A Mourre estimate for a Schrödinger operator on a binary tree

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Abstract

Let G be a binary tree with vertices V and let H be a Schrödinger operator acting on $\ell^2(V)$. A decomposition of the space $\ell^2(V)$ into invariant subspaces is exhibited yielding a conjugate operator A for use in the Mourre estimate. We show that for potentials q satisfying a first order difference decay condition, a Mourre estimate for H holds.

Introduction

Let $G = (V, E)$ be a graph with vertices V and edges E. The Laplace operator acts on functions defined on V. If $\phi : V \to \mathbb{C}$ is such a function, then $\Delta\phi$ is the function defined by

$$(\Delta\phi)(v) = \sum_{w:w-v} (\phi(w) - \phi(v)),$$

where $w - v$ means that v and w are connected by an edge. We are interested in the spectral theory of $-\Delta$ and perturbations $-\Delta + q$, acting in the Hilbert space $\ell^2(V)$ of square summable functions on V. This is the space of functions ϕ satisfying

$$\sum_{v\in V} |\phi(v)|^2 < \infty$$

with inner product given by

$$\langle \phi, \psi \rangle = \sum_{v\in V} \overline{\phi}(v)\psi(v).$$

Let L denote the off-diagonal part of Δ. Thus

$$(L\phi)(v) = \sum_{w:w-v} \phi(w).$$

If $d(v)$ denotes the number of edges joined to the vertex v then

$$\Delta = L - d$$

where d is the operator of multiplication by $d(v)$. The degree term d can be included in the potential as a perturbation, hence $-\Delta + q = -L + d + q$ can be considered as a perturbation of L. Both Δ and L are symmetric on $\ell^2(V)$. When $d(v)$ is bounded, then both operators are also bounded operators, hence self-adjoint.

The goal of this paper is to prove a Mourre estimate and related bounds for the Schrödinger operator $-\Delta + q$ when the underlying graph is a binary tree. Here q denotes multiplication by a potential function that tends to zero at infinity. For a binary tree, $d = 3 - d_0$, where d_0 is the potential with $d_0(v) = 1$ at the root of the tree and 0 otherwise. Hence $-\Delta + q = -L + 3 - d_0 + q$ and the spectrum of $-\Delta + q$ is the same as that of $L - q + d_0$ up to a shift and a reflection about zero. In considering the Mourre estimate, the d_0 term can be absorbed in q and the sign of the potential changed, since $-q + d_0$ satisfies our decay assumptions whenever q does. Hence we aim at obtaining a Mourre estimate for $L + q$.

The operator L can be diagonalized explicitly. Its spectrum is absolutely continuous and equal $\sigma(L) = \sigma_{\mathrm{ac}}(L) = [-2\sqrt{2}, 2\sqrt{2}]$. This is also the essential spectrum of L. Since q is a compact operator, perturbation by q does not change the essential spectrum, and so $\sigma_{\mathrm{ess}}(L + q) = [-2\sqrt{2}, 2\sqrt{2}]$.

We will define a self-adjoint conjugate operator A such that, under appropriate conditions on the potential q

(i) $[L + q, iA]$ is bounded

(ii) $[[L + q, iA], iA]$ is bounded

(iii) $L + q$ and A satisfy a Mourre estimate at every point in $(-2\sqrt{2}, 2\sqrt{2})$

By definition (iii) means that for every $\lambda \in (-2\sqrt{2}, 2\sqrt{2})$, there exist an interval I containing λ such that

$$E_I[L + q, iA]E_I \geq \alpha E_I^2 + K$$

Here $E_I = E_I(L + q)$ denotes the (possibly smoothed) spectral projection corresponding to the interval I, α is a positive number, and K is a compact operator. Precise statements can be found in Lemma 5, Lemma 6, Lemma 7 and Theorem 9 below.

The estimates (i) (ii) and (iii) together with the abstract Mourre theory (see for example [1]), have the following consequences:

(1) Eigenvalues of $L + q$ not equal to $\pm 2\sqrt{2}$ have finite multiplicity and can only accumulate at $\pm 2\sqrt{2}$.

(2) The operator $L + q$ has no singular continuous spectrum.

(3) Scattering for the pair L and $L + q$ is asymptotically complete, see [2].

