Couplings and asymptotic exponentiality of exit times

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Abstract. The goal of this note is simply to call attention to the resulting simplification in the proof of asymptotic exponentiality of exit times in Freidlin-Wentzell regime (as proved in [MOS]) by using the coupling proposed by T. Lindvall and C. Rogers (cf. [LR]).

Key words: Exit times, Exponentiality, Metastability, Couplings.

Introduction.

In this note we examine a classical problem in the framework of the theory of small random perturbations of dynamical systems: the first exit from a domain G positively invariant with respect to the unperturbed flow.

In particular, for a class of Itô equations, we address the question of the asymptotic exponentiality, in the limit of small noise, of the suitably normalized first exit time from G. We are interested in the general case of G containing many attractors of the unperturbed system. This problem is, on one side, interesting in itself; it amounts to considerably strengthen the classical Freidlin-Wentzell results on the asymptotics of the first exit time from a domain G. It is, on the other side, also related to the so-called metastable behavior of the particular stochastic dynamics described by our Itô equations, in the framework of the pathwise approach to metastability introduced in [CGOV].

From a probabilistic point of view the asymptotic exponentiality (or asymptotic unpredictability) of the exit time is related to a particular exit mechanism: the repetition

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of a large number of almost independent trials. Among the various different ways the large deviation theory is able to select a particularly efficient one. So the heuristic explanation of the asymptotic exponentiality is based on a long sequence of recurrences inside G together with a loss of memory and eventually a successful exit attempt. In [GOV], very sophisticated analytical results due to Day (cf.[D]) were used to extend to a "tunneling" problem the previous results relative to the case of a domain G completely attracted by a unique asymptotically stable point. In [MOS], for a general class of domains G, the analytical methods of Day were replaced by probabilistic arguments based on contraction properties of the stochastic map (depending on the noise) which associates to the initial datum of our stochastic equation the solution at a given time T. Again the ingredient of loss of memory, necessary for the asymptotic exponentiality is based on delicate and highly non-trivial arguments developed in [MS].

In the present note, in the general [MOS] context, we give another proof of the asymptotic exponentiality by using a simple and beautiful coupling argument due to Lindvall and Rogers (cf. [LR]). The goal is to stress the resulting simplicity.

Coupling methods have been also successfully used to show asymptotic exponentiality for an infinite dimensional case, as the stochastically perturbed non-linear heat equation, also considered in [MOS]. Using a coupling introduced by Mueller in [M], Brassesco (cf. [B]) was able to treat escape times which were not treatable with the techniques considered in [MOS].

The result. Let $X_t^{x,\varepsilon}$ be the Markov process obtained as the unique solution of the following Itô equation:

$$\begin{aligned} dX_t^{x,\varepsilon} &= b(X_t^{x,\varepsilon})dt + \varepsilon\, dW_t \\ X_0^{x,\varepsilon} &= x \end{aligned} \tag{1}$$

where (W_t) is a standard d-dimensional Brownian motion, $x \in \mathbb{R}^d$, $\varepsilon > 0$, and the vector field b is assumed to be globally Lipschitz. Let us in fact, and to simplify, assume b to be of class C^1 with bounded gradient. In particular, as it is well known, this implies strong uniqueness of the solution of (1), for any given Brownian motion (W_t), as well as the strong Markov property for $(X_t^{x,\varepsilon})_t$. Of course, more general assumptions on the field b can be taken, and an extension to varying diffusion coefficients is also possible, cf. Remark 4.

Notation: Though everything is done on any probability space $(\Omega, \mathcal{A}, P)$ where (W_t) is defined, through a pathwise (and continuous) transformation, sometimes it is more convenient to relax the notation, eliminating the superscript x on $X^{x,\varepsilon}$ and using P_x to denote the condition $X_0^\varepsilon = x$.

