomials, the distribution of zeros approaches uniformity.

4.2. SU$(m+1)$ **polynomials.** We now turn to the case of polynomials in several variables. An 'SU$(m+1)$ polynomial of degree N' is an element of the probability space of homogeneous polynomials of degree N on $\mathbb{C}^{m+1}$ with an SU$(m+1)$-invariant Gaussian probability measure. Recall that this space can be identified with $H^0(\mathbb{CP}^m, \mathcal{O}(N))$. We give $H^0(\mathbb{CP}^m, \mathcal{O}(N))$ the standard inner product. A basis for $H^0(\mathbb{CP}^m, \mathcal{O}(N))$ is given by the monomials

$$\sigma_J = z_0^{j_0} \cdots z_m^{j_m}, \qquad J = (j_0, \dots, j_m),\ |J| = N .$$

One easily sees that the σ_J are orthogonal. We compute

$$|\sigma_J|^2 = \int_{\mathbb{CP}^m} \frac{|\sigma_J(z)|^2}{|z|^{2N}} \omega_{FS}^m = \int_{S^{2m+1}} |\sigma_J(z)|^2 d\mu^{2m+1} = \frac{m! j_0! \cdots j_m!}{(N+m)!} \tag{30}$$

(where μ^{2m+1} is Haar probability measure on S^{2m+1}), by writing

$$\int_{\mathbb{C}^{m+1}} e^{-|z|^2} |\sigma_J(z)|^2 dz = \left(\int_{\mathbb{C}} e^{-|z_0|^2} |z_0|^{2j_0} dz_0 \right) \cdots \left(\int_{\mathbb{C}} e^{-|z_m|^2} |z_m|^{2j_m} dz_m \right) .$$

Therefore, the sections

$$S_J^N := \left[\frac{(N+m)!}{m! j_0! \cdots j_m!} \right]^{\frac{1}{2}} z^J$$

form an orthonormal basis for $H^0(\mathbb{CP}^m, \mathcal{O}(N))$. Furthermore

$$\sum_{|J|=N} \|S_J^N\|^2 \equiv \binom{N+m}{m} , \tag{31}$$

since the sum is SU$(m+1)$ invariant, hence constant, and the integral of the left side equals $\dim H^0(\mathbb{CP}^m, \mathcal{O}(N))$.

In our results on zeros, we can replace the unit sphere $SH^0(M, L^N)$ with the complex d_N-dimensional vector space $H^0(M, L^N)$ with the Gaussian probability measure $\frac{1}{\pi^{d_N}} e^{-|s|^2} ds$ (where ds means $2d_N$-dimensional Lebesgue measure). (We continue to use the inner product structure on $H^0(M, L^N)$ introduced in §2.1.) The space of SU$(m+1)$ polynomials of degree N is by definition the space $H^0(\mathbb{CP}^m, \mathcal{O}(N))$ of homogeneous polynomials of degree N in $m+1$ variables (or equivalently, polynomials in m variables of degree $\leq N$) with this Gaussian measure. We can use (30) to describe the space of SU$(m+1)$ polynomials explicitly as follows. For $P \in H^0(\mathbb{CP}^m, \mathcal{O}(N))$, we write

$$P(z_0, \dots, z_m) = \sum_{|J|=N} \frac{a_J}{\sqrt{j_0! \cdots j_m!}} z_0^{j_0} \cdots z_m^{j_m} . \tag{32}$$

The Gaussian measure on $H^0(\mathbb{CP}^m, \mathcal{O}(N))$ is then given by

$$\frac{1}{\pi^{d_N}} e^{-|A|^2} dA , \qquad A = (a_J) \in \mathbb{C}^{d_N} ,$$

where $d_N = \binom{N+m}{m}$.

Lemma 3.1 and (31) now tell us that if P is a polynomial given by (32), with the a_J being independent Gaussian random variables with mean 0 and variance 1, then the expected zero current Z_P equals $N\omega_{FS}$. (This fact, which is the higher dimensional analogue of

Theorem 4.1, is extended to cover simultaneous zeros in Proposition 4.5 below.) Furthermore, Theorem 1.1 yields the following:

PROPOSITION **4.2.** *Suppose we have a sequence of polynomials*

$$P_N(z_0,\dots,z_m)=\sum_{|J|=N}\frac{a^N_J}{\sqrt{j_0!\cdots j_m!}}z_0^{j_0}\cdots z_m^{j_m}\ ,$$

where the a^N_J are independent Gaussian random variables with mean 0 and variance 1. Then

$$\frac{1}{N}Z_{P_N}\to\omega_{FS}\qquad \text{almost surely}$$

(weakly in the sense of measures).

4.3. **Expected distribution of simultaneous zeros.** We take a brief detour now to generalize Lemma 3.1 and Proposition 3.2 to simultaneous zero sets of holomorphic sections. This yields a generalization (Proposition 4.5) of Theorem 4.1 to the case of simultaneous zeros of polynomials in several variables. In particular, the 0-dimensional case of Proposition 4.5 says that the simultaneous zeros of m random $\mathrm{SU}(m+1)$ polynomials are uniformly distributed on $\mathbb{CP}^m$ with respect to the volume $\omega_{\rm FS}^m$.