Although we only treat the binary tree, the same method can be applied to related graphs, for example the Bethe Lattice or trees with k-fold branching. Schrödinger operators on the Bethe Lattice are of interest in solid state physics, where they serve as a model for tightly bound electrons. Much effort has gone into studying operators with random potentials, and it is interesting to note that although the existence of dense point spectrum near the band edges has been proven in many situations, the Bethe Lattice is the only model where it has been proved that for weak disorder, some absolutely continuous spectrum remains in the middle of the band [3]. From the purely mathematical point of view, the Bethe Lattice is the Cayley graph of a free group. It would be most interesting to be able to say something about the continuous spectrum of the Laplace operator on the Cayley graph for a finitely generated group that is not free, and to relate properties of the spectrum to properties of the group.

The subspace decomposition and conjugate operator used in this paper bear some similarity to the ones used in [4] in the case of exponentially large manifolds. However, the details are quite different. In particular, the calculation of the matrix elements of A and the method of estimating $[q, iA]$ have no analogue. These results first appeared in [2].

The operators Π and R

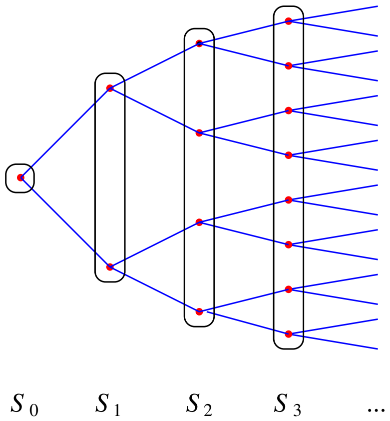
In this section we will let (V, E) be an arbitrary graph and introduce polar co-ordinates and some associated operators. Choose some $0 \in V$ to be the origin. Define $|v|$ to be the distance in the graph from 0 to v. In other words, $|v|$ is the length of the shortest path in the graph joining 0 to v. Define S_r, the sphere of radius r, to be the set of all vertices with $|v| = r$. Then V is a disjoint union

$$V = \bigcup_{r=0}^{\infty} S_r$$

and

$$\ell^2(V) = \bigoplus_{r=0}^{\infty} \ell^2(S_r)$$

In the case of the binary tree see figure below.



Spheres in a binary tree

We will write $v \to w$ if v and w are connected by an edge and $|w| = |v| + 1$. Define

$$(\Pi\phi)(v) = \sum_{w:w\to v} \phi(w)$$

The adjoint of Π can be computed by calculating

$$\langle \psi, \Pi\phi \rangle = \sum_{v} \sum_{w:w\to v} \overline{\psi}(v)\phi(w)$$

This can be interpreted as a sum over all edges joining neighboring spheres, where $\overline{\psi}$ and ϕ are evaluated at the right and left endpoint of the edge respectively. We have chosen to label the edges by their right endpoint v, and the sum over $w : w \to v$ accounts for the possibility of several edges having the same right endpoint. If we choose to label the edges by their left endpoints instead we find that the same sum can be written as

$$\langle \psi, \Pi\phi \rangle = \sum_{v} \sum_{w:v\to w} \overline{\psi}(w)\phi(v)$$

This shows that

$$(\Pi^*\phi)(v) = \sum_{w:v\to w} \phi(w)$$

Notice that $\Pi^*\Pi$ and $\Pi\Pi^*$ leave each $\ell^2(S_r)$ invariant. The action of $\Pi^*\Pi$ is given by

$$(\Pi^*\Pi\phi)(v) = \sum_w \phi(w) \tag{1}$$

where the sum is extended over $w \in S_{|v|}$ that are joined to v by a path in the graph of length two going from v to some element in $S_{|v|+1}$ and then back to $S_{|v|}$. The formula for $\Pi\Pi^*$ is analogous, except that the path goes in the other direction to $S_{|v|-1}$ and back.

We will denote by R the operator of multiplication by $|v|$. When restricted to $\ell^2(S_r)$, the operator R is multiplication by r. An easy calculation shows that

$$[R, \Pi] = \Pi \tag{2}$$

We may write L in terms of Π and Π^* by breaking the sum in the definition of L into three pieces. We obtain

$$L = \Pi + \Pi^* + L_S$$

where the spherical Laplacian L_S is defined by

$$(L_S\phi)(v) = \sum_{w: \substack{w-v \\ |w|=|v|}} \phi(w).$$

Diagonalization of L and definition of A for a Binary Tree

In this section we will exhibit a diagonalization of the off-diagonal Laplacian L on a binary tree.