Our goal is to discuss the asymptotic behavior, as $\varepsilon \to 0$, of the first exit time $\tau^\varepsilon \stackrel{\text{def}}{=} \inf\{t > 0; X_t^\varepsilon \notin G\}$, when $X_0^\varepsilon = x \in G$, and where G is a bounded domain verifying certain conditions. A possible set of assumptions would be, similarly to [MS] and [MOS]:

(a1) G is a bounded domain in $\mathbb{R}^d$, with a smooth boundary ∂G, taken as of class C^2. If $I_{0,T}(\varphi)$ denotes the rate functional

$$I_{0,T}(\varphi) = \begin{cases} \dfrac{1}{2}\displaystyle\int_0^T |\dot{\varphi}_t - b(\varphi_t)|^2\, dt & \text{if } \varphi \text{ is absolutely continuous} \\ +\infty & \text{otherwise} \end{cases} \tag{2}$$

defined on the space $C([0,T], \mathbb{R}^d)$ and corresponding to the large deviation principle associated to the family of laws of $\left(X_t^{x,\varepsilon}\right)_{0\le t\le T}$ on this space, $V(x,y)$ is the associated quasi potential of Freidlin and Wentzell:

$$V(x,y) = \inf_{\substack{\varphi:\varphi(0)=x,\varphi(T)=y \\ T>0}} \{I_{0,T}(\varphi)\} \tag{3}$$

and one considers the equivalence relation

$$x \sim y \quad \text{iff} \quad V(x,y) = V(y,x) = 0 \tag{4}$$

then one assumes:

(a2) There are finitely many compact sets $K_1, \ldots, K_m$, equivalence classes for $\sim$, and such that:

(i) each w-limit set of the deterministic system given by $\dot{x}(t) = b(x(t))$ is contained in some K_i.

(ii) The stable classes are $K_1, \ldots, K_\ell$ $(\ell < m)$ and each of them consist of a fixed point of the deterministic system. These are denoted by x_i, $i =$

$1, \ldots, \ell$. Here the notion of a "stable" class is that coming from Freidlin and Wentzell theory:

Definition 1: An equivalence class K is said to be stable if $V(x,y) > 0$ for all $x \in K$, all $y \notin K$.

We know that $V(x,y)$ is constant for all $x \in K_i$, all $y \in K_j$.
Let $V_{i,j}$ denote this constant, so that K_i is stable iff

$$\inf_{j \neq i} V_{i,j} > 0.$$

Let $1 \leq k \leq \ell$ be such that $\{x_1, \ldots, x_\ell\} \cap G = \{x_1, \ldots, x_k\}$ and let $\delta > 0$ such that $B_\delta(x_i)$ is contained in the basin of attraction of x_i as well as in G, for $i = 1, \ldots, k$. Let

$$D_i = B_\delta(x_i) \quad D = \bigcup_{i=1}^{k} D_i\,.$$

We assume further

(a3) Among $x_1, \ldots, x_k$ at least one of them is a hyperbolic fixed point i.e. there exists $i_0 \in \{1, \ldots, k\}$ such all the eigenvalues of the Jacobian matrix $\left(\frac{\partial b_r}{\partial x^s}\right)_{r,s}\Big|_{x=x_{i_0}}$ have negative real part.

The last assumption concerns the "cycle" property:

(a4) Let $V = \max_{i,j \leq k} V_{i,j}$ and $V_G = \min_{1 \leq i \leq k} \min_{y \in \partial G} V(x_i, y)$.
We assume that $V_G > V$.

We may now state

Theorem 1. Under above assumptions, and if we define β_ε through the relation

$$\sup_{x \in D} P_x(\tau^\varepsilon > \beta_\varepsilon) = e^{-1} \tag{5}$$

then:

i) $\lim_{\varepsilon \to 0} P_x(\tau^\varepsilon > t\beta_\varepsilon) = e^{-t}$,

for each $x \in D$, each $t > 0$

ii) $\lim_{\varepsilon \to 0} \dfrac{E_x(\tau^\varepsilon)}{\beta_\varepsilon} = 1, \quad \forall\, x \in D.$

Remark 1. If G is confining, i.e., $\langle b(x), n(x)\rangle < 0$ for each $x \in \partial G$, where $n(x)$ indicates the outward unit normal vector to ∂G, at the point x, then we may take any $x \in G$ in (i) and (ii) of Theorem 1.