Let $1\le\ell\le m$, and consider the Grassmannian of ℓ-dimensional subspaces of $H^0(M,L^N)$, which we denote $G_\ell H^0(M,L^N)$. For an element $\mathcal{S}=\mathrm{Span}\{s_1,\dots,s_\ell\}\in G_\ell H^0(M,L^N)$, we let $Z_{\mathcal{S}}\in\mathcal{D}'^{\ell,\ell}$ denote the current of integration over the set $\{z\in M: s_1(z)=\cdots=s_\ell(z)=0\}$. Note that this definition is independent of the choice of basis $\{s_j\}$ of $\mathcal{S}$; furthermore by Bertini's theorem (see [GH]), the zero sets Z_{s_j} are smooth and intersect transversely for almost all $\mathcal{S}$, so we can ignore multiplicities if we wish. As before, we consider the normalized current

$$\widetilde Z^N_{\mathcal{S}}=\frac{1}{N^\ell}Z_{\mathcal{S}}\ ,$$

which we regard as a random current with $\mathcal{S}$ varying over the probability space $G_\ell H^0(M,L^N)$ with unit-mass Haar measure. The expected value of $\widetilde Z^N_{\mathcal{S}}$ is then given by the following elementary formula:

LEMMA **4.3.** *For N sufficiently large so that Φ_N is defined, we have:*

$$E(\widetilde Z^N_{\mathcal{S}})=\omega_N^\ell\ .$$

Proof: Using our fixed orthonormal basis $\{S^N_j\}$, we can write $s_k=\sum_{j=1}^{d_N}a^j_kS^N_j$. Let

$$\mathcal{S}^\perp=\{w\in\mathbb{CP}^{d_N-1}:\sum_{j=1}^{d_N}a^j_kw_j=0,\quad k=1,\dots,\ell\}\ .$$

We let $[\mathcal{S}^\perp]$ denote the current of integration over $\mathcal{S}^\perp$, regarded as a $\mathcal{D}'^{\ell,\ell}(\mathbb{CP}^{d_N-1})$-valued random variable. Since $\widetilde Z^N_{\mathcal{S}}=\frac{1}{N^\ell}\Phi_N^*[\mathcal{S}^\perp]$, we then have

$$E(\widetilde Z^N_{\mathcal{S}})=\frac{1}{N^\ell}\Phi_N^*E([\mathcal{S}^\perp])\ ,$$

where

$$E([\mathcal{S}^\perp])=\int_{G_\ell\mathbb{C}^{d_N}}[\mathcal{S}^\perp]d\mathcal{S}\ .$$

We note that $E([\mathcal{S}^\perp])$ is $\mathrm{U}(d_N)$-invariant. It is well-known that the only (ℓ,ℓ)-currents on projective space that are invariant under the unitary group are multiples of $\omega_{\mathrm{FS}}^\ell$; see [Sh, Lemma 3.3]. Since $(E([\mathcal{S}^\perp]),\omega^{m-\ell})=1$, we conclude that $E([\mathcal{S}^\perp])=\omega_{\mathrm{FS}}^\ell$ and thus

$$E(\widetilde{Z}_{\mathcal{S}}^N)=\frac{1}{N^\ell}\Phi_N^*\omega_{\mathrm{FS}}^\ell=\omega_N^\ell\ .$$

□

Applying Corollary 2.2, we obtain the following generalization of Proposition 3.2:

PROPOSITION **4.4.** *Let $\mathcal{S}$ be a random element of $G_\ell H^0(M,L^N)$, where $1\le\ell\le m$. Then*

$$E(\widetilde{Z}_{\mathcal{S}}^N)=\omega^\ell+O(\frac{1}{N})\ .$$

We now apply Lemma 4.3 to random $\mathrm{SU}(m+1)$ polynomials to obtain:

PROPOSITION **4.5.** *Choose an ℓ-tuple $\mathcal{P}=(P_1,\dots,P_\ell)$ of $\mathrm{SU}(m+1)$ polynomials of degree N at random. Then*

$$E(Z_{\mathcal{P}})=N^\ell\omega_{FS}^\ell\ ,$$

and in particular

$$E\left(\mathrm{Vol}_{2m-2\ell}\{z\in U:P_1(z)=\cdots=P_\ell(z)=0\}\right)=\frac{m!}{(m-\ell)!}N^\ell\mathrm{Vol}_{2m}(U)$$

for all open subsets U of $\mathbb{CP}^m$ (where Vol_k denotes the Riemannian k-volume in (M,ω)).

