DUBROVIN EQUATIONS AND INTEGRABLE SYSTEMS ON HYPERELLIPTIC CURVES

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Abstract. We introduce the most general version of Dubrovin-type equations for divisors on a hyperelliptic curve $\mathcal{K}_{\mathfrak{g}}$ of arbitrary genus $\mathfrak{g} \in \mathbb{N}$, and provide a new argument for linearizing the corresponding completely integrable flows. Detailed applications to completely integrable systems, including the KdV, AKNS, Toda, and the combined sine-Gordon and mKdV hierarchies, are made. These investigations uncover a new principle for $1+1$-dimensional integrable soliton equations in the sense that the Dubrovin equations, combined with appropriate trace formulas, encode all hierarchies of soliton equations associated with hyperelliptic curves. In other words, completely integable hierarchies of soliton equations determine Dubrovin equations and associated trace formulas and, vice versa, Dubrovin-type equations combined with trace formulas permit the construction of hierarchies of soliton equations.

1. Introduction

The purpose of this paper is to re-examine Dubrovin equations for divisors on hyperelliptic Riemann surfaces and to underscore their exceptional role in connection with completely integrable hierarchies of soliton equations.

Starting from four representative hierarchies, the Korteweg–de Vries (KdV), Ablowitz–Kaup–Newell–Segur (AKNS), Toda lattice (Tl), and the combined sine-Gordon and mKdV (sGmKdV) hierarchy, we derive a new argument for linearizing the corresponding completely integrable flows. As a result of these investigations we show that a proper combination of Dubrovin equations and trace formulas involving auxiliary divisors on hyperelliptic curves encodes all information on the underlying completely integrable hierarchy of soliton equations.

In Section 2 we briefly review some basic facts on hyperelliptic curves and establish the notation used throughout this paper. Section 3 provides a "crash course" into the four different hierarchies closely following the detailed accounts in [11], [33], [34], [35], and [36]. In particular, we outline an elementary polynomial, recursive approach to these hierarchies, as originally developed by S. I. Al'ber [1], [2] in the KdV context, and introduce the corresponding divisors on $\mathcal{K}_{\mathfrak{g}}$ associated with their algebro-geometric solutions. Section 4 relates the polynomial recursion relation approach with elementary symmetric functions (and functions derived from them) of $\mu_1(x,t_r),\dots,\mu_{\mathfrak{g}}(x,t_r)$, where $\mu_j(x,t_r)$ are certain analogs of Dirichlet-type eigenvalues of the corresponding Lax operator. In Section 5 we study Dubrovin equations and, based on the results of Section 4, provide a new proof of the fundamental fact that a change of coordinates effected by the Abel map straightens out the flows on the Jacobi variety $J(\mathcal{K}_{\mathfrak{g}})$ of $\mathcal{K}_{\mathfrak{g}}$. In Section 6 we briefly recall the theta function representations of algebro-geometric solutions and Section 7 illustrates our results by deriving interesting connections between the $\mathrm{KdV}_{\mathfrak{g}}$ and sG equations, and AKNS and Toda hierarchies, respectively. These connections establish the fundamental role played by the Dubrovin equations as the common underlying principle for hierarchies of soliton equations. In particular, our formalism establishes an isomorphism between the class of algebro-geometric solutions of these integrable systems. Finally, Appendix A collects some useful results in connection with Lagrange interpolation formulas.

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We emphasize that our results are not necessarily restricted to hyperelliptic curves. In particular, the approach of this paper applies to Boussinesq-type curves using the polynomial recursion formalism for the Boussinesq hierarchy developed in [19], [20].

Depending perhaps a bit on one's taste, the results of this paper may at first sight appear somewhat discouraging as they clearly shift the emphasis from individual hierarchies of soliton equations toward Dubrovin-type equations. On the positive side, however, they establish the Dubrovin equations as a universal object underlying all hierarchies.

We note that our approach to completely integrable soliton equations is close in spirit to that developed by M. S. Al'ber and S. I. Al'ber in a series of papers (see, e.g., [1]–[7] and the references therein). While their approach focuses on algebraically integrable systems and hence on a Hamiltonian formalism with associated action and angle variables, our approach concentrates on how a combination of elementary symmetric functions of $\mu_1(x,t_1),\dots,\mu_{\mathfrak{g}}(x,t_r)$ and certain trace formulas generate completely integrable hierarchies of soliton equations and their algebro-geometric solutions.

A different series of papers closely related to our investigations and focusing on characterizing real-valued solutions of various soliton equations was published by Chen, Chin, Lee, Neil, Ting, and Tracy [59]–[61], [63]–[65]. These authors, however, appear to be unaware of the prior work of M. S. Al'ber and S. J. Al'ber in this field.

Finally, we stress that the use of elementary symmetric functions and hence of trace formulas in terms of Dirichlet eigenvalues has a long history in the context of integrable equations. In fact, as early as 1975, Flaschka [30] characterized the real-valued periodic potentials q with finitely many stability intervals of the associated Schrödinger operator $-d^2/dx^2+q$ in $L^2(\mathbb{R};dx)$ as stationary solutions of the KdV hierarchy using (regularized) trace relations for Dirichlet eigenvalues associated with q and the underlying periodicity interval.

2. Hyperelliptic curves

Fix $N\in\mathbb{N}_0$. We briefly review hyperelliptic Riemann surfaces of the type,

$$\mathcal{F}_N(z,y)=y^2-R_{N+1}(z)=0,\quad R_{N+1}(z)=\prod_{m=0}^{N}(z-E_m),$$
$$\{E_m\}_{m=0,\dots,N}\subset\mathbb{C},\quad E_m\neq E_{m'} \text{ for } m\neq m'. \tag{2.1}$$

The material of this section is standard and can be found, for instance, in [29]. The curve (2.1) is compactified by adding one point P_∞ at infinity if N is even, and two points P_{∞_+} and P_{∞_-} if N is odd.

One introduces an appropriate set of[1] $\lfloor N+1\rfloor/2$ nonintersecting cuts $\mathcal{C}_j$ joining $E_{m(j)}$ and $E_{m'(j)}$ and $\mathcal{C}_\infty$ joining E_N and ∞ if N is even. Denote

$$\mathcal{C}=\bigcup_{j\in J\cup\{\infty\}}\mathcal{C}_j,\quad \mathcal{C}_j\cap\mathcal{C}_k=\emptyset,\quad j\neq k, \tag{2.2}$$

where $J\subseteq\{1,\dots,\lfloor N+1\rfloor/2\}$. Define the cut plane

$$\Pi=\mathbb{C}\setminus\mathcal{C}, \tag{2.3}$$

and introduce the holomorphic function

$$R_{N+1}(\cdot)^{1/2}\colon\Pi\to\mathbb{C},\quad z\mapsto\bigg(\prod_{m=0}^{N}(z-E_m)\bigg)^{1/2} \tag{2.4}$$

on Π with an appropriate choice of the square root branch in (2.4). Define

$$\mathcal{M}_{\mathfrak{g}}=\{(z,\sigma R_{N+1}(z)^{1/2})\mid z\in\mathbb{C},\ \sigma\in\{\pm1\}\}\cup\begin{cases}\{P_\infty\} & \text{for } N \text{ even},\\ \{P_{\infty_+},P_{\infty_-}\} & \text{for } N \text{ odd},\end{cases} \tag{2.5}$$

[1] $\lfloor x\rfloor=\sup\{y\in\mathbb{Z}\mid y\leq x\}$.

by extending $R_{N+1}(\cdot)^{1/2}$ to $\mathcal{C}$. The hyperelliptic curve $\mathcal{K}_{\mathfrak{g}}$ is then the set $\mathcal{M}_{\mathfrak{g}}$ with its natural complex structure obtained upon gluing the two sheets of $\mathcal{M}_{\mathfrak{g}}$ crosswise along the cuts. Finite points P on $\mathcal{K}_{\mathfrak{g}}$ are denoted by $P=(z,y)$, where $y(P)$ denotes the meromorphic function on $\mathcal{K}_{\mathfrak{g}}$ satisfying $\mathcal{F}_N(z,y)=y^2-R_{N+1}(z)=0$; $\mathcal{K}_{\mathfrak{g}}$ has genus $\mathfrak{g}=\lfloor N+1\rfloor/2$.

A basis of $\mathfrak{g}$ linearly independent holomorphic differentials on $\mathcal{K}_{\mathfrak{g}}$ is given by $z^{\ell-1}\,dz/y(P)$ for $\ell=1,\dots,\mathfrak{g}$, and we introduce

$$d\omega_j(P)=\sum_{\ell=1}^{\mathfrak{g}} c_{j,\ell}\frac{z^{\ell-1}\,dz}{y(P)}, \quad j=1,\dots,\mathfrak{g}, \tag{2.6}$$

with normalization,

$$\int_{a_k} d\omega_j=\delta_{j,k}, \tag{2.7}$$

where $\{a_j,b_j\}_{j=1}^{\mathfrak{g}}$ is a homology basis for $\mathcal{K}_{\mathfrak{g}}$.

Define the matrix $\tau=(\tau_{j,\ell})$ by

$$\tau_{j,\ell}=\int_{b_j} d\omega_\ell. \tag{2.8}$$

Then $\operatorname{Im}(\tau)>0$ and $\tau_{j,\ell}=\tau_{\ell,j}$.

The Riemann theta function associated with $\mathcal{K}_{\mathfrak{g}}$ and the given homology basis $\{a_j,b_j\}_{j=1}^{\mathfrak{g}}$, by definition, reads

$$\theta(\underline{z})=\sum_{\underline{n}\in\mathbb{Z}^{\mathfrak{g}}}\exp\big(2\pi i(\underline{n},\underline{z})+\pi i(\underline{n},\tau\underline{n})\big), \quad \underline{z}\in\mathbb{C}^{\mathfrak{g}}. \tag{2.9}$$

We fix a base point P_0 on $\mathcal{K}_{\mathfrak{g}}$ and define the Abel map $\underline{A}_{P_0}$ by

$$\underline{A}_{P_0}(P)=\Big(\int_{P_0}^P d\omega_1,\dots,\int_{P_0}^P d\omega_{\mathfrak{g}}\Big) \pmod{L_{\mathfrak{g}}}, \quad P\in\mathcal{K}_{\mathfrak{g}}, \tag{2.10}$$

with period lattice

$$L_{\mathfrak{g}}=\{\underline{n}+\tau\underline{m}\mid \underline{n},\underline{m}\in\mathbb{Z}^{\mathfrak{g}}\}. \tag{2.11}$$

Similarly, we introduce

$$\underline{\alpha}_{P_0}\colon \operatorname{Div}(\mathcal{K}_{\mathfrak{g}})\to J(\mathcal{K}_{\mathfrak{g}}), \quad \mathcal{D}\mapsto\underline{\alpha}_{P_0}(\mathcal{D})=\sum_{P\in\mathcal{K}_{\mathfrak{g}}}\mathcal{D}(P)\underline{A}_{P_0}(P), \tag{2.12}$$

where $\operatorname{Div}(\mathcal{K}_{\mathfrak{g}})$ and $J(\mathcal{K}_{\mathfrak{g}})=\mathbb{C}^{\mathfrak{g}}/L_{\mathfrak{g}}$ denote the set of divisors on $\mathcal{K}_{\mathfrak{g}}$ and the Jacobi variety of $\mathcal{K}_{\mathfrak{g}}$, respectively.

In connection with divisors on $\mathcal{K}_{\mathfrak{g}}$ we shall employ the following (additive) notation,

$$\mathcal{D}_{Q_0\underline{Q}}=\mathcal{D}_{Q_0}+\mathcal{D}_{\underline{Q}}, \quad \mathcal{D}_{\underline{Q}}=\mathcal{D}_{Q_1}+\cdots+\mathcal{D}_{Q_n}, \text{ etc.,}$$
$$\text{for } \underline{Q}=(Q_1,\dots,Q_n)\in\sigma^n\mathcal{K}_{\mathfrak{g}}, \tag{2.13}$$

where for any $Q\in\mathcal{K}_{\mathfrak{g}}$,

$$\mathcal{D}_Q\colon\mathcal{K}_{\mathfrak{g}}\to\mathbb{N}_0, \quad P\mapsto\mathcal{D}_Q(P)=\begin{cases}1 & \text{for } P=Q,\\ 0 & \text{for } P\in\mathcal{K}_{\mathfrak{g}}\setminus\{Q\},\end{cases} \tag{2.14}$$

and $\sigma^n\mathcal{K}_{\mathfrak{g}}$ denotes the nth symmetric product of $\mathcal{K}_{\mathfrak{g}}$.

3. The hierarchies

We give a brief presentation of the KdV, AKNS, sGmKdV, and Toda hierarchies based on a polynomial, recursive approach. The material of this section originated with work of S. I. Al'ber [1], [2] (see also [3]–[7]). It has been further developed in [11], [33], [34], [35], and [36] and we closely follow the latter sources. Common to all these hierarchies is that one can naturally associate with each one of them a hyperelliptic curve as described in the previous section.

The KdV hierarchy. The Lax pair consists of a second-order linear differential expression L of Schrödinger-type,

$$L(t_{\mathfrak{g}}) = -\frac{d^2}{dx^2} + V(x,t_{\mathfrak{g}}), \quad t_{\mathfrak{g}} \in \mathbb{R} \tag{3.1}$$

and a differential expression $P_{2\mathfrak{g}+1}(t_{\mathfrak{g}})$ of order $2\mathfrak{g}+1$ defined recursively as follows. Let $\{f_j\}_{j\in\mathbb{N}_0}$ be given by

$$f_0 = 1, \quad f_{j,x} = -\frac{1}{4} f_{j-1,xxx} + V f_{j-1,x} + \frac{1}{2} V_x f_{j-1}, \quad j \in \mathbb{N}. \tag{3.2}$$

Explicitly ($f_j = f_j(x,t_{\mathfrak{g}})$, $(x,t_{\mathfrak{g}}) \in \mathbb{R}^2$),

$$f_1 = \tfrac{1}{2}V + c_1, \quad f_2 = -\tfrac{1}{8}V_{xx} + \tfrac{3}{8}V^2 + c_1\tfrac{1}{2}V + c_2, \quad \text{etc.}, \tag{3.3}$$

where $c_j \in \mathbb{C}$ are integration constants. Then one defines

$$P_{2\mathfrak{g}+1}(t_{\mathfrak{g}}) = \sum_{j=0}^{\mathfrak{g}} \big(f_j(t_{\mathfrak{g}})\frac{d}{dx} - \frac{1}{2}f_{j,x}(t_{\mathfrak{g}})\big)L(t_{\mathfrak{g}})^{\mathfrak{g}-j}, \quad \mathfrak{g} \in \mathbb{N}_0, \tag{3.4}$$

and using the definition of f_j in (3.2), one finds that the commutator of $L(t_{\mathfrak{g}})$ and $P_{2\mathfrak{g}+1}(t_{\mathfrak{g}})$ is in fact a multiplication operator. Indeed, the Lax commutator representation reads

$$L_{t_{\mathfrak{g}}}(t_{\mathfrak{g}}) - [P_{2\mathfrak{g}+1}(t_{\mathfrak{g}}), L(t_{\mathfrak{g}})] = V_{t_{\mathfrak{g}}} - 2f_{\mathfrak{g}+1,x}(x,t_{\mathfrak{g}}) = \mathrm{KdV}_{\mathfrak{g}}(V) = 0. \tag{3.5}$$

Explicitly, one obtains for the first few KdV equations,

$$\begin{aligned}
&\mathrm{KdV}_0(V) = V_{t_0} - V_x = 0,\\
&\mathrm{KdV}_1(V) = V_{t_1} + \tfrac{1}{4}V_{xxx} - \tfrac{3}{2}VV_x - c_1V_x = 0,\\
&\mathrm{KdV}_2(V) = V_{t_2} - \tfrac{1}{16}V_{xxxxx} + \tfrac{5}{8}VV_{xxx} + \tfrac{5}{4}V_xV_{xx}\\
&\qquad\qquad - \tfrac{15}{8}V^2V_x - c_2V_x + c_1\big(\tfrac{1}{4}V_{xxx} - \tfrac{3}{2}VV_x\big) = 0, \text{ etc.},
\end{aligned} \tag{3.6}$$

where, of course, $\mathrm{KdV}_1(V)$ is *the* KdV equation.