Remark 2. Contrarily to what happens in the case of a domain contained in the basin of attraction of a single fixed point or a periodic orbit, we do not always have asymptotic equivalence (even logarithmically) between a quantile of the d istribution of τ^ε under P_x $(x \in D)$ and $E_x\,\tau^\varepsilon$. For a counterexample see eg. [FW] pg. 197.

Nevertheless, if β_ε is defined through equation (5), as observed in [MOS], the bound

$$\sup_{x\in D} \frac{E_x\,\tau^\varepsilon}{\beta_\varepsilon} \le C < +\infty \tag{6}$$

for some finite constant C, holds independently of (i) of Theorem 1.

Moreover, from equation (6) and the known results of Freidlin and Wentzell on the asymptotic behaviour of $\varepsilon^2 \log E_x\,\tau_\varepsilon$, we get

$$\varliminf_{\varepsilon\to 0} \varepsilon^2 \log \beta_\varepsilon \ge V_G\,. \tag{7a}$$

On the other side, and this is the reason for the name "cycle", if $x \in D_i$ one has

$$\varlimsup_{\varepsilon\to 0} \varepsilon^2 \log E_x\,\tau^\varepsilon(D_j) \le V. \tag{7b}$$

For convenience of the reader let us recall the verification of equation (6), as in [MOS]:

$$\begin{aligned} \frac{E_x\tau^\varepsilon}{\beta_\varepsilon} &= \frac{1}{\beta_\varepsilon}\int_0^{+\infty} P_x(\tau^\varepsilon > t)\,dt \\ &= \int_0^{+\infty} P_x(\tau^\varepsilon > t\beta_\varepsilon)\,dt \\ &\le \int_0^{+\infty} g_\varepsilon(t)\,dt \end{aligned} \tag{8}$$

where $g_\varepsilon(t) \overset{\text{def}}{=} \sup_{x\in G} P_x(\tau^\varepsilon > t\beta_\varepsilon)$. But the Markov property implies that

$$g_\varepsilon(t+s) \le g_\varepsilon(t) g_\varepsilon(s)$$

so that

$$g_\varepsilon(2k) \le (g_\varepsilon(2))^k.$$

As in [MOS] we can see that $g_\varepsilon(2) \le r < 1$ for ε small, and so we get (6).

In fact,

$$\begin{aligned} g_\varepsilon(2) \le & \sup_{x\in G} P_x(\tau_\varepsilon(D) > \beta_\varepsilon) \\ & + \sup_{x\in D} P_x(\tau^\varepsilon > \beta_\varepsilon) \end{aligned} \tag{9}$$

the second term on the r.h.s. of equation (9) is e^{-1}, and using Freidlin and Wentzell estimates we see that the first term goes to zero, so that we get the claimed upper bound.

Remark 3. The argument just described allows also to make use of the Dominated Convergence Theorem and, from equation (8), to get (ii) of Theorem 1, once part (i) is proved.

Moreover, and as in [GOV], for the proof of part (i) in Theorem 1 in the case $x = x_{i_0}$, it suffices to prove the following:

Lemma 1. Under the assumptions of Theorem 1, with $\beta_\varepsilon > 0$ given by equation (5) and letting

$$f_\varepsilon(t) = P_{x_{i_0}}(\tau^\varepsilon > t\beta_\varepsilon)$$

for $t > 0$, $\varepsilon > 0$, then there exist positive numbers δ_ε, which tend to zero as $\varepsilon \to 0$, and such that for each $s, t > 0$:

$$f_\varepsilon(s+\delta_\varepsilon) f_\varepsilon(t+\delta_\varepsilon) - o_t(1) \le f_\varepsilon(t+s) \le f_\varepsilon(s) f_\varepsilon(t-\delta_\varepsilon) + o_t(1), \tag{10}$$

where $o_t(1)$ is a function of t and ε, which tends to zero as $\varepsilon \to 0$, uniformly on $t \ge t_0$, for any given $t_0 > 0$.