Proof: An ℓ-tuple of $\mathrm{SU}(m+1)$ polynomials is an element of the probability space

$$\left(\left[H^0(\mathbb{P}^m,\mathcal{O}(N))\right)^\ell,d\mathcal{G}\right)\ ,$$

where $d\mathcal{G}$ the ℓ-fold self-product of the Gaussian measure on $H^0(\mathbb{P}^m,\mathcal{O}(N))$ (which, of course, is itself a Gaussian measure). By (31), we conclude as before that $\omega_N=\omega$. Let

$$\Omega=\left\{(W_1,\dots,W_\ell)\in\left[H^0(\mathbb{P}^m,\mathcal{O}(N))\right]^\ell:W_1\wedge\cdots\wedge W_\ell\neq0\right\}\ ,$$

and let $\gamma:\Omega\to G_\ell H^0(\mathbb{P}^m,\mathcal{O}(N))$ be the natural map. The conclusion follows from Lemma 4.3 by noting that $\gamma_*(d\mathcal{G})$ equals Haar measure on $G_\ell H^0(\mathbb{P}^m,\mathcal{O}(N))$. □

5. Ergodic orthonormal bases and sections

We now turn to the distribution of zeros of sections which form an 'ergodic orthonormal basis'. As will be explained below, eigenfunctions of quantum ergodic maps form such a basis. So do random orthonormal bases. Both of these facts belong to now familiar genres of results in quantum chaos. Let us briefly recall the basic definitions and results and then prove the principal new results, Theorem 1.3 and Lemma 1.4. Proofs of the background results on ergodic bases are given in the Appendix.

5.1. **The ergodic property.** The weak*-convergence hypothesis of Lemma 1.4 is closely related to the following 'ergodic property':

Definition: We say that $\mathbf{S} \in \mathcal{ONB}$ has the ergodic property if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \frac{1}{d_n} \sum_{j=1}^{d_n} \left| \int_M \varphi(z) \|S_j^n(z)\|_{h_n}^2 dV - \bar{\varphi} \right|^2 = 0 \,, \quad \forall \varphi \in \mathcal{C}(M) \,. \tag{$\mathcal{EP}$}$$

Here, $\bar{\varphi} = \int_M \varphi dV$ denotes the average value of a function f over M.

As is well-known (see, for example [Z2, §1]), this property may be rephrased in the following way: Let $\mathbf{S} = \{(S_1^N, \dots, S_{d_N}^N) : N = 1, 2, \dots\} \in \mathcal{ONB}$. Then the ergodic property $(\mathcal{EP})$ is equivalent to the following weak* convergence property: *There exists a subsequence $\{S_1', S_2', \dots\}$ of relative density one of the sequence $\{S_1^1, \dots, S_{d_1}^1, \ \dots, \ S_1^N, \dots, S_{d_N}^N, \ \dots\}$ such that*

$$\int_M \varphi(z) \|S_n'(z)\|^2 dV \to \bar{\varphi} \,, \quad \forall \varphi \in \mathcal{C}(M) \,. \tag{$\mathcal{EP}'$}$$

A subsequence $\{a_{k_n}\}$ of a sequence $\{a_n\}$ is said to have relative density one if $\lim_{n\to\infty} n/k_n = 1$. The equivalence of $(\mathcal{EP})$ and $(\mathcal{EP}')$ is a consequence of the fact that if

$$\{a_1, a_2, a_3, \dots\} = \{A_1^1, \dots, A_{d_1}^1, \ \dots, \ A_1^n, \dots, A_{d_n}^n, \ \dots\}$$

is a sequence of non-negative real numbers, then the following are equivalent:

i) there exists a subsequence $\{a_{k_n}\}$ of relative density one such that $\lim_{n\to\infty} a_{k_n} \to 0$.
ii) $\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} a_n \to 0$.
iii) $\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \frac{1}{d_n} \sum_{j=1}^{n} A_j^n \to 0$.

The equivalence of (i) and (ii) is given in [W, Theorem 1.20]. (By a diagonalization argument, one can pick a subsequence independent of φ satisfying $(\mathcal{EP}')$.) For the equivalence of (ii) and (iii), which depends on the fact that $d_n \sim n^m$, see [Z2, §1.3].

We first have:

THEOREM **5.1. (a)** *A random $\mathbf{S} \in \mathcal{ONB}$ has the ergodic property $(\mathcal{EP})$, or equivalently, $(\mathcal{EP}')$. In fact, in complex dimensions $m \geq 2$, a random $\mathbf{S} \in \mathcal{ONB}$ has the property*

$$\lim_{N\to\infty} \frac{1}{d_N} \sum_{j=1}^{d_N} \left| \int_M \varphi \|S_j^N\|^2 dV - \bar{\varphi} \right|^2 = 0 \,, \quad \forall \varphi \in \mathcal{C}(M) \,,$$

or equivalently, for each N there exists a subset $\Lambda_N \subset \{1, \dots, d_N\}$ such that $\frac{\#\Lambda_N}{d_N} \to 1$ and

$$\lim_{N\to\infty, j\in\Lambda_N} \int_M \varphi \|S_j^N\|^2 dV = \bar{\varphi}.$$

(b) *A random sequence of sections $\mathbf{s} = \{s_1, s_2, \dots\} \in \mathcal{S}$ has a subsequence $\{s_{N_k}\}$ of relative density 1 such that*

$$\int_M \varphi(z) \|s_{N_k}(z)\|^2 dV \to \bar{\varphi} \,, \quad \forall \varphi \in \mathcal{C}(M) \,.$$

In complex dimensions $m \geq 2$, the entire sequence has this property.