Choose the origin to be the base of the tree and introduce polar co-ordinates. Since there are no edges that connect vertices within each sphere, $L_S = 0$, and

$$L = \Pi + \Pi^*.$$

We now construct invariant subspaces M_n for Π. Let $Q_{0,0} = \ell^2(S_0)$ and define $Q_{0,r} = \Pi^r Q_{0,0}$. Let

$$M_0 = \bigoplus_{r=0}^{\infty} Q_{0,r}.$$

To define $Q_{n,r}$ and M_n for $n > 0$ we proceed recursively. Suppose that $Q_{m,s}$ have been defined whenever $m < n$ and $s \geq m$. Let $Q_{n,n}$ be the orthogonal complement in $\ell^2(S_n)$ of all previously defined subspaces,

$$Q_{n,n} = \ell^2(S_n) \ominus (Q_{0,n} \oplus \cdots \oplus Q_{n-1,n}).$$

For $r = n + j$, $j > 0$, define $Q_{n,r} = \Pi^j Q_{n,n}$, and

$$M_n = \bigoplus_{r=n}^{\infty} Q_{n,r} = \bigoplus_{j=0}^{\infty} Q_{n,n+j}.$$

Schematically this gives:

$$
\begin{array}{c|ccccc}
 & \ell^2(S_0) & \ell^2(S_1) & \ell^2(S_2) & \ell^2(S_3) & \\
\hline
M_0 & Q_{0,0} & Q_{0,1} & Q_{0,2} & Q_{0,3} & \cdots \\
M_1 & & Q_{1,1} & Q_{1,2} & Q_{1,3} & \cdots \\
M_2 & & & Q_{2,2} & Q_{2,3} & \cdots \\
\vdots & & & & \vdots &
\end{array}
$$

Orthogonal Subspace decomposition

Lemma 1 *The Hilbert space $\ell^2(V)$ can be written as an orthogonal direct sum*

$$\ell^2(V) = \bigoplus_{n=0}^{\infty} M_n = \bigoplus_{n=0}^{\infty}\bigoplus_{r=n}^{\infty} Q_{n,r}.$$

The subspaces M_n are invariant for L

Proof: Since Π maps $\ell^2(S_r)$ to $\ell^2(S_{r+1})$ it follows that each $Q_{n,r}$ is contained in $\ell^2(S_r)$. Thus, if $r \neq s$, then $Q_{n,r}$ and $Q_{m,s}$ are orthogonal for all n and m.

For a binary tree, it follows from (1) that

$$\Pi^*\Pi = 2I \tag{3}$$

This implies that if ϕ and ψ are orthogonal, then $\Pi\phi$ and $\Pi\psi$ are orthogonal too. Since $Q_{n,n}$ is orthogonal to $Q_{m,n}$, for $m < n$ by construction, it follows that $Q_{n,r}$ and $Q_{m,r}$, for $r \geq n$ are orthogonal too. By construction $\bigoplus_{l=0}^{r} Q_{l,r} = \ell^2(S_r)$, so it is clear that the subspaces add up to $\ell^2(V)$.

By construction, each M_n is invariant for Π. That they are invariant for Π^* follows from (3) as follows. It suffices to show that each $Q_{n,r}$ is mapped to M_n under Π^*. Suppose that $\phi \in Q_{n,r}$ for $r = n+j$, $j \geq 1$. Then $\phi = \Pi^j\chi$ for $\chi \in Q_{n,n}$. Hence $\Pi^*\phi = 2\Pi^{j-1}\chi \in Q_{n,r-1}$. On the other hand, if $\phi \in Q_{n,n}$, then $\Pi^*\phi \in \ell^2(S_{n-1})$. Suppose that $\psi \in Q_{l,n-1}$ for some $l \leq n-1$. Since $\Pi\psi \in Q_{l,n}$ for some $l \leq n-1$ and $\phi \in Q_{n,n}$, we have $\langle\psi, \Pi^*\phi\rangle = \langle\Pi\psi, \phi\rangle = 0$. Hence $\Pi^*\phi$ is orthogonal to each $Q_{l,n-1}$, which implies that $\Pi^*\phi = 0$. Thus each M_n is invariant for Π and Π^*, and hence for L. $\square$

Since each M_n is an invariant subspace for L we can decompose $L = \oplus_{n=0}^{\infty} L_n$, where L_n is the restriction of L to M_n. We now diagonalize L_n.