The proof of Lemma 1, as presented below, is similar to that of Lemma 4 in [GOV] with assumption (a4) and the Freidlin and Wentzell theory being used to control the time needed to arrive to a suitably small neighborhood of x_{i_0} , and using the coup ling method proposed by Lindvall and Rogers (cf. [LR] sections 2 and 3) to ensure the loss of memory.

For this, let $\delta_0 > 0$ be taken so that if $x, x' \in B_{2\delta_0}(x_{i_0})$ are distinct then

$$\langle x - x', b(x) - b(x') \rangle < 0. \tag{11}$$

($\langle \cdot, \cdot \rangle$ denotes the euclidean scalar product.)

To achieve this we need to recall assumption (a3) and the fact that $b(\cdot)$ is assumed of class C^1.

The coupling proposed in [LR] may thus be used to replace the analytical results of [Day] used in [GOV], or the exponential joining proposed by [MS], and used in [MOS], and allows us to write the following

Lemma 2. If $x \in B_{\delta_0}(x_{i_0})$ then

$$P_x(\tau^\varepsilon > t\beta_\varepsilon) - P_{x_{i_0}}(\tau^\varepsilon > t\beta_\varepsilon) \to 0 \qquad \text{as } \varepsilon \to 0 \tag{12}$$

uniformly on $t \geq t_0$, for any given $t_0 > 0$.

Proof. For the proof of (12) it suffices to present a coupling of the two processes $X_t^{x,\varepsilon}$ and $X_t^{x_{i_0},\varepsilon}$ in such a way that with probability tending to one they will meet before leaving $B_{2\delta_0}(x_{i_0})$, and this in time of order shorter than β_ε .

In order to do so, we consider the coupling proposed by Lindvall and Rogers (sections 2 and 3 of [LR]), which is particularly simple in the case of constant diffusion coefficient (example 5 of [LR]). The processes $X_t^{x,\varepsilon}$ and $X_t^{x_{i_0},\varepsilon}$ are constructed using the same noise, as follows: take $X_t^{x,\varepsilon}$ and $X_t^{x_{i_0},\varepsilon}$ as solutions of the Itô equations

$$\begin{aligned} dX_t^{x,\varepsilon} &= b(X_t^{x,\varepsilon})dt + \varepsilon\, dW_t; \qquad X_0^{x,\varepsilon} = x \\ dX_t^{x_{i_0},\varepsilon} &= b(X_t^{x_{i_0},\varepsilon})dt + \varepsilon H(X_t^{x,\varepsilon}, X_t^{x_{i_0},\varepsilon})\, dW_t; \qquad X_0^{x_{i_0},\varepsilon} = x_{i_0}, \end{aligned} \tag{13}$$

where W_t is a standard d-dimensional Brownian motion and $H(x,y)$ is the $d \times d$ orthogonal matrix with determinant -1 given by

$$H(x,y) = \mathbb{I} - 2\,\frac{(x-y)}{|x-y|}\Big[\frac{(x-y)}{|x-y|}\Big]^T. \tag{14}$$

(We are using ^{T} for transposition, and $\mathbb{I}$ to denote the identity $d \times d$ matrix .)

The geometric idea behind this construction is clear: Consider $x \neq y \in \mathbb{R}^d$. From (14), we have

$$H(x,y)\left(\frac{x-y}{|x-y|}\right) = -\left(\frac{x-y}{|x-y|}\right), \tag{15}$$

and, acting on vectors that belong to the plane orthogonal to $x-y$, $H(x,y)$ is just the identity. Thus, $H(x,y)$ is simply the specular reflexion through the plane (by the origin) orthogonal to the vector $x-y$, and has determinant -1. Then, given $z \in \mathbb{R}^d$, $z = x+b$, consider $z' = H(x,y)b+y$. Then, z' is the reflexion of z by the plane orthogonal to $x-y$, that passes by the middle point between x and y. In particular, if Z_t is a d-dimensional Brownian motion starting at x, then Z_t' as obtained by the above described reflexion (for each point in the path), is a d-dimensional Brownian motion starting at y.