Theorem 5.1(a) is the line-bundle analogue of Theorem (b) in [Z2] on random orthonormal combinations of eigenfunctions of positive elliptic operators with periodic bicharacteristic flow. The proof of Theorem 5.1 closely parallels those of [Z1, Z2] and strengthens them in dimensions $m \geq 2$. Details will be given the Appendix below.

The second setting in which ergodic orthonormal bases appear is that of quantum ergodicity. We recall the following result from [Z3, Theorem B-Corollary B], which together with Lemma 1.4 yields Theorem 1.3.

THEOREM **5.2.** [Z3] *Let $\{S_j^N\}$ be an orthonormal basis of eigenfunctions of an ergodic quantum map $U_{\chi,N}$ on $H^0(M, L^N)$ (as described in Theorem 1.3). Then $\{S_j^N\}$ has the ergodic property $(\mathcal{EP})$, or equivalently, $(\mathcal{EP}')$.*

Theorem 5.2 belongs to a long line of results originating in the work of A. Shnirelman [Shn1] in 1974 (see also [Shn2]) on eigenfunctions of the Laplacian on compact Riemannian manifolds with ergodic geodesic flow. The definition of 'quantum map' and the proof of ergodicity of eigenfunctions for ergodic quantum maps over compact Kähler manifolds is contained in [Z3], where further references can be found to the literature of quantum ergodicity.

We now complete the proofs of Theorems 1.2 and 1.3 by verifying Lemma 1.4.

5.2. **Proof of Lemma 1.4.** Let $(L, h) \to (M, \omega)$ and $s_N \in H^0(M, L^N)$, $N = 1, 2, \dots,$ be as in the hypotheses of Lemma 1.4. We write

$$u_N = \frac{1}{N} \log \|s_N(z)\|_{h_N} \,.$$

First we observe that it suffices to show that $u_N \to 0$ in $L^1(M)$. Indeed, if that is the case, then for any smooth test form $\varphi \in \mathcal{D}^{m-1,m-1}(M)$, we have by the Poincaré-Lelong formula (3),

$$\left(\frac{1}{N} Z_N - \omega, \varphi\right) = \left(u_N, \frac{\sqrt{-1}}{\pi} \partial\bar{\partial}\varphi\right) \to 0 \,.$$

Since by(4),

$$\left(\frac{1}{N} Z_N, \varphi\right) \leq \frac{c_1(L)^m}{m!} \sup|\varphi| \,,$$

the conclusion of the lemma holds for all $\mathcal{C}^0$ test forms φ.

Next, we observe that:

i) the functions u_N are uniformly bounded above on M;

ii) $\limsup_{N\to\infty} u_N \leq 0$.

Indeed, since $\|s_N\|^2$ converges weakly to 1, we have

$$|s_N|^2 = \int_M \|s_N\|_{h_N}^2 dV \to 1; .$$

Choose orthonormal bases $\{S_j^N\}$ and write $s_N = \sum_j a_j S_j^N$, so that $\sum |a_j|^2 = |s_N|^2$. By Theorem 2.1, we have

$$\|s_N(z)\|_{h_N}^2 \leq |s_N|^2 \sum_{j=1}^{d_N} \|S_j^N(z)\|_{h_N}^2 = \left(\frac{c_1(L)^m}{m!} + O(1/N)\right) N^m.$$

Hence $\|s_N(z)\|_{h_N} \leq C N^{m/2}$ for some $C < \infty$ and taking the logarithm gives both statements.

Let e_L be a local holomorphic frame for L over $U \subset M$ and let e_L^N be the corresponding frame for L^N. Let $g(z) = \|e_L(z)\|_h$ so that $\|e_L^N(z)\|_{h_N} = g(z)^N$. Then we may write $s_N = f_N e_L^N$ with $f_N \in \mathcal{O}(U)$ and $\|s_N\|_{h_N} = |f_N| g^N$. It is useful to consider the function

$$v_N = \frac{1}{N} \log |f_N| = u_N - \log g \ ,$$

which is plurisubharmonic on U. (For the properties of plurisubharmonic functions used here, see for example, [Kl].)