Next define the polynomial[2] of degree $\mathfrak{g}$ in z

$$F_{\mathfrak{g}}(z,x,t_{\mathfrak{g}}) = \sum_{j=0}^{\mathfrak{g}} f_{\mathfrak{g}-j}(x,t_{\mathfrak{g}})z^j = \prod_{j=1}^{\mathfrak{g}}(z - \mu_j(x,t_{\mathfrak{g}})) \tag{3.7}$$

implying

$$-2V_{t_{\mathfrak{g}}} = F_{\mathfrak{g},xxx} - 4(V-z)F_{\mathfrak{g},x} - 2V_xF_{\mathfrak{g}}. \tag{3.8}$$

In the special stationary case, defined by $V_{t_{\mathfrak{g}}} = 0$, this integrates to

$$\frac{1}{2}F_{\mathfrak{g},xx}F_{\mathfrak{g}} - \frac{1}{4}F_{\mathfrak{g},x}^2 - (V-z)F_{\mathfrak{g}}^2 = R_{2\mathfrak{g}+1}(z). \tag{3.9}$$

Here $R_{2\mathfrak{g}+1}$ is a monic polynomial of degree $2\mathfrak{g}+1$ with zeros $\{E_0, \dots, E_{2\mathfrak{g}}\}$. Hence,

$$R_{2\mathfrak{g}+1}(z) = \prod_{m=0}^{2\mathfrak{g}}(z - E_m), \quad \{E_m\}_{m=0,\dots,2\mathfrak{g}} \subset \mathbb{C}. \tag{3.10}$$

[2]The zeros $\mu_j(x,t_{\mathfrak{g}})$ of $F_{\mathfrak{g}}$ turn out to be eigenvalues associated with $L(t_{\mathfrak{g}})$ and a Dirichlet boundary condition at the point $x \in \mathbb{R}$.

The hyperelliptic curve $\mathcal{K}_{\mathfrak{g}}$ is defined in terms of the stationary KdV hierarchy obtained by considering a $t_{\mathfrak{g}}$-independent function $V = V(x)$, resulting in

$$[P_{2\mathfrak{g}+1}, L] = 2f_{\mathfrak{g}+1,x} = 0. \tag{3.11}$$

The classical Burchnall–Chaundy theorem [12], [13], [14] (see also [56], [66]) states that commuting differential operators are algebraically related. In the present context one finds

$$P_{2\mathfrak{g}+1}^2 = R_{2\mathfrak{g}+1}(L) \tag{3.12}$$

and thus the hyperelliptic curve $\mathcal{K}_{\mathfrak{g}}$ of genus $\mathfrak{g}$ is of the type $y^2 = R_{2\mathfrak{g}+1}(z)$, with $N = 2\mathfrak{g}$ even when compared to Section 2.

For later purpose we quote the following asymptotic high-energy expansion[3] (see [31], [36], [57])

$$\frac{iF_{\mathfrak{g}}(z,x,t_{\mathfrak{g}})}{2R_{2\mathfrak{g}+1}(z)^{1/2}} \underset{z\to i\infty}{=} \frac{i}{2z^{1/2}} \sum_{j=0}^{\infty} \hat{f}_j(x,t_{\mathfrak{g}}) z^{-j}, \tag{3.13}$$

where $\hat{f}_j$ denotes the homogeneous coefficients f_j with all integration constants equal to zero, $c_\ell = 0$, $\ell \geq 1$, that is,

$$\hat{f}_0 = f_0 = 1, \quad \hat{f}_j = f_j|_{c_\ell=0}, \quad \ell = 1,\dots,j,\ j \in \mathbb{N}. \tag{3.14}$$

We also introduce the following fundamental meromorphic function on $\mathcal{K}_{\mathfrak{g}}$,

$$\phi(P,x,t_{\mathfrak{g}}) = \frac{iy(P) + \frac{1}{2}F_{\mathfrak{g},x}(z,x,t_{\mathfrak{g}})}{F_{\mathfrak{g}}(z,x,t_{\mathfrak{g}})} = \frac{-H_{\mathfrak{g}+1}(z,x,t_{\mathfrak{g}})}{iy(P) - \frac{1}{2}F_{\mathfrak{g},x}(z,x,t_{\mathfrak{g}})}, \quad (x,t_{\mathfrak{g}}) \in \mathbb{R}^2 \tag{3.15}$$

(the second equality in (3.15) serving as a definition of the polynomial $H_{\mathfrak{g}+1}$ of degree $\mathfrak{g}+1$ with respect to z) and the time-dependent Baker–Akhiezer function $\psi(P,x,x_0,t_{\mathfrak{g}},t_{0,\mathfrak{g}})$ on $\mathcal{K}_{\mathfrak{g}} \setminus \{P_\infty\}$,

$$\begin{aligned}&\psi(P,x,x_0,t_{\mathfrak{g}},t_{0,\mathfrak{g}}) \\ &\quad = \exp\bigg(\int_{t_{0,\mathfrak{g}}}^{t_{\mathfrak{g}}} ds\, \big(F_{\mathfrak{g}}(z,x_0,s)\phi(P,x_0,s) - \tfrac{1}{2}F_{\mathfrak{g},x}(z,x_0,s)\big) + \int_{x_0}^{x} dx'\, \phi(P,x',t_{\mathfrak{g}})\bigg), \\ &\qquad (x,x_0,t_{\mathfrak{g}},t_{0,\mathfrak{g}}) \in \mathbb{R}^4.\end{aligned} \tag{3.16}$$

The divisor $(\phi(P,x,t_{\mathfrak{g}}))$ of $\phi(P,x,t_{\mathfrak{g}})$ is given by

$$(\phi(P,x,t_{\mathfrak{g}})) = \mathcal{D}_{\hat{\nu}_0(x,t_{\mathfrak{g}})\underline{\hat{\nu}}(x,t_{\mathfrak{g}})} - \mathcal{D}_{P_\infty \underline{\hat{\mu}}(x,t_{\mathfrak{g}})}, \tag{3.17}$$

where

$$\begin{aligned}&\underline{\hat{\mu}}(x,t_{\mathfrak{g}}) = (\hat{\mu}_1(x,t_{\mathfrak{g}}),\dots,\hat{\mu}_{\mathfrak{g}}(x,t_{\mathfrak{g}})) \in \sigma^{\mathfrak{g}}\mathcal{K}_{\mathfrak{g}}, \\ &\qquad \hat{\mu}_j(x,t_{\mathfrak{g}}) = (\mu_j(x,t_{\mathfrak{g}}), -\tfrac{1}{2}F_{\mathfrak{g},x}(\mu_j(x,t_{\mathfrak{g}}),x,t_{\mathfrak{g}})), \quad j=1,\dots,\mathfrak{g}\end{aligned} \tag{3.18}$$

denote the Dirichlet divisors[2] and

$$\begin{aligned}&\hat{\nu}_0(x,t_{\mathfrak{g}})\underline{\hat{\nu}}(x,t_{\mathfrak{g}}) = (\hat{\nu}_0(x,t_{\mathfrak{g}}),\dots,\hat{\nu}_{\mathfrak{g}}(x,t_{\mathfrak{g}})) \in \sigma^{\mathfrak{g}+1}\mathcal{K}_{\mathfrak{g}}, \\ &\qquad \hat{\nu}_\ell(x,t_{\mathfrak{g}}) = (\nu_\ell(x,t_{\mathfrak{g}}), \tfrac{1}{2}F_{\mathfrak{g},x}(\nu_\ell(x,t_{\mathfrak{g}}),x,t_{\mathfrak{g}})), \quad \ell=0,\dots,\mathfrak{g}\end{aligned} \tag{3.19}$$

abbreviate the Neumann divisors[4] derived from the zeros of $H_{\mathfrak{g}+1}(z,x,t_{\mathfrak{g}})$,

$$H_{\mathfrak{g}+1}(z,x,t_{\mathfrak{g}}) = \prod_{\ell=0}^{\mathfrak{g}} (z - \nu_\ell(x,t_{\mathfrak{g}})). \tag{3.20}$$

The importance of ϕ in connection with divisors on hyperelliptic curves was recognized by Jacobi [41] and applied to the KdV case by Mumford [53], Sect. IIIa.1 and McKean [49] (see also [28], [56]).

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[3] For an appropriate choice of the sign of the square roots in (3.13), the left-hand side of (3.13) equals the Green's function on the diagonal (i.e., for $x = x'$) of $L(t_{\mathfrak{g}})$.

[4] The zeros $\nu_\ell(x,t_{\mathfrak{g}})$ of $H_{\mathfrak{g}+1}$ turn out to be eigenvalues associated with $L(t_{\mathfrak{g}})$ and a Neumann boundary condition at the point $x \in \mathbb{R}$.

The AKNS hierarchy. The Lax pair consists of a Dirac-type matrix-valued differential expression

$$M(t_{\mathfrak{g}}) = i \begin{pmatrix} \frac{d}{dx} & -q(x,t_{\mathfrak{g}}) \\ p(x,t_{\mathfrak{g}}) & -\frac{d}{dx} \end{pmatrix}, \quad t_{\mathfrak{g}} \in \mathbb{R}, \tag{3.21}$$

and a matrix-valued differential operator $Q_{\mathfrak{g}+1}(t_{\mathfrak{g}})$ of order $\mathfrak{g}+1$. To define $Q_{\mathfrak{g}+1}(t_{\mathfrak{g}})$ we proceed as follows. Define $\{f_\ell(x,t_{\mathfrak{g}})\}_{\ell\in\mathbb{N}_0}$, $\{g_\ell(x,t_{\mathfrak{g}})\}_{\ell\in\mathbb{N}_0}$, and $\{h_\ell(x,t_{\mathfrak{g}})\}_{\ell\in\mathbb{N}_0}$ recursively by $((x,t_{\mathfrak{g}})\in\mathbb{R}^2)$,

$$\begin{aligned} &f_0 = -iq, && g_0 = 1, && h_0 = ip, \\ &f_{\ell+1} = \frac{i}{2} f_{\ell,x} - iqg_{\ell+1}, && g_{\ell+1,x} = pf_\ell + qh_\ell, && h_{\ell+1} = -\frac{i}{2} h_{\ell,x} + ipg_{\ell+1}, \quad \ell\in\mathbb{N}_0. \end{aligned} \tag{3.22}$$

The 2×2 matrix $Q_{\mathfrak{g}+1}(t_{\mathfrak{g}})$ is then defined by

$$Q_{\mathfrak{g}+1}(t_{\mathfrak{g}}) = i \sum_{\ell=0}^{\mathfrak{g}+1} \begin{pmatrix} -g_{\mathfrak{g}+1-\ell}(t_{\mathfrak{g}}) & f_{\mathfrak{g}-\ell}(t_{\mathfrak{g}}) \\ -h_{\mathfrak{g}-\ell}(t_{\mathfrak{g}}) & g_{\mathfrak{g}+1-\ell}(t_{\mathfrak{g}}) \end{pmatrix} M(t_{\mathfrak{g}})^\ell, \quad \mathfrak{g}\in\mathbb{N}_0, \quad f_{-1} = h_{-1} = 0, \tag{3.23}$$

and one verifies that the commutator of $Q_{\mathfrak{g}+1}(t_{\mathfrak{g}})$ and $M(t_{\mathfrak{g}})$ becomes

$$[Q_{\mathfrak{g}+1}(t_{\mathfrak{g}}), M(t_{\mathfrak{g}})] = \begin{pmatrix} 0 & -2if_{\mathfrak{g}+1}(t_{\mathfrak{g}}) \\ 2ih_{\mathfrak{g}+1}(t_{\mathfrak{g}}) & 0 \end{pmatrix}, \quad \mathfrak{g}\in\mathbb{N}_0. \tag{3.24}$$

Consequently, the Lax commutator representation for the AKNS hierarchy reads

$$\frac{d}{dt_{\mathfrak{g}}} M(t_{\mathfrak{g}}) - [Q_{\mathfrak{g}+1}(t_{\mathfrak{g}}), M(t_{\mathfrak{g}})] = \begin{pmatrix} p_{t_{\mathfrak{g}}}(x,t_{\mathfrak{g}}) - 2h_{\mathfrak{g}+1}(x,t_{\mathfrak{g}}) \\ q_{t_{\mathfrak{g}}}(x,t_{\mathfrak{g}}) - 2f_{\mathfrak{g}+1}(x,t_{\mathfrak{g}}) \end{pmatrix} = \mathrm{AKNS}_{\mathfrak{g}}(p,q) = 0, \quad \mathfrak{g}\in\mathbb{N}_0. \tag{3.25}$$

The first few equations equal,

$$\begin{aligned} \mathrm{AKNS}_0(p,q) &= \begin{pmatrix} p_{t_0} - p_x + c_1(-2ip) \\ q_{t_0} - q_x + c_1(2iq) \end{pmatrix} = 0, \\ \mathrm{AKNS}_1(p,q) &= \begin{pmatrix} p_{t_1} + \frac{i}{2} p_{xx} - ip^2q + c_1(-p_x) + c_2(-2ip) \\ q_{t_1} - \frac{i}{2} q_{xx} + ipq^2 + c_1(-q_x) + c_2(2iq) \end{pmatrix} = 0, \text{ etc.} \end{aligned} \tag{3.26}$$

Next, define polynomials $F_{\mathfrak{g}}$, $G_{\mathfrak{g}+1}$, and $H_{\mathfrak{g}}$ with respect to $z\in\mathbb{C}$,

$$\begin{aligned} F_{\mathfrak{g}}(z,x,t_{\mathfrak{g}}) &= \sum_{\ell=0}^{\mathfrak{g}} f_{\mathfrak{g}-\ell}(x,t_{\mathfrak{g}}) z^\ell = -iq(x,t_{\mathfrak{g}}) \prod_{j=1}^{\mathfrak{g}} (z - \mu_j(x,t_{\mathfrak{g}})), \\ G_{\mathfrak{g}+1}(z,x,t_{\mathfrak{g}}) &= \sum_{\ell=0}^{\mathfrak{g}+1} g_{\mathfrak{g}+1-\ell}(x,t_{\mathfrak{g}}) z^\ell, \\ H_{\mathfrak{g}}(z,x,t_{\mathfrak{g}}) &= \sum_{\ell=0}^{\mathfrak{g}} h_{\mathfrak{g}-\ell}(x,t_{\mathfrak{g}}) z^\ell = ip(x,t_{\mathfrak{g}}) \prod_{j=1}^{\mathfrak{g}} (z - \nu_j(x,t_{\mathfrak{g}})). \end{aligned} \tag{3.27}$$

In the special stationary case, where $p_{t_{\mathfrak{g}}} = q_{t_{\mathfrak{g}}} = 0$, we infer from the recursion (3.22) that

$$\left(G_{\mathfrak{g}+1}^2 - F_{\mathfrak{g}} H_{\mathfrak{g}}\right)_x = 0 \tag{3.28}$$

and hence

$$G_{\mathfrak{g}+1}^2 - F_{\mathfrak{g}} H_{\mathfrak{g}} = R_{2\mathfrak{g}+2}(z), \tag{3.29}$$

where $R_{2\mathfrak{g}+2}$ is a monic polynomial of degree $2\mathfrak{g}+2$ with zeros $\{E_0,\dots,E_{2\mathfrak{g}+1}\}$. Thus,

$$R_{2\mathfrak{g}+2}(z) = \prod_{m=0}^{2\mathfrak{g}+1} (z - E_m), \quad \{E_m\}_{m=0,\dots,2\mathfrak{g}+1} \subset \mathbb{C}. \tag{3.30}$$

The stationary case determines the hyperelliptic curve $\mathcal{K}_{\mathfrak{g}}$ of genus $\mathfrak{g}$ of the type $y^2 = R_{2\mathfrak{g}+1}(z)$, with $N = 2\mathfrak{g}+1$ when compared to Section 2. If $p = p(x)$ and $q = q(x)$ are stationary solutions of the AKNS equation,

$$[Q_{\mathfrak{g}+1}, M] = 0, \text{ that is, } f_{\mathfrak{g}+1} = h_{\mathfrak{g}+1} = 0, \tag{3.31}$$