Then, the processes $X_t^{x,\varepsilon}$ and $X_t^{x_{i_0},\varepsilon}$ are both solutions of our original Itô equation, and if one considers the function $g : \mathbb{R}^{2d} \to \mathbb{R}$, $g(x,y) = |x-y|$, then, Itô's formula (which is valid as long as $|x-y| > 0$), yields for the one-dimensional process Y_t, given by

$$Y_t = |X_t^{x,\varepsilon} - X_t^{x_{i_0},\varepsilon}| :$$

$$\begin{aligned} dY_t &= \langle b(X_t^{x,\varepsilon}) - b(X_t^{x_{i_0},\varepsilon}), \frac{(X_t^{x,\varepsilon} - X_t^{x_{i_0},\varepsilon})}{|X_t^{x,\varepsilon} - X_t^{x_{i_0},\varepsilon}|}\rangle\, dt \\ &+ \varepsilon\langle [Id - H(X_t^{x,\varepsilon}, X_t^{x_{i_0},\varepsilon})]\frac{(X_t^{x,\varepsilon} - X_t^{x_{i_0},\varepsilon})}{|X_t^{x,\varepsilon} - X_t^{x_{i_0},\varepsilon}|}, dW_t\rangle\, ; \qquad Y_0 = |x - x_{i_0}| \end{aligned} \tag{16}$$

Recall that from (11) it follows that the drift part in (16) is negative, as long as $X_t^{x,\varepsilon}$ and $X_t^{x_{i_0},\varepsilon}$ remain in $B_{2\delta_0}(x_{i_0})$. From (15),

$$\langle [Id - H(X_t^{x,\varepsilon}, X_t^{x_{i_0},\varepsilon})]\frac{X_t^{x,\varepsilon} - X_t^{x_{i_0},\varepsilon}}{|X_t^{x,\varepsilon} - X_t^{x_{i_0},\varepsilon}|}, dW_t\rangle = 2\langle \frac{X_t^{x,\varepsilon} - X_t^{x_{i_0},\varepsilon}}{|X_t^{x,\varepsilon} - X_t^{x_{i_0},\varepsilon}|}, dW_t\rangle$$

which implies that the process Y_t satisfies

$$Y_t = Y_0 + \int_0^t \langle b(X_s^{x,\varepsilon}) - b(X_s^{x_{i_0},\varepsilon}), \frac{X_s^{x,\varepsilon} - X_s^{x_{i_0},\varepsilon}}{|X_s^{x,\varepsilon} - X_s^{x_{i_0},\varepsilon}|}\rangle\, ds + 2\varepsilon B_t,$$

for B_t a standard one dimensional Brownian motion. Next, let S^ε be the coupling time, $T^\varepsilon(y)$ the exit time from $B_{2\delta_0}(x_{i_0})$ of the solution $X_t^{y,\varepsilon}$ and $\tilde{S}^\varepsilon$ the time it takes for $Y(0) + 2\varepsilon B_t$ to hit zero:

$$
\begin{aligned}
S^\varepsilon &= \inf\{t \geq 0 : |Y_t| = 0\} \\
T^\varepsilon(y) &= \inf\{t \geq 0 \ : X_t^{y,\varepsilon} \notin B_{2\delta_0}(x_{i_0})\} \\
\tilde{S}^\varepsilon &= \inf\{t \geq 0 : Y_0 + 2\varepsilon B_t = 0\}
\end{aligned}
$$