To finish the proof, we follow the potential-theoretic approach used by Fornaess and Sibony [FS] in their proof of the Brolin-Lyubich theorem on the dynamics of rational functions. Let U' be a relatively compact, open subset of U. We must show that $u_N \to 0$ (or equivalently, $v_N \to -\log g$) in $L^1(U')$. Suppose on the contrary that $u_N \not\to 0$ in $L^1(U')$. Then we can find a subsequence $\{u_{N_k}\}$ with $\|u_{N_k}\|_{L^1(U')} \geq \delta > 0$. By a standard result on subharmonic functions (see [Ho, Theorem 4.1.9]), we know that the sequence $\{v_{N_k}\}$ either converges uniformly to $-\infty$ on U' or else has a subsequence which is convergent in $L^1(U')$. Let us now rule out the first possibility. If it occurred, there would exist $K > 0$ such that for $k \geq K$,

$$\frac{1}{N_k} \log \|s_{N_k}(z)\|_{h_{N_k}} \leq -1. \tag{33}$$

However, (33) means that

$$\|s_{N_k}(z)\|^2_{h_{N_k}} \leq e^{-2N_k} \quad \forall z \in U' \ ,$$

which is inconsistent with the hypothesis that $\|s_{N_k}(z)\|^2_{h_{N_k}} \to 1$ in the weak* sense.

Therefore there must exist a subsequence, which we continue to denote by $\{v_{N_k}\}$, which converges in $L^1(U')$ to some $v \in L^1(U')$. By passing if necessary to a further subsequence, we may assume that $\{v_{N_k}\}$ converges pointwise almost everywhere in U' to v, and hence

$$v(z) = \limsup_{k\to\infty} u_{N_k}(z) - \log g \leq -\log g \qquad \text{(a.e)} \ .$$

Now let

$$v^*(z) := \limsup_{w\to z} v(w) \leq -\log g$$

be the upper-semicontinuous regularization of v. Then v^* is plurisubharmonic on U' and $v^* = v$ almost everywhere.

Since $\|v_{N_k} + \log g\|_{L^1(U')} = \|u_{N_k}\|_{L^1(U')} \geq \delta > 0$, we know that $v^* \not\equiv -\log g$. Hence, for some $\epsilon > 0$, the open set $U_\epsilon = \{z \in U' : v^* < -\log g - \epsilon\}$ is non-empty. Let U'' be a non-empty, relatively compact, open subset of U_ϵ; by Hartogs' Lemma, there exists a positive integer K such that $v^* \leq -\log g - \epsilon/2$ for $z \in U''$, $k \geq K$; i.e.,

$$\|s_{N_k}(z)\|^2_{h_{N_k}} \leq e^{-\epsilon N_k}, \qquad z \in U'', \ \ k \geq K, \tag{34}$$

which contradicts the weak convergence to 1. $\square$

6. Appendix

In this Appendix, we give a proof of Theorem 5.1, closely following the proof of Proposition 2.1.4(b) in [Z2].

To simplify things, we write

$$A^{\varphi}_{nj}(\mathbf{S}) = \left|\int_M \varphi(z)\|S^n_j(z)\|^2_{h_n} dV - \bar{\varphi}\right|^2 . \tag{35}$$

In view of the isomorphism $H^0(M, L^N) \cong H^2_N(X)$, we may identify $\mathcal{ONB}$ with the space of orthonormal bases of eigenfunctions for the operator $\frac{1}{i}\frac{\partial}{\partial\theta}$ generating the S^1 action on X. Assume without loss of generality that φ is real-valued, and consider the Toeplitz operators $T^{\varphi}_N = \Pi_N M_{\varphi} \Pi_N = \Pi_N M_{\varphi} : H^2_N(X) \to H^2_N(X)$, where M_{φ} is multiplication by the lift of φ to X, and $\Pi_N : \mathcal{L}^2(X) \to H^2_N(X)$ is the orthogonal projection. Then T^{φ}_N is a self-adjoint operator on $H^2_N(X)$, which can be identified with a Hermitian $d_N \times d_N$ matrix via the fixed basis $\mathbf{e}$. We then have

$$A^{\varphi}_{nj}(\mathbf{S}) = \left|(\varphi S^n_j, S^n_j) - \bar{\varphi}\right|^2 = \left|(T^{\varphi}_n S^n_j, S^n_j) - \bar{\varphi}\right|^2 = \left|(U^*_n T^{\varphi}_n U_n e^n_j, e^n_j) - \bar{\varphi}\right|^2 , \tag{36}$$

where $\mathbf{S} = \{U_N\},\ U_N \in \mathrm{U}(d_N) \equiv \mathcal{ONB}_N$.

We have

$$\bar{\varphi} = \frac{1}{d_n}\int_M \sum_{j=1}^{d_n} \|e^n_j\|^2 \varphi dV + \int_M \left[1 - \frac{1}{d_n}\sum_{j=1}^{d_n} \|e^n_j\|^2\right]\varphi dV = \frac{1}{d_n}\mathrm{Tr}\ T^{\varphi}_n + O(\frac{1}{n}) , \tag{37}$$

where the last equality is by Theorem 2.1. Therefore,

$$A^{\varphi}_{nj}(\mathbf{S}) = \widetilde{A}^{\varphi}_{nj}(\mathbf{S}) + O(\frac{1}{n}) , \tag{38}$$

where

$$\widetilde{A}^{\varphi}_{nj}(\mathbf{S}) = \left|(U^*_n T^{\varphi}_n U_n e^n_j, e^n_j) - \frac{1}{d_n}\mathrm{Tr}\ T^{\varphi}_n\right|^2 . \tag{39}$$

(The bound for the $O(\frac{1}{n})$ term in (38) is independent of $\mathbf{S}$.)