We begin by writing a vector ϕ in M_n as

$$\phi = \oplus_{j=0}^{\infty}\phi_{n+j}$$

where $\phi_{n+j} \in Q_{n,n+j}$. We want to obtain an isomorphism between M_n and $l^2(\mathbb{Z}^+, Q_{n,n})$ the space of $Q_{n,n}$ valued sequences.

We first note that $(\frac{1}{\sqrt{2}}\Pi)^j$ (not Π^j) defines an isometry between $Q_{n,n}$ and $Q_{n,n+j}$ for all j and that any $\phi \in M_n$ can be written as

$$\phi = \oplus_{j=0}^{\infty} \left(\frac{1}{\sqrt{2}}\Pi\right)^j \chi_{n+j} \tag{4}$$

for a sequence of vectors $\chi_n, \chi_{n+1}, \ldots \in Q_{n,n}$.

Under this representation M_n and $l^2(\mathbb{Z}^+, Q_{n,n})$ are isomorphic, since

$$\langle \phi, \phi \rangle = \sum_{j=0}^{\infty} \langle \phi_{n+j}, \phi_{n+j} \rangle = \sum_{j=0}^{\infty} \langle \chi_{n+j}, \chi_{n+j} \rangle = \langle W\phi, W\phi \rangle$$

by the above isometry where W denotes the isomorphism.

In this representation, the operator $\frac{1}{\sqrt{2}}\Pi$ acts as a shift to the right, while $\frac{1}{\sqrt{2}}\Pi^*$ is a shift to the left with kernel $Q_{n,n}$.

Now let

$$U : M_n \cong \ell^2(\mathbb{Z}^+, Q_{n,n}) \to L^2_{\text{odd}}([-\pi, \pi], d\theta)$$

denote the unitary map defined by

$$U((\alpha_n, \alpha_{n+1}, \ldots)) = \frac{1}{\sqrt{\pi}} \sum_{j=0}^{\infty} \alpha_{n+j} \sin((j+1)\theta)$$

Lemma 2

$$UL_nU^* = 2\sqrt{2}\cos(\theta)$$

Proof: The proof is a straightforward calculation. □

This lemma shows that the spectrum of L_n is $[-2\sqrt{2}, 2\sqrt{2}]$, and is absolutely continuous. Thus the spectrum of L is also $[-2\sqrt{2}, 2\sqrt{2}]$ with infinite multiplicity.

This representation motivates the choice of a conjugate operator. For a general multiplication operator $\omega(\theta)$, a natural conjugate operator is $A_\omega = \frac{i}{2}(\omega' \frac{d}{d\theta} + \frac{d}{d\theta}\omega')$, since

$$[\omega, iA_\omega] = \omega'^2$$

which is positive away from the critical points of ω.

In the present case the natural conjugate operator is therefore

$$UA_nU^* = -i\sqrt{2}\left(\sin(\theta)\frac{d}{d\theta} + \frac{d}{d\theta}\sin(\theta)\right)$$

and a calculation now shows that on M_n

$$iA_n = U^*\sqrt{2}\left(\sin(\theta)\frac{d}{d\theta} + \frac{d}{d\theta}\sin(\theta)\right)U = (R - n + \frac{1}{2})\Pi - \Pi^*(R - n + \frac{1}{2})$$

Therefore a natural conjugate operator for L is $\oplus_{n=0}^{\infty} A_n$. If we let P_n denote the projection onto M_n, and define

$$N = \sum_{n=1}^{\infty} nP_n,$$

the conjugate operator for L can be written as

$$iA = (R - N + \frac{1}{2})\Pi - \Pi^*(R - N + \frac{1}{2})$$

Matrix elements of A

We will need estimates on the matrix elements of A. Let δ_w denote the standard basis element in $\ell^2(V)$ defined by $\delta_w(v) = \delta_{w,v}$. We wish to estimate the matrix elements $\langle \delta_v, iA\delta_w \rangle$.