Now, from (11) and the above remarks ,

$$
\begin{aligned}
P\Big(S^\varepsilon < \varepsilon^{-3}\Big) &\geq P\Big(S^\varepsilon < \varepsilon^{-3}, T^\varepsilon(x) \wedge T^\varepsilon(x_{i_0}) > \varepsilon^{-4}\Big) \\
&\geq P\Big(\tilde{S}^\varepsilon < \varepsilon^{-3}, T^\varepsilon(x) \wedge T^\varepsilon(x_{i_0}) > \varepsilon^{-4}\Big) \\
&\geq P\Big(\tilde{S}^\varepsilon < \varepsilon^{-3}\Big) - P\Big(T^\varepsilon(x) \wedge T^\varepsilon(x_{i_0}) \leq \varepsilon^{-4}\Big), \qquad (17)
\end{aligned}
$$

where we denoted by $t \wedge s$ the minumum between t and s. But, from the Freidlin and Wentzell theory we know that there exists $a, b > 0$ so that

$$
P\Big(T^\varepsilon(x) \wedge T^\varepsilon(x_{i_0}) \leq \varepsilon^{-4}\Big) \leq 2 \sup_{y \in B_{\delta_0}(x_{i_0})} P\Big(T^\varepsilon(y) \leq \exp^{-a/\varepsilon^2}\Big) \leq \exp^{-b/\varepsilon^2} \qquad (18)
$$

For the other term, we have

$$
P\Big(\tilde{S} < \varepsilon^{-3}\Big) = 1 - \int \mathbf{1}_{\{|x| \leq Y_0 \varepsilon^{1/2}\}} \frac{\exp^{-x^2/2}}{\sqrt{2\pi}} \geq 1 - \delta_0 \varepsilon^{1/2}, \qquad (19)
$$

¿From (17), (18) and (19), it follows that $P(S^\varepsilon < \varepsilon^{-3}) \to 1$ as $\varepsilon \to 0$, which implies Lemma 2 from (7a).

Proof of Lemma 1. As in [GOV], the point is to show the existence of $\eta_\varepsilon > 0$ such that $\eta_\varepsilon / \beta_\varepsilon \to 0$ and such that

$$
\lim_{\varepsilon \to 0} \sup_{x \in G} P_x(\tau^\varepsilon > \eta_\varepsilon, \tau_\varepsilon(B_{\delta_0}(x_{i_0})) > \eta_\varepsilon) = 0. \qquad (20)
$$

To verify (20) let us take

$$
V < \alpha < V_G
$$

and let $\eta_\varepsilon = e^{\alpha/\varepsilon^2}$.

Since $\alpha > 0$, $G \backslash D$ is bounded, and all stable classes are contained in D, from Freidlin and Wentzell theory we know that

$$\lim_{\varepsilon \to 0} \sup_{x \in G \backslash D} P_x(\tau_\varepsilon(G^c \cup D) > \eta_\varepsilon) = 0.$$

On the other side, by assumption (a4) and since $V < \alpha$, Freidlin and Wentzell theory implies that

$$\sup_{x \in D} P_x(\tau_\varepsilon(B_{\delta_0}(x_{i_0})) > \eta_\varepsilon/2) \to 0.$$

Thus we get:

$$\begin{aligned}
&\sup_{x \in G} P_x(\tau^\varepsilon > \eta_\varepsilon, \tau_\varepsilon(B_{\delta_0}(x_{i_0})) > \eta_\varepsilon) \\
&\leq \sup_{x \in G} P_x(\tau^\varepsilon > \eta_\varepsilon, \tau_\varepsilon(D) > \eta_\varepsilon/2) \\
&+ \sup_{x \in D} P_x(\tau_\varepsilon(B_{\delta_0}(x_{i_0})) > \eta_\varepsilon/2)
\end{aligned}$$

which both tend to zero, yielding equation (20).