Note that iT^{φ}_N can be identified with an element of the Lie algebra $\mathfrak{u}(d_N)$ of $\mathrm{U}(d_N)$. Let $\mathfrak{t}(d)$ denote the Cartan subalgebra of diagonal elements in $\mathfrak{u}(d)$, and let $\|\cdot\|^2$ denote the Euclidean inner product on $\mathfrak{t}(d)$. Also let

$$J_d : i\mathfrak{u}(d) \to i\mathfrak{t}(d)$$

denote the orthogonal projection (extracting the diagonal). Finally, let

$$\bar{J}_d(H) = \left(\frac{1}{d}\mathrm{Tr}\ H\right)\mathrm{Id}_d ,$$

for Hermitian matrices $H \in i\mathfrak{u}(d)$. (Thus, $H = H^0 + \bar{J}_d(H)$, with H^0 traceless, gives us the decomposition $\mathfrak{u}(d) = \mathfrak{su}(d) \oplus \mathbb{R}$.)

We introduce the random variables:

$$Y^{\varphi}_n : \mathcal{ONB} \to [0, +\infty)$$
$$Y^{\varphi}_n(\mathbf{S}) := \|J_{d_n}(U^*_n T^{\varphi}_n U_n) - \bar{J}_{d_n}(T^{\varphi}_n)\|^2$$

By (38)

$$\frac{1}{d_n} Y^{\varphi}_n(\mathbf{S}) = \frac{1}{d_n}\sum_{j=1}^{d_n} \widetilde{A}^{\varphi}_{nj}(\mathbf{S}) = \frac{1}{d_n}\sum_{j=1}^{d_n} A^{\varphi}_{nj}(\mathbf{S}) + O(\frac{1}{n}) \tag{40}$$

(where the $O(\frac{1}{n})$ term is independent of $\mathbf{S}$). Thus, $(\mathcal{EP})$ is equivalent to:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \frac{1}{d_n} Y_n^\varphi(\mathbf{S}) = 0 \ , \quad \forall \varphi \in \mathcal{C}(M) \ . \tag{41}$$

The main part of the proof of (41) is to show the following asymptotic formula for the expected values of the Y_n^φ.

Lemma **6.1.** $E(Y_n^\varphi) = \overline{\varphi^2} - (\bar\varphi)^2 + o(1)$.

Assume Lemma 6.1 for the moment. The lemma implies that (41) holds on the average; i.e.,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} E\left(\frac{1}{d_n} Y_n^\varphi\right) = 0 \ . \tag{42}$$

Next we note that

$$\mathrm{Var}\left(\frac{1}{d_n} Y_n^\varphi\right) \le \sup\left(\frac{1}{d_n} Y_n^\varphi\right)^2 \le \max_j \ \sup(\widetilde{A}_{nj}^\varphi)^2 \ .$$

By (39),

$$\widetilde{A}_{nj}^\varphi(\mathbf{S}) \le 4(U_n^* T_n^\varphi U_n e_j^n, e_j^n)^2 \le 4 \sup \varphi^2 \ ,$$

and therefore

$$\mathrm{Var}\left(\frac{1}{d_n} Y_n^\varphi\right) \le 16 \sup \varphi^4 < +\infty \ . \tag{43}$$

Since the variances of the independent random variables $\frac{1}{d_n} Y_n^\varphi$ are bounded, (41) follows from (42) and the Kolmogorov strong law of large numbers, which gives part (a) for general dimensions. In dimensions $m \ge 2$, we obtain the improved conclusion as follows: From the fact that $E(\frac{1}{d_N} Y_N^\varphi) = O(\frac{1}{N^m})$ it follows that $E\left(\sum_{N=1}^\infty \frac{1}{d_N} Y_N^\varphi\right) < +\infty$ and thus $\frac{1}{d_N} Y_N^\varphi \to 0$ almost surely when $m \ge 2$. The quantity we are interested in is

$$X_N^\varphi := \frac{1}{d_N} \sum_{j=1}^{d_N} \left| \int_M \varphi \|S_j^N\|^2 dV - \bar\varphi \right|^2 = \frac{1}{d_N} \sum_{j=1}^{d_N} A_{Nj}^\varphi.$$

However, by (40),

$$\sup_{\mathcal{ONB}} |X_N^\varphi - \frac{1}{d_N} Y_N^\varphi| = O(\frac{1}{N}).$$

Hence also $X_N^\varphi \to 0$ almost surely.