Burchnall–Chaundy's theorem implies that

$$Q_{\mathfrak{g}+1}^2 + R_{2\mathfrak{g}+2}(M) = 0. \tag{3.32}$$

By studying the Green's matrix of M one finds the following asymptotic high-energy expansion [35],

$$\frac{F_{\mathfrak{g}}(z,x,t_{\mathfrak{g}})}{R_{2\mathfrak{g}+2}(z)^{1/2}} \underset{z\to i\infty}{=} \frac{1}{z}\sum_{k=0}^{\infty} \hat{f}_k(x,t_{\mathfrak{g}})z^{-k}, \tag{3.33}$$

$$\frac{H_{\mathfrak{g}}(z,x,t_{\mathfrak{g}})}{R_{2\mathfrak{g}+2}(z)^{1/2}} \underset{z\to i\infty}{=} \frac{1}{z}\sum_{k=0}^{\infty} \hat{h}_k(x,t_{\mathfrak{g}})z^{-k} \tag{3.34}$$

for an appropriate determination of the square roots in (3.33) and (3.34). Here $\hat{f}_j$ and $\hat{h}_j$ denote the homogeneous quantities with vanishing integration constants c_ℓ, $\ell \geq 1$, that is,

$$\hat{f}_0 = f_0 = -iq, \quad \hat{f}_j = f_j|_{c_\ell=0}, \quad \ell = 1,\dots,j,\ j\in\mathbb{N}, \tag{3.35}$$

$$\hat{h}_0 = h_0 = ip, \quad \hat{h}_j = h_j|_{c_\ell=0}, \quad \ell = 1,\dots,j,\ j\in\mathbb{N}. \tag{3.36}$$

We also record the functions

$$\phi(P,x,t_{\mathfrak{g}}) = \frac{y(P)+G_{\mathfrak{g}+1}(z,x,t_{\mathfrak{g}})}{F_{\mathfrak{g}}(z,x,t_{\mathfrak{g}})} = \frac{-H_{\mathfrak{g}}(z,x,t_{\mathfrak{g}})}{y(P)-G_{\mathfrak{g}+1}(z,x,t_{\mathfrak{g}})}, \qquad P=(z,y)\in\mathcal{K}_{\mathfrak{g}} \tag{3.37}$$

and the Baker-Akhiezer vector,

$$\Psi(P,x,x_0,t_{\mathfrak{g}},t_{0,\mathfrak{g}}) = \begin{pmatrix}\psi_1(P,x,x_0,t_{\mathfrak{g}},t_{0,\mathfrak{g}})\\ \psi_2(P,x,x_0,t_{\mathfrak{g}},t_{0,\mathfrak{g}})\end{pmatrix}, \tag{3.38}$$

$$\begin{aligned}\psi_1(P,x,x_0,t_{\mathfrak{g}},t_{0,\mathfrak{g}}) &= \exp\bigg(\int_{x_0}^{x} dx'\,(-iz+q(x',t_{\mathfrak{g}})\phi(P,x',t_{\mathfrak{g}}))\\ &\qquad + i\int_{t_{0,\mathfrak{g}}}^{t_{\mathfrak{g}}} ds(F_{\mathfrak{g}}(z,x_0,s)\phi(P,x_0,s)-G_{\mathfrak{g}+1}(z,x_0,s))\bigg),\\ \psi_2(P,x,x_0,t_{\mathfrak{g}},t_{0,\mathfrak{g}}) &= \phi(P,x,t_{\mathfrak{g}})\psi_1(P,x,x_0,t_{\mathfrak{g}},t_{0,\mathfrak{g}}),\\ &\qquad P\in\mathcal{K}_{\mathfrak{g}}\setminus\{P_{\infty_\pm}\},\ (x,t_{\mathfrak{g}})\in\mathbb{R}^2.\end{aligned} \tag{3.39}$$

The divisor of $\phi(P,x,t_{\mathfrak{g}})$ is given by

$$(\phi(P,x,t_{\mathfrak{g}})) = \mathcal{D}_{P_{\infty_+}\underline{\hat{\nu}}(x,t_{\mathfrak{g}})} - \mathcal{D}_{P_{\infty_-}\underline{\hat{\mu}}(x,t_{\mathfrak{g}})}, \tag{3.40}$$

where

$$\begin{aligned}&\underline{\hat{\mu}}(x,t_{\mathfrak{g}}) = (\hat{\mu}_1(x,t_{\mathfrak{g}}),\dots,\hat{\mu}_{\mathfrak{g}}(x,t_{\mathfrak{g}}))\in\sigma^{\mathfrak{g}}\mathcal{K}_{\mathfrak{g}},\\ &\qquad \hat{\mu}_j(x,t_{\mathfrak{g}}) = (\mu_j(x,t_{\mathfrak{g}}),G_{\mathfrak{g}+1}(\mu_j(x,t_{\mathfrak{g}}),x,t_{\mathfrak{g}})),\quad j=1,\dots,\mathfrak{g}\end{aligned} \tag{3.41}$$

and

$$\begin{aligned}&\underline{\hat{\nu}}(x,t_{\mathfrak{g}}) = (\hat{\nu}_1(x,t_{\mathfrak{g}}),\dots,\hat{\nu}_{\mathfrak{g}}(x,t_{\mathfrak{g}}))\in\sigma^{\mathfrak{g}}\mathcal{K}_{\mathfrak{g}},\\ &\qquad \hat{\nu}_j(x,t_{\mathfrak{g}}) = (\nu_j(x,t_{\mathfrak{g}}),-G_{\mathfrak{g}+1}(\nu_j(x,t_{\mathfrak{g}}),x,t_{\mathfrak{g}})),\quad j=1,\dots,\mathfrak{g}.\end{aligned} \tag{3.42}$$

The Toda hierarchy. Let $(S^{\pm}f)(n) = f^{\pm}(n) = f(n\pm1)$, $n\in\mathbb{Z}$ denote the shift operation on the lattice $\mathbb{Z}$. The equations in the Toda hierarchy are continuous in time and discrete in the space variable. The Lax pair consists of the second-order difference operator

$$L(t_{\mathfrak{g}}) = a(t_{\mathfrak{g}})S^+ + a^-(t_{\mathfrak{g}})S^- - b(t_{\mathfrak{g}}), \quad t_{\mathfrak{g}}\in\mathbb{R} \tag{3.43}$$

and a difference operator $P_{2\mathfrak{g}+2}(t_\mathfrak{g})$ of order $2\mathfrak{g}+1$

$$P_{2\mathfrak{g}+2}(t_\mathfrak{g}) = -L(t_\mathfrak{g})^{\mathfrak{g}+1} + \sum_{j=0}^{\mathfrak{g}} (g_j(t_\mathfrak{g}) + 2a(t_\mathfrak{g}) f_j(t_\mathfrak{g}) S^+) L(t_\mathfrak{g})^{\mathfrak{g}-j} + f_{\mathfrak{g}+1}(t_\mathfrak{g}), \quad \mathfrak{g} \in \mathbb{N}_0. \tag{3.44}$$

Here $\{f_j(n,t_\mathfrak{g})\}_{j\in\mathbb{N}_0}$ and $\{g_j(n,t_\mathfrak{g})\}_{j\in\mathbb{N}_0}$ satisfy the recursion relations,

$$\begin{aligned} & f_0 = 1, \quad g_0 = -c_1, \\ & 2f_{j+1} + g_j + g_j^- + 2bf_j = 0, \quad j \in \mathbb{N}_0, \\ & g_{j+1} - g_{j+1}^- + 2\big(a^2 f_j^+ - (a^-)^2 f_j^-\big) + b\big(g_j - g_j^-\big) = 0, \quad j \in \mathbb{N}_0. \end{aligned} \tag{3.45}$$

The Lax commutator representation of the Toda hierarchy then reads

$$L_{t_\mathfrak{g}}(t_\mathfrak{g}) - [P_{2\mathfrak{g}+2}(t_\mathfrak{g}), L(t_\mathfrak{g})] = \mathrm{Tl}_\mathfrak{g}(a,b)_1 S^+ - \mathrm{Tl}_\mathfrak{g}(a,b)_2 + \mathrm{Tl}_\mathfrak{g}(a^-,b^-)_1 S^- = 0, \quad \mathfrak{g} \in \mathbb{N}_0, \tag{3.46}$$

where

$$\begin{aligned} \mathrm{Tl}_\mathfrak{g}(a,b)_1 &= a_{t_\mathfrak{g}} + a(g_\mathfrak{g}^+ + g_\mathfrak{g} + f_{\mathfrak{g}+1}^+ + f_{\mathfrak{g}+1} + 2b^+ f_\mathfrak{g}^+) = 0, \\ \mathrm{Tl}_\mathfrak{g}(a,b)_2 &= b_{t_\mathfrak{g}} + 2\big(b(g_\mathfrak{g} + f_{\mathfrak{g}+1}) + a^2 f_\mathfrak{g}^+ - (a^-)^2 f_\mathfrak{g}^- + b^2 f_\mathfrak{g}\big) = 0. \end{aligned} \tag{3.47}$$

This is equivalent to

$$\mathrm{Tl}_\mathfrak{g}(a,b) = (\mathrm{Tl}_\mathfrak{g}(a,b)_1, \mathrm{Tl}_\mathfrak{g}(a,b)_2)^t = 0, \quad \mathfrak{g} \in \mathbb{N}_0. \tag{3.48}$$

The first few equations of the Toda hierarchy equal,

$$\mathrm{Tl}_0(a,b) = \begin{pmatrix} a_{t_1} - a(b - b^+) \\ b_{t_1} - 2\left((a^-)^2 - a^2\right) \end{pmatrix} = 0, \tag{3.49}$$

$$\mathrm{Tl}_1(a,b) = \begin{pmatrix} a_{t_2} - a\left((a^+)^2 - (a^-)^2 + (b^+)^2 - b^2\right) \\ b_{t_2} - 2a^2(b^+ + b) + 2(a^-)^2(b + b^-) \end{pmatrix} + c_1 \begin{pmatrix} -a(b-b^+) \\ -2\left((a^-)^2 - a^2\right) \end{pmatrix} = 0,$$

etc.

Next, define

$$F_\mathfrak{g}(z,n,t_\mathfrak{g}) = \sum_{j=0}^{\mathfrak{g}} z^j f_{\mathfrak{g}-j}(n,t_\mathfrak{g}) = \prod_{j=1}^{\mathfrak{g}} (z - \mu_j(n,t_\mathfrak{g})), \tag{3.50}$$

$$G_{\mathfrak{g}+1}(z,n,t_\mathfrak{g}) = -z^{\mathfrak{g}+1} + \sum_{j=0}^{\mathfrak{g}} z^j g_{\mathfrak{g}-j}(n,t_\mathfrak{g}) + f_{\mathfrak{g}+1}(n,t_\mathfrak{g}). \tag{3.51}$$

In the special stationary case, defined by $a_{t_\mathfrak{g}} = b_{t_\mathfrak{g}} = 0$, the recursion formulas (3.45) then imply

$$G_{\mathfrak{g}+1} - 4a^2 F_\mathfrak{g} F_\mathfrak{g}^+ = G_{\mathfrak{g}+1}^- - 4(a^-)^2 F_\mathfrak{g}^- F_\mathfrak{g} = R_{2\mathfrak{g}+2}(z), \tag{3.52}$$

where $R_{2\mathfrak{g}+2}(z)$ is a lattice constant. By inspection, $R_{2\mathfrak{g}+2}(z)$ is a polynomial in z of degree $2\mathfrak{g}+2$ with zeros $\{E_0,\dots,E_{2\mathfrak{g}+1}\}$, that is,

$$R_{2\mathfrak{g}+2}(z) = \prod_{m=0}^{2\mathfrak{g}+1} (z - E_m), \quad \{E_m\}_{m=0,\dots,2\mathfrak{g}+1} \subset \mathbb{C}. \tag{3.53}$$

Consider now the stationary hierarchy where $a = a(n)$ and $b = b(n)$ satisfy $[P_{2\mathfrak{g}+2}, L] = 0$, or

$$\begin{aligned} g_\mathfrak{g}^+ + g_\mathfrak{g} + f_{\mathfrak{g}+1}^+ + f_{\mathfrak{g}+1} + 2b^+ f_\mathfrak{g}^+ = 0, \\ b(g_\mathfrak{g} + f_{\mathfrak{g}+1}) + a^2 f_\mathfrak{g}^+ - (a^-)^2 f_\mathfrak{g}^- + b^2 f_\mathfrak{g} = 0. \end{aligned} \tag{3.54}$$

Burchnall–Chaundy's theorem then states that

$$P_{2\mathfrak{g}+2}^2 = R_{2\mathfrak{g}+2}(L). \tag{3.55}$$

Hence the hyperelliptic curve $\mathcal{K}_{\mathfrak{g}}$ of genus $\mathfrak{g}$ reads $y^2 = R_{2\mathfrak{g}+2}(z)$, and thus $N = 2\mathfrak{g}+1$ is odd in the terminology of Section 2.

Studying the diagonal Green's function of $L(t_{\mathfrak{g}})$ yields the high-energy expansion [11]

$$\frac{F_{\mathfrak{g}}(z,n,t_{\mathfrak{g}})}{R_{2\mathfrak{g}+2}(z)^{1/2}} = \sum_{j=0}^{\infty} \hat{f}_j(n,t_{\mathfrak{g}}) z^{-j-1} \text{ for } |z| > \|L\|, \tag{3.56}$$

with an appropriate choice of the radical in (3.56). Here $\hat{f}_j$ and similarly $\hat{g}_j$, denote the homogeneous coefficients f_j and g_j with vanishing integration constants $c_\ell = 0$, $\ell \geq 1$, that is,

$$\hat{f}_0 = 1, \quad \hat{f}_j = f_j|_{c_\ell=0}, \quad \ell = 1,\dots,j,\, j \in \mathbb{N}, \tag{3.57}$$

$$\hat{g}_0 = 0, \quad \hat{g}_j = g_j|_{c_\ell=0}, \quad \ell = 1,\dots,j+1,\, j \in \mathbb{N}. \tag{3.58}$$

Furthermore, define the meromorphic function $\phi(P,n,t_{\mathfrak{g}})$ on $\mathcal{K}_{\mathfrak{g}}$ by

$$\phi(P,n,t_{\mathfrak{g}}) = \frac{-G_{\mathfrak{g}+1}(z,n,t_{\mathfrak{g}}) + y(P)}{2a(n,t_{\mathfrak{g}})F_{\mathfrak{g}}(z,n,t_{\mathfrak{g}})} = \frac{-2a(n,t_{\mathfrak{g}})F_{\mathfrak{g}}(z,n+1,t_{\mathfrak{g}})}{G_{\mathfrak{g}+1}(z,n,t_{\mathfrak{g}}) + y(P)}, \quad P = (z,y) \tag{3.59}$$

using relation (3.52). With the help of $\phi(P,n,t_{\mathfrak{g}})$ we define another meromorphic function $\psi(P,n,n_0,t_{\mathfrak{g}},t_{0,\mathfrak{g}})$ on $\mathcal{K}_{\mathfrak{g}}$, the Baker–Akhiezer function, by

$$\begin{aligned}\psi(P,n,n_0,t_{\mathfrak{g}},t_{0,\mathfrak{g}}) &\\ = \exp\bigg(\int_{t_{0,\mathfrak{g}}}^{t} ds\big(2a(n_0,s)F_{\mathfrak{g}}(z,n_0,s)\phi(P,n_0,s) + G_{\mathfrak{g}+1}(z,n_0,s)\big)\bigg) &\times \\ \times \begin{cases} \prod_{m=n_0}^{n-1} \phi(P,m,t_{\mathfrak{g}}) & \text{for } n \geq n_0+1, \\ 1 & \text{for } n = n_0, \\ \prod_{m=n}^{n_0-1} \phi(P,m,t_{\mathfrak{g}})^{-1} & \text{for } n \leq n_0-1. \end{cases}&\end{aligned} \tag{3.60}$$

The divisor $(\phi(P,n,t_{\mathfrak{g}}))$ of $\phi(P,n,t_{\mathfrak{g}})$ is given by

$$(\phi(P,n,t_{\mathfrak{g}})) = \mathcal{D}_{P_{\infty_+}\underline{\hat{\mu}}(n+1,t_{\mathfrak{g}})} - \mathcal{D}_{P_{\infty_-}\underline{\hat{\mu}}(n,t_{\mathfrak{g}})}, \tag{3.61}$$

where

$$\underline{\hat{\mu}}(m,t_{\mathfrak{g}}) = (\hat{\mu}_1(m,t_{\mathfrak{g}}),\dots,\hat{\mu}_{\mathfrak{g}}(m,t_{\mathfrak{g}})) \in \sigma^{\mathfrak{g}}\mathcal{K}_{\mathfrak{g}}, \quad m \in \mathbb{Z}, \tag{3.62a}$$

$$\hat{\mu}_j(n,t_{\mathfrak{g}}) = (\mu_j(n,t_{\mathfrak{g}}), -G_{\mathfrak{g}+1}(\mu_j(n,t_{\mathfrak{g}}),n,t_{\mathfrak{g}})), \quad j = 1,\dots,\mathfrak{g}, \tag{3.62b}$$

$$\hat{\mu}_j(n+1,t_{\mathfrak{g}}) = (\mu_j(n+1,t_{\mathfrak{g}}), G_{\mathfrak{g}+1}(\mu_j(n+1,t_{\mathfrak{g}}),n,t_{\mathfrak{g}})), \quad j = 1,\dots,\mathfrak{g}. \tag{3.62c}$$