To complete the proof of Lemma 1, we proceed as in [GOV]: Let $s > 0$ and

$$R^s = \inf\{u > s\beta_\varepsilon : X_u^\varepsilon \in B_{\delta_0}(x_{i_0})\}$$

then

$$\sup_{x \in G} P_x(\tau^\varepsilon > s\beta_\varepsilon + \eta_\varepsilon,\ R^s > s\beta_\varepsilon + \eta_\varepsilon) \leq \sup_{x \in G} P_x(\tau^\varepsilon > \eta_\varepsilon, \tau_\varepsilon(B_{\delta_0}(x_{i_0})) > \eta_\varepsilon)$$

which tends to zero as $\varepsilon \to 0$.

Due to equation (6) (cf. Remark 2) and the choice of α, $\eta_\varepsilon/\beta_\varepsilon \to 0$, and we have that uniformly on $(s,t) \in [0, +\infty) \times [t_0, +\infty)$ for any given $t_0 > 0$:

$$\sup_{x \in G} P_x(\tau^\varepsilon > (s+t)\beta_\varepsilon, R^s \geq s\beta_\varepsilon + \eta_\varepsilon) \to 0. \tag{21}$$

But, as in equation (2.16) of [GOV]:

$$\begin{aligned}
&P_{x_{i_0}}(\tau^\varepsilon > (s+t)\beta_\varepsilon, R^s \leq s\beta_\varepsilon + \eta_\varepsilon) \\
&\quad \leq P_{x_{i_0}}(\tau^\varepsilon > s\beta_\varepsilon) \sup_{y \in B_{\delta_0}(x_{i_0})} P_y(\tau^\varepsilon > t\beta_\varepsilon - \eta_\varepsilon)
\end{aligned} \tag{22}$$

and

$$
(23)\qquad \begin{aligned} &P_{x_{i_0}}(\tau^\varepsilon > (s+t)\beta_\varepsilon, R^s \le s\beta_\varepsilon + \eta_\varepsilon) \\ &\qquad \ge P_{x_{i_0}}(\tau^\varepsilon > s\beta_\varepsilon + \eta_\varepsilon, R^s \le s\beta_\varepsilon + \eta_\varepsilon) \inf_{y\in B_{\delta_0}(x_{i_0})} P_y(\tau^\varepsilon > t\beta_\varepsilon) \end{aligned}
$$

so that Lemma 1 follows easily from equations (21)-(23), and Lemma 2.

Proof of Theorem 1.

As already noticed it suffices us to prove part (i).

Also if $x = x_{i_0}$, (i) follows at once from Lemma 1. Using Lemma 2 we extend to any $x \in B_{\delta_0}(x_{i_0})$. To conclude we need to recall, as in equation (7b)

$$
\overline{\lim_{\varepsilon\to 0}}\, \varepsilon^2\, \ell n\, E_x\, \tau_\varepsilon(B_{\delta_0}(x_{i_0})) \le V
$$

so that if $V < \alpha < V_G$ and $\eta_\varepsilon = e^{\alpha/\varepsilon^2}$ then

$$
P_x(\tau_\varepsilon(B_{\delta_0}(x_{i_0})) < \eta_\varepsilon) \to 1.
$$

Using the strong Markov property at $\tau_\varepsilon(B_{\delta_0}(x_{i_0}))$ we then conclude the proof as before.

Remark 4. The case of constant diffusion coefficient and $b(\,\cdot\,)$ satisfying equation (11) makes the coupling time of the two processes $X^{x,\varepsilon}_{\cdot}$, $X^{y,\varepsilon}_{\cdot}$ – if we use the coupling designed in [LR] – particularly easy t o evaluate, and directly comparable with $\tilde S$ where $\tilde S$ is the time for a one dimensional Brownian motion starting at some point $r = \frac{|x-y|}{2\varepsilon}$ to reach the origin.

On the other hand if $\sigma(\,\cdot\,)$ is not constant, one needs to examine condition (23) of [LR] to verify if coupling occurs. Since we not only want to see the finiteness of the coupling time, but also its ε-dependence, we need to make a further

comparison, and we do not enter this.

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