To verify part (b), we note that since $E(\widetilde{A}_{nj}^\varphi) = E(\widetilde{A}_{n1}^\varphi)$, for all j, it follows from (40) that $E(\widetilde{A}_{n1}^\varphi) = E(\frac{1}{d_n} Y_n^\varphi)$. Thus,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \widetilde{A}_{n1}^\varphi = 0 \ , \tag{44}$$

or equivalently,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} A_{n1}^\varphi = 0 \ . \tag{45}$$

Part (b) then follows from (45) exactly as before.

It remains to prove Lemma 6.1. Denote the eigenvalues of T_n^φ by $\lambda_1, \dots, \lambda_{d_n}$ and write

$$\mathcal{S}_k(\lambda_1, \dots, \lambda_{d_n}) = \sum_{j=1}^{d_n} \lambda_j^k .$$

Note that

$$\mathrm{Tr}\,(T_n^\varphi)^k = \mathcal{S}_k(\lambda_1, \dots, \lambda_{d_n}) . \tag{46}$$

We shall use the following 'Szego limit theorem' due to Boutet de Monvel and Guillemin [BG, Theorem 13.13]:

Lemma **6.2.** [BG] *For $k \in \mathbb{Z}^+$, we have*

$$\lim_{N\to\infty} \frac{1}{d_N} \mathrm{Tr}\,(T_N^\varphi)^k = \overline{\varphi^k} .$$

Lemma 6.1 is an immediate consequence of Lemma 6.2 and the following formula:

$$\int_{\mathrm{U}(d)} \|J_d(U^* D(\vec\lambda) U) - \bar J_d(D(\vec\lambda))\|^2 dU = \frac{\mathcal{S}_2(\vec\lambda)}{d+1} - \frac{\mathcal{S}_1(\vec\lambda)^2}{d(d+1)} , \tag{47}$$

where $\vec\lambda = (\lambda_1, \dots, \lambda_d) \in \mathbb{R}^d$, $D(\vec\lambda)$ denotes the diagonal matrix with entries equal to the λ_j, and integration is with respect to Haar probability measure on $\mathrm{U}(d)$.

A proof of the identity (47) is given in [Z1, pp. 68–69] (see also [Z2]). For completeness, we provide here a simplified proof of (47) following the methods of [Z1, Z2]. Let $\mathcal{E}(\vec\lambda)$ denote the left side of (47). Since $\mathcal{E}(\vec\lambda)$ is a homogeneous, degree 2, symmetric polynomial in $\vec\lambda$, we can write

$$\mathcal{E}(\vec\lambda) = c_d \mathcal{S}_2(\vec\lambda) + c'_d \mathcal{S}_1(\vec\lambda)^2 . \tag{48}$$

Substituting $\vec\lambda = (1, \dots, 1)$ in (48) and using the fact that $\mathcal{E}(1, \dots, 1) = 0$, we conclude that $c'_d = -c_d/d$. To find c_d, we substitute $\vec\lambda = (1, 0, \dots, 0)$, and write $D = D(1, 0, \dots, 0)$. For $U = (u_{jk}) \in \mathrm{U}(d)$, we have

$$(U^* D U)_{jj} = |u_{1j}|^2 , \quad \bar J_d D = \frac{1}{d} \mathrm{Id}_d .$$

Therefore,

$$\begin{aligned}
\mathcal{E}(1, 0, \dots, 0) &= \int_{\mathrm{U}(d)} \sum_{j=1}^d \left(|u_{1j}|^2 - \frac{1}{d} \right)^2 dU = \int_{S^{2d-1}} \sum_{j=1}^d \left(|a_j|^2 - \frac{1}{d} \right)^2 d\mu^{2d-1}(a) \\
&= \int_{S^{2d-1}} \Big(\sum_{j=1}^d |a_j|^4 - \frac{1}{d} \Big) d\mu^{2d-1}(a) = -\frac{1}{d} + d \int_{S^{2d-1}} |a_1|^4 d\mu^{2d-1}(a) ,
\end{aligned}$$

where $a = (a_1, \dots, a_d) \in S^{2d-1}$ and μ^{2d-1} is unit-mass Haar measure on S^{2d-1}. By (30),

$$\int_{S^{2d-1}} |a_1|^4 d\mu^{2d-1}(a) = \frac{2}{d(d+1)} ,$$

and therefore

$$\mathcal{E}(1,0,\dots,0) = \frac{d-1}{d(d+1)}\ . \tag{49}$$

Substituting (49) into (48) with $c'_d = -c_d/d$, we conclude that

$$c_d = \frac{1}{d+1}\ .$$

□

References

[BD] P. Bleher and X. Di, Correlations between zeros of a random polynomial, *J. Stat . Phys.* 88 (1997), 269–305.

[BP] A. Bloch and G. Polya, On the roots of certain algebraic equations, *Proc. London Math. Soc.* 33 (1932), 102–114.