Using the formula

$$\Pi\delta_w = \sum_{z:w\to z} \delta_z$$

we find that

$$\begin{aligned} \langle \delta_v, iA\delta_w \rangle &= \langle \delta_v, \left((R - N + \frac{1}{2})\Pi - \Pi^*(R - N + \frac{1}{2}) \right) \delta_w \rangle \\ &= \begin{cases} (|v| + \frac{1}{2})\delta(w \to v) - \sum_{z:w\to z} \langle \delta_v, N\delta_z \rangle & \text{if} \quad |w| = |v| - 1 \\ -(|w| + \frac{1}{2})\delta(v \to w) + \sum_{z:v\to z} \langle \delta_z, N\delta_w \rangle & \text{if} \quad |w| = |v| + 1 \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{5}$$

Here $\delta(w \to v)$ is equal to 1 if $w \to v$ and 0 otherwise.

To estimate the matrix elements of N appearing in this formula, we introduce at this point an explicit basis for each $Q_{n,r}$. When $n = 0$ we have that $Q_{0,r} = \Pi^r \ell^2(S_0)$ is one dimensional and consists of all vectors $\phi(v)$ in $\ell^2(S_r)$ such that $\phi(v)$ has the same value for all v. An orthonormal basis for $Q_{0,r}$ is therefore the single vector

$$\rho_{0,r,0} = 2^{-r/2}[1, 1, \ldots, 1]$$

The space $Q_{1,1}$ is the orthogonal complement in $\ell^2(S_1)$ of $Q_{0,1}$. Thus $Q_{1,1}$ is also one dimensional and has orthonormal basis

$$\rho_{1,1,0} = 2^{-1/2}[1, -1]$$

Pushing the vector forward along the tree using Π and normalizing gives

$$\rho_{1,r,0} = 2^{-r/2}[1, 1, \ldots, 1, -1, -1, \ldots, -1]$$

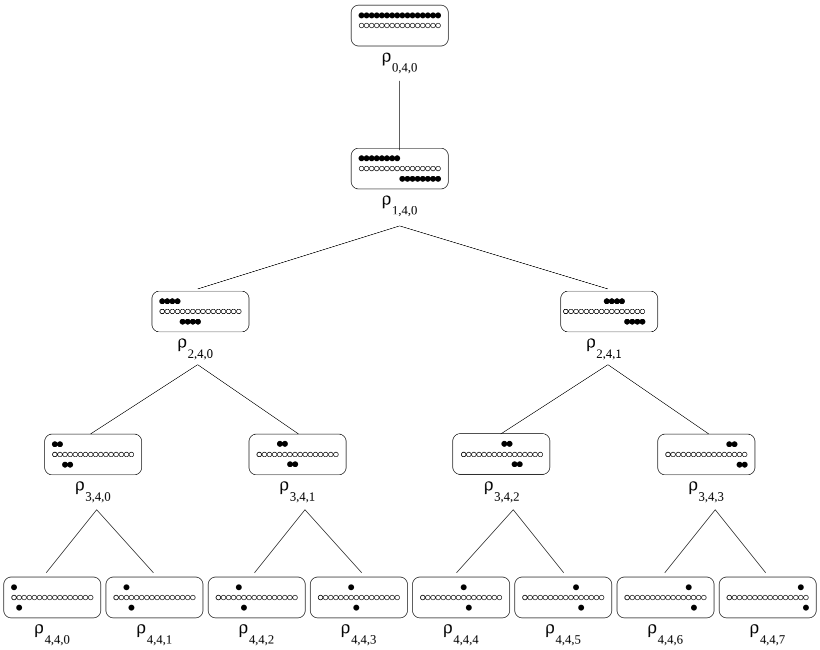
as a basis for $Q_{1,r}$ where half the entries are 1 and the other half -1.

The space $Q_{2,2}$ is the orthogonal complement in $\ell^2(S_2)$ of $Q_{0,1} \oplus Q_{1,1}$. Since $\dim(\ell^2(S_2)) = 4$ the space $Q_{2,2}$ is two dimensional, it has orthogonal basis

$$\rho_{2,2,0} = 2^{-1/2}[1, -1, 0, 0] \qquad \rho_{2,2,1} = 2^{-1/2}[0, 0, 1, -1]$$

Pushing these vectors forward along the tree using Π and normalizing yields $\rho_{2,r,0}$ and $\rho_{2,r,1}$.