The sGmKdV hierarchy. The combined sine-Gordon and mKdV hierarchy is defined in terms of a zero curvature formalism as follows. Introduce the 2×2 matrices

$$U(z,x,t_{\mathfrak{g}}) = -i\begin{pmatrix} \frac{1}{2}u_x(x,t_{\mathfrak{g}}) & 1 \\ z & -\frac{1}{2}u_x(x,t_{\mathfrak{g}}) \end{pmatrix}, \quad (x,t_{\mathfrak{g}}) \in \mathbb{R}^2, \tag{3.63}$$

and

$$V_{\mathfrak{g}}(z,x,t_{\mathfrak{g}}) = \begin{pmatrix} -G_{\mathfrak{g}-1}(z,x,t_{\mathfrak{g}}) & \frac{1}{z}F_{\mathfrak{g}}(z,x,t_{\mathfrak{g}}) \\ H_{\mathfrak{g}}(z,x,t_{\mathfrak{g}}) & G_{\mathfrak{g}-1}(z,x,t_{\mathfrak{g}}) \end{pmatrix}, \quad (x,t_{\mathfrak{g}}) \in \mathbb{R}^2,\, \mathfrak{g} \in \mathbb{N}_0. \tag{3.64}$$

Then the zero curvature relation reads

$$U_{t_{\mathfrak{g}}} - V_{\mathfrak{g},x} + [U,V_{\mathfrak{g}}] = 0, \quad \mathfrak{g} \in \mathbb{N}_0, \tag{3.65}$$

resulting in the equations

$$u_{xt_{\mathfrak{g}}}(x,t_{\mathfrak{g}}) = -2iG_{\mathfrak{g}-1,x}(x,t_{\mathfrak{g}}) - 2(H_{\mathfrak{g}}(x,t_{\mathfrak{g}}) - F_{\mathfrak{g}}(x,t_{\mathfrak{g}})), \tag{3.66a}$$

$$F_{\mathfrak{g},x}(x,t_{\mathfrak{g}}) = -iu_x(x,t_{\mathfrak{g}})F_{\mathfrak{g}}(x,t_{\mathfrak{g}}) - 2izG_{\mathfrak{g}-1}(x,t_{\mathfrak{g}}), \tag{3.66b}$$

$$H_{\mathfrak{g},x}(x,t_{\mathfrak{g}}) = iu_x(x,t_{\mathfrak{g}})H_{\mathfrak{g}}(x,t_{\mathfrak{g}}) + 2izG_{\mathfrak{g}-1}(x,t_{\mathfrak{g}}). \tag{3.66c}$$

Making the following polynomial ansatz

$$F_{\mathfrak{g}}(z,x,t_{\mathfrak{g}}) = \sum_{j=0}^{\mathfrak{g}} f_{\mathfrak{g}-j}(x,t_{\mathfrak{g}})z^j = \prod_{j=1}^{\mathfrak{g}}(z-\mu_j(x,t_{\mathfrak{g}})), \tag{3.67a}$$

$$H_{\mathfrak{g}}(z,x,t_{\mathfrak{g}}) = \sum_{j=0}^{\mathfrak{g}} h_{\mathfrak{g}-j}(x,t_{\mathfrak{g}})z^j = \prod_{j=1}^{\mathfrak{g}}(z-\nu_j(x,t_{\mathfrak{g}})), \tag{3.67b}$$

$$G_{-1}(z,x,t_{\mathfrak{g}}) = 0, \quad G_{\mathfrak{g}-1}(z,x,t_{\mathfrak{g}}) = \sum_{j=0}^{\mathfrak{g}-1} g_{\mathfrak{g}-1-j}(x,t_{\mathfrak{g}})z^j, \tag{3.67c}$$

one concludes[5] (see [33] for a detailed discussion) ($f_j = f_j(x,t_{\mathfrak{g}})$, etc.)

$$f_{\mathfrak{g}} = \alpha e^{-iu}, \qquad h_{\mathfrak{g}} = \beta e^{iu}, \quad \alpha,\beta\in\mathbb{C},\ \mathfrak{g}\in\mathbb{N}_0. \tag{3.68a}$$

$$f_0 = 1, \qquad f_{j,x} = -\frac{1}{4}f_{j-1,xxx} + w_+ f_{j-1,x} + \frac{1}{2}w_{+,x}f_{j-1}, \quad j=1,\dots,\mathfrak{g},\ \mathfrak{g}\in\mathbb{N}, \tag{3.68b}$$

$$h_0 = 1, \qquad h_{j,x} = -\frac{1}{4}h_{j-1,xxx} + w_- h_{j-1,x} + \frac{1}{2}w_{-,x}h_{j-1}, \quad j=1,\dots,\mathfrak{g},\ \mathfrak{g}\in\mathbb{N}, \tag{3.68c}$$

where

$$w_\pm = -\frac{1}{4}(u_x^2 \pm 2iu_{xx}), \tag{3.69}$$

and

$$g_{-1} = 0, \quad g_j = \frac{i}{2}(f_{j,x} + iu_x f_j) = \frac{i}{2}(-h_{j,x} + iu_x h_j), \quad j=0,\dots,\mathfrak{g}-1,\ \mathfrak{g}\in\mathbb{N}. \tag{3.70}$$

Explicitly,

$$g_0 = -\tfrac{1}{2}u_x, \quad g_1 = \tfrac{1}{16}u_x^3 + \tfrac{1}{8}u_{xxx} - \tfrac{c_1}{2}u_x, \text{ etc.} \tag{3.71}$$

We also list a few coefficients in the homogeneous case where all integration constants c_ℓ, $\ell\geq 1$ vanish,

$$\begin{aligned} &\hat f_0 = f_0 = 1, && \hat g_0 = g_0 = -\tfrac{1}{2}u_x, && \hat h_0 = h_0 = 1, \quad \mathfrak{g}\in\mathbb{N},\\ &\hat f_j = f_j\big|_{c_\ell=0}, && \hat g_j = g_j\big|_{c_\ell=0}, && \hat h_j = h_j\big|_{c_\ell=0},\\ &&&&& \ell=1,\dots,j,\ j=1,\dots,\mathfrak{g}-1,\ \mathfrak{g}\geq 2, \end{aligned} \tag{3.72}$$

$$\hat f_{\mathfrak{g}} = f_{\mathfrak{g}} = \alpha e^{-iu}, \qquad \hat h_{\mathfrak{g}} = h_{\mathfrak{g}} = \beta e^{iu}, \qquad \mathfrak{g}\in\mathbb{N}_0. \tag{3.73}$$

Remark 3.1. The recursion for the sGmKdV hierarchy is anomalous compared to the other hierarchies studied in this paper. For the KdV, AKNS as well as the Tl hierarchies the functions f_j (and g_j and h_j where applicable) are defined by the same recursion formula for all $j\in\mathbb{N}$ irrespective of the given genus $\mathfrak{g}$. However, for the sG hierarchy $f_{\mathfrak{g}}$ and $h_{\mathfrak{g}}$ are always given by (3.68a) for $\mathfrak{g}\in\mathbb{N}_0$. This raises a compatibility problem in the recursion formalism. A proof of the solvability of the recursion can be found in [33], Appendix C.

The $\mathfrak{g}$th sGmKdV equation is then defined by

$$\text{sGmKdV}_{\mathfrak{g}}(u(x,t_{\mathfrak{g}})) = u_{xt_{\mathfrak{g}}}(x,t_{\mathfrak{g}}) + 2ig_{\mathfrak{g}-1,x}(x,t_{\mathfrak{g}}) + 2(\beta e^{iu(x,t_{\mathfrak{g}})} - \alpha e^{-iu(x,t_{\mathfrak{g}})}) = 0, \quad \mathfrak{g}\in\mathbb{N}_0. \tag{3.74}$$

Explicitly, the first few equations read

$$\begin{aligned} &\text{sGmKdV}_0(u) = u_{xt_0} + 2(\beta e^{iu} - \alpha e^{-iu}) = 0,\\ &\text{sGmKdV}_1(u) = u_{xt_1} - iu_{xx} + 2(\beta e^{iu} - \alpha e^{-iu}) = 0,\\ &\text{sGmKdV}_2(u) = u_{xt_2} + \tfrac{i}{8}(u_x^3 + 2u_{xxx})_x - c_1 iu_{xx} + 2(\beta e^{iu} - \alpha e^{-iu}) = 0, \text{ etc.} \end{aligned} \tag{3.75}$$

[5] Observe that the recursions (3.68b) and (3.68c) are identical with the KdV recursion replacing V by $w_\pm$. Hence explicit formulas for the first few can be read off from (3.3) upon changing V into $w_\pm$.

Observe that $\alpha = \beta = i/4$ and $\mathfrak{g} = 0$ yields the well-known sine-Gordon equation in light-cone coordinates. Appropriate choices of α and β in (3.74) include the sinh-Gordon hierarchy, the corresponding elliptic equations, the Liouville model, as well as the modified KdV hierarchy (taking $\alpha = \beta = 0$).

In the stationary case, where $u_{xt_\mathfrak{g}} = 0$, we find

$$\frac{d}{dx}\Big(zG_{\mathfrak{g}-1}(z,x)^2 + F_\mathfrak{g}(z,x)H_\mathfrak{g}(z,x)\Big) = 0 \tag{3.76}$$

and hence

$$zG_{\mathfrak{g}-1}(z,x)^2 + F_\mathfrak{g}(z,x)H_\mathfrak{g}(z,x) = P_{2\mathfrak{g}}(z), \tag{3.77}$$

where $P_{2\mathfrak{g}}(z)$ is x-independent. It is more convenient to define $R_{2\mathfrak{g}+1}(z) = zP_{2\mathfrak{g}}(z)$ so that (3.77) becomes

$$z^2G_{\mathfrak{g}-1}(z,x)^2 + zF_\mathfrak{g}(z,x)H_\mathfrak{g}(z,x) = R_{2\mathfrak{g}+1}(z), \tag{3.78}$$

where $R_{2\mathfrak{g}+1}$ is a monic polynomial in z of degree $2\mathfrak{g}+1$ of the form

$$R_{2\mathfrak{g}+1}(z) = \prod_{m=0}^{2\mathfrak{g}}(z - E_m), \quad E_0 = 0,\, E_1, \dots, E_{2\mathfrak{g}} \in \mathbb{C}. \tag{3.79}$$

This polynomial defines the hyperelliptic curve $\mathcal{K}_\mathfrak{g}$ of genus $\mathfrak{g}$ by the relation $y^2 - R_{2\mathfrak{g}+1}(z) = 0$ and hence $N = 2\mathfrak{g}$ in the terminology of Section 2. $\mathcal{K}_\mathfrak{g}$ is compactified by adding a point P_∞.

Remark 3.2. Observe that the sGmKdV-curve is a special case of the KdV curve with the additional constraint $E_0 = 0$.

Remark 3.3. In the stationary case the choice of α and β is constrained by the relation

$$\alpha\beta = \prod_{j=1}^{2\mathfrak{g}} E_j, \tag{3.80}$$

as can be seen by inserting $z = 0$ in (3.78), using (3.68a) and (3.79).

We now return to the time-dependent formalism. Let $\phi(P,x,t_\mathfrak{g})$ be the meromorphic function on $\mathcal{K}_\mathfrak{g}$ given by

$$\phi(P,x,t_\mathfrak{g}) = \frac{y(P) - zG_{\mathfrak{g}-1}(z,x,t_\mathfrak{g})}{F_\mathfrak{g}(z,x,t_\mathfrak{g})} = \frac{zH_\mathfrak{g}(z,x,t_\mathfrak{g})}{y(P) + zG_{\mathfrak{g}-1}(z,x,t_\mathfrak{g})}, \quad (x,t_\mathfrak{g}) \in \mathbb{R}^2,\, P = (z,y) \in \mathcal{K}_\mathfrak{g} \setminus \{P_\infty\}. \tag{3.81}$$

Hence the divisor $(\phi(P,x,t_\mathfrak{g}))$ of $\phi(P,x,t_\mathfrak{g})$ reads

$$(\phi(P,x,t_\mathfrak{g})) = \mathcal{D}_{Q_0\underline{\nu}(x,t_\mathfrak{g})} - \mathcal{D}_{P_\infty\underline{\mu}(x,t_\mathfrak{g})}, \tag{3.82}$$

with

$$\begin{aligned}
\underline{\hat\mu}(x,t_\mathfrak{g}) &= (\hat\mu_1(x,t_\mathfrak{g}), \dots, \hat\mu_\mathfrak{g}(x,t_\mathfrak{g})) \in \sigma^\mathfrak{g}\mathcal{K}_\mathfrak{g}, && (3.83a)\\
\hat\mu_j(x,t_r) &= (\mu_j(x,t_r), -\mu_j(x,t_r)G_{\mathfrak{g}-1}(\mu_j(x,t_r),x,t_r)) \in \mathcal{K}_\mathfrak{g}, \quad j = 1,\dots,\mathfrak{g}, &&\\
\underline{\hat\nu}(x,t_\mathfrak{g}) &= (\hat\nu_1(x,t_\mathfrak{g}), \dots, \hat\nu_\mathfrak{g}(x,t_\mathfrak{g})) \in \sigma^\mathfrak{g}\mathcal{K}_\mathfrak{g}, && (3.83b)\\
\hat\nu_j(x,t_r) &= (\nu_j(x,t_r), \nu_j(x,t_r)G_{\mathfrak{g}-1}(\nu_j(x,t_r),x,t_r)) \in \mathcal{K}_\mathfrak{g}, \quad j = 1,\dots,\mathfrak{g}. &&
\end{aligned}$$

The time-dependent Baker–Akhiezer function

$$\Psi(P,x,x_0,t_\mathfrak{g},t_{0,\mathfrak{g}}) = \begin{pmatrix}\psi_1(P,x,x_0,t_\mathfrak{g},t_{0,\mathfrak{g}})\\ \psi_2(P,x,x_0,t_\mathfrak{g},t_{0,\mathfrak{g}})\end{pmatrix} \tag{3.84}$$

is defined by

$$\psi_1(P,x,x_0,t_\mathfrak{g},t_{0,\mathfrak{g}}) = \exp\Big(-\frac{i}{2}(u(x,t_\mathfrak{g}) - u(x_0,t_\mathfrak{g})) \tag{3.85}$$

$$
\begin{aligned}
&+ i\int_{x_0}^{x} dx'\,\phi(P,x',t_{\mathfrak{g}}) - \int_{t_{0,\mathfrak{g}}}^{t_{\mathfrak{g}}} ds\,\Big(\frac{1}{z}\widetilde F_{\mathfrak{g}}(z,x_0,s)\phi(P,x_0,s) + \widetilde G_{\mathfrak{g}-1}(z,x_0,s)\Big)\bigg),\\
&\psi_2(P,x,x_0,t_{\mathfrak{g}},t_{0,\mathfrak{g}}) = -\psi_1(P,x,x_0,t_{\mathfrak{g}},t_{0,\mathfrak{g}})\phi(P,x,t_{\mathfrak{g}}), \qquad (3.86)\\
&\qquad\qquad P\in\mathcal{K}_{\mathfrak{g}}\setminus\{P_\infty\},\ (x,x_0,t_{\mathfrak{g}},t_{0,\mathfrak{g}})\in\mathbb{R}^4.
\end{aligned}
$$

Combining relations (3.67a), (3.67b), and (3.68a) one concludes

$$
u(x,t_{\mathfrak{g}}) = i\ln\bigg((-1)^{\mathfrak{g}}\alpha^{-1}\prod_{j=1}^{\mathfrak{g}}\mu_j(x,t_{\mathfrak{g}})\bigg) = -i\ln\bigg((-1)^{\mathfrak{g}}\beta^{-1}\prod_{j=1}^{\mathfrak{g}}\nu_j(x,t_{\mathfrak{g}})\bigg). \tag{3.87}
$$