[BBL] E. Bogomolny, O. Bohigas, and P. Leboeuf, Quantum chaotic dynamics and random polynomials, *J. Stat. Phys.* 85 (1996), 639–679.

[BG] L. Boutet de Monvel and V. Guillemin, *The Spectral Theory of Toeplitz Operators*, Ann. Math. Studies 99, Princeton Univ. Press, Princeton, 1981.

[D] S. Donaldson, Symplectic submanifolds and almost complex geometry, *J. Diff. Geom.* 44 (1996), 666–705.

[ET] P. Erdos and P. Turan, On the distribution of roots of polynomials, *Ann. Math.* 51 (1950), 105–119.

[FS] J. E. Fornaess and N. Sibony, Complex dynamics in higher dimensions, II, *Modern methods in Complex Analysis (Princeton, NJ, 1992)*, Ann. of Math. Stud. 137, Princeton Univ. Press, Princeton, NJ, 1995, pp. 135–182.

[Gr] H. Grauert, Über Modifikationen und exzeptionelle analytische Mengen, *Math. Annalen* 146 (1962), 331–368.

[GH] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, Wiley-Interscience, New York, 1978.

[Ha] J. H. Hannay, Chaotic analytic zero points: exact statistics for those of a random spin state, *J. Phys. A: Math. Gen.* 29 (1996), 101–105.

[Ho] L. Hörmander, *The Analysis of Linear Partial Differential Operators*, Grund. Math. Wiss. 256, Springer-Verlag, New York, 1983.

[Ka] M. Kac, On the average number of real roots of a random algebraic equation, *Bull. Amer. Math. Soc.* 49 (1943), 314–320.

[Kl] M. Klimek, *Pluripotential Theory*, Clarendon Press, Oxford, 1991.

[LS] P. Leboeuf and P. Shukla, Universal fluctuations of zeros of chaotic wavefunctions, *J. Phys. A: Math. Gen.* 29 (1996), 4827-4835.

[LO] J. Littlewood and A. Offord, On the number of real roots of random algebraic equations I, II, III, *J. London Math. Soc.* 13 (1938), 288–295; *Proc. Camb. Phil. Soc.* 35 (1939), 133–148; *Math. Sborn.* 12 (1943), 277–286.

[NV] S. Nonnenmacher and A. Voros, Chaotic eigenfunctions in phase space, (preprint 1997).

[RSh] A. Russakovskii and B. Shiffman, Value distribution for sequences of rational mappings and complex dynamics, *Indiana Univ. Math. J.* 46 (1997), 897–932.

[RSo] A. Russakovskii and M. Sodin, Equidistribution for sequences of polynomial mappings, *Indiana Univ. Math. J.* 44 (1995), 851–882.

[SV] L. A. Shepp and R. J. Vanderbei, The complex zeros of random polynomials, *Trans. Amer. Math. Soc.* 347 (1995),4365–4384.

[Sh] B. Shiffman, Applications of geometric measure theory to value distribution theory for meromorphic maps, *Value-Distribution Theory, Part A*, 63-96, Marcel-Dekker, New York, 1974.

[SS] B. Shiffman and A. J. Sommese, *Vanishing theorems on complex manifolds*, Progress in Math. 56, Birkhäuser, Boston, 1985.

[Shn1] A. I. Shnirelman, Ergodic properties of eigenfunctions, *Usp. Mat. Nauk.* 29/6 (1974), 181–182.

[Shn2] A. I. Shnirelman, On the asymptotic properties of eigenfunctions in the region of chaotic motion, addendum to V. F. Lazutkin, *KAM Theory and Semiclassical Approximations to Eigenfunctions*, Springer-Verlag, New York, 1993.

[So] M. L. Sodin, Value distribution of sequences of rational mappings, *Entire and Subharmonic Functions*, B. Ya. Levin, ed., Advances in Soviet Math. 11, 1992.

[T] G. Tian, On a set of polarized Kähler metrics on algebraic manifolds, *J. Diff. Geometry* 32 (1990), 99–130.

[V] J. M. VanderKam, L^∞ norms and quantum ergodicity on the sphere, *Int. Math. Res. Notices* 7 (1997), 329–347.

[W] P. Walters, *An Introduction to Ergodic Theory*, Springer-Verlag, New York, 1981.

[Z1] S. Zelditch, Quantum ergodicity on the sphere, *Comm. Math. Phys.* 146 (1992), 61–71.

[Z2] S. Zelditch, A random matrix model for quantum mixing, *Int. Math. Res. Notices* 3 (1996), 115–137.

[Z3] S. Zelditch, Index and dynamics of quantized contact transformations, *Annales de l'Institut Fourier (Grenoble)* 47 (1997), 305–363.

[Z4] S. Zelditch, Szego kernels and a theorem of Tian, *Int. Math. Res. Notices*, to appear.

Department of Mathematics, Johns Hopkins University, Baltimore, MD 21218, USA

E-mail address: `shiffman@math.jhu.edu` *(first author)*, `zel@math.jhu.edu` *(second author)*