Continuing in this fashion we define the orthogonal basis $\rho_{n,r,k}$ with $k = 0, \ldots, 2^{\max\{n-1,0\}} - 1$. When we fix the second index r, the vectors $\rho_{n,r,k}$ are the Haar basis for $\ell^2(S_r)$. Upon defining a partial order on the Haar basis elements for $\ell^2(S_r)$ using inclusion of supports, the Haar basis functions, as illustrated on the next page for $r = 4$, naturally form a binary tree with r levels, extended by an extra vertex at its base.



The Haar basis

Lemma 3 *Let z,$w \in S_r$. If $z \neq w$, let $N(z,w)$ denote the largest value of n for which both z and w lie in the support of a single basis function. Then*

$$\langle \delta_z, N\delta_w \rangle = \begin{cases} r - 1 + 2^{-r} & \text{if} \quad z = w \\ -2^{N(z,w)-r} + 2^{-r} & \text{if} \quad z \neq w \end{cases}$$

Proof: Fix r and label the elements of the Haar basis by the vertices α in the associated (extended) basis binary tree. Then

$$N\delta_w = \sum_\alpha n(\alpha)\rho_\alpha(w)\rho_\alpha$$

where $n(\alpha)$ denotes the level of α in the tree and the sum is taken over the one α at each level for which $\rho_\alpha(w) \neq 0$

Suppose $z = w$. Then, for the one α at level $n(\alpha) = n$, for which $\rho_\alpha(w) \neq 0$, we have $\rho_\alpha(w)^2 = 2^{-r+n-1}$. Thus

$$\langle \delta_w, N\delta_w \rangle = \sum_{n=1}^{r} n 2^{-r+n-1} = r - 1 + 2^{-r}$$

Now suppose $z \neq w$. Then for the one α at level n for which $\rho_\alpha(w) \neq 0$ we have

$$\rho_\alpha(z) = \begin{cases} -\rho_\alpha(w) & \text{if } n = N(z,w) \\ \rho_\alpha(w) & \text{if } n < N(z,w) \\ 0 & n > N(z,w) \end{cases}$$

Hence

$$\begin{aligned} \langle \delta_z, N\delta_w \rangle &= \sum_{n=1}^{N(z,w)-1} n 2^{-r+n-1} - N(z,w) 2^{-r+N(z,w)-1} \\ &= -2^{-r+N(z,w)} + 2^{-r} \end{aligned}$$

□

Lemma 4
$$\sum_w |\langle \delta_v, iA\delta_w\rangle| = O(|v|)$$

Proof: Using (5) we find

$$\begin{aligned}\sum_w |\langle \delta_v, iA\delta_w\rangle| \;\le\; & \sum_{w:|w|=|v|-1}\left(|v|+\frac{1}{2}\right)\delta(w\to v) + \sum_{w:|w|=|v|-1}\sum_{z:w\to z}|\langle \delta_v, N\delta_z\rangle| \\ & + \sum_{w:|w|=|v|+1}\left(|w|+\frac{1}{2}\right)\delta(v\to w) + \sum_{w:|w|=|v|+1}\sum_{z:v\to z}|\langle \delta_z, N\delta_w\rangle|\end{aligned} \tag{6}$$

Since there is only one w with $w\to v$ we have

$$\sum_{|w|=|v|-1}\left(|v|+\frac{1}{2}\right)\delta(w\to v) = |v|+\frac{1}{2}.$$

Since there are exactly two w with $v\to w$,

$$\sum_{|w|=|v|+1}\left(|w|+\frac{1}{2}\right)\delta(v\to w) = 2(|v|+1)+1.$$

To estimate the remaining two terms in (6), we begin with

$$\begin{aligned}\sum_{z:|z|=|v|}|\langle \delta_v, N\delta_z\rangle| &= |\langle \delta_v, N\delta_v\rangle| + \sum_{z:\substack{|z|=|v|\\ z\neq v}}|\langle \delta_v, N\delta_z\rangle| \\ &= |v|-1+2^{-|v|} + \sum_{z:\substack{|z|=|v|\\ z\neq v}} 2^{N(v,z)-|v|} - 2^{-|v|} \\ &\le |v| + \sum_{z:\substack{|z|=|v|\\ z\neq v}} 2^{N(v,z)-|v|}\end{aligned} \tag{7}$$