We will also need the following asymptotic high-energy expansion

$$
\frac{F_{\mathfrak{g}}(z,x,t_{\mathfrak{g}})}{R_{2\mathfrak{g}+1}(z)^{1/2}} \underset{z\to i\infty}{=} \frac{1}{z^{1/2}}\sum_{j=0}^{\infty}\hat f_j(x,t_{\mathfrak{g}})z^{-j}. \tag{3.88}
$$

4. Symmetric functions

Let $\mathfrak{g}\in\mathbb{N}$ be fixed and define

$$
\mathcal{S}_k = \{\underline{\ell}=(\ell_1,\dots,\ell_k)\in\mathbb{N}^k \mid \ell_1<\cdots<\ell_k\le\mathfrak{g}\},\quad k\le\mathfrak{g}, \tag{4.1a}
$$

$$
\mathcal{I}_k^{(j)} = \{\underline{\ell}=(\ell_1,\dots,\ell_k)\in\mathcal{S}_k \mid \ell_m\neq j\},\quad k\le\mathfrak{g}-1. \tag{4.1b}
$$

Define

$$
\Psi_0(\underline{\mu})=1,\quad \Psi_k(\underline{\mu}) = (-1)^k\sum_{\underline{\ell}\in\mathcal{S}_k}\mu_{\ell_1}\cdots\mu_{\ell_k},\quad k\le\mathfrak{g}, \tag{4.2a}
$$

$$
\Phi_0^{(j)}(\underline{\mu})=1,\quad \Phi_k^{(j)}(\underline{\mu}) = (-1)^k\sum_{\underline{\ell}\in\mathcal{I}_k^{(j)}}\mu_{\ell_1}\cdots\mu_{\ell_k},\quad k\le\mathfrak{g}-1,\quad \Phi_{\mathfrak{g}}^{(j)}(\underline{\mu})=0, \tag{4.2b}
$$

where $\underline{\mu}=(\mu_1,\dots,\mu_{\mathfrak{g}})\in\mathbb{C}^{\mathfrak{g}}$. One recognizes the simple pattern,

$$
\Psi_1(\underline{\mu}) = -\sum_{\ell=1}^{\mathfrak{g}}\mu_\ell,\quad \Psi_2(\underline{\mu}) = \sum_{\substack{\ell_1,\ell_2=1\\ \ell_1<\ell_2}}^{\mathfrak{g}}\mu_{\ell_1}\mu_{\ell_2},\ \text{etc.}, \tag{4.3a}
$$

$$
\Phi_1^{(j)}(\underline{\mu}) = -\sum_{\substack{\ell=1\\ \ell\neq j}}^{\mathfrak{g}}\mu_\ell,\quad \Phi_2^{(j)}(\underline{\mu}) = \sum_{\substack{\ell_1,\ell_2=1\\ \ell_1,\ell_2\neq j\\ \ell_1<\ell_2}}^{\mathfrak{g}}\mu_{\ell_1}\mu_{\ell_2},\ \text{etc.} \tag{4.3b}
$$

Let $E_0,\dots,E_N$ be $N+1$ complex numbers, where $N=2\mathfrak{g}$ or $N=2\mathfrak{g}+1$ depending on the underlying hierarchy of soliton equations. For brevity we introduce

$$
\underline{E}=(E_0,\dots,E_N). \tag{4.4}
$$

We will need the following elementary result.

Lemma 4.1. *For $z\in\mathbb{C}$ such that $|z|>\max\{|E_0|,\dots,|E_N|\}$ we have*

$$
\bigg(\prod_{m=0}^{N}\big(1-\frac{E_m}{z}\big)\bigg)^{-1/2} = \sum_{k=0}^{\infty}c_k(\underline{E})z^{-k}, \tag{4.5}
$$

where[6]

$$
c_0(\underline{E})=1,\quad c_k(\underline{E}) = \sum_{\substack{j_0,\dots,j_N=0\\ j_0+\cdots+j_N=k}}^{k}\frac{(2j_0-1)!!\cdots(2j_N-1)!!}{2^k j_0!\cdots j_N!}E_0^{j_0}\cdots E_N^{j_N},\ k\in\mathbb{N}. \tag{4.6}
$$

Proof. It suffices to apply the binomial expansion. □

[6] $(2n-1)!!=1\cdot3\cdots(2n-1)$, and $(-1)!!=1$.

The few first terms read

$$c_1(\underline{E}) = \frac{1}{2}\sum_{j=0}^{N} E_j, \quad c_2(\underline{E}) = \frac{1}{4}\sum_{\substack{j,k=0\\ j\neq k}}^{N} E_j E_k + \frac{3}{8}\sum_{j=0}^{N} E_j^2, \text{ etc.} \tag{4.7}$$

Next, assuming $\mu_j \neq \mu_{j'}$ for $j \neq j'$, introduce the $\mathfrak{g}\times\mathfrak{g}$ matrix $U_{\mathfrak{g}}(\underline{\mu})$ by

$$U_1(\underline{\mu}) = 1, \quad U_{\mathfrak{g}}(\underline{\mu}) = \left(\frac{\mu_k^{j-1}}{\prod_{m\neq k}^{\mathfrak{g}}(\mu_k - \mu_m)}\right)_{j,k=1}^{\mathfrak{g}}. \tag{4.8}$$

Lemma 4.2. *Suppose $\mu_j \neq \mu_{j'}$ for $j \neq j'$. Then*

$$U_{\mathfrak{g}}(\underline{\mu})^{-1} = \left(\Phi_{\mathfrak{g}-k}^{(j)}(\underline{\mu})\right)_{j,k=1}^{\mathfrak{g}}. \tag{4.9}$$

Proof. Observe that we may write

$$U_{\mathfrak{g}}(\underline{\mu}) = \left(\frac{\mu_k^{j-1}}{F_{\mathfrak{g}}'(\mu_k)}\right)_{j,k=1}^{\mathfrak{g}}. \tag{4.10}$$

Using Lagrange's interpolation result, Theorem A.1 (replacing k by $\mathfrak{g}-k$ in (A.3)), proves the result. ◻

Of crucial importance for our approach is the fact that we are able to express f_j and F_r in terms of elementary symmetric functions of $\mu_1,\dots,\mu_{\mathfrak{g}}$. The expression is given below for the homogeneous case only, denoted by $\hat{f}_j$ and $\widehat{F}_r$, where the integration constants c_ℓ for $\ell\in\mathbb{N}$ vanish. We start with $\hat{f}_j$.

Lemma 4.3. *Let $c_j(\underline{E})$ be defined as in* (4.6)*. Then we infer the following results for the KdV and the Toda hierarchies*[7]*,*

$$\hat{f}_j = \sum_{k=0}^{j\wedge\mathfrak{g}} c_{j-k}(\underline{E})\Psi_k(\underline{\mu}). \tag{4.11}$$

For the AKNS hierarchy we obtain

$$\hat{f}_j = -iq\sum_{k=0}^{j\wedge\mathfrak{g}} c_{j-k}(\underline{E})\Psi_k(\underline{\mu}), \quad \hat{h}_j = ip\sum_{k=0}^{j\wedge\mathfrak{g}} c_{j-k}(\underline{E})\Psi_k(\underline{\nu}), \tag{4.12}$$

where $\underline{\nu} = (\nu_1,\dots,\nu_{\mathfrak{g}})$.
In the sGmKdV case we have

$$\hat{f}_j = \sum_{k=0}^{j} c_{j-k}(\underline{E})\Psi_k(\underline{\mu}), \quad \hat{h}_j = \sum_{k=0}^{j} c_{j-k}(\underline{E})\Psi_k(\underline{\nu}), \quad j = 0,\dots,\mathfrak{g}-1,\ \mathfrak{g}\in\mathbb{N}, \tag{4.13}$$

and

$$\hat{f}_{\mathfrak{g}} = \Psi_{\mathfrak{g}}(\underline{\mu}), \quad \hat{h}_{\mathfrak{g}} = \Psi_{\mathfrak{g}}(\underline{\nu}), \quad \mathfrak{g}\in\mathbb{N}_0. \tag{4.14}$$

Proof. The proof is identical in all cases, and is based on the high-energy expansion of the Green's function of the corresponding linear operator L or M in the Lax pair or zero curvature formulation of the hierarchy considered. We provide the details for the KdV hierarchy only.

Using Lemma 4.1 we find

$$\begin{aligned}\frac{F_{\mathfrak{g}}(z)}{R_{2\mathfrak{g}+1}(z)^{1/2}} &= \frac{\prod_{j=1}^{\mathfrak{g}}(z-\mu_j)}{R_{2\mathfrak{g}+1}(z)^{1/2}} = \frac{1}{z^{1/2}}\frac{\prod_{j=1}^{\mathfrak{g}}(1-\frac{\mu_j}{z})}{\prod_{m=0}^{2\mathfrak{g}}(1-\frac{E_m}{z})^{1/2}}\\ &= \frac{1}{z^{1/2}}\Big(\sum_{j=0}^{\mathfrak{g}}\Psi_j(\underline{\mu})z^{-j}\Big)\Big(\sum_{m=0}^{\infty}c_m(\underline{E})z^{-m}\Big) = \frac{1}{z^{1/2}}\sum_{m=0}^{\infty}z^{-m}\sum_{k=0}^{m\wedge\mathfrak{g}}c_{m-k}(\underline{E})\Psi_k(\underline{\mu}).\end{aligned}$$

Combining this result with the high-energy expansion (3.13) proves the result. ◻

[7] $n\wedge m = \min\{n,m\}$.

Remark 4.4. Observe that the right-hand side of (4.11) is defined for all $x \in \mathbb{R}$, but when we sample it at integer values $x = n \in \mathbb{Z}$, it coincides with $\hat{f}_j(n)$ for the Toda lattice. Thus we have in some sense a continuous extension of the Toda hierarchy (cf. also Lemma 5.6).

Theorem 4.5. *Let $r \in \mathbb{N}_0$. For both the KdV and the Tl case one derives*[8]

$$\widehat{F}_r(\mu_j) = \sum_{p=(r-\mathfrak{g})\vee 0}^{r} c_p(\underline{E})\Phi^{(j)}_{r-p}(\underline{\mu}). \tag{4.15}$$

For the AKNS hierarchy one infers

$$\widehat{F}_r(\mu_j) = -iq \sum_{p=(r-\mathfrak{g})\vee 0}^{r} c_p(\underline{E})\Phi^{(j)}_{r-p}(\underline{\mu}), \quad \widehat{H}_r(\nu_j) = ip \sum_{p=(r-\mathfrak{g})\vee 0}^{r} c_p(\underline{E})\Phi^{(j)}_{r-p}(\underline{\nu}). \tag{4.16}$$

For the sGmKdV hierarchy one concludes[9]

$$\begin{aligned} \frac{\widehat{F}_r(\mu_j)}{\mu_j} &= \sum_{p=(r-1-\mathfrak{g})\vee 0}^{r-1} c_p(\underline{E})\Phi^{(j)}_{r-1-p}(\underline{\mu}) - \frac{\tilde{\alpha}}{\alpha}\Phi^{(j)}_{\mathfrak{g}-1}(\underline{\mu}), \\ \frac{\widehat{H}_r(\nu_j)}{\nu_j} &= \sum_{p=(r-1-\mathfrak{g})\vee 0}^{r-1} c_p(\underline{E})\Phi^{(j)}_{r-1-p}(\underline{\nu}) - \frac{\tilde{\beta}}{\beta}\Phi^{(j)}_{\mathfrak{g}-1}(\underline{\nu}). \end{aligned} \tag{4.17}$$

Proof. It suffices to consider the KdV and sG cases. By definition

$$\widehat{F}_r(z) = \sum_{\ell=0}^{r} \hat{f}_{r-\ell} z^\ell = \sum_{\ell=0}^{r} z^\ell \sum_{m=0}^{(r-\ell)\wedge\mathfrak{g}} \Psi_m(\underline{\mu}) c_{r-\ell-m}(\underline{E}). \tag{4.18}$$

Consider first the case $r \le \mathfrak{g}$. Then

$$\widehat{F}_r(z) = \sum_{p=0}^{r} c_p(\underline{E}) \sum_{\ell=0}^{r-p} z^\ell \Psi_{r-\ell-p}(\underline{\mu}) \tag{4.19}$$

and hence

$$\widehat{F}_r(\mu_j) = \sum_{p=0}^{r} c_p(\underline{E})\Phi^{(j)}_{r-p}(\underline{\mu}), \tag{4.20}$$

using (A.13). In the case when $r \ge \mathfrak{g}+1$ we find

$$\begin{aligned} \widehat{F}_r(z) &= \sum_{m=0}^{\mathfrak{g}} \Psi_m(\underline{E}) \sum_{p=0}^{r-m} z^{r-m-p} c_p(\underline{E}) \\ &= \sum_{p=0}^{r-\mathfrak{g}} c_p(\underline{E}) \Big(\sum_{\ell=0}^{\mathfrak{g}} \Psi_\ell(\underline{\mu}) z^{\mathfrak{g}-\ell}\Big) z^{r-\mathfrak{g}-p} + \sum_{p=r-\mathfrak{g}+1}^{r} c_p(\underline{E}) \sum_{\ell=0}^{r-p} \Psi_\ell(\underline{\mu}) z^{r-p-\ell} \\ &= F_{\mathfrak{g}}(z) \sum_{p=0}^{r-\mathfrak{g}} c_p(\underline{E}) z^{r-\mathfrak{g}-p} + \sum_{p=r-\mathfrak{g}+1}^{r} c_p(\underline{E}) \sum_{\ell=0}^{r-p} \Psi_\ell(\underline{\mu}) z^{r-p-\ell} \\ &= F_{\mathfrak{g}}(z) \sum_{p=0}^{r-\mathfrak{g}} c_p(\underline{E}) z^{r-\mathfrak{g}-p} + \sum_{p=r-\mathfrak{g}+1}^{r} c_p(\underline{E}) \sum_{m=0}^{r-p} \Psi_{r-p-m}(\underline{\mu}) z^m. \end{aligned} \tag{4.21}$$

Hence

$$\widehat{F}_r(\mu_j) = \sum_{p=r-\mathfrak{g}+1}^{r} c_p(\underline{E})\Phi^{(j)}_{r-p}(\underline{\mu}), \tag{4.22}$$

by using (A.13) again.

---

[8] $n \vee m = \max\{n, m\}$.

[9] Since r is independent of $\mathfrak{g}$, one obtains $\hat{f}_r = \tilde{\alpha} e^{-iu}$, $\hat{h}_r = \tilde{\beta} e^{iu}$ with $\tilde{\alpha}, \tilde{\beta} \in \mathbb{C}$ independent of α, β, and $\hat{f}_q, \hat{h}_q$, $q = 1, \dots, r-1$ constructed as in (3.72).

In the sGmKdV case we first observe the identity

$$\widehat{F}_r(z) = z\widehat{F}_{r-1}(z) + \hat{f}_r, \tag{4.23}$$

which implies

$$\frac{\widehat{F}_r(\mu_j)}{\mu_j} = \widehat{F}_{r-1}(\mu_j) + \frac{\hat{f}_r}{\mu_j} = \sum_{p=(r-1-\mathfrak{g})\vee 0}^{r-1} c_p(\underline{E})\Phi^{(j)}_{r-1-p}(\underline{\mu}) - \frac{\tilde{\alpha}}{\alpha}\Phi^{(j)}_{\mathfrak{g}-1}(\underline{\mu}), \tag{4.24}$$

using $\tilde{f}_r = \tilde{\alpha}e^{-iu}$ and the trace relation (3.87).

□

5. Dubrovin equations and linearized flows

Dubrovin [21] made the fundamental observation that the Dirichlet divisors for the KdV equation satisfy a first-order system of differential equations. Solving this system can then be used to recover the function V by appealing to a trace formula (cf. (5.28)).

Before we state the Dubrovin equations we need some notation. Let $\mathfrak{g} \in \mathbb{N}$. We start by constructing the hierarchies as explained in Section 3. In particular, we construct the function $F_{\mathfrak{g}}$ with its zeros $\underline{\mu} = (\mu_1, \dots, \mu_{\mathfrak{g}})$, and define the corresponding hyperelliptic curve $\mathcal{K}_{\mathfrak{g}}$. (In the AKNS case we also construct the function $H_{\mathfrak{g}}$.) Next, fix an $r \in \mathbb{N}_0$, and construct the function F_r. The integration constants in the definition of F_r are assumed to be independent of those used to construct $F_{\mathfrak{g}}$, and to emphasize this fact we denote it by $\widetilde{F}_r$ and the corresponding constants by $\tilde{c}_\ell$. The Dubrovin equations give the evolution of $\underline{\mu} = (\mu_1, \dots, \mu_{\mathfrak{g}})$ in terms of the deformation (time) parameter t_r according to the rth equation in the hierarchy considered.

The KdV hierarchy. In our setting the Dubrovin equations for the KdV hierarchy read [16], [18], Sect. 12.3, [32], [36], [45], Chs. 10, 12, [47], Ch. 4,

$$\frac{\partial}{\partial x}\mu_j(x,t_r) = -2i\,\frac{y(\hat{\mu}_j(x,t_r))}{\prod_{\ell\neq j}^{\mathfrak{g}}(\mu_j(x,t_r)-\mu_\ell(x,t_r))}, \tag{5.1a}$$

$$\begin{aligned}\frac{\partial}{\partial t_r}\mu_j(x,t_r) &= \widetilde{F}_r(\mu_j(x,t_r),x,t_r)\frac{\partial}{\partial x}\mu_j(x,t_r) \\ &= -2i\,\frac{y(\hat{\mu}_j(x,t_r))}{\prod_{\ell\neq j}^{\mathfrak{g}}(\mu_j(x,t_r)-\mu_\ell(x,t_r))}\widetilde{F}_r(\mu_j(x,t_r),x,t_r),\end{aligned} \tag{5.1b}$$

for $j = 1, \dots, \mathfrak{g}$. The initial data for (5.1) on $\mathcal{K}_{\mathfrak{g}}$ equal

$$\underline{\hat{\mu}}(x,t_{0,r}) = \underline{\hat{\mu}}^{(0)}(x), \tag{5.2}$$

where $\underline{\hat{\mu}} = (\hat{\mu}_1, \dots, \hat{\mu}_{\mathfrak{g}})$ denotes

$$\hat{\mu}_j(x,t_r) = (\mu_j(x,t_r), -\frac{i}{2}F_{\mathfrak{g},x}(\mu_j(x),x,t_r)) \in \mathcal{K}_{\mathfrak{g}}, \quad j = 1, \dots, \mathfrak{g}. \tag{5.3}$$

We remark that (5.1a) is an immediate consequence of (3.7) and (3.18), while (5.1b) follows from (3.18) and

$$F_{\mathfrak{g},t_r} = \widetilde{F}_r F_{\mathfrak{g},x} - \widetilde{F}_{r,x}F_{\mathfrak{g}} \tag{5.4}$$

upon taking $z = \mu_j(x,t_r)$.

The AKNS hierarchy. In this case the Dubrovin equations for $\underline{\hat{\mu}} = (\hat{\mu}_1, \dots, \hat{\mu}_{\mathfrak{g}})$ are given by [15], [35],

$$\frac{\partial}{\partial x}\mu_j(x,t_r) = -2i\,\frac{y(\hat{\mu}_j(x,t_r))}{\prod_{\ell\neq j}^{\mathfrak{g}}(\mu_j(x,t_r)-\mu_\ell(x,t_r))}, \tag{5.5a}$$

$$\begin{aligned}\frac{\partial}{\partial t_r}\mu_j(x,t_r) &= -\frac{\widetilde{F}_r(\mu_j(x,t_r),x,t_r)}{iq(x,t_r)}\frac{\partial}{\partial x}\mu_j(x,t_r) \\ &= 2\,\frac{y(\hat{\mu}_j(x,t_r))}{q(x,t_r)\prod_{\ell\neq j}^{\mathfrak{g}}(\mu_j(x,t_r)-\mu_\ell(x,t_r))}\widetilde{F}_r(\mu_j(x,t_r),x,t_r),\end{aligned} \tag{5.5b}$$

for $j = 1, \dots, \mathfrak{g}$, with initial data on $\mathcal{K}_\mathfrak{g}$

$$\underline{\hat{\mu}}(x, t_{0,r}) = \underline{\hat{\mu}}^{(0)}(x), \tag{5.6}$$

where

$$\hat{\mu}_j(x, t_r) = (\mu_j(x, t_r), G_{\mathfrak{g}+1}(\mu_j(x), x, t_r)) \in \mathcal{K}_\mathfrak{g}, \quad j = 1, \dots, \mathfrak{g}. \tag{5.7}$$

For the corresponding evolution of $\underline{\hat{\nu}} = (\hat{\nu}_1, \dots, \hat{\nu}_\mathfrak{g})$ we have

$$\frac{\partial}{\partial x}\nu_j(x, t_r) = -2i \, \frac{y(\hat{\nu}_j(x, t_r))}{\prod_{\ell \neq j}^{\mathfrak{g}}(\nu_j(x, t_r) - \nu_\ell(x, t_r))}, \tag{5.8a}$$

$$\begin{aligned} \frac{\partial}{\partial t_r}\nu_j(x, t_r) &= \widetilde{H}_r(\nu_j(x, t_r), x, t_r)\frac{\partial}{\partial x}\nu_j(x, t_r) \\ &= -2 \, \frac{y(\hat{\nu}_j(x, t_r))}{p(x, t_r)\prod_{\ell \neq j}^{\mathfrak{g}}(\nu_j(x, t_r) - \nu_\ell(x, t_r))}\widetilde{H}_r(\nu_j(x, t_r), x, t_r), \end{aligned} \tag{5.8b}$$

for $j = 1, \dots, \mathfrak{g}$, with initial data on $\mathcal{K}_\mathfrak{g}$

$$\underline{\hat{\nu}}(x, t_{0,r}) = \underline{\hat{\nu}}^{(0)}(x), \tag{5.9}$$

where

$$\hat{\nu}_j(x, t_r) = (\nu_j(x, t_r), -G_{\mathfrak{g}+1}(\nu_j(x), x, t_r)) \in \mathcal{K}_\mathfrak{g}, \quad j = 1, \dots, \mathfrak{g}. \tag{5.10}$$

The Toda hierarchy. Here the Dubrovin equations for $\underline{\hat{\mu}} = (\hat{\mu}_1, \dots, \hat{\mu}_\mathfrak{g})$ read [11], [17], [51], [62], Ch.4,

$$\frac{\partial}{\partial t_r}\mu_j(n, t_r) = 2 \, \frac{y(\hat{\mu}_j(n, t_r))}{\prod_{\ell \neq j}^{\mathfrak{g}}(\mu_j(n, t_r) - \mu_\ell(n, t_r))}\widetilde{F}_r(\mu_j(n, t_r), n, t_r), \tag{5.11}$$

for $j = 1, \dots, \mathfrak{g}$, with initial data on $\mathcal{K}_\mathfrak{g}$

$$\underline{\hat{\mu}}(n, t_{0,r}) = \underline{\hat{\mu}}^{(0)}(n), \tag{5.12}$$

where

$$\hat{\mu}_j(n, t_r) = (\mu_j(n, t_r), -G_{\mathfrak{g}+1}(\mu_j(n), n, t_r)) \in \mathcal{K}_\mathfrak{g}, \quad j = 1, \dots, \mathfrak{g}. \tag{5.13}$$

We note that (5.5a) and (5.8a) formally coincide with (5.1a). The case of the Toda hierarchy, however, is quite different since (5.11) concerns the t_r-dependence of $\underline{\hat{\mu}}(n, t_r)$ and no analogous first-order nonlinear difference equation concerning the n-dependence of $\underline{\hat{\mu}}(n, t_r)$ (i.e., an analog of (5.1a) or (5.5a)) appears to be known. In this context we refer the reader to Lemma 5.6, where we continue this discussion.

The sGmKdV hierarchy. Finally, in the case of the sGmKdV hierarchy the equations for $\underline{\mu}$ read [33]

$$\mu_{j,x}(x, t_r) = -2i\frac{y(\hat{\mu}_j(x, t_r))}{\prod_{\substack{\ell=1 \\ \ell \neq j}}^{\mathfrak{g}}(\mu_j(x, t_r) - \mu_\ell(x, t_r))}, \tag{5.14}$$

$$\mu_{j,t_r}(x, t_r) = 2\frac{\widetilde{F}_r(\mu_j(x, t_r), x, t_r)}{\mu_j(x, t_r)}\frac{y(\hat{\mu}_j(x, t_r))}{\prod_{\substack{\ell=1 \\ \ell \neq j}}^{\mathfrak{g}}(\mu_j(x, t_r) - \mu_\ell(x, t_r))}, \tag{5.15}$$

$$j = 1, \dots, \mathfrak{g}, \ (x, t_r) \in \Omega,$$

with initial data

$$\hat{\mu}_j(x_0, t_{0,r}) \in \mathcal{K}_\mathfrak{g}, \quad j = 1, \dots, \mathfrak{g}. \tag{5.16}$$

The corresponding equations for $\underline{\nu}$ equal

$$\nu_{j,x}(x, t_r) = -2i\frac{y(\hat{\nu}_j(x, t_r))}{\prod_{\substack{\ell=1 \\ \ell \neq j}}^{\mathfrak{g}}(\nu_j(x, t_r) - \nu_\ell(x, t_r))}, \tag{5.17}$$

$$\nu_{j,t_r}(x,t_r) = 2\frac{\widetilde{H}_r(\nu_j(x,t_r),x,t_r)}{\nu_j(x,t_r)}\frac{y(\hat{\nu}_j(x,t_r))}{\prod_{\substack{\ell=1\\ \ell\neq j}}^{\mathfrak{g}}(\nu_j(x,t_r)-\nu_\ell(x,t_r))}, \tag{5.18}$$
$$j=1,\dots,\mathfrak{g},\ (x,t_r)\in\Omega,$$

with initial conditions

$$\hat{\nu}_j(x_0,t_{0,r})\in\mathcal{K}_{\mathfrak{g}},\quad j=1,\dots,\mathfrak{g}. \tag{5.19}$$

Next we will prove that the Abel map provides a clever change of coordinates that linearizes the Dubrovin flows. This will turn out to be a consequence of the fact that $\widetilde{F}_r(\mu_j)$ can be expressed as a linear combination of the functions $\Phi_k^{(j)}$. Using Theorem 4.5 it is immediate that this is not only the case for the KdV hierarchy, but also for all the other hierarchies discussed in this paper.

Theorem 5.1. *Suppose $\underline{\mu}(x,t_r)=(\mu_1(x,t_r),\dots,\mu_{\mathfrak{g}}(x,t_r))$ satisfies the Dubrovin equations* (5.1) *and assume that $\mu_j\neq\mu_{j'}$ for $j\neq j'$. Let $r\in\mathbb{N}_0$ and introduce*

$$\widetilde{F}_r(\mu_j)=\sum_{k=0}^{r\wedge\mathfrak{g}} d_{r,k}\Phi_k^{(j)}(\underline{\mu}),\quad d_{r,0},\dots,d_{r,r\wedge\mathfrak{g}}\in\mathbb{C}. \tag{5.20}$$

Then the Abel map

$$\underline{A}_{P_0}(\hat{\mu}_j(x,t_r))=(\underline{A}_{P_0,1}(\hat{\mu}_j(x,t_r)),\dots,\underline{A}_{P_0,\mathfrak{g}}(\hat{\mu}_j(x,t_r))) \tag{5.21}$$

linearizes the Dubrovin flows (5.1) *in the sense that*

$$\frac{\partial}{\partial t_r}\sum_{j=1}^{\mathfrak{g}}\underline{A}_{P_0,k}(\hat{\mu}_j(x,t_r)) = -2i\sum_{\ell=1\vee(\mathfrak{g}-r)}^{\mathfrak{g}} c_{k,\ell}d_{r,\mathfrak{g}-\ell} \tag{5.22}$$

and hence[10]

$$\underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)})=\underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat{\mu}}(x_0,t_{0,r})})-2i(x-x_0)c_{k,\mathfrak{g}}d_{0,0}-2i(t_r-t_{0,r})\sum_{\ell=1\vee(\mathfrak{g}-r)}^{\mathfrak{g}} c_{k,\ell}d_{r,\mathfrak{g}-\ell}. \tag{5.23}$$

Proof. One computes,

$$\begin{aligned}
&\frac{\partial}{\partial t_r}\sum_{j=1}^{\mathfrak{g}}\underline{A}_{P_0,k}(\hat{\mu}_j(x,t_r)) = \frac{\partial}{\partial t_r}\sum_{j=1}^{\mathfrak{g}}\int_{P_0}^{\hat{\mu}_j(x,t_r)} d\omega_k\\
&=\frac{\partial}{\partial t_r}\sum_{j=1}^{\mathfrak{g}}\sum_{\ell=1}^{\mathfrak{g}}c_{k,\ell}\int_{P_0}^{\hat{\mu}_j(x,t_r)}\frac{z^{\ell-1}\,dz}{y(P)}\\
&=\sum_{j=1}^{\mathfrak{g}}\sum_{\ell=1}^{\mathfrak{g}}c_{k,\ell}\frac{\mu_j(x,t_r)^{\ell-1}}{y(\hat{\mu}_j(x,t_r))}\frac{\partial}{\partial t_r}\mu_j(x,t_r)\\
&=-2i\sum_{j=1}^{\mathfrak{g}}\sum_{\ell=1}^{\mathfrak{g}}c_{k,\ell}\frac{\mu_j(x,t_r)^{\ell-1}}{y(\hat{\mu}_j(x,t_r))}\frac{y(\hat{\mu}_j(x,t_r))}{\prod_{m\neq j}^{\mathfrak{g}}(\mu_j(x,t_r)-\mu_m(x,t_r))}\widetilde{F}_r(\mu_j(x,t_r))\\
&=-2i\sum_{j=1}^{\mathfrak{g}}\sum_{\ell=1}^{\mathfrak{g}}c_{k,\ell}U_{\mathfrak{g}}(\underline{\mu}(x,t_r))_{\ell,j}\widetilde{F}_r(\mu_j(x,t_r)) = -2i\sum_{\ell=1\vee(\mathfrak{g}-r)}^{\mathfrak{g}}c_{k,\ell}d_{r,\mathfrak{g}-\ell},
\end{aligned} \tag{5.24}$$

using Lemma 4.2 in the final step. As for the x-variation, we observe that t_0-derivative of μ_j coincides with the x-derivative in (5.1), and hence it is a special case of (5.24). This proves the theorem. ☐

Corollary 5.2. *The Abel map linearizes the Dubrovin flows for the KdV, AKNS, Tl, as well as the sGmKdV hierarchies.*

[10] The situation here resembles the one in classical mechanics where, by a canonical change to cyclic coordinates, the momentum p_j becomes a constant of motion and thus $q_j(t)=q_j(t_0)+p_j(t-t_0)$ is linear in time.

Proof. Theorem 4.5 shows that $\widetilde{F}_r(\mu_j)$ (and $\widetilde{H}_r(\nu_j)$ in the AKNS case) indeed satisfies the assumption (5.20) of Theorem 5.1, and hence the key calculation (5.24) carries over to the AKNS, Tl, and sGmKdV systems. The special case $r=0$ gives the x-variation in all but the sGmKdV case which, however can easily be verified by explicit computation. □

Remark 5.3. We provide a few more details in the AKNS case. Suppose $\underline{\mu}(x,t_r)$ satisfies (5.5) and similarly, $\underline{\nu}(x,t_r)$ satifies (5.8), with $\mu_j \neq \mu_{j'}$ and $\nu_j \neq \nu_{j'}$ for $j \neq j'$. Let $r \in \mathbb{N}_0$ and introduce

$$\widetilde{F}_r(\mu_j) = -iq \sum_{k=0}^{r\wedge\mathfrak{g}} d_{r,k}\Phi_k^{(j)}(\underline{\mu}), \quad \widetilde{H}_r(\nu_j) = ip \sum_{k=0}^{r\wedge\mathfrak{g}} e_{r,k}\Phi_k^{(j)}(\underline{\nu}). \tag{5.25}$$

Then (5.22) and (5.23) hold. In addition, one obtains the following results for the analog of Neumann divisors $\underline{\nu}(x,t_r)$.

$$\frac{\partial}{\partial t_r} \sum_{j=1}^{\mathfrak{g}} \underline{A}_{P_0,k}(\hat{\nu}_j(x,t_r)) = -2i \sum_{\ell=1\vee(\mathfrak{g}-r)}^{\mathfrak{g}} c_{k,\ell} e_{r,\mathfrak{g}-\ell} \tag{5.26}$$

and hence

$$\underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat{\nu}}(x,t_r)}) = \underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat{\nu}}(x_0,t_{0,r})}) - 2i(x-x_0)c_{k,\mathfrak{g}}e_{0,0} - 2i(t_r - t_{0,r}) \sum_{\ell=1\vee(\mathfrak{g}-r)}^{\mathfrak{g}} c_{k,\ell} e_{r,\mathfrak{g}-\ell}. \tag{5.27}$$

Necessary and sufficient conditions on Lax pairs to linearize the flow $t \to L_t$ on $J(C)$, where $\{L_t\}$ represents a dymamical system on the Jacobi variety $J(C)$, with C the underlying spectral curve, have been considered by Griffiths [37]. While he considers Lax equations within a cohomological framework, our approach is much more modest in scope but in turn reduces the linearization problem to an elementary exercise in symmetric functions.