Since there are $2^{|v|-N(v,z)}$ z's associated to each value of $N(v,z)=n$

$$\begin{aligned}\sum_{z:|z|=|v|}|\langle \delta_v, N\delta_z\rangle| &\le |v| + \sum_{n=1}^{|v|} 2^{n-|v|}\cdot \sum_{z:|z|=|v|,z\neq v,N(v,z)=n} 1 \\ &= |v| + \sum_{n=1}^{|v|} 2^{n-|v|}2^{|v|-n} \\ &= 2|v|\end{aligned}$$

Therefore
$$\sum_{w:|w|=|v|-1}\sum_{z:w\to z}|\langle \delta_v, N\delta_z\rangle| = \sum_{z:|z|=|v|}|\langle \delta_v, N\delta_z\rangle| \le 2|v|$$

and
$$\sum_{w:|w|=|v|+1}\sum_{z:v\to z}|\langle \delta_z, N\delta_w\rangle| = \sum_{z:v\to z}\sum_{w:|w|=|z|}|\langle \delta_z, N\delta_w\rangle| \le \sum_{z:v\to z} 2|v| = 4|v|.$$

Thus each term in (6) is $O(|v|)$ and the proof is complete. □

The Mourre estimate

We begin with the commutator formula for L. This is just a disguised form of the formula

$$[2\sqrt{2}\cos(\theta), \sqrt{2}(\sin(\theta)\frac{d}{d\theta} + \frac{d}{d\theta}\sin(\theta))] = 8\sin^2(\theta).$$

Lemma 5

$$[L, iA] = 8 - L^2$$

Proof: Since Π and Π^* commute with N, we have

$$\begin{aligned}
[L, iA] &= [\Pi + \Pi^*, (R - N + \tfrac{1}{2})\Pi] + \text{ adjoint} \\
&= [\Pi, (R - N + \tfrac{1}{2})]\Pi + [\Pi^*, (R - N + \tfrac{1}{2})]\Pi + (R - N + \tfrac{1}{2})[\Pi^*, \Pi] + \text{ adjoint} \\
&= [\Pi, R]\Pi + [\Pi^*, R]\Pi + (R - N + \tfrac{1}{2})[\Pi^*, \Pi] + \text{ adjoint} \\
&= -\Pi^2 + \Pi^*\Pi + (R - N + \tfrac{1}{2})[\Pi^*, \Pi] + \text{ adjoint}
\end{aligned}$$

where adjoint applies to all the previous terms. Here we used (2). Now notice that $[\Pi^*, \Pi]$ is a projection onto the sum of the initial subspaces $Q_{n,n}$ and that is precisely the kernel of the operator $R - N$ since on this sum $R = N$. Thus $(R - N)[\Pi^*, \Pi] = 0$ and,

$$\begin{aligned}
[L, iA] &= -\Pi^2 + \Pi^*\Pi + \tfrac{1}{2}[\Pi^*, \Pi] + \text{ adjoint} \\
&= -\Pi^2 - (\Pi^*)^2 + 3\Pi^*\Pi - \Pi\Pi^* \\
&= 4\Pi^*\Pi - (\Pi + \Pi^*)^2 \\
&= 8 - L^2
\end{aligned}$$

□

Lemma 6 *Suppose that*

$$\sup_{w:|w|=|v|\pm 1} |q(v) - q(w)| = o(|v|^{-1}),$$

as $|v| \to \infty$, then $[q, iA]$ is compact.

Proof: Let Λ_n denote the projection onto $\oplus_{r=0}^n l^2(S_r)$. We will show that $||[q, iA] - [q, iA]\Lambda_n|| = ||[q, iA](1 - \Lambda_n)|| \to 0$ as $n \to \infty$. This shows that $[q, iA]$ is approximated in norm by the finite rank operator $[q, iA]\Lambda_n$, and hence compact.

The matrix elements of $[q, iA](1 - \Lambda_n)$ are given by

$$\langle \delta_v, [q, iA](1 - \Lambda_n)\delta_w \rangle = (q(v) - q(w))\langle \delta_v, iA\delta_w \rangle$$

provided $|w| > n$, and 0 if $|w| \leq n$. Using Schur's lemma (the $\ell^1 - \ell^\infty$ bound), the fact that the matrix element of $\langle \delta_v, iA\delta_w \rangle$ are non-zero only for $|w| = |v| \pm 1$, the decay hypothesis on q, and Lemma 4 we find that

$$\begin{aligned}
||[q, iA](1 - \Lambda_n)|| &\leq \sup_v \sum_{w:|w|>n} |q(v) - q(w)||\langle \delta_v, iA\delta_w \rangle| \\
&\leq \sup_{v:|v|>n-1} o(|v|^{-1}) \sum_w |\langle \delta_v, iA\delta_w \rangle| \\
&\leq \sup_{v:|v|>n-1} o(|v|^{-1}) O(|v|)
\end{aligned}$$

which tends to zero for large n. □

Lemma 7 *Suppose that*

$$\sup_{\substack{w:|w|=|v|\pm 1\\ z:|z|=|v|\pm 2}} |q(v)+q(z)-2q(w)| = O(|v|^{-2}),$$

as $|v|\to\infty$*, then* $[[q,iA],iA]$ *is bounded.*

Proof: The matrix elements of $[[q,iA],iA]$ are given by

$$\begin{aligned}\langle\delta_v,[[q,iA],iA]\delta_z\rangle &= \langle\delta_v,[q,iA]iA-iA[q,iA]\delta_z\rangle\\ &= \sum_w \langle\delta_v,[q,iA]\delta_w\rangle\langle\delta_w,iA\delta_z\rangle-\langle\delta_v,iA\delta_w\rangle\langle\delta_w,[q,iA]\delta_z\rangle\\ &= \sum_w (q(v)+q(z)-2q(w))\langle\delta_v,iA\delta_w\rangle\langle\delta_w,iA\delta_z\rangle\end{aligned}.$$

Thus as in Lemma 6,

$$\begin{aligned}\|[[q,iA],iA]\| &\le \sup_v \sum_z\sum_w |q(v)+q(z)-2q(w)||\langle\delta_v,iA\delta_w\rangle||\langle\delta_w,iA\delta_z\rangle|\\ &\le \sup_v O(|v|^{-2})\sum_w|\langle\delta_v,iA\delta_w\rangle|\sum_z|\langle\delta_w,iA\delta_z\rangle|\\ &= \sup_v O(|v|^{-2})O(|v|)O(|v|)\le C\end{aligned}$$

□

Lemma 8 *Suppose that* $q(v)\to 0$ *as* $|v|\to\infty$*. Let* E *denote a smoothed out spectral projection. Then* $E(L)-E(L+q)$ *is compact.*

Proof: This follows from the compactness of $(L-z)^{-1}-(L+q-z)^{-1}=(L-z)^{-1}q(L+q-z)^{-1}$ and a Stone-Weierstrass approximation argument (see [1]). □

Now we can prove the Mourre estimate for $L+q$ and A.

Theorem 9 *Suppose that* $q(v)\to 0$ *as* $|v|\to\infty$*. Assume that*

$$\sup_{w:|w|=|v|\pm 1}|q(v)-q(w)| = o(|v|^{-1}).$$

as $|v|\to\infty$*. Let* E *denote a smoothed out spectral projection whose support is properly contained in the interval* $(-2\sqrt{2},2\sqrt{2})$*. Then there exists a compact operator* K *and a positive number* α *such that*

$$E(L+q)[L+q,iA]E(L+q)\ge\alpha E^2(L+q)+K.$$

Proof: By compactness of $[q,iA]$ and $E(L+q)-E(L)$ we have

$$\begin{aligned}E(L+q)[L+q,iA]E(L+q) &= E(L)[L,iA]E(L)+K\\ &= E(L)(8-L^2)E(L)+K\end{aligned}$$

On the support of $E(L)$, $8-L^2\ge\alpha$ for some positive α, which gives

$$E(L+q)[L+q,iA]E(L+q)\ge\alpha E^2(L+q)+K$$

where the compact term $E^2(L)-E^2(L+q)$ has been added into K, this completes the proof. □

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