Solving these equations we can recover the solution of the integrable equation using *trace formulas.* For the KdV hierarchy we have the classical trace formula

$$V(x,t_r) = \sum_{m=0}^{2\mathfrak{g}} E_m - 2\sum_{j=1}^{\mathfrak{g}} \mu_j(x,t_r). \tag{5.28}$$

For the AKNS hierarchy we have

$$\frac{p_x(x,t_r)}{p(x,t_r)} = i\sum_{m=0}^{2\mathfrak{g}+1} E_m - 2i\sum_{j=1}^{\mathfrak{g}} \nu_j(x,t_r), \tag{5.29a}$$

$$\frac{q_x(x,t_r)}{q(x,t_r)} = -i\sum_{m=0}^{2\mathfrak{g}+1} E_m + 2i\sum_{j=1}^{\mathfrak{g}} \mu_j(x,t_r), \tag{5.29b}$$

while for the Toda hierarchy one obtains

$$\begin{aligned} a(n,t_r)^2 = &-\frac{1}{2}\sum_{j=1}^{\mathfrak{g}} \frac{G_{\mathfrak{g}+1}(\mu_j(n,t_r),n,t_r)}{\prod_{k\neq j}^{\mathfrak{g}}(\mu_j(n,t_r)-\mu_k(n,t_r))} \\ &-\frac{1}{4}b(n,t_r)^2 - \frac{1}{4}\sum_{j=1}^{\mathfrak{g}} \mu_j(n,t_r)^2 + \frac{1}{8}\sum_{m=0}^{2\mathfrak{g}+1} E_m^2, \end{aligned} \tag{5.30a}$$

$$b(n,t_r) = -\frac{1}{2}\sum_{m=0}^{2\mathfrak{g}+1} E_m + \sum_{j=1}^{\mathfrak{g}} \mu_j(n,t_r). \tag{5.30b}$$

The "trace" relation for the sGmKdV hierarchy (it would be more appropriate to call this a "determinant" relation) was given in (3.87),

$$u(x,t_{\mathfrak{g}}) = i\ln\Big((-1)^{\mathfrak{g}}\alpha^{-1}\prod_{j=1}^{\mathfrak{g}} \mu_j(x,t_{\mathfrak{g}})\Big) = -i\ln\Big((-1)^{\mathfrak{g}}\beta^{-1}\prod_{j=1}^{\mathfrak{g}} \nu_j(x,t_{\mathfrak{g}})\Big). \tag{5.31}$$

Remark 5.4. It is important to observe that if one *postulates* the Dubrovin equations (5.1), and *defines* V using the trace formula (5.28), one could show by a long and tedious calculation that V indeed satisfies the rth KdV equation with the correct initial condition. The same applies, of course, to the AKNS, Toda, and sGmKdV hierarchies.

Remark 5.5. For simplicity we assumed $\mu_j(x,t_r) \neq \mu_{j'}(x,t_r)$ for $j \neq j'$ in Theorem 5.1. In the self-adjoint cases, where $\{E_m\}_{m=0,\dots,N} \subset \mathbb{R}$, this condition is automatically fulfilled for all $(x,t_r) \in \mathbb{R}^2$ since then all $\mu_j(x,t_r)$ are separated from each other by spectral gaps of $L(t_r)$ or $M(t_r)$. In the general nonself-adjoint case this is no longer true and collisions between the μ_j's become possible. Nevertheless the Dubrovin equations, properly desingularized near such collision points, stay meaningful as demonstrated in detail by Birnir [9], [10] in the case of complex-valued KdV solutions. In particular, (5.23) (and (5.27)) remain valid in the presence of such collisions due to the continuity of $\underline{\alpha}_{P_0}(\cdot)$.

We already mentioned in the paragraph following (5.13) that the Dubrovin equations for the Toda hierarchy differ from the ones associated with the remaining soliton hierarchies in the sense that they do not seem to govern the n-dependence of $\underline{\hat{\mu}}(n,t_r)$. We now show how to use Theorem 5.1 to obtain a first-order Dubrovin system for $\underline{\hat{\mu}}(x,t_r)$ in x, whose solution coincides with $\underline{\hat{\mu}}(n,t_r)$ at the integer points $x = n \in \mathbb{Z}$. Since the t_r-dependence of $\underline{\hat{\mu}}$ plays no role for this argument, we ignore this dependence in the following result.

Lemma 5.6. *Abbreviate* $\underline{A} = (A_1,\dots,A_{\mathfrak{g}}) = \underline{A}_{P_{\infty_-}}(P_{\infty_+})$ *and* $\underline{\hat{\mu}}(n_0) = \underline{\hat{\mu}}^{(0)}$, *with* $\underline{\hat{\mu}}(n)$ *defined in* (3.62b). *Consider the Dubrovin-type system*

$$\frac{\partial}{\partial x}\mu_j(x) = \frac{2y(\hat{\mu}_j(x))}{\prod_{\ell\neq j}^{\mathfrak{g}}(\mu_j(x)-\mu_\ell(x))}\sum_{m=1}^{\mathfrak{g}}\Phi^{(j)}_{\mathfrak{g}-m}(\underline{\mu}(x))\sum_{n=1}^{\mathfrak{g}} c^{-1}_{m,n}A_n, \quad j=1,\dots,\mathfrak{g}, \tag{5.32a}$$

$$\underline{\hat{\mu}}(n_0) = \underline{\hat{\mu}}^{(0)}, \tag{5.32b}$$

with $c_{m,n}$ *defined in* (2.6). *Denote the solution of* (5.32) *by* $\underline{\hat{\mu}}_0(x)$. *Then* $\underline{\hat{\mu}}(n)$ *coincides with* $\underline{\hat{\mu}}_0(x)$ *at integer values* $x = n \in \mathbb{Z}$, *that is,*

$$\underline{\hat{\mu}}(n) = \underline{\hat{\mu}}_0(n), \quad n \in \mathbb{Z}. \tag{5.33}$$

Proof. First we recall the well-known result (see, e.g., [11], Sect. 3)

$$\underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat{\mu}}(n)}) - \underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat{\mu}}(n_0)}) = (n-n_0)\underline{A}_{P_{\infty_-}}(P_{\infty_+}) = (n-n_0)\underline{A}. \tag{5.34}$$

In order to complete the proof we only need to stablish that $\underline{\hat{\mu}}_0(x)$ satisfies

$$\underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat{\mu}}_0(x)}) - \underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat{\mu}}_0(x_0)}) = (x-x_0)\underline{A}_{P_{\infty_-}}(P_{\infty_+}) = (x-x_0)\underline{A}, \tag{5.35}$$

that is, we need to show

$$\frac{\partial}{\partial x}\underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat{\mu}}_0(x)}) = \underline{A}. \tag{5.36}$$

But equation (5.36) follows immediately from (5.11) (identifying t_r with x and ignoring its n-dependence), (5.20), (5.24) (multiplied by i), and (5.32). ☐

Thus, the solution $\underline{\hat{\mu}}_0(x)$ of (5.32) provides a continuous interpolation for $\underline{\hat{\mu}}(n)$. In fact, it was our attempt to prove a result like Lemma 5.6 which led us to reconsider Dubrovin equations and ultimately resulted in Theorem 4.5, the explicit connection between $\Phi^{(j)}_k(\underline{\mu})$ and the polynomials $\widehat{F}_r(z,x)$ defining the hierarchy in question.

Toda systems as integrable discretizations of continuous systems are also studied in [3].

6. Theta function representations

The fundamental problem in the construction of algebro-geometric solutions of soliton hierarchies is the following. Fix $\mathfrak{g} \in \mathbb{N}$, and pick a stationary solution of the $\mathfrak{g}$th equation in the hierarchy. Then solve the rth time-dependent equation (for any $r \in \mathbb{N}$) with the given stationary solution as initial datum and express the solution in terms of the Riemann theta function associated with $\mathcal{K}_{\mathfrak{g}}$. In the KdV case this procedure has been pioneered by Its and Matveev in their celebrated paper [40]. Subsequently, the algebro-geometric approach to integrable equations was developed in papes by Date, Dubrovin, Krichever, Matveev, Novikov, Tanaka, and others [16], [17], [22], [23], [24], [26], [42], [43], [44]. We briefly recall the results for the hierarchies studied in this paper. Detailed discussions for the KdV, AKNS, and Toda hierarchies as well as the sG equation and other completely integrable systems can be found, for instance, in [8], Chs. 3, 4, 6, [11], [18], Sect. 12.4, [25], [33], [34], [35], [36], [38], [39], [46], [47], Sect. 4.4, [48], [50], [52], [54], [55], [58], [62], Ch. 4 and the references therein. (Without explicitly repeating this in each case below, we exclude collision points for $\mu_j(x,t_r)$, that is, we will always assume $\mu_j \neq \mu_{j'}$ for $j \neq j'$.)

The KdV hierarchy. Let $V^{(0)}(x)$ be a stationary solution of the $\mathfrak{g}$th KdV equation associated with divisor $\mathcal{D}_{\underline{\hat\mu}^{(0)}(x)}$. We seek the rth KdV flow

$$\mathrm{KdV}_r(V) = 0, \quad V(x,t_{0,r}) = V^{(0)}(x), \quad x \in \mathbb{R}. \tag{6.1}$$

The solution is given by the Its–Matveev formula [40]

$$V(x,t_r) = \Lambda - 2\partial_x^2 \ln\big(\theta(\underline{z}(\underline{\hat\mu}(x,t_r)), \quad (x,t_r) \in \mathbb{R}^2, \tag{6.2}$$

where

$$\underline{\hat\mu}(x,t_{0,r}) = \underline{\hat\mu}^{(0)}(x) \tag{6.3}$$

and

$$\underline{z}(\underline{\hat\mu}(x,t_r)) = \underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat\mu}(x,t_r)}) - \underline{\widehat{A}}_{P_0}(P_\infty) + \underline{\Xi}_{P_0}, \tag{6.4}$$

with $\underline{\Xi}_{P_0}$ the vector of Riemann constants and Λ a $\mathcal{K}_{\mathfrak{g}}$-dependent constant.

The AKNS hierarchy. Let $(p^{(0)}(x), q^{(0)}(x))$ be a stationary solution of the $\mathfrak{g}$th AKNS equation associated with divisors $\mathcal{D}_{\underline{\hat\mu}^{(0)}(x)}$ and $\mathcal{D}_{\underline{\hat\nu}^{(0)}(x)}$. We want to construct the rth AKNS flow

$$\mathrm{AKNS}_r(p,q) = 0, \quad (p(x,t_{0,r}), q(x,t_{0,r})) = (p^{(0)}(x), q^{(0)}(x)), \quad x \in \mathbb{R}. \tag{6.5}$$

The solution reads

$$\begin{aligned} p(x,t_r) &= p(x_0,t_{0,r}) \frac{\theta(\underline{z}_-(\underline{\hat\nu}(x_0,t_{0,r})))}{\theta(\underline{z}_+(\underline{\hat\nu}(x_0,t_{0,r})))} \frac{\theta(\underline{z}_+(\underline{\hat\nu}(x,t_r)))}{\theta(\underline{z}_-(\underline{\hat\nu}(x,t_r)))} \times \qquad (6.6a)\\ &\quad \times \exp(-2i(x-x_0)e_0 - 2i(t_r - t_{0,r})\tilde e_r), \\ q(x,t_r) &= q(x_0,t_{0,r}) \frac{\theta(\underline{z}_+(\underline{\hat\mu}(x_0,t_{0,r})))}{\theta(\underline{z}_-(\underline{\hat\mu}(x_0,t_{0,r})))} \frac{\theta(\underline{z}_-(\underline{\hat\mu}(x,t_r)))}{\theta(\underline{z}_+(\underline{\hat\mu}(x,t_r)))} \times \qquad (6.6b)\\ &\quad \times \exp(2i(x-x_0)e_0 + 2i(t_r - t_{0,r})\tilde e_r), \\ &\qquad\qquad (x,t_r) \in \mathbb{R}^2, \end{aligned}$$

where $\tilde e_r$ is a $\mathcal{K}_{\mathfrak{g}}$-dependent constant, and

$$\underline{\hat\mu}(x,t_{0,r}) = \underline{\hat\mu}^{(0)}(x), \quad \underline{\hat\nu}(x,t_{0,r}) = \underline{\hat\nu}^{(0)}(x) \tag{6.7}$$

and

$$\underline{z}_\pm(\underline{Q}) = \underline{\alpha}_{P_0}(\mathcal{D}_{\underline{Q}}) - \underline{A}_{P_0}(P_{\infty_\pm}) + \underline{\Xi}_{P_0}, \quad \underline{Q} = (Q_1, \dots, Q_{\mathfrak{g}}), \tag{6.8}$$

with e_0, $\tilde e_r$ are $\mathcal{K}_{\mathfrak{g}}$-dependent constants.

The Toda hierarchy. Let $(a^{(0)}, b^{(0)})$ be a stationary solution of the $\mathfrak{g}$th Toda equation associated with divisor $\mathcal{D}_{\underline{\hat\mu}^{(0)}(n)}$. We are interested in the solution of the Tl_r flow

$$\mathrm{Tl}_r(a,b) = 0, \quad (a(n,t_{0,r}), b(n,t_{0,r})) = (a^{(0)}(n), b^{(0)}(n)), \; n \in \mathbb{Z}. \tag{6.9}$$

Here the solution equals

$$a(n,t_r)=\tilde a\left(\frac{\theta(\underline{z}_+(\hat{\underline{\mu}}(n-1,t_r)))\theta(\underline{z}_+(\hat{\underline{\mu}}(n+1,t_r)))}{\theta(\underline{z}_+(\hat{\underline{\mu}}(n,t_r)))^2}\right)^{1/2}, \tag{6.10a}$$

$$b(n,t_r)=\sum_{j=1}^{\mathfrak{g}}\lambda_j-\frac{1}{2}\sum_{m=0}^{2\mathfrak{g}+1}E_m-\sum_{j=1}^{\mathfrak{g}}c_j(\mathfrak{g})\frac{\partial}{\partial w_j}\ln\left(\frac{\theta(\underline{w}+\underline{z}_+(\hat{\underline{\mu}}(n,t_r)))}{\theta(\underline{w}+\underline{z}_+(\hat{\underline{\mu}}(n-1,t_r)))}\right)\Bigg|_{\underline{w}=\underline{0}}, \tag{6.10b}$$
$$(n,t_r)\in\mathbb{Z}\times\mathbb{R},$$

where

$$\hat{\underline{\mu}}(n,t_{0,r})=\hat{\underline{\mu}}^{(0)}(n) \tag{6.11}$$

and

$$\underline{z}_+(\hat{\underline{\mu}}(n,t_r))=\underline{\alpha}_{P_0}(\mathcal{D}_{\hat{\underline{\mu}}(n,t_r)})-\underline{A}_{P_0}(P_{\infty_+})+\underline{\Xi}_{P_0}, \tag{6.12}$$

with $\tilde a\neq 0$ a $\mathcal{K}_{\mathfrak{g}}$-dependent constant.

The sGmKdV hierarchy. Let $u^{(0)}(x)$ be the solution of the $\mathfrak{g}$th stationary sGmKdV equation, that is,

$$g_{\mathfrak{g}-1,x}-i(\beta e^{iu^{(0)}}-\alpha e^{-iu^{(0)}})=0,\quad \mathfrak{g}\in\mathbb{N}, \tag{6.13}$$

subject to the constraints

$$\alpha\beta=\prod_{j=1}^{2\mathfrak{g}}E_j,\quad E_0=0. \tag{6.14}$$

Let $r\in\mathbb{N}_0$. We are seeking the solution u of $\mathrm{sGmKdV}_r(u)=0$ with $u(x,t_{0,r})=u^{(0)}(x)$. Here the solution reads

$$\begin{aligned}u(x,t_r)&=u(x_0,t_r)\\&\quad+2i\ln\left(\frac{\theta(z(\hat{\underline{\mu}}(x,t_r))+\underline{\Delta})\theta(z(\hat{\underline{\mu}}(x_0,t_{0,r})))}{\theta(z(\hat{\underline{\mu}}(x_0,t_{0,r}))+\underline{\Delta})\theta(z(\hat{\underline{\mu}}(x,t_r)))}\exp\big(-ie_0(x-x_0)\big)\right),\end{aligned} \tag{6.15}$$

where $P_0=(0,0)$, $\underline{\Delta}$ is a half-period,

$$\underline{\Delta}=\underline{A}_{P_0}(P_\infty), \tag{6.16}$$

e_0 a $\mathcal{K}_{\mathfrak{g}}$-dependent constant, and

$$z(\underline{Q})=\underline{\alpha}_{P_0}(\mathcal{D}_{\underline{Q}})-\underline{A}_{P_0}(P_\infty)+\underline{\Xi}_{P_0}. \tag{6.17}$$

The linear equivalence of $\mathcal{D}_{P_\infty\hat{\underline{\mu}}(x,t_r)}$ and $\mathcal{D}_{Q_0\hat{\underline{\nu}}(x,t_r)}$, that is,

$$\underline{\alpha}_{P_0}(\mathcal{D}_{\hat{\underline{\nu}}(x,t_r)})=\underline{\alpha}_{P_0}(\mathcal{D}_{\hat{\underline{\mu}}(x,t_r)})+\underline{\Delta}, \tag{6.18}$$

shows that

$$z(\hat{\underline{\nu}}(x,t_r))=z(\hat{\underline{\mu}}(x,t_r))+\underline{\Delta}. \tag{6.19}$$

In each of the theta function representations of this section one should keep in mind that the change of coordinates effected by the Abel map straightens out all Dirichlet and Neumann flows[10] on the Jacobi variety $J(\mathcal{K}_{\mathfrak{g}})$ of $\mathcal{K}_{\mathfrak{g}}$, that is, $z(\hat{\underline{\mu}}(x,t_r))$ and $z(\hat{\underline{\nu}}(x,t_r))$ in (6.2), (6.6), (6.10) and (6.15) are linear in x (respectively, n) and t_r.

7. Examples

KdV and sG. Pick $E_0 = 0$ and $E_1, \dots, E_{2\mathfrak{g}} \in \mathbb{C}$, $E_m \neq E_{m'}$ for $m \neq m'$ and solve

$$\frac{\partial}{\partial x}\mu_j(x,t_{\mathfrak{g}}) = -2i \frac{y(\hat{\mu}_j(x,t_{\mathfrak{g}}))}{\prod_{\ell\neq j}^{\mathfrak{g}}(\mu_j(x,t_{\mathfrak{g}}) - \mu_\ell(x,t_{\mathfrak{g}}))}, \tag{7.1a}$$

$$\frac{\partial}{\partial t_{\mathfrak{g}}}\mu_j(x,t_{\mathfrak{g}}) = \frac{\partial}{\partial x}\mu_j(x,t_{\mathfrak{g}}) \frac{1}{16Q^{1/2}} \prod_{\ell\neq j}^{\mathfrak{g}} \mu_\ell, \quad j = 1, \dots, \mathfrak{g}, \tag{7.1b}$$

with $Q = \prod_{m=1}^{2\mathfrak{g}} E_m$ and $R_{2\mathfrak{g}+1}(z) = z\prod_{m=1}^{2\mathfrak{g}}(z - E_m)$. Define

$$u(x,t_{\mathfrak{g}}) = i\ln(Q^{-1/2}\prod_{j=1}^{\mathfrak{g}} \mu_j(x,t_{\mathfrak{g}})), \tag{7.2}$$

$$V(x,t_{\mathfrak{g}}) = \sum_{m=0}^{2\mathfrak{g}} E_m - 2\sum_{j=1}^{\mathfrak{g}} \mu_j(x,t_{\mathfrak{g}}). \tag{7.3}$$

Then u and V satisfy the sG equation and $\mathfrak{g}$th KdV equation, respectively, that is,

$$4u_{x,t_{\mathfrak{g}}} = \sin(u), \quad \mathrm{KdV}_{\mathfrak{g}}(V) = 0 \tag{7.4}$$

for the following choice of $\tilde{c}_\ell$,

$$\tilde{c}_0 = 1, \quad \tilde{c}_1 = \frac{(-1)^{\mathfrak{g}-1}}{16Q^{1/2}} - c_1(\underline{E}), \quad \tilde{c}_\ell = -\sum_{p=0}^{\ell-1} \tilde{c}_p c_{\ell-p}(\underline{E}), \quad \ell = 2, \dots, \mathfrak{g}. \tag{7.5}$$

The isomorphism between algebro-geometric $\mathrm{KdV}_{\mathfrak{g}}$ and sG equations is of course well-known and has been discussed, for instance, in [5], [6].

AKNS and Tl. Pick $E_0, \dots, E_{2\mathfrak{g}+1} \in \mathbb{C}$, $E_m \neq E_{m'}$ for $m \neq m'$ and solve

$$\frac{\partial}{\partial x}\mu_j(x,t_r) = -2i \frac{y(\hat{\mu}_j(x,t_r))}{\prod_{\ell\neq j}^{\mathfrak{g}}(\mu_j(x,t_r) - \mu_\ell(x,t_r))}, \tag{7.6a}$$

$$\frac{\partial}{\partial t_r}\mu_j(x,t_r) = -\frac{\partial}{\partial x}\mu_j(x,t_r) \sum_{n=(r-\mathfrak{g})\vee 0}^{r} d_n \Phi_n^{(j)}(\underline{\mu}), \quad j = 1, \dots, \mathfrak{g} \tag{7.6b}$$

and

$$\frac{\partial}{\partial x}\nu_j(x,t_r) = -2i \frac{y(\hat{\nu}_j(x,t_r))}{\prod_{\ell\neq j}^{\mathfrak{g}}(\nu_j(x,t_r) - \nu_\ell(x,t_r))}, \tag{7.7a}$$

$$\frac{\partial}{\partial t_r}\nu_j(x,t_r) = -\frac{\partial}{\partial x}\nu_j(x,t_r) \sum_{n=(r-\mathfrak{g})\vee 0}^{r} d_n \Phi_n^{(j)}(\underline{\nu}), \quad j = 1, \dots, \mathfrak{g}, \tag{7.7b}$$

where $R_{2\mathfrak{g}+2}(z) = \prod_{m=0}^{2\mathfrak{g}+1}(z - E_m)$ and $d_n \in \mathbb{C}$. Define

$$\frac{p_x(x,t_r)}{p(x,t_r)} = i\sum_{m=0}^{2\mathfrak{g}+1} E_m - 2i\sum_{j=1}^{\mathfrak{g}} \nu_j(x,t_r), \tag{7.8a}$$

$$\frac{q_x(x,t_r)}{q(x,t_r)} = -i\sum_{m=0}^{2\mathfrak{g}+1} E_m + 2i\sum_{j=1}^{\mathfrak{g}} \mu_j(x,t_r) \tag{7.8b}$$

and

$$\begin{aligned} a(n,t_r)^2 = &\frac{1}{2}\sum_{j=1}^{\mathfrak{g}} \frac{y(\hat{\mu}_j(n,t_r))}{\prod_{k\neq j}^{\mathfrak{g}}(\mu_j(n,t_r) - \mu_k(n,t_r))} \\ &- \frac{1}{4}b(n,t_r)^2 - \frac{1}{4}\sum_{j=1}^{\mathfrak{g}} \mu_j(n,t_r)^2 + \frac{1}{8}\sum_{m=0}^{2\mathfrak{g}+1} E_m^2, \end{aligned} \tag{7.9a}$$

$$b(n,t_r) = -\frac{1}{2}\sum_{m=0}^{2\mathfrak{g}+1} E_m + \sum_{j=1}^{\mathfrak{g}} \mu_j(n,t_r). \tag{7.9b}$$

Then (p,q) and (a,b) satisfy the rth AKNS equation and the rth Toda lattice (Tl) equation, respectively, that is,

$$\mathrm{AKNS}_r(p,q) = 0, \quad \mathrm{Tl}_r(a,b) = 0 \tag{7.10}$$

for the same choice of c_ℓ, $\ell = 1,\dots,r$ in both equations (7.10) (depending on the choice of d_n in (7.6b), (7.7b)).

Remark 7.1. These examples provide interesting connections between the $\mathrm{KdV}_{\mathfrak{g}}$ and sG equation (where N is even and $0 \in \{E_m\}_{m=0,\dots,N}$), and AKNS and Toda hierarchies (where N is odd), respectively, and illustrate the fundamental role of the Dubrovin equations as the common underlying principle for hierarchies of soliton equations. In particular, our approach establishes an isomorphism between the classes of algebro-geometric solutions of these pairs of integrable systems. Indeed, once the hyperelliptic curve $\mathcal{K}_{\mathfrak{g}}$ is fixed, algebro-geometric solutions of the $\mathrm{KdV}_{\mathfrak{g}}$ and sG (respectively, algebro-geometric solutions of the rth AKNS and rth Tl equation) are just certain symmetric functions (i.e., "trace" relations) of the solutions $\mu_1(x,t_{\mathfrak{g}}),\dots,\mu_{\mathfrak{g}}(x,t_{\mathfrak{g}})$ (resp. $\mu_1(x,t_r),\dots,\mu_{\mathfrak{g}}(x,t_r),\nu_1(x,t_r),\dots,\nu_{\mathfrak{g}}(x,t_r)$) of the corresponding Dubrovin equations on $\mathcal{K}_{\mathfrak{g}}$. Analagous considerations apply to the nonlinear Schrödinger equation and the (continuum) Heisenberg chain (see, e.g., [27]). The interesting problem which types of symmetric functions of $\mu_1,\dots,\mu_{\mathfrak{g}}$ (i.e., which types of "trace" formulas) actually lead to completely integrable hierarchies is currently under investigation.

Appendix A. Lagrange interpolation formulas

In the following we suppress the (x,t_r)-dependence as it will be of no importance in this appendix.

Fix $\mathfrak{g} \in \mathbb{N}$ and recall that

$$F_{\mathfrak{g}}(z) = \prod_{j=1}^{\mathfrak{g}} (z-\mu_j), \tag{A.1}$$

which implies that ($F'_{\mathfrak{g}} = \partial F_{\mathfrak{g}}/\partial z$)

$$F'_{\mathfrak{g}}(\mu_k) = \prod_{\substack{j=1\\ j\neq k}}^{\mathfrak{g}} (\mu_k-\mu_j). \tag{A.2}$$

The general form of Lagrange's interpolation theorem then reads as follows. (For convenience of the reader we supply its proof even though the result is well-known.)

Theorem A.1. *Assume that $\mu_1,\dots,\mu_{\mathfrak{g}}$ are $\mathfrak{g}$ distinct complex numbers. Then*

$$\sum_{j=1}^{\mathfrak{g}} \frac{\mu_j^{m-1}}{F'_{\mathfrak{g}}(\mu_j)} \Phi_k^{(j)}(\underline{\mu}) = \delta_{m,\mathfrak{g}-k} - \Psi_{k+1}(\underline{\mu})\delta_{m,\mathfrak{g}+1},$$
$$m = 1,\dots,\mathfrak{g}+1, \quad k = 0,\dots,\mathfrak{g}-1. \tag{A.3}$$

Proof. Let C_R be a circle with center at the origin and radius R that contains the zeros μ_j of the polynomial $F_{\mathfrak{g}}$ and which is oriented clockwise. Cauchy's theorem then yields

$$\frac{1}{2\pi i}\oint_{C_R} d\zeta\, \frac{\zeta^{m-1}}{F_{\mathfrak{g}}(\zeta)(\zeta-z)} = \frac{z^{m-1}}{F_{\mathfrak{g}}(z)} + \sum_{k=1}^{\mathfrak{g}} \frac{\mu_k^{m-1}}{F'_{\mathfrak{g}}(\mu_j)(\mu_j-z)},$$
$$z \neq \mu_1,\dots,\mu_{\mathfrak{g}}, \quad m = 1,\dots,\mathfrak{g}+1. \tag{A.4}$$

However, by letting $R\to\infty$ we infer that

$$\frac{1}{2\pi i}\oint_{C_R} d\zeta\, \frac{\zeta^{m-1}}{F_{\mathfrak{g}}(\zeta)(\zeta-z)} = \lim_{R\to\infty}\frac{R^{m-1}}{F_{\mathfrak{g}}(R)} = \delta_{m,\mathfrak{g}+1}, \quad m = 1,\dots,\mathfrak{g}+1, \tag{A.5}$$

which implies

$$z^{m-1} - \sum_{k=1}^{\mathfrak{g}} \frac{\mu_k^{m-1} F_{\mathfrak{g}}(z)}{F'_{\mathfrak{g}}(\mu_j)(z-\mu_j)} = F_{\mathfrak{g}}(z)\delta_{m,\mathfrak{g}+1}. \tag{A.6}$$

Using the symmetric functions Ψ_j we may write

$$F_{\mathfrak{g}}(z) = \sum_{j=0}^{\mathfrak{g}} z^{\mathfrak{g}-j}\Psi_j(\underline{\mu}) \tag{A.7}$$

and

$$\frac{F_{\mathfrak{g}}(z)}{z-\mu_j} = \sum_{k=0}^{\mathfrak{g}-1} z^{\mathfrak{g}-1-k}\Phi_k^{(j)}(\underline{\mu}). \tag{A.8}$$

Expanding both sides of equation (A.6) in powers in z, using (A.7) on the right-hand side and (A.8) on the left-hand side, proves (A.3). $\square$

The simplest Lagrange interpolation formula reads in the case $k=0$,

$$\sum_{j=1}^{\mathfrak{g}} \frac{\mu_j^{m-1}}{F'_{\mathfrak{g}}(\mu_j)} = \delta_{m,\mathfrak{g}}, \quad m = 1,\dots,\mathfrak{g}. \tag{A.9}$$

For use in the main text we finally observe the following equalities. Adding (A.7) to μ_j times (A.8) we find

$$F_{\mathfrak{g}}(z) + \mu_j \frac{F_g(z)}{z-\mu_j} = \sum_{k=0}^{\mathfrak{g}-1} z^{\mathfrak{g}-k-1}(\Psi_{k+1} + \mu_j\Phi_k^{(j)}) + z^{\mathfrak{g}}. \tag{A.10}$$

However, we also have

$$F_{\mathfrak{g}}(z) + \mu_j \frac{F_{\mathfrak{g}}(z)}{z-\mu_j} = z\frac{F_{\mathfrak{g}}(z)}{z-\mu_j} = \sum_{k=0}^{\mathfrak{g}-1} z^{\mathfrak{g}-k-1}\Phi_{k+1}^{(j)} + z^{\mathfrak{g}}, \tag{A.11}$$

using (A.8) and recalling $\Phi_{\mathfrak{g}}^{(j)} = 0$. Thus we conclude

$$\Psi_{k+1}(\underline{\mu}) + \mu_j\Phi_k^{(j)}(\underline{\mu}) = \Phi_{k+1}^{(j)}(\underline{\mu}), \quad k = 0,\dots,\mathfrak{g}-1. \tag{A.12}$$

Finally, we will show

$$\sum_{\ell=0}^{k} \mu_j^{\ell}\Psi_{k-\ell}(\underline{\mu}) = \Phi_k^{(j)}(\underline{\mu}), \quad k = 0,\dots,\mathfrak{g}, \tag{A.13}$$

by induction. Equation (A.13) clearly holds for $k=0$; next assume that

$$\sum_{\ell=0}^{k-1} \mu_j^{\ell}\Psi_{k-1-\ell} = \Phi_{k-1}^{(j)} \tag{A.14}$$

holds. Then

$$\begin{aligned}\sum_{\ell=0}^{k} \mu_j^{\ell}\Psi_{k-\ell} &= \Psi_k + \mu_j \sum_{\ell=1}^{k} \mu_j^{\ell-1}\Psi_{k-\ell} \\ &= \Psi_k + \mu_j \sum_{\ell=0}^{k-1} \mu_j^{\ell}\Psi_{k-1-\ell} = \Psi_k + \mu_j\Phi_{k-1}^{(j)} = \Phi_k^{(j)},\end{aligned} \tag{A.15}$$

using first the induction hypothesis and then (A.12